Explicit Remainder Terms for Some Asymptotic Series

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Summary of results and notation. AIREY and STIELTJES have indicated how to write asymptotic series (not always of the Poincaré type) for the remainder terms of the well known asymptotic series for the incomplete gamma function and related functions. As an illustration of some general techniques explained in the early part of our text, we give bounds on the errors for the series of AIREY and STIELTJES; in addition, we provide improved techniques for computing higher terms of the series.

We use t, u, v, w, x as variables to denote complex numbers, and j, k, l, m, n as variables (or parameters) to denote integers. M is the parameter of (1) or (24) in terms of which one seeks an asymptotic series. We preempt $r, A, B, h(x)$, and $H(x)$ for their roles in (1) and (24); ϕ for the unique constant determined by assumption (c) of Section 2; y, z , and Y as the parameters of the incomplete gamma function (first formula of Section 3) and of (48); $N = m + 1 - z$; $g_n(y)$ defined by (31) and (32); $\gamma = y/N$; $h_n(x)$ defined by (34) and (35); $\delta = \gamma + \log \gamma$; $\gamma(t)$ defined by (41); $C(w)$ defined by (54); α defined by (59); $x(\alpha)$ defined by (63) so that x as a function of α shall satisfy (59); A_j defined by (78); D_j defined by 82. There are also many transient notations that we have not listed, because they appear only in limited contexts.

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1. The exponential case. Consider

$$(1) \quad \int_0^\infty f(x) x^r e^{-Mx} dx,$$

where the real part of r is greater than -1 , and $f(x)$ is analytic in the neighborhood of the origin and has a behavior as $x \rightarrow \infty$ such that (1) is absolutely integrable for all M with real part greater than some M_0 . To get an asymptotic series in M for (1) the standard procedure is to expand $f(x)$ as a power series in x and integrate term-by-term, getting the formal result

$$(2) \quad \int_0^\infty f(x) x^r e^{-Mx} dx = \sum_{m=0}^\infty \frac{\Gamma(r+m+1)}{M^{r+m+1}} \frac{f^{(m)}(0)}{m!}.$$

With this procedure one can estimate the error committed in using several terms of the series in (2) by first estimating the error committed in using several terms of the power series for $f(x)$. Indeed, one can demonstrate by this means that the series in (2) is asymptotic in the sense of POINCARÉ (slightly generalized if r is not an integer). If $f(x)$ is one of the elementary functions, one can usually get quite satisfactory estimates by this procedure, and without much complication. For more complicated functions, this procedure necessarily involves some complications (see Section 5 of this paper, for example) and can be extremely complicated (see [1], for example). Accordingly we present two alternative procedures which have advantages in various cases.

For the first alternative procedure, we define

$$(3) \quad \phi_0(x) = x^r e^{-x},$$

$$(4) \quad \phi_{n+1}(x) = \int_x^\infty \phi_n(t) dt.$$

For M in a suitable angle containing the positive real axis, we can make a change of variable and then rotate the line of integration to get

$$\int_0^\infty f(x) x^r e^{-Mx} dx = M^{-r-1} \int_0^\infty f(t/M) \phi_0(t) dt.$$

Successive integrations by parts give

$$(5) \quad \begin{aligned} \int_0^\infty f(x) x^r e^{-Mx} dx = M^{-r-1} & \left\{ f(0) \phi_1(0) + \frac{f'(0) \phi_2(0)}{M} + \cdots + \frac{f^{(m-1)}(0) \phi_m(0)}{M^{m-1}} \right\} \\ & + M^{-m-r-1} \int_0^\infty f^{(m)}(t/M) \phi_m(t) dt. \end{aligned}$$

To determine the coefficients of this series, one establishes the relation

$$n\phi_{n+1}(x) = -(x - r - n)\phi_n(x) + x\phi_{n-1}(x)$$

for $n \geq 1$ by induction on n . Then one verifies that the values of $\phi_n(0)$ are such that (5) has the same coefficients as the series in (2).

To estimate the remainder in (5), one must determine the behavior of $\phi_m(x)$, which may be complicated if r is complex. However, in the very common case that $-1 < r \leq 0$, an integration by parts gives for $0 < x$

$$\phi_1(x) \leq \phi_0(x),$$

whence by induction

$$(6) \quad \phi_{n+1}(x) \leq \phi_n(x)$$

holds for $0 < x$ and $0 \leq n$, so that for $0 < x$ and $0 \leq n$,

$$(7) \quad \phi_n(x) \leq x^r e^{-x}.$$

By (6) and (4), we have for $0 < x$ and $0 \leq n$

$$\frac{d}{dx} \{e^x \phi_{n+1}(x)\} \leq 0.$$

Thus for $n \geq 1$ and $0 \leq x$,

$$(8) \quad \phi_n(x) \leq e^{-x} \phi_n(0).$$

For the second alternative procedure, we define

$$(9) \quad f_0(x) = f(x),$$

$$(10) \quad f_{n+1}(x) = x^{-r} \frac{d}{dx} \{f_n(x) - f_n(0)\} x^r.$$

Clearly, since $f(x)$ is analytic in the neighborhood of the origin, so is each of $f_n(x)$. An integration by parts gives the recurrence relation

$$\begin{aligned} \int_0^\infty f_n(x) x^r e^{-Mx} dx &= f_n(0) \int_0^\infty x^r e^{-Mx} dx + \int_0^\infty \{f_n(x) - f_n(0)\} x^r e^{-Mx} dx \\ &= f_n(0) \frac{\Gamma(r+1)}{M^{r+1}} + M^{-1} \int_0^\infty f_{n+1}(x) x^r e^{-Mx} dx. \end{aligned}$$

Thus we get

$$\begin{aligned} (11) \quad \int_0^\infty f(x) x^r e^{-Mx} dx &= \frac{\Gamma(r+1)}{M^{r+1}} \left\{ f_0(0) + \frac{f_1(0)}{M} + \dots + \frac{f_{m-1}(0)}{M^{m-1}} \right\} \\ &\quad + M^{-m} \int_0^\infty f_m(x) x^r e^{-Mx} dx. \end{aligned}$$

This series has the same coefficients as the series in (2), as one verifies by determining the series expansion for each $f_n(x)$.

If $r = 0$, then $\phi_n(x) = e^{-x}$ and $f_n(x) = f^{(n)}(x)$, so that the two alternative procedures are identical.

If $r \neq 0$ and $f(x)$ is an elementary function whose derivatives are known and rather simple, then $f_m(x)$ will be more complicated than $f^{(m)}(x)$, and the first alternative procedure is to be preferred when r and M are real. However, in more complicated situations, or when r and M are complex, the second alternative procedure is often preferable, as we indicate below. Before proceeding to these more complicated situations, we take a quick look at the case of

$$(12) \quad \int_0^\infty f(x) e^{-Mx^2} dx.$$

If $f(x)$ is either an even function or an odd function, this may be treated as a special case of (1) with $r = -\frac{1}{2}$ or 0 respectively. Thus one could reduce (12) to two instances of (1) by writing $f(x)$ as the sum of an odd and even function. However, it is easier to proceed directly.

One has of course the standard procedure of expanding $f(x)$ as a power series in x and integrating term-by-term. One has also the analogue of our first alternative procedure. Define

$$(13) \quad \theta_0(x) = e^{-x^2},$$

$$(14) \quad \theta_{n+1}(x) = \int_0^\infty \theta_n(t) dt.$$

Then integration by parts gives

$$(15) \quad \begin{aligned} \int_0^\infty f(x) e^{-Mx^2} dx &= \frac{f(0)\theta_1(0)}{M^{\frac{1}{2}}} + \frac{f'(0)\theta_2(0)}{M} + \dots + \frac{f^{(m-1)}(0)\theta_m(0)}{M^{m/2}} \\ &\quad + M^{-(m+1)/2} \int_0^\infty f^{(m)}(tM^{-\frac{1}{2}})\theta_m(t) dt. \end{aligned}$$

To get bounds for $\theta_m(t)$, we integrate by parts to conclude

$$(16) \quad \theta_1(x) \leq \theta_0(x)/(2x)$$

for $0 < x$. Then an induction gives for $0 < x$ and $0 \leq n$

$$(17) \quad \theta_{n+1}(x) \leq \theta_n(x)/(2x),$$

so that for $0 < x$ and $0 \leq n$

$$(18) \quad \theta_n(x) \leq e^{-x^2}(2x)^{-n}.$$

Then for $0 < x$ and $0 \leq n$

$$\frac{d}{dx} \{e^{x^2} \theta_{n+1}(x)\} \leq 0,$$

so that for $0 \leq x$ and $0 \leq n$

$$(19) \quad \theta_n(x) \leq e^{-x^2} \theta_n(0).$$

For $n \geq 1$, one has also the relation

$$n\theta_{n+1}(x) = -x\theta_n(x) + \frac{1}{2}\theta_{n-1}(x)$$

for determining the values of $\theta_n(0)$. For $n \geq 0$ one can verify the relation

$$(20) \quad \theta_{n+1}(x) = \frac{(-1)^n}{2^n n!} e^{-x^2} \frac{d^n}{dx^n} \left\{ e^{x^2} \int_x^\infty e^{-t^2} dt \right\}.$$

The bounds furnished by (18) and (19) would suffice in many cases. However, if more precise information about $\theta_n(x)$ is desired, it may be obtained from (20) by referring to Section 8 of [2], where a thorough study is made of the derivative appearing on the right side of (20). Specifically, if M is complex, one may need to modify (15), and then one will require bounds on $\theta_m(t)$ for complex t .

For an analogue of the second alternative procedure, we define

$$(21) \quad f_0(x) = f(x),$$

$$(22) \quad f_{n+1}(x) = \frac{d}{dx} \left\{ \frac{f_n(x) - f_n(0)}{2x} \right\}.$$

Then an integration by parts gives the recurrence relation

$$\begin{aligned} \int_0^\infty f_n(x) e^{-Mx^2} dx &= f_n(0) \int_0^\infty e^{-Mx^2} dx + \int_0^\infty \left\{ \frac{f_n(x) - f_n(0)}{2x} \right\} 2xe^{-Mx^2} dx \\ &= \frac{1}{2}(\pi/M)^{\frac{1}{2}} f_n(0) + f'_n(0)/(2M) + M^{-1} \int_0^\infty f_{n+1}(x) e^{-Mx^2} dx. \end{aligned}$$

From this one gets

$$\begin{aligned} (23) \quad \int_0^\infty f(x) e^{-Mx^2} dx &= \frac{1}{2}(\pi/M)^{\frac{1}{2}} \left\{ f_0(0) + \frac{f_1(0)}{M} + \cdots + \frac{f_m(0)}{M^m} \right\} \\ &+ (2M)^{-1} \left\{ f'_0(0) + \frac{f'_1(0)}{M} + \cdots + \frac{f'_m(0)}{M^m} \right\} + M^{-m-1} \int_0^\infty f_{m+1}(x) e^{-Mx^2} dx, \end{aligned}$$

which can be verified to have the same coefficients as in (15).

2. The general case. Consider

$$(24) \quad \int_A^B h(x)H^M(x) dx.$$

We shall assume both $h(x)$ and $H(x)$ analytic for $A \leq x \leq B$. Actually, much less is required for our procedures, but we leave consideration of the various possible exceptional cases to those persons who actually encounter such cases. We also make certain additional assumptions on $H(x)$, namely:

(a) For $A \leq x \leq B$, we have $0 \leq H(x)$.

(b) There is at most one value of x with $A \leq x \leq B$ for which $H'(x) = 0$.

If $H(x)$ is real and these conditions are not satisfied, one can subdivide the interval (A, B) and replace (24) by the sum of several integrals, each of which satisfies assumptions (a) and (b).

The cases of real interest (and difficulty) are those for which there is an x with $A \leq x \leq B$ for which $H'(x) = 0$. However, for completeness, we give a quick résumé of the case where there is no such x . This will also give an illustration of the sorts of techniques available. In this case $H(x)$ must take its maximum value in the interval (A, B) at either A or B ; let us say at A . We can introduce the new variable

$$\xi = \log (H(A)/H(x)).$$

Expressing x in terms of ξ , say $x = x(\xi)$, we have

$$\int_A^B h(x)H^M(x) dx = H^M(A) \int_0^J \left\{ h(x(\xi)) \frac{dx}{d\xi} \right\} e^{-M\xi} d\xi,$$

where $J = \log (H(A)/H(B))$. We are thus reduced to (1) with $r = 0$. Actually, for application of our alternative procedures for treating (1) (which are the same in this case), it is not necessary to use ξ as the variable of integration. One can instead define

$$f_0(x) = -\frac{H(x)}{H'(x)} h(x),$$

$$f_{n+1}(x) = -\frac{H(x)}{H'(x)} \frac{d}{dx} f_n(x).$$

Because $H'(x) \neq 0$ for $A \leq x \leq B$, each $f_n(x)$ is analytic for $A \leq x \leq B$. Then

$$\int_A^B h(x)H^M(x) dx = -\int_A^B f_0(x)H'(x)H^{M-1}(x) dx,$$

and an integration by parts gives the recurrence relation

$$\begin{aligned} - \int_A^B f_n(x) H'(x) H^{M-1}(x) dx \\ = \frac{f_n(A) H^M(A)}{M} - \frac{f_n(B) H^M(B)}{M} - M^{-1} \int_A^B f_{n+1}(x) H'(x) H^{M-1}(x) dx. \end{aligned}$$

Repeated application of this recurrence relation generates an asymptotic series with an explicit remainder term.

For the more difficult case, we add the assumptions:

(c) There is exactly one ϕ with $A \leq \phi \leq B$ such that $H'(\phi) = 0$.

(d) $H''(\phi) < 0$.

Our assumption (d) excludes the case where $H(x)$ takes its minimum value in the interval (A, B) at the point ϕ . This case is rather trivial, as one can then separate the interval (A, B) into three intervals; the integral over the interval containing ϕ will be negligible for large M because $H(x)$ is small in this interval, while the integrals over the other two intervals can be treated as above. Similar remarks apply if $H(x)$ has an inflection point at ϕ . Thus the only case of real interest excluded by assumption (d) is the case where $H(x)$ has a maximum at ϕ but $H'(\phi) = 0$. This case seldom occurs, and if it should occur, one could use straightforward generalizations of the procedures given below.

Using our assumptions (c) and (d), we introduce the new variable

$$\eta = (\log (H(\phi)/H(x)))^{\frac{1}{2}},$$

where we use the positive square root when $x > \phi$ and the negative square root when $x < \phi$. Expressing x in terms of η , say $x = x(\eta)$, we have

$$\int_A^B h(x) H^M(x) dx = H^M(\phi) \int_C^D \left\{ h(x(\eta)) \frac{dx}{d\eta} \right\} e^{-M\eta^2} d\eta,$$

where C and D are the values of η corresponding to $x = A$ and $x = B$. We thus have a generalized version of (12), and can apply the procedures suggested for (12) (see [1], for example). For the two alternative procedures it is not necessary to use η as the variable of integration. We shall indicate how to carry out the second alternative procedure directly, with x as the variable of integration, since this seems to involve as little complication as the first alternative procedure. We note that by our various assumptions η is an analytic function of x for $A \leq x \leq B$. Moreover,

$$2\eta \frac{d\eta}{dx} = - \frac{H'(x)}{H(x)},$$

and

$$\lim_{x \rightarrow \phi} \frac{d\eta}{dx} = \sqrt{\frac{-H''(\phi)}{2H(\phi)}} > 0.$$

Accordingly, we can conclude that $f_n(x)$ is analytic for $A \leq x \leq B$ if we define $f_n(x)$ by

$$f_0(x) = h(x),$$

$$f_{n+1}(x) = \frac{d}{dx} \left\{ -\frac{H(x)}{H'(x)} f_n(x) - \frac{f_n(\phi)}{2\eta} \sqrt{\frac{2H(\phi)}{-H''(\phi)}} \right\}.$$

Then we have the recurrence relation

$$\begin{aligned} \int_A^B f_n(x) H^M(x) dx &= \int_A^B f_n(\phi) \sqrt{\frac{2H(\phi)}{-H''(\phi)}} \frac{d\eta}{dx} H^M(x) dx \\ &\quad + \int_A^B \left\{ -\frac{H(x)}{H'(x)} f_n(x) - \frac{f_n(\phi)}{2\eta} \sqrt{\frac{2H(\phi)}{-H''(\phi)}} \right\} H'(x) H^{M-1}(x) dx \\ &= f_n(\phi) \sqrt{\frac{2H(\phi)}{-H''(\phi)}} \int_c^D e^{-M\eta^2} d\eta + \frac{H^M(A) f'_{n+1}(A)}{M} - \frac{H^M(B) f'_{n+1}(B)}{M} \\ &\quad + M^{-1} \int_A^B f_{n+1}(x) H^M(x) dx. \end{aligned}$$

Repeated application of this recurrence relation generates an asymptotic series with an explicit remainder term.

We need hardly stress that in general the successive functions $f_n(x)$ will become more complicated very rapidly with increasing n . Thus we should prefer a simpler procedure. In case the value of

$$\int_A^B H^M(x) dx$$

is known, such a simpler procedure is available. We define

$$(25) \quad h_0(x) = h(x),$$

$$(26) \quad h_{n+1}(x) = \frac{d}{dx} \left\{ \frac{h_n(x) - h_n(\phi)}{-H'(x)} H(x) \right\}.$$

Clearly each $h_n(x)$ is analytic for $A \leq x \leq B$, and we have the recurrence relation

$$\begin{aligned} \int_A^B h_n(x) H^M(x) dx &= h_n(\phi) \int_A^B H^M(x) dx + \frac{H^M(A) h'_{n+1}(A)}{M} - \frac{H^M(B) h'_{n+1}(B)}{M} \\ &\quad + M^{-1} \int_A^B h_{n+1}(x) H^M(x) dx. \end{aligned}$$

In a succeeding section, we shall give an instance of this procedure.

We remark that for this procedure one can weaken assumption (d) and assume

merely $H''(\phi) \neq 0$. If $H''(\phi) = 0$ but $H'''(\phi) \neq 0$, and if the value of

$$\int_A^B (x - \phi) H^M(x) dx$$

is known, one could proceed by using instead of (26) the definition

$$h_{n+1}(x) = \frac{d}{dx} \left\{ \frac{h_n(x) - h_n(\phi) - h'_n(\phi)(x - \phi)}{-H'(x)} H(x) \right\}.$$

If $H''(\phi) = H'''(\phi) = 0$, but $H^{iv}(\phi) \neq 0$, one could use a similar device.

A procedure that is only very slightly different from the sort of procedure just proposed is considered in [6] for a special $H(x)$ and a special class of $h(x)$'s.

3. Airey's converging factor. Let us consider the incomplete gamma function

$$\int_y^\infty t^{z-1} e^{-t} dt.$$

By allowing complex values of y in some cases, this can be specialized to give such useful transcendental functions as the error function, the exponential integral, the sine integral, the cosine integral, and the Fresnel integrals. Therefore it is worthy of note that in our subsequent developments we can permit both y and z to be complex.

Repeated integrations by parts give

$$\begin{aligned} \int_y^\infty t^{z-1} e^{-t} dt &= e^{-y} \{ y^{z-1} + (z-1)y^{z-2} + (z-1)(z-2)y^{z-3} \\ &\quad + \cdots + (z-1)(z-2) \cdots (z-m)y^{z-m-1} \} \\ &\quad + (z-1)(z-2) \cdots (z-m)(z-m-1) \int_y^\infty t^{z-m-2} e^{-t} dt. \end{aligned}$$

AIREY proposed a scheme for estimating the remainder (see [3]). Writing $N = m + 1 - z$, we wish to estimate

$$\int_y^\infty t^{-N-1} e^{-t} dt.$$

Proceeding formally by repeated integration by parts, we get

$$\begin{aligned} y^{N+1} e^y \int_y^\infty t^{-N-1} e^{-t} dt &= 1 - \frac{N+1}{y} + \frac{(N+1)(N+2)}{y^2} - \cdots \\ &= 1 - \frac{N}{y} \left(1 + \frac{1}{N} \right) + \left(\frac{N}{y} \right)^2 \left(1 + \frac{1}{N} \right) \left(1 + \frac{2}{N} \right) - \cdots. \end{aligned}$$

Putting

$$\beta = N/y,$$

we have

$$1 - \beta \left(1 + \frac{1}{N}\right) + \beta^2 \left(1 + \frac{1}{N}\right) \left(1 + \frac{2}{N}\right) - \dots.$$

Calling this $F(\beta)$ and carrying out formal operations with power series in β , we get

$$(27) \quad (1 + \beta)F(\beta) + \frac{\beta}{N} \frac{d}{d\beta} (\beta F(\beta)) = 1.$$

Consider $F(\beta)$ expanded in inverse powers of N

$$F(\beta) = F_0(\beta) + \frac{F_1(\beta)}{N} + \frac{F_2(\beta)}{N^2} + \dots.$$

Substituting this into (27) and equating powers of N gives

$$(28) \quad F_0(\beta) = \frac{1}{1 + \beta},$$

$$(29) \quad F_{n+1}(\beta) = -\frac{1}{1 + \beta} \frac{d}{d\beta} (\beta F_n(\beta)).$$

Then

$$F_1(\beta) = -\frac{\beta}{(1 + \beta)^3},$$

$$F_2(\beta) = -\frac{\beta^2(\beta - 2)}{(1 + \beta)^5},$$

$$F_3(\beta) = -\frac{\beta^3(\beta^2 - 8\beta + 6)}{(1 + \beta)^7},$$

etc.

AIREY used a somewhat different notation, but one can read off the coefficients for $F_n(\beta)$ for $n \leq 10$ from results listed in [3]. AIREY did not have the relation (27), nor did he have explicitly the relation (29). However, he observed empirically a numerical relationship between the coefficients of F_n and F_{n+1} which is equivalent to (29).

Using the values we have listed above for F_0 , F_1 , F_2 , and F_3 gives formally

$$(30) \quad \int_y^\infty t^{-N-1} e^{-t} dt = y^{-N} e^{-y} \cdot \left\{ \frac{1}{y+N} - \frac{y}{(y+N)^3} - \frac{y(N-2y)}{(y+N)^5} - \frac{y(N^2-8Ny+6y^2)}{(y+N)^7} - \dots \right\}.$$

In special cases, (30) can be compared with known values, and is very successful. Thus, taking $y = N = 5$, and using a very accurate value for the integral on the left of (30), we see that the value represented by the curly brackets should come out to be 0.10474 41741. Adding up thirteen terms of the series gives 0.10474 41725. For other illustrations, see [3].

If we write the right side of (30) as

$$y^{-N} e^{-y} \{g_0(y) + g_1(y) + \dots\},$$

we see from the relations (28) and (29) that

$$(31) \quad g_0(y) = \frac{1}{y+N},$$

$$(32) \quad g_{n+1}(y) = \frac{y}{y+N} \frac{d}{dy} g_n(y).$$

One thus has two methods of computing the coefficients occurring in $g_n(y)$; one method is to compute the $F_n(\beta)$ by (28) and (29) and then form $g_n(y)$ by the relation

$$g_n(y) = F_n(N/y)/yN^n,$$

while the other method is to apply (31) and (32) directly. Different arithmetic operations are involved in the two methods, so that one has available a useful arithmetical check.

4. An explicit remainder term for the series in (30). We verify

$$\int_y^\infty t^{-N-1} e^{-t} dt = \frac{1}{\Gamma(N+1)} \int_0^\infty \frac{w^N e^{-(1+w)y}}{1+w} dw$$

by observing that both sides have the same derivative with respect to y . Thus, writing $\gamma = y/N$, we have

$$(33) \quad \int_y^\infty t^{-N-1} e^{-t} dt = y^{-N} e^{-y} \frac{N^N}{\Gamma(N+1)} \int_0^\infty \frac{x^N e^{-Nx}}{\gamma+x} dx.$$

So we seek a series for

$$\int_0^{\infty} \frac{x^N e^{-Nx}}{\gamma + x} dx.$$

We reduce this to the form (24) by setting

$$\begin{aligned} A &= 0, \\ B &= \infty, \\ M &= N, \\ H(x) &= xe^{-x}, \\ h(x) &= \frac{1}{\gamma + x}. \end{aligned}$$

Then our assumptions (a) through (d) are satisfied with $\phi = 1$. Furthermore we have the known result

$$\int_A^B H^M(x) dx = \int_0^{\infty} x^N e^{-Nx} dx = \Gamma(N+1)N^{-N-1},$$

so that we can apply the simplified procedure suggested at the end of Section 2. Accordingly we define

$$(34) \quad h_0(x) = \frac{1}{\gamma + x},$$

$$(35) \quad h_{n+1}(x) = \frac{d}{dx} \left\{ \frac{h_n(x) - h_n(1)}{x - 1} x \right\}.$$

We have the following recurrence relation when the real part of N is greater than zero:

$$\int_0^{\infty} h_n(x) x^N e^{-Nx} dx = \frac{\Gamma(N+1)}{N^{N+1}} h_n(1) + N^{-1} \int_0^{\infty} h_{n+1}(x) x^N e^{-Nx} dx.$$

Then by (33)

$$\begin{aligned} (36) \quad \int_y^{\infty} t^{-N-1} e^{-t} dt &= y^{-N} e^{-y} \left\{ \frac{h_0(1)}{N} + \frac{h_1(1)}{N^2} + \cdots + \frac{h_{m-1}(1)}{N^m} \right\} \\ &\quad + \frac{y^{-N} e^{-y}}{\Gamma(N+1)N^{m-N}} \int_0^{\infty} h_m(x) x^N e^{-Nx} dx. \end{aligned}$$

If this is to furnish an estimate for the remainder in (30) one must establish

that

$$(37) \quad \frac{h_n(1)}{N^{n+1}} = g_n(y)$$

with $g_n(y)$ defined by (31) and (32), and one must give a bound for $h_m(x)$. We shall do this for general m , but we first note that with a specific m it is quite practicable to proceed directly.

A straightforward application of (35) soon leads to quite tedious differentiations for the larger values of n . However, matters can be simplified by the following device. Let us put

$$W = \frac{\gamma + 1}{\gamma + x}.$$

Then for $m \geq 1$

$$\begin{aligned} \frac{d}{dx} \left\{ (W^m - 1) \frac{x}{x - 1} \right\} \\ = (\gamma + 1)^{-2} \{ -m\gamma W^{m+1} + (m - 1)W^m + (m - 2)W^{m-1} + \cdots + 2W^3 + W^2 \}. \end{aligned}$$

Using this, we get

$$\begin{aligned} h_0(x) &= \frac{W}{\gamma + 1}, \\ h_1(x) &= - \frac{\gamma}{(\gamma + 1)^3} W^2, \\ h_2(x) &= - \frac{\gamma}{(\gamma + 1)^5} \{ -2\gamma W^3 + W^2 \}, \\ h_3(x) &= - \frac{\gamma}{(\gamma + 1)^7} \{ 6\gamma^2 W^4 - 6\gamma W^3 + (1 - 2\gamma)W^2 \}, \\ h_4(x) &= - \frac{\gamma}{(\gamma + 1)^9} \{ -24\gamma^3 W^5 + 36\gamma^2 W^4 \\ &\quad - (14\gamma - 16\gamma^2)W^3 + (1 - 8\gamma + 6\gamma^2)W^2 \}, \\ &\text{etc.} \end{aligned}$$

Recalling that $\gamma = y/N$, one easily verifies that (37) holds for $0 \leq n \leq 3$. Also, if γ is in the right half plane, then for $0 \leq x$

$$|W| \leq \frac{|\gamma + 1|}{|\gamma|}$$

so that

$$|h_4(x)| \leq \frac{24|\gamma+1|^3 + 36|\gamma+1|^2 + |14-16\gamma||\gamma+1| + |1-8\gamma+6\gamma^2|}{|\gamma(\gamma+1)^7|}.$$

If γ is positive, then for $0 \leq x$ each of W^2 and

$$-24\gamma^3 W^3 + 36\gamma^2 W^2 - (14\gamma - 16\gamma^2)W + (1 - 8\gamma + 6\gamma^2)$$

takes its maximum absolute value at $x = 0$. Thus, for x and γ positive, $|h_4(x)| \leq |h_4(0)|$. That is, for $0 \leq x$ and $0 < \gamma$

$$|h_4(x)| \leq \frac{(1+4\gamma)(1+2\gamma+6\gamma^2)}{\gamma(\gamma+1)^7}.$$

To handle the general case, we define

$$(38) \quad \delta = \gamma + \log \gamma,$$

and define $\gamma(t)$ to be the solution of the differential equation

$$(39) \quad \gamma'(t) = \frac{\gamma(t)}{\gamma(t) + 1}$$

that satisfies the boundary condition

$$(40) \quad \gamma(\delta) = \gamma.$$

Then for $\gamma \neq -1$ there is a circle with center at δ inside which $\gamma(t)$ is analytic and satisfies the condition

$$(41) \quad t = \gamma(t) + \log \gamma(t).$$

Theorem 1. *If C is a curve enclosing the point δ and such that on and within C $\gamma(t)$ is analytic and different from both $-x$ and -1 , then*

$$(42) \quad h_n(x) = \frac{n!}{2\pi i} \int_C \frac{dt}{(\gamma(t) + x)(t - \delta)^{n+1}}.$$

Proof. We use induction on n . For $n = 0$, our result is valid by Cauchy's Theorem because of (34) and (40). So assume that (42) holds for n . Then by (35)

$$h_{n+1}(x) = \frac{n!}{2\pi i} \int_C \frac{-\gamma(t) dt}{(\gamma(t) + x)^2(\gamma(t) + 1)(t - \delta)^{n+1}}.$$

Because of (39), we can integrate by parts, establishing (42) for $n + 1$.

Corollary 1.

$$(43) \quad h_{n+1}(x) = \frac{\gamma}{\gamma + 1} \frac{\partial}{\partial \gamma} h_n(x),$$

This is obvious by (38) and (42).

This gives an alternative method of computing $h_n(x)$, which furnishes a useful check on the determination of the various coefficients.

Corollary 2.

$$\frac{h_n(1)}{N^{n+1}} = g_n(y).$$

This is seen by putting $x = 1$ in (43), recalling that $\gamma = y/N$, and referring to (34), (31), and (32). Thus (37) is verified.

Corollary 3. For $0 \leq n$

$$h_n(1) = \frac{d^{n+1}}{d\delta^{n+1}} (\delta - \gamma(\delta)).$$

Proof. By (42)

$$h_n(1) = \frac{d^n}{d\delta^n} \left(\frac{1}{\gamma(\delta) + 1} \right).$$

But by (39)

$$\frac{d}{d\delta} (\delta - \gamma(\delta)) = \frac{1}{\gamma(\delta) + 1}.$$

In order to be able to get a bound for $h_n(x)$ from (42) by the methods of complex variable theory, we investigate the properties of $\gamma(t)$. Assuming $\gamma(t)$ to be defined by (41), let us consider the semicircles in the γ -plane located above the real axis and having center at the origin and radius less than or equal to unity. Their images in the t -plane are non-overlapping curves with negative slope, with one end on the real axis and the other end on the horizontal half-line that goes left from the point $t = -1 + \pi i$. Moreover, these images fill the space between the real axis, the half-line just mentioned and the curve

$$(44) \quad t = \cos \theta + i(\theta + \sin \theta) \quad (0 \leq \theta \leq \pi)$$

which is the map of the semicircle of radius unity. Now, in that part of the γ -plane above the real axis and above the upper semicircle with center at the origin and radius unity, let us consider vertical lines. Their images in the t -plane are non-overlapping curves with positive slope and these images fill the space above the curve consisting of the half-line mentioned above, the curve

(44), and the real axis to the right of the point $t = 1$. We thus arrive at the following conclusion:

Theorem 2. *In the t -plane make two cuts extending horizontally leftward respectively from the two points $t = -1 \pm \pi i$. In the γ -plane make a cut on the negative real axis. Then in the cut t -plane (41) defines $\gamma(t)$ uniquely. Also $\gamma(t)$ is analytic in the cut t -plane, and maps the cut t -plane in a one-to-one fashion onto the cut γ -plane.*

Thus for any γ not on the negative real axis, if one defines δ by (38), then δ will be in the cut t -plane but not on one of the cuts. Accordingly, in (42) we can take for C a circle with center at δ and radius less than the distance to the nearest point on either cut. Then for t on and within C , $\gamma(t)$ is analytic and never takes a value on the negative real axis. Then for x real and non-negative, the conditions for Theorem 1 are satisfied. In

$$\int_0^{\infty} h_m(x) x^N e^{-Nx} dx$$

the path of integration for x is the positive real axis, so that x is real and non-negative. So for each x in the path of integration, one can apply (42) after determining a minimum for $|\gamma(t) + x|$ on C . If γ is close to the negative real axis, one can often effect a considerable improvement in the minimum of $|\gamma(t) + x|$ on C by modifying the path of integration for x .

The case of γ positive is particularly simple. We have:

Lemma 1. *If γ is positive and δ and $\gamma(t)$ are defined by (38) and (41) respectively and C is a circle in the t -plane with center at δ and radius $\gamma + \frac{1}{2}\pi$, then $\gamma(t)$ maps C into a curve in the γ -plane lying entirely to the right of the imaginary axis.*

Proof. Let $t = \rho + i\sigma$. Then the imaginary axis of the γ -plane is the map by $\gamma(t)$ of the two curves

$$(45) \quad \sigma = \pm(e^{\rho} + \tfrac{1}{2}\pi).$$

So we must show that the circle in question lies entirely between the two curves (45). Note that for real u

$$2e^{-u} - 2ue^{-u} + 2 - \pi e^{-u} > 0$$

because the quantity considered is positive at its minimum point, which is at $u = (4 - \pi)/2$. Integrating, we conclude that for $u > 0$

$$2ue^{-u} + 2u + \pi e^{-u} - \pi > 0.$$

Then for $u \geq 0$

$$(u - \gamma e^{-u})^2 + 2\gamma u e^{-u} + 2\gamma u + \pi \gamma e^{-u} > \pi \gamma.$$

Putting $v = u + \gamma$, we conclude that for $v \geq \gamma$

$$v^2 + \gamma^2 e^{2(\gamma-v)} + \pi \gamma e^{\gamma-v} > \gamma^2 + \pi \gamma.$$

However, this holds trivially for $v < \gamma$. So

$$\gamma e^{\gamma-v} + \frac{1}{2}\pi > ((\gamma + \frac{1}{2}\pi)^2 - v^2)^{\frac{1}{2}}.$$

As the circle in question has the parametric equations

$$\begin{aligned}\rho &= \gamma + \log \gamma - v, \\ \sigma &= ((\gamma + \frac{1}{2}\pi)^2 - v^2)^{\frac{1}{2}},\end{aligned}$$

we can conclude that this circle lies between the curves (45).

Using this lemma with (42), we can conclude:

Theorem 3. *If x is in the right half plane and γ is positive, then*

$$(46) \quad |h_n(x)| \leq \frac{n!}{(\gamma + \frac{1}{2}\pi)^n \Re(x)}.$$

Corollary 1. *If N and y are each positive, then*

$$(47) \quad \left| \int_0^\infty h_n(x) x^N e^{-Nx} dx \right| \leq \frac{\Gamma(N+1)n!}{N^{N+1-n}(y + \frac{1}{2}\pi N)^n}.$$

5. The case of γ near the negative real axis. We adapt a procedure of STELTJES (see [4] and [5]). It is best to start afresh from the form of integral which would lead to consideration of a γ near the negative real axis. Let Y be near the positive real axis, and consider

$$(48) \quad \int_0^Y t^{z-1} e^t dt.$$

For Y near the origin, one easily evaluates (48) by expanding e^t into a power series in t and integrating term-by-term. Alternatively, integrating by parts gives

$$(49) \quad \int_0^Y t^{z-1} e^t dt = \frac{-e^Y}{\Gamma(1-z)} \sum_{n=0}^\infty Y^{z+n} \Gamma(-z-n).$$

This series is more convenient for computational purposes, since each term can be computed from the preceding term by a very simple operation so that it is not a great chore to compute numerical approximations for a large number of terms of the series.

We note that the series on the right of (49) converges for each non-zero Y and for each non-integral z . Thus we can use (49) to give a meaning to (48) for

z in the left half plane. Indeed, if we make a cut along the negative real axis in the z -plane, then for each fixed non-zero Y (49) expresses (48) as an analytic function of z for each z in the cut z -plane except for integer values of z . Moreover, for each such z and each non-negative integer m we can choose a $C_m(z)$ dependent only on m and z so that the following result holds:

$$\begin{aligned}
 \int_0^Y t^{z-1} e^t dt &= \Gamma(z) \cos \pi z + \frac{C_m(z)}{\Gamma(1-z)} + e^Y \{ Y^{z-1} + (1-z) Y^{z-2} \\
 (50) \quad &+ (1-z)(2-z) Y^{z-3} + \cdots + (1-z) \cdots (m-1-z) Y^{z-m} \} \\
 &+ (1-z) \cdots (m-z) \int_{m+1-z}^Y t^{z-m-1} e^t dt.
 \end{aligned}$$

To see this, we observe that both sides have the same derivative with respect to Y , so that in effect $C_m(z)$ plays the role of a constant of integration.

We undertake the determination of $C_m(z)$. For this we temporarily let the real part of z be between zero and unity and the real part of Y be positive. Then

$$(51) \quad \int_0^Y t^{z-1} e^t dt = \frac{1}{\Gamma(1-z)} \int_0^\infty \frac{e^{Y(1-x)} - 1}{x^z(1-x)} dx,$$

as one can verify by differentiating both sides with respect to Y .

Using Cauchy principal values, we can write this in the form

$$(52) \quad \int_0^Y t^{z-1} e^t dt = \Gamma(z) \cos \pi z + \frac{1}{\Gamma(1-z)} \text{P.V.} \int_0^\infty \frac{e^{Y(1-x)}}{x^z(1-x)} dx.$$

Then

$$\begin{aligned}
 (53) \quad \int_0^Y t^{z-1} e^t dt &= \Gamma(z) \cos \pi z + \frac{1}{\Gamma(1-z)} \text{P.V.} \int_0^\infty \frac{x^m e^{Y(1-x)}}{x^z(1-x)} dx \\
 &+ e^Y \{ Y^{z-1} + (1-z) Y^{z-2} + \cdots + (1-z) \cdots (m-1-z) Y^{z-m} \}.
 \end{aligned}$$

By comparing this with (50) and putting $Y = N = m + 1 - z$ we infer

$$C_m(z) = \text{P.V.} \int_0^\infty \frac{x^m e^{N(1-x)}}{x^z(1-x)} dx.$$

This prompts us to make the definition

$$(54) \quad C(w) = \text{P.V.} \int_0^\infty \frac{x^w e^{w(1-x)}}{x(1-x)} dx$$

so that we can conclude

$$(55) \quad C_m(z) = C(m + 1 - z) = C(N).$$

By (54), $C(w)$ is analytic in the right half plane and for $n \geq 1$

$$\frac{d^n}{dw^n} C(w) = \int_0^\infty \frac{x^w e^{w(1-x)} (1-x+\log x)^n}{x(1-x)} dx.$$

By putting $Y = w$ and $z = 1 - w$ in (52) we infer that for $0 < \Re(w) < 1$

$$(56) \quad C(w) = \pi \cot \pi w + \Gamma(w) \int_0^w t^{-w} e^t dt.$$

This can be used to continue $C(w)$ analytically into the left half plane. We infer that $C(w)$ has a branch point at the origin; moreover, if we make a cut along the negative real axis, then $C(w)$ is analytic in the cut w -plane.

By putting $Y = w$ and $z = 1 - w$ in (49) and putting the result into (56), we infer

$$(57) \quad C(w) = \pi \cot \pi w - e^w \sum_{n=1}^{\infty} w^{n-w} \Gamma(w-n).$$

This can be used to determine $C(w)$ for w in the cut z -plane except when w is a positive integer. To get a determination of $C(w)$ for this case, we combine (50) and (55) to get

$$\begin{aligned} \int_0^Y t^{z-1} e^t dt - \int_{-\infty}^0 |t|^{z-1} e^t dt &= \Gamma(z)(\cos \pi z - 1) + \frac{C(m+1-z)}{\Gamma(1-z)} \\ &+ e^Y \{ \quad \} + (1-z) \cdots (m-z) \int_{m+1-z}^Y t^{z-m-1} e^t dt. \end{aligned}$$

Letting $z \rightarrow 0$ gives

$$(58) \quad \text{P.V.} \int_{-\infty}^Y t^{-1} e^t dt = C(m+1) + e^Y \sum_{j=1}^m (j-1)! Y^{-j} + m! \int_{m+1}^Y t^{-m-1} e^t dt.$$

The function on the left of (58) is the well known exponential integral, which has been extensively tabulated, especially for integral values of Y . Putting $Y = m+1$ enables us to get accurate determinations of $C(w)$ for w a positive integer unless w is quite large.

We now seek an asymptotic series for $C(w)$. We define

$$(59) \quad \alpha = (2(x-1-\log x))^{\frac{1}{2}},$$

so that

$$xe^{1-x} = e^{-\alpha^2/2}.$$

We then introduce α as the variable of integration in (54) so that it takes the

form

$$(60) \quad C(w) = \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x(1-x)} \frac{dx}{d\alpha} e^{-w\alpha^{2/2}} d\alpha.$$

By (59) we get

$$\frac{d}{d\alpha} (x-1)^2 = 2\alpha + 2\alpha(x-1)$$

and

$$(61) \quad \frac{1}{x(1-x)} \frac{dx}{d\alpha} = -\frac{\alpha}{(x-1)^2}.$$

Using these and writing

$$\begin{aligned} \frac{dx}{d\alpha} &= \sum_{j=0}^{\infty} D_j \alpha^j, \\ \frac{1}{x(1-x)} \frac{dx}{d\alpha} &= \sum_{j=0}^{\infty} A_j \alpha^{j-1}, \end{aligned}$$

we get $D_0 = 1$, $D_1 = \frac{2}{3}$, $2!D_2 = \frac{1}{6}$, $3!D_3 = -\frac{4}{45}$, $4!D_4 = \frac{1}{36}$, $5!D_5 = \frac{8}{189}$, $6!D_6 = -\frac{139}{1080}$ and $A_0 = -1$, $A_1 = \frac{2}{3}$, $2!A_2 = -\frac{5}{6}$, $3!A_3 = \frac{23}{45}$, $4!A_4 = -\frac{62}{135}$, $5!A_5 = \frac{115}{84}$.

Putting the second series into (60) and integrating term-by-term gives the asymptotic series

$$(62) \quad C(w) = (2\pi/w)^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(2j)! A_{2j+1}}{j!(2w)^j}.$$

In [4], STIELTJES essentially derived the special case of this for w a positive integer.

Using a scheme suggested by STIELTJES in [4], the following values were computed:

$$\begin{aligned} A_1 &= \frac{2}{3}, \\ A_3 &= \frac{23}{270}, \\ \frac{4!A_5}{2!2^2} &= \frac{23}{3024}, \\ \frac{6!A_7}{3!2^3} &= -\frac{259}{77760}, \end{aligned}$$

$$\frac{8!A_9}{4!2^4} = -\frac{20\,16373}{18475\,77600},$$

$$\frac{10!A_{11}}{5!2^5} = \frac{45\,68387}{40758\,68160}.$$

As an instance of the efficiency of the formula (62), we take $z = \frac{1}{2}$. Then we have the results listed in Table 1, where the values listed under $C(w)$ were computed by means of (57).

TABLE 1

u	Sum of 4 terms	Sum of 5 terms	Sum of 6 terms	$C(w)$
$\frac{1}{2}$	1.682	1.647	1.718	1.690
$\frac{3}{2}$	0.83814 0	0.83789 1	0.83806 1	0.83806 3
$\frac{5}{2}$	0.62765 94	0.62763 44	0.62764 46	0.62764 54
$\frac{7}{2}$	0.52276 15	0.52275 60	0.52275 76	0.52275 78
$\frac{9}{2}$	0.45729 050	0.45728 873	0.45728 913	0.45728 918

To estimate the error in (62), we first investigate the properties of x as a function $x(\alpha)$ of α , defined by (59). By means of (41), we see that we can satisfy (59) by putting

$$(63) \quad x = e^{-\pi i} \gamma(-\tfrac{1}{2}\alpha^2 - 1 + \pi i),$$

fixing the choice of branches of the Riemann surface for γ by saying that

$$(64) \quad x(\sqrt{2\pi}e^{-3\pi i/4}) = \tau e^{-\pi i},$$

where τ is the positive real solution of

$$(65) \quad \tau + 1 + \log \tau = 0.$$

To use (63) over all the parts of the α -plane that we shall require, we need to know what $\gamma(t)$ does if t goes onto other branches of the Riemann surface for $\gamma(t)$. However this is easily determined by observing from (41) that if we start on the principal branch at $t = -2$ and keep t to the left of the vertical line through $t = -1$, then $\gamma(t)$ is periodic with period $2\pi i$. Thus we can prove:

Theorem 4. *The function $x(\alpha)$ defined by (59) has branch points at $\alpha = 2\sqrt{\pi}e^{\pm 3\pi i/4}$. If we make cuts from these points extending radially away from the origin, then in the cut α -plane, $x(\alpha)$ is analytic, and equals unity only at $\alpha = 0$.*

Our estimations of the error in using (62) will be based on the following obvious result.

Theorem 5. *Let L be a straight line in the α -plane passing through the origin and making an angle θ with the real axis, where $|\theta| < \pi/4$. Let s be the real part of $we^{2\theta i}$, and suppose that s is positive. Let $g(\alpha)$ be a function which is continuous on the line L and has there the property*

$$(66) \quad \left| g(\alpha) - \sum_{k=0}^{n-1} B_k \alpha^{2k} \right| \leq K_n |\alpha^{2n}|.$$

Then

$$(67) \quad \left| \int_L g(\alpha) e^{-w\alpha^{2/2}} d\alpha - (2\pi/w)^{\frac{1}{2}} \sum_{k=0}^{n-1} \frac{(2k)! B_k}{k!(2w)^k} \right| \leq (2\pi/s)^{\frac{1}{2}} \frac{(2n)! K_n}{n!(2s)^n}.$$

To apply this theorem to the estimation of $C(w)$, we use (61) in (60), getting

$$C(w) = \text{P.V.} \int_{-\infty}^{\infty} \frac{-\alpha e^{-w\alpha^{2/2}}}{(x-1)^2} d\alpha.$$

Replacing α by $-\alpha$ in the integral and adding gives

$$(68) \quad C(w) = \int_{-\infty}^{\infty} \frac{1}{2} \left\{ \frac{\alpha}{(x(-\alpha)-1)^2} - \frac{\alpha}{(x(\alpha)-1)^2} \right\} e^{-w\alpha^{2/2}} d\alpha.$$

Then in Theorem 5 we will take

$$(69) \quad g(\alpha) = \frac{1}{2} \left\{ \frac{\alpha}{(x(-\alpha)-1)^2} - \frac{\alpha}{(x(\alpha)-1)^2} \right\},$$

$$(70) \quad B_k = A_{2k+1},$$

since in (68) one can rotate the path of integration to the line L . In order to establish a result of the form (66), we will need estimates of the A_j . They will be based on the following result.

Theorem 6. *Let $j(x)$ be a polynomial in x of order p with real coefficients. Let*

$$(71) \quad \frac{\alpha^2 j(x)}{(x-1)^2} = \sum_{j=0}^{\infty} \frac{C_j \alpha^j}{(4\pi)^{j/2}}.$$

For some non-negative integral l , m , and n , let

$$(72) \quad \left| \frac{1}{2} \left\{ \frac{\alpha j(x(\alpha))}{(x(\alpha)-1)^2} - \frac{\alpha j(x(-\alpha))}{(x(-\alpha)-1)^2} \right\} - \sum_{k=0}^{n-1} \frac{C_{2k+1} \alpha^{2k}}{(4\pi)^{(2k+1)/2}} \right| \\ \leq L_l |\alpha^{2l}| + M_m |\alpha^{2m+1}|$$

hold for all real α . Then for $j > \max(2l + 1, 2m + 2, 2p - 2)$

$$\begin{aligned}
 (73) \quad & \left| C_j + C_0 \cos\left(\frac{3\pi j}{4}\right) \right. \\
 & \quad \left. - \frac{1}{\pi} \sum_{k=0}^{n-1} C_{2k+1} \sin\left(\frac{(3j+2k+1)\pi}{4}\right) B\left(\frac{2k+1}{2}, \frac{j-2k-1}{2}\right) \right| \\
 & \leq \frac{(4\pi)^{(2l+1)/2} L_l}{2\pi} B\left(\frac{2l+1}{4}, \frac{j-2l-1}{4}\right) \\
 & \quad + \frac{(4\pi)^{m+1} M_m}{2\pi} B\left(\frac{m+1}{2}, \frac{j-2m-2}{4}\right).
 \end{aligned}$$

Proof. By Theorem 4, the expansion (71) is valid for $|\alpha^2| < 4\pi$. Then if C is a small circle in the α -plane enclosing the origin

$$C_j = \frac{(4\pi)^{j/2}}{2\pi i} \int_C \frac{j(x) d\alpha}{(x-1)^2 \alpha^{j-1}}.$$

So by (61)

$$C_j = \frac{(4\pi)^{j/2}}{2\pi i} \int_C \frac{j(x)}{x(x-1)} \frac{dx}{d\alpha} \frac{d\alpha}{\alpha^j}.$$

So if C is a small circle in the x -plane enclosing the point $x = 1$,

$$C_j = \frac{(4\pi)^{j/2}}{2\pi i} \int_C \frac{j(x) dx}{x(x-1)\alpha^j}.$$

Now we deform the contour of integration into the following contour. Let V and v be positive real numbers, with V large and v small. We start at $x = V$, proceed counterclockwise halfway around the circle $|x| = V$ to $x = -V$, proceed to the right along the negative real axis to $x = -v$, proceed clockwise completely around the circle $|x| = v$ back to $x = -v$, proceed to the left along the negative real axis to $x = -V$, and then proceed counterclockwise halfway around the circle $|x| = V$ back to $x = V$. Since $j > \max(2, 2p - 2)$, one can let $V \rightarrow \infty$ and $v \rightarrow 0$, so that (since the coefficients of $j(x)$ are real) one gets

$$(74) \quad C_j = \frac{(4\pi)^{j/2}}{\pi} \oint_Q \frac{j(x) dx}{x(x-1)\alpha^j}$$

where Q is the negative real axis traversed from left to right and we use the determination of α got by using (59) and proceeding counterclockwise around through π radians from the positive real axis. We now swing Q around counterclockwise to the position of the positive real axis, temporarily avoiding the

point $x = 1$ by going below it on a small semicircle. For x on the positive real axis, we have by (59)

$$\alpha^2 = 2(x - 1 - \log x - 2\pi i).$$

That is

$$(75) \quad \alpha = 2\sqrt{-\pi}e^{3\pi i/4} \left\{ 1 + \frac{x - 1 - \log x}{2\pi} i \right\}^{\frac{1}{2}}.$$

Then, in (74) we shrink the semicircle around the point $x = 1$ to zero, and infer

$$C_j = -j(1) \cos(3\pi j/4) - \mathcal{J} \text{ P.V. } \frac{1}{\pi} \int_0^\infty \frac{e^{-3\pi i j/4} j(x) dx}{x(x-1)\{1 + ((x-1) - \log x)i/2\pi\}^{i/2}}.$$

Using (59), we can write this as

$$C_j = -j(1) \cos(3\pi j/4) - \mathcal{J} \text{ P.V. } \frac{1}{\pi} \int_0^\infty \frac{e^{-3\pi i j/4} j(x) dx}{x(x-1)\{1 + (\alpha^2 i/4\pi)\}^{i/2}}$$

where we now use the principal determination of α on the positive real axis of x . We change to α as the variable of integration, and use (61). This gives

$$(76) \quad C_j = -j(1) \cos(3\pi j/4) - \mathcal{J} \text{ P.V. } \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{-3\pi i j/4} j(x) \alpha dx}{(x-1)^2 \{1 + (\alpha^2 i/4\pi)\}^{i/2}}.$$

Putting $\alpha = 0$ in (71) gives

$$(77) \quad j(1) = C_0.$$

Using this and (72) in (76) gives (73) because the C_j 's are all real.

Lemma 2. *If m, n_1 , and n_2 are positive and $n_1 < n_2$, then $B(m, n_2) < B(m, n_1)$.*

Proof. Use the relation

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

Lemma 3. *Let*

$$(78) \quad \frac{1}{x(1-x)} \frac{dx}{d\alpha} = \sum_{j=0}^\infty A_j \alpha^{j-1}$$

and

$$(79) \quad g(\alpha) = \frac{1}{2} \left\{ \frac{\alpha}{(x(-\alpha) - 1)^2} - \frac{\alpha}{(x(\alpha) - 1)^2} \right\}.$$

If for $j \geq 0$,

$$|A_j| \leq K(4\pi)^{-j/2},$$

then for real α

$$|g(\alpha) - \frac{2}{3}| \leq (0.5 + (0.007)K) |\alpha|.$$

Proof. By (61)

$$g(\alpha) = \sum_{k=0}^{\infty} A_{2k+1} \alpha^{2k},$$

and by Theorem 4 this holds for $|\alpha^2| < 4\pi$. Using the known values of A_{2k+1} for $0 \leq k \leq 2$, we see that for $|\alpha| \leq 2$

$$\begin{aligned} |g(\alpha) - \frac{2}{3}| &\leq \left\{ \frac{2^3}{2^7 5} |\alpha| + \frac{2^3}{9 \cdot 6^7 2} |\alpha^3| \right\} |\alpha| + K |\alpha| \sum_{k=3}^{\infty} \frac{|\alpha|^{2k-1}}{(4\pi)^{(2k+1)/2}} \\ &\leq \left\{ \frac{2^3}{1^3 5} + \frac{2^3}{1^3 4} \right\} |\alpha| + K |\alpha| \sum_{k=3}^{\infty} \frac{2^{2k-1}}{(4\pi)^{(2k+1)/2}} \\ &\leq (0.2 + (0.007)K) |\alpha|. \end{aligned}$$

For $\alpha \geq 2$, we have $x(\alpha) > 4$ and $x(-\alpha) < 0.053$ by (59). So for $2 \leq \alpha \leq 11$

$$\left| \frac{1}{(x(-\alpha) - 1)^2} - \frac{1}{(x(\alpha) - 1)^2} - \frac{4}{3\alpha} \right| < 1.$$

However, this result also holds for $11 < \alpha$, since by (59)

$$\begin{aligned} x(\alpha) &> \alpha^2/2, \\ x(-\alpha) &< e^{-\alpha^2/2}. \end{aligned}$$

So for $2 \leq \alpha$

$$|g(\alpha) - \frac{2}{3}| \leq \frac{1}{2} |\alpha|.$$

As $g(\alpha)$ is an even function of α , this result also holds for $\alpha \leq -2$.

Theorem 7. For A_j as in (78) and for $j \geq 0$,

$$(80) \quad |A_j| \leq (3.8)(4\pi)^{-j/2}.$$

Proof. Taking $g(\alpha)$ as defined by (79), we see that it is analytic for $|\alpha^2| < 4\pi$. Thus there is a constant L_0 such that $|g(\alpha)| \leq L_0$ for $|\alpha| \leq 2$. Then, reasoning as in the proof of Lemma 3, we conclude that for real α

$$|g(\alpha)| \leq L_0 + (0.56) |\alpha|.$$

Then in Theorem 6 we take $j(x) = -1$, $n = 0$, $l = 0$, $m = 0$, $M_0 = 0.56$, and conclude by Lemma 2 that $|C_j|$ is bounded for $j \geq 3$. But with $j(x) = -1$, we have by (61)

$$\frac{\alpha j(x)}{(x-1)^2} = \frac{1}{x(1-x)} \frac{dx}{d\alpha}$$

so that by (71) and (78),

$$(81) \quad C_j = A_j(4\pi)^{j/2}.$$

Let K be the least upper bound of $|C_j|$ for $0 \leq j$. Then we wish to prove that $K \leq 3.8$. By calculation from the known values of A_j , we conclude that

$$|C_j| \leq 3.8$$

for $0 \leq j \leq 5$. We now use Lemma 3, and then in Theorem 6 we take $j(x) = -1$, $n = 1$, $l = 0$, $m = 0$, $L_0 = 0$, and $M_0 = 0.5 + (0.007)K$. Then for $3 \leq j$

$$\begin{aligned} \left| C_j - \cos\left(\frac{3\pi j}{4}\right) - \frac{4}{\varepsilon\sqrt{\pi}} \sin\left(\frac{(3j+1)\pi}{4}\right) B\left(\frac{1}{2}, \frac{j+1}{2}\right) \right| \\ \leq \frac{4\pi}{2\pi} (0.5 + (0.007)K) B\left(\frac{1}{2}, \frac{j-2}{4}\right). \end{aligned}$$

So

$$|C_6| \leq 2.7 + (0.028)K,$$

$$|C_7| \leq 3.3 + (0.025)K,$$

and, by Lemma 2, if $8 \leq j$ then

$$|C_j| \leq 3.4 + (0.022)K.$$

As K is the least upper bound of the $|C_j|$, these inequalities enable us to conclude $K \leq 3.8$.

Theorem 8. *Let*

$$(82) \quad \frac{dx}{d\alpha} = \sum_{j=0}^{\infty} D_j \alpha^j.$$

Then for $j \geq 0$

$$(83) \quad |D_j| \leq (2.4)(4\pi)^{-j/2}.$$

Proof. By (61)

$$\frac{dx}{d\alpha} = \frac{\alpha x(x-1)}{(x-1)^2}.$$

So we take $j(x) = x(x-1)$ in Theorem 6, and reason as for the proof of Theorem 7.

Theorem 9. *Let θ and L be as in Theorem 5 and let $g(\alpha)$ and B_j be as in (69) and (70). If for α on L and $|\alpha^2| \geq 12$ we have*

$$|g(\alpha)| \leq k(\theta) |\alpha|$$

then for all α on L and for each positive integer n

$$(84) \quad \left| g(\alpha) - \sum_{j=0}^{n-1} B_j \alpha^{2j} \right| \leq \frac{(3.5)k(\theta) + 24}{12^n} |\alpha^{2n}|.$$

Proof. By (70) and Theorem 7

$$|B_j| \leq (3.8)(4\pi)^{-(2j+1)/2}.$$

As for $|\alpha^2| < 4\pi$

$$g(\alpha) = \sum_{j=0}^{\infty} B_j \alpha^{2j},$$

it follows that for $|\alpha^2| \leq 12$

$$\begin{aligned} \left| g(\alpha) - \sum_{j=0}^{n-1} B_j \alpha^{2j} \right| &\leq (3.8) \sum_{j=n}^{\infty} \frac{|\alpha|^{2j}}{(4\pi)^{(2j+1)/2}} \\ &\leq \frac{(3.8) |\alpha^{2n}|}{(4\pi)^{(2n+1)/2}} \sum_{k=0}^{\infty} \frac{12^k}{(4\pi)^k} \\ &\leq \frac{24 |\alpha^{2n}|}{(4\pi)^n}. \end{aligned}$$

Also for α on L and $|\alpha^2| \geq 12$

$$\begin{aligned} \left| g(\alpha) - \sum_{j=0}^{n-1} B_j \alpha^{2j} \right| &\leq |\alpha^{2n}| \left\{ \frac{k(\theta)}{|\alpha^{2n-1}|} + \frac{3.8}{\sqrt{4\pi}} \sum_{j=0}^{n-1} \frac{|\alpha^{2j-2n}|}{(4\pi)^j} \right\} \\ &\leq |\alpha^{2n}| \left\{ \frac{\sqrt{12} k(\theta)}{12^n} + \frac{3.8}{\sqrt{4\pi}} \frac{1}{12^n} \sum_{j=0}^{n-1} \frac{12^j}{(4\pi)^j} \right\} \\ &\leq |\alpha^{2n}| \frac{(3.5)k(\theta) + 24}{12^n}. \end{aligned}$$

Once we have determined $k(\theta)$, we may use this result in Theorem 5 to get estimates for the error in using (62), or we may use it in Theorem 6 to get more accurate estimates for A_j for large j .

Lemma 4. *If α is real and $|\alpha^2| \geq 12$, then*

$$(85) \quad |g(\alpha)| \leq \frac{1}{2} |\alpha|.$$

Proof as for Lemma 3.

Lemma 5. *Let θ and L be as in Theorem 5 and let $g(\alpha)$ be as in (69). Let q be the solution of*

$$(86) \quad 6 \cos 2\theta = q - 1 - \log q$$

with $0 < q < 1$. Then for α on L and $|\alpha^2| \geq 12$

$$(87) \quad |g(\alpha)| \leq (0.03 + \tfrac{1}{2}(1-q)^{-2}) |\alpha|.$$

To prove this, one must find the curve in the x -plane which is the image of L in the α -plane. If $\theta = 0$, we can use Lemma 4. So let $\theta \neq 0$. As α moves from right to left on the line L , x comes from the right on a somewhat exponential type curve. When α gets to the origin, x reaches the value unity. As α proceeds onward to the left, x spirals around and around the origin, each loop of the spiral being inside the previous one.

For $|\alpha^2| \geq 12$ and α on the right half of L , one easily verifies that $|x(\alpha) - 1|^{-2} \leq 0.06$. This accounts for the term 0.03 in (87). To get a bound for $|x(-\alpha) - 1|^{-2}$ for α on the right half of L is the same as getting a bound for $|x(\alpha) - 1|^{-2}$ for α on the left half of L . Here we have x on the spiral. Writing

$$\begin{aligned} \alpha &= se^{i\theta}, \\ x &= a + ib, \end{aligned}$$

with $s^2 \geq 12$, we see by (59) that

$$\frac{1}{2}s^2 \cos 2\theta = a - 1 - \frac{1}{2} \log(a^2 + b^2).$$

So

$$\begin{aligned} a - 1 - \log a &\geq \frac{1}{2}s^2 \cos 2\theta \\ &\geq 6 \cos 2\theta. \end{aligned}$$

As $a < 1$ for x on the spiral, we conclude that $a \leq q$. So $|x - 1|^{-2} \leq (1 - q)^{-2}$.

We remark that for θ near $\pi/4$ one has approximately

$$\frac{1}{2(1-q)^2} = \frac{1}{12(\pi - 4\theta)}.$$

Thus one can let θ approach quite close to $\pi/4$ without seriously enlarging the bounds obtained by Theorem 5.

We conclude that for $-\pi + \epsilon \leq \arg(w) \leq \pi - \epsilon$, the series in (62) is asymptotic in the sense of POINCARÉ (slightly generalized).

Turning back to (50), we see that we require a determination of

$$(88) \quad \int_N^Y t^{-N} e^t dt.$$

This equals

$$e^N N^{1-N} \int_1^{Y/N} (xe^{1-x})^{-N} dx$$

so that if we define

$$(89) \quad E = \left(2 \left(\frac{Y}{N} - 1 - \log \frac{Y}{N} \right) \right)^{\frac{1}{2}}$$

then

$$\int_N^Y t^{-N} e^t dt = e^N N^{1-N} \int_0^E \frac{dx}{d\alpha} e^{N\alpha^2/2} d\alpha.$$

If $|E^2| < 4\pi$, then by Theorem 8 we can expand $dx/d\alpha$ into a power series and integrate term-by-term. This gives

$$(90) \quad \int_N^Y t^{-N} e^t dt = \frac{1}{2} e^N N^{1-N} \sum_{j=0}^{\infty} D_j(2/N)^{(j+1)/2} \int_0^{NE^2/2} t^{(j-1)/2} e^t dt.$$

The integrals on the right are of the form (48), so that it might appear that our attempt to evaluate (48) has led us into a vicious circle. This is not necessarily so. Often when Y is quite large, so that it is not practicable to evaluate (48) by means of the series in (49), one will find the series on the right of (90) rapidly convergent, with the individual terms easily computed by means of (49).

Admittedly, the procedure we have been indicating for evaluating (48) is not of universal application. For one thing, we wish to have $|E^2|$ considerably less than 4π , to insure rapid convergence of the series on the right of (90). Although we have some control over the value of N (since the parameter m is at our disposal), we cannot guarantee that for each value of Y one can arrange to have $|E^2|$ small.

However, let us recall that the procedure developed in this section is intended only as a supplement to the procedure of the preceding section. There are cases where the procedure of the preceding section would be of little value but where the procedure of the present section is quite successful. Such a case is when Y and z are both near the positive real axis. Then, by proper choice of m , one can bring N near to Y . Thus by (89) one has approximately

$$E^2 = (Y - N)^2/N^2.$$

Consequently (since we are disregarding the cases where Y is near the origin), E^2 is quite small.

There still remains the objection that the integrals on the right of (90) have the same form as (48). However, this is not serious. In (48), the exponent of t is arbitrary, whereas in the integrals on the right of (90), the exponents of t are multiples of $\frac{1}{2}$. For the even multiples, one can integrate immediately in closed form, and for the odd multiples one has a type of integral that has been extensively studied, and for which many methods of computation are available (see [2], for example).

Another significant point is that in many cases that actually arise, the con-

tribution of (88) to the value of (48) is very small, percentagewise. Thus one often needs only to get a poor approximation for (88) to be enabled to compute a good approximation for (48).

In some cases, an adaptation of a procedure of STIELTJES (see [5]) may be more convenient for evaluating (88). Writing

$$X = Y - N,$$

we have

$$\begin{aligned} \int_N^Y t^{-N} e^t dt &= \int_0^X (N + v)^{-N} e^{N+v} dv \\ &= e^N N^{-N} \int_0^X \exp \left(v - N \log \left(1 + \frac{v}{N} \right) \right) dv. \end{aligned}$$

Expanding in power series and integrating term-by-term, we get

$$(91) \quad \int_N^Y t^{-N} e^t dt = e^N N^{-N} \left\{ X + \frac{1}{N} \frac{X^3}{3!} + \frac{1}{N^2} \left(\frac{3X^5}{5!} - \frac{2!X^4}{4!} \right) + \cdots \right\}.$$

To get a convenient means for writing down more terms of this series, define

$$(92) \quad G(v) = \exp \left(v - N \log \left(1 + \frac{v}{N} \right) \right),$$

$$(93) \quad G(v) = \sum_{n=0}^{\infty} G_n(v) N^{-n}.$$

One easily verifies

$$G(v) = \left(1 + \frac{N}{v} \right) \frac{\partial G(v)}{\partial v},$$

which gives

$$(94) \quad G_{n+1}(v) = \int_0^v v \left\{ G_n(v) - \frac{d}{dv} G_n(v) \right\} dv.$$

Then one quickly establishes

$$G_0(v) = 1,$$

$$G_1(v) = \frac{v^2}{2},$$

$$G_2(v) = \frac{v^4}{2 \cdot 4} - \frac{v^3}{3},$$

$$G_3(v) = \frac{v^6}{2 \cdot 4 \cdot 6} - \frac{1}{3} \frac{v^5}{2} + \frac{v^4}{4},$$

etc.

It is probably not possible to derive a general formula for $G_n(v)$. However, it is perfectly straightforward (though tedious) to derive such partial results as

$$(95) \quad G_n(v) = \frac{v^{2n}}{n!2^n} - \frac{2}{3} \frac{v^{2n-1}}{(n-2)!2^{n-1}} + \frac{4n-3}{18} \frac{v^{2n-2}}{(n-3)!2^{n-2}} \\ - \frac{20n^2 - 45n + 22}{405} \frac{v^{2n-3}}{(n-4)!2^{n-3}} + \dots$$

Also, one has the result:

Lemma 6. *If we write*

$$G_n(v) = \sum_{j=0}^{\infty} \frac{(-v)^{n+j} H_j(n)}{n+j},$$

then for $0 \leq j$ and $0 \leq n$

$$H_0(n) = 0,$$

$$H_{j+1}(n) = \sum_{k=0}^{n-1} \frac{H_j(k)}{k+j},$$

where we make the determination

$$\frac{0}{0} = 1.$$

Thus for $n \geq 1$

$$G_n(v) = \frac{(-v)^{n+1}}{n+1} + \frac{(-v)^{n+2}}{n+2} \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) + \dots$$

6. A remark about the asymptotic expansion for the gamma function. Writing

$$\Gamma(N+1) = N^{N+1} e^{-N} \int_0^{\infty} (xe^{1-x})^N dx,$$

we have by (82) the formal result

$$(96) \quad \Gamma(N+1) = N^{N+1} e^{-N} \int_{-\infty}^{\infty} \frac{dx}{d\alpha} e^{-N\alpha^{2/2}} d\alpha \\ = \sqrt{2\pi N} N^N e^{-N} \sum_{j=0}^{\infty} \frac{(2j)! D_{2j}}{j!(2N)^j}.$$

This will be recognized as the usual asymptotic expansion for the gamma function. By combining Theorem 8 and Theorem 5, one can estimate the error in (96) by the sort of reasoning used in Theorem 9. We believe this gives as efficient an estimate as any available.

Of more interest is the series

$$(97) \quad D_{j-1} = \pi^{-1} \sum_{k=0} D_{2k} (4\pi)^{(2k+1-j)/2} \sin\left(\frac{(3j+2k+1)\pi}{4}\right) \\ \cdot B\left(\frac{2k+1}{2}, \frac{j-2k-1}{2}\right).$$

By Theorem 6, we can use this to compute an approximation to D_{j-1} for large j . Note that only the D 's with even subscripts occur on the right of (97). These are the same ones that occur on the right in (96). Thus one can get exact values of D_{2k} for the smaller values of k from the coefficients of the asymptotic series for the gamma function, for which a good many coefficients have been computed. Then by (97), one can get approximations to D_{2k} for larger k , and thus one can give approximate values for the later coefficients of the asymptotic series for the gamma function.

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