Approximately Planar Rigid Body Motion

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INTRODUCTION

Many cases of rigid body motion which are of practical interest consist in the spinning at high, substantially constant speed of a symmetrical body about an axis (angular velocity vector) which remains close both to the axis of symmetry of the body and to some fixed direction in space. Such a motion we shall call an approximately planar rigid body motion.

In this case, an approximate linearization of the equations of motion can be made. This linearization reduces the equations of motion to a single second-order linear differential equation and thus makes the integration trivial. The theory is sufficiently accurate to lead to a detailed prediction of many of the well-known phenomena of rigid body motion.

This paper first develops the equation and then applies it to some well-known classical cases.

I. GENERAL THEORY

A. Euler's equations. We begin by recalling Euler's equations for the motion of a rigid body having one point fixed. The derivation of Euler's equations is available in the standard texts on classical mechanics.

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, a moving orthonormal system of vectors, be the principal

axes of the body. Let ω_a , ω_b , and ω_c , the components of the angular velocity vector, be defined by,

$$\omega_a = \dot{\mathbf{B}} \cdot \mathbf{C} = -\dot{\mathbf{C}} \cdot \mathbf{B}, \quad (1.1)$$

$$\omega_b = \dot{\mathbf{C}} \cdot \mathbf{A} = -\dot{\mathbf{A}} \cdot \mathbf{C}, \quad (1.2)$$

$$\omega_c = \dot{\mathbf{A}} \cdot \mathbf{B} = -\dot{\mathbf{B}} \cdot \mathbf{A}. \quad (1.3)$$

Let $\mathbf{T}$ be the total torque on the body, and let I_a , I_b , and I_c be the principal moments of inertia of the body.

Then

$$\mathbf{T} \cdot \mathbf{A} = I_a \dot{\omega}_a - (I_b - I_c) \omega_b \omega_c, \quad (2.1)$$

$$\mathbf{T} \cdot \mathbf{B} = I_b \dot{\omega}_b - (I_c - I_a) \omega_c \omega_a, \quad (2.2)$$

$$\mathbf{T} \cdot \mathbf{C} = I_c \dot{\omega}_c - (I_a - I_b) \omega_a \omega_b. \quad (2.3)$$

(In sequences of equations (1) and (2), dots between letters indicate dot products; dots above letters signify time differentiation.)

When C is an axis of symmetry of the body so that $I_a = I_b$, we may rewrite Euler's equations:

Let $I_a = I_b = I_T$, let $I_c = I$, let $I/I_T = \rho$, and let $\omega_c = s$. Then we may call I_T the transverse moment of inertia, I the spin moment of inertia, ρ the nutation ratio, and s the spin speed. Euler's equations now become:

$$\mathbf{T} \cdot \mathbf{A} = I_T [\dot{\omega}_a + (\rho - 1)s\omega_b], \quad (3.1)$$

$$\mathbf{T} \cdot \mathbf{B} = I_T [\dot{\omega}_b - (\rho - 1)s\omega_a], \quad (3.2)$$

$$\mathbf{T} \cdot \mathbf{C} = I\dot{s}. \quad (3.3)$$

Equation (3.3) tells how spin speed varies with time. Since we are assuming that spin speed remains substantially constant as time passes, equation (3.3) is of interest only for the purpose of checking the truth of this assumption.

By the use of complex numbers, equations (3.1) and (3.2) can be combined.

Let $i = \sqrt{-1}$, and define ω , a complex-valued function of time, by

$$\omega = \omega_a + i\omega_b. \quad (4)$$

Then, multiplying (3.2) by i and adding to (3.1), we obtain

$$(\mathbf{A} + i\mathbf{B}) \cdot \mathbf{T} = I_T [\dot{\omega} - i(\rho - 1)s\omega]. \quad (5)$$

This equation is the basis of what is to follow. We note here that (1.1) and (1.2) give

$$\begin{aligned} \omega &= -(\mathbf{B} \cdot \dot{\mathbf{C}}) + i(\mathbf{A} \cdot \dot{\mathbf{C}}) \\ &= i(\mathbf{A} + i\mathbf{B}) \cdot \dot{\mathbf{C}}. \end{aligned} \quad (6)$$

B. A coordinate system. In order to use equation (5), we must introduce a coordinate system. The system we shall introduce is approximate.

Let $\{\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0\}$ be a fixed orthonormal system such that $\mathbf{C}$ remains close to $\mathbf{C}_0$ during the motion. Then we may write:

$$\mathbf{A} = \cos \alpha \mathbf{A}_0 + \sin \alpha \mathbf{B}_0 - u \mathbf{C}_0, \quad (7.1)$$

$$\mathbf{B} = -\sin \alpha \mathbf{A}_0 + \cos \alpha \mathbf{B}_0 - v \mathbf{C}_0, \quad (7.2)$$

$$\mathbf{C} = x \mathbf{A}_0 + y \mathbf{B}_0 + \mathbf{C}_0. \quad (7.3)$$

Here we are assuming that $x^2 + y^2$ is small enough to be ignored compared to unity. It is easy to verify that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is an orthonormal system to the goodness of the approximation, provided that x, y, u, v , are related by the formula:

$$u + iv = e^{-i\alpha}(x + iy). \quad (8)$$

From (1.3) we have $\omega_c = \mathbf{s} = \dot{\mathbf{A}} \cdot \mathbf{B}$. Then (7.1) and (7.2) give:

$$s = \dot{\alpha}, \quad (9)$$

after we ignore uv as being small compared to $\dot{\alpha}$.

If we define the complex number Z by $Z = x + iy$, (10)
then (7.1) and (7.2) can be combined by means of (8) to give

$$\mathbf{A} + i\mathbf{B} = e^{-i\alpha}(\mathbf{A}_0 + i\mathbf{B}_0 - Z\mathbf{C}_0). \quad (11)$$

From (6) we have $\omega = i(\mathbf{A} + i\mathbf{B}) \cdot \dot{\mathbf{C}}$, so that, using (7.3), (6), (10), and (11), we obtain

$$\omega = ie^{-i\alpha} \dot{Z}. \quad (12)$$

Thus,

$$\dot{\omega} = ie^{-i\alpha}(\ddot{Z} - is\dot{Z}). \quad (13)$$

C. The equation of motion. If now we use (12) and (13) in (5), we obtain

$$(\mathbf{A} + i\mathbf{B}) \cdot \mathbf{T} = I_T ie^{-i\alpha}(\ddot{Z} - ips\dot{Z}). \quad (14)$$

This equation, together with the solution to (9),

$$\alpha = \alpha_0 + st, \quad (15)$$

determines the motion completely.

A little thought shows that we may interpret Z as follows: Let P be the plane through the tip of $\mathbf{C}_0$ and orthogonal to $\mathbf{C}_0$. Then, any point in plane P can be represented in the form $x\mathbf{A}_0 + y\mathbf{B}_0 + \mathbf{C}_0$. With such a point, we may associate

the complex number Z , where $Z = x + iy$. Thus, a graph of the complex number Z as a function of time tells us just how the axis of symmetry, the *spin axis* of the body, moves as time passes.

D. The angular velocity vector. The angular velocity vector is $\mathbf{\Omega}$ defined by

$$\mathbf{\Omega} = \omega_a \mathbf{A} + \omega_b \mathbf{B} + \omega_c \mathbf{C}. \quad (16)$$

It is easy to show that

$$\mathbf{\Omega} = \mathbf{C} \times \dot{\mathbf{C}} + s\mathbf{C}. \quad (17)$$

From (7.3) we obtain

$$\mathbf{\Omega} = (-\dot{y} + sx)\mathbf{A}_0 + (\dot{x} + sy)\mathbf{B}_0 + s\mathbf{C}_0. \quad (18)$$

Thus, $\mathbf{\Omega}/s$ is again in the plane P and, if w is the complex number associated with it, we have

$$w = Z + \frac{i}{s} \dot{Z}. \quad (19)$$

From (17), we can determine the path of the angular velocity vector when we know the path of Z .

II. SPECIAL CASES

A. The free-spinning body. A free-spinning body is one on which no torque is exerted. For this case, (14) becomes

$$\ddot{Z} - i\rho s \dot{Z} = 0, \quad (20)$$

and we obtain by inspection

$$Z = k_1 + k_2 e^{i\rho s t}. \quad (21)$$

The quantities k_1 and k_2 are constants determined by the initial conditions. From (20) we see that Z travels in a circle at frequency $\rho s/(2\pi)$. This frequency is called the nutation frequency and shows up naturally as the resonance frequency of the system.

Using (18) we obtain

$$w = k_1 + k_2(1 - \rho)e^{i\rho s t}. \quad (22)$$

Thus, we see that the angular velocity point w also travels in a circle having the same center. However, its radius is $(1 - \rho)$ $\times$ the radius of the circle in which Z travels.

We see, then, that when $\rho = 1$ so that the body is dynamically spherical, the angular velocity point w remains fixed, so that we may say that a sphere spinning under no forces must spin about a fixed axis.

B. Free spinning body with friction. If we wish to include friction in the above equations, several ways of doing so present themselves. It is most natural to assume that the frictional torque is proportional to the angular velocity vector and directly opposed to it, so that we may write

$$\mathbf{T}_f = -\lambda s I_T \boldsymbol{\Omega}, \quad (23)$$

where λ is a friction constant. Then,

$$(\mathbf{A} + i\mathbf{B}) \cdot \mathbf{T}_f = -\lambda s I_T \omega \quad (24)$$

$$= -i\lambda s I_T e^{-i\alpha} \dot{Z} \quad (25)$$

by (12). Using this in (14), we obtain

$$-\lambda s \dot{Z} = \ddot{Z} - i\rho s \dot{Z}, \quad (26)$$

or $\ddot{Z} - is(\rho + i\lambda)\dot{Z} = 0$. The solution is now,

$$Z = k_1 + k_2 e^{is(\rho + i\lambda)t}. \quad (27)$$

Because of the presence of λ , the motion is now in a spiral and the point Z ultimately comes to rest at k_1 . Thus, friction moves the resonance frequency off the real axis. Now,

$$w = k_1 + k_2(\rho - 1 + i\lambda)e^{is(\rho + i\lambda)t}, \quad (28)$$

so that w also spirals in to k_1 . When $\rho = 1$, w is 90° out of phase with Z .

When $\rho = 1$ the friction term we have assumed is somewhat unsatisfactory because the $\mathbf{C}$ -axis now plays a distinguished role in the equations, since we have assumed that $\boldsymbol{\Omega} \cdot \mathbf{C}$ remains constant. This shows up in the solution in the statement that the angular velocity vector always returns to $\mathbf{C}$ after the transient dies out.

For this reason, we are led to consider the effect of a friction term specifically opposed to nutation (that is, to rapid movements of the angular velocity vector). Such a term might be generated by a force operating on the tip of the angular velocity vector and opposing its motion. This would give rise to a frictional torque $\mathbf{T}_f$ given by

$$\mathbf{T}_f = -\mu \frac{I_T}{s} (\boldsymbol{\Omega} \times \dot{\boldsymbol{\Omega}}).$$

Using $\mathbf{\Omega} = \omega_a \mathbf{A} + \omega_b \mathbf{B} + s\mathbf{C}$, $\dot{\mathbf{\Omega}} = \dot{\omega}_a \mathbf{A} + \dot{\omega}_b \mathbf{B}$, we have

$$\mathbf{\Omega} \times \dot{\mathbf{\Omega}} = -s\dot{\omega}_b \mathbf{A} + s\dot{\omega}_a \mathbf{B} + (\omega_a \dot{\omega}_b - \omega_b \dot{\omega}_a) \mathbf{C}.$$

Thus

$$(\mathbf{A} + i\mathbf{B}) \cdot \mathbf{T}_f = -\mu i I_T \dot{\omega}.$$

Using (13) and (15), we obtain

$$-\mu i (\ddot{Z} - i s \dot{Z}) = \ddot{Z} - i \rho s \dot{Z}, \quad (29)$$

or

$$\ddot{Z} - i s \frac{(\rho + i\mu)}{(1 + i\mu)} \dot{Z} = 0. \quad (30)$$

Now we observe that when $\rho = 1$, no damping operates. For ρ less than 1, it is easy to show that the motion becomes unstable. That is, when this type of damping is present, a rod-like body will not spin stably about its axis of symmetry. The reasonableness of this result leads us to infer that this damping term is a more realistic one than the simpler one first assumed. It will be noted that for ρ greater than 1, both damping terms yield the same type of results.

C. Precession. Now we shall consider the case in which a torque $\mathbf{T}$ is exerted on the spinning body by means of a force on the tip of the $\mathbf{C}$ -axis.

Let the force be given by

$$\mathbf{F} = f_1 \mathbf{A} + f_2 \mathbf{B}.$$

Then,

$$\begin{aligned} \mathbf{T} &= \mathbf{C} \times \mathbf{F}, \\ &= -f_2 \mathbf{A} + f_1 \mathbf{B}. \end{aligned}$$

Letting $f_1 + i f_2 = f$, we have for (5),

$$f = I_T e^{-i\alpha} (\ddot{Z} - i \rho s \dot{Z}), \quad (31)$$

or $f e^{i\alpha} = \ddot{Z} - i \rho s \dot{Z}$. Taking $f e^{-i\alpha}$ to be constant (so that the force will now be in a constant direction), we may write

$$k = I_T (\ddot{Z} - i \rho s \dot{Z}). \quad (32)$$

For the solution, we have

$$Z = Z_0 + \frac{ik}{\rho s I_T} t + k_1 e^{i \rho s t}. \quad (33)$$

Because of damping, we may expect that the last term will eventually disappear. Then, the path of Z will be at right angles to the applied force k , as (33) clearly shows.

D. Frictionless top. A frictionless top is a symmetric body spinning with one point fixed. The force acting on it is the force of gravity. We may take the force to be $-mgC_0$, where m is the mass of the top and g is the acceleration due to gravity. The force acts at lC , where l is the distance from the point of support to the center of gravity, so that

$$\mathbf{T}_g = -mgl(\mathbf{C} \times \mathbf{C}_0).$$

Then

$$\begin{aligned} (\mathbf{A} + i\mathbf{B}) \cdot \mathbf{T}_g &= -mgl(\mathbf{A} + i\mathbf{B}) \cdot (\mathbf{C} \times \mathbf{C}_0) \\ &= -mgl\mathbf{C}_0 \cdot [(\mathbf{A} + i\mathbf{B}) \times \mathbf{C}] \\ &= -mgl i \mathbf{C}_0 \cdot (\mathbf{A} + i\mathbf{B}) \\ &= imgle^{-i\alpha} Z \end{aligned}$$

by (11). Thus, $mglZ = I_\tau(\ddot{Z} - i\rho s\dot{Z})$,

$$\ddot{Z} - i\rho s\dot{Z} - \frac{mgl}{I_\tau} Z = 0.$$

The solution to this equation is

$$Z = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t},$$

where λ_1 and λ_2 are the roots of

$$\lambda^2 - i\rho s\lambda - \frac{mgl}{I_\tau} = 0.$$

Since the spinning is rapid, we may consider mgl/I_τ to be small compared to ρs and take,

$$\lambda_1 = i\rho s.$$

Then, dividing mgl/I_τ by $i\rho s$, we find

$$\lambda_2 = \frac{imgl}{\rho s I_\tau}.$$

The root λ_1 corresponds to the nutational part of the motion. The root λ_2 , which is small compared to λ_1 , shows that the top precesses in the direction of the spin and that the frequency with which the top traces out its precessional circle is independent of initial conditions.

E. Top with friction. We may include the effect of friction at the point where the top touches the table by postulating a frictional torque $-\lambda \mathbf{C}_0$. Then,

$$-\lambda(\mathbf{A} + \mathbf{B}i) \cdot \mathbf{C}_0 = \lambda Z e^{-i\alpha}$$

and the equation of motion becomes

$$\ddot{Z} - i\rho s \dot{Z} - \frac{mgl + i\lambda}{I_T} Z = 0.$$

The discussion is similar to the previous case except that now λ_2 has a real part, so that, as the top precesses, it spirals inward until the axis of symmetry of the top is vertical. This phenomenon is known as the rising of the top.

Note. Professor G. HAMEL has kindly pointed out to me that a paper by R. GRAMMEL, "Über einige Bewegungen des unsymmetrischen schweren Kreisels" (Jber. dtsh. Math. Ver. 1920), uses methods related to those of the present paper. GRAMMEL's paper makes use of the assumption that the angular velocity vector makes a small angle with a principal axis of the body; also, GRAMMEL uses a first order theory in his development.

APPENDIX

Professor TRUESDALL has suggested that a rigorous estimate of the errors committed in the use of the approximate equations in the body of the paper would be of interest. We shall give this estimate for the case of a free spinning body, for which a reasonably simple exact solution is available for comparison.

Any real orthonormal system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) can be specified exactly in terms of two parameters Z and α (where Z is a complex number and α is real) by the following scheme: Let

$$\begin{aligned} \mathbf{u}_0 &= \frac{1}{\sqrt{2}} (\mathbf{A}_0 + i\mathbf{B}_0), & \bar{\mathbf{u}}_0 &= \frac{1}{\sqrt{2}} (\mathbf{A}_0 - i\mathbf{B}_0), \\ \mathbf{u} &= \frac{e^{-i\alpha}}{1 + \frac{1}{4}|Z|^2} \left\{ \mathbf{u}_0 - \frac{1}{4}Z^2 \bar{\mathbf{u}}_0 - \frac{Z}{\sqrt{2}} \mathbf{C}_0 \right\}, \\ \bar{\mathbf{u}} &= \frac{e^{i\alpha}}{1 + \frac{1}{4}|Z|^2} \left\{ -\frac{1}{4}\bar{Z}^2 \mathbf{u}_0 + \bar{\mathbf{u}}_0 - \frac{\bar{Z}}{\sqrt{2}} \mathbf{C}_0 \right\}, \\ \mathbf{C} &= \frac{1}{1 + \frac{1}{4}|Z|^2} \left\{ \frac{\bar{Z}}{\sqrt{2}} \mathbf{u}_0 + \frac{Z}{\sqrt{2}} \bar{\mathbf{u}} + (1 - \frac{1}{4}|Z|^2) \mathbf{C}_0 \right\}, \\ \mathbf{A} &= \frac{1}{\sqrt{2}} (\mathbf{u} + \bar{\mathbf{u}}), & \mathbf{B} &= \frac{1}{\sqrt{2}i} (\mathbf{u} - \bar{\mathbf{u}}). \end{aligned}$$

When $|Z|^2$ is small compared to unity, these equations reduce to the approximate equations of this paper.

If

$$\mathbf{u}_1 = \frac{e^{-i\alpha_1}}{1 + \frac{1}{4}|Z_1|^2} \left\{ \mathbf{u}_0 - \frac{1}{4}Z_1^2 \bar{\mathbf{u}}_0 - \frac{Z_1}{\sqrt{2}} \mathbf{C}_0 \right\},$$

and

$$\mathbf{u}_2 = \frac{e^{-i\alpha_2}}{1 + \frac{1}{4}|Z_1|^2} \left\{ \mathbf{u}_1 - \frac{1}{4}Z_2^2 \bar{\mathbf{u}}_1 - \frac{Z_2}{\sqrt{2}} \mathbf{C}_1 \right\}$$

and

$$\mathbf{u}_2 = \frac{e^{-i\alpha_3}}{1 + \frac{1}{4}|Z_3|^2} \left\{ \mathbf{u}_0 - \frac{1}{4}Z_3^2 \bar{\mathbf{u}}_0 - \frac{Z_3}{\sqrt{2}} \mathbf{C}_0 \right\},$$

then

$$e^{-i\alpha_3} = e^{-i(\alpha_1+\alpha_2)} \frac{(1 - \frac{1}{4}Z_1 \bar{Z}_2 e^{i\alpha_2})^2}{|1 - \frac{1}{4}Z_1 \bar{Z}_2 e^{i\alpha_2}|^2},$$

$$Z_3 = \frac{Z_2 + Z_1 e^{i\alpha_2}}{1 - \frac{1}{4}Z_1 \bar{Z}_2 e^{i\alpha_2}}. \quad (34)$$

Further

$$\omega = i\xi \dot{Z}$$

$$s = i(\dot{\xi}/\xi + \frac{1}{2}\dot{Z}\bar{Z}/|\xi|)$$

$$\dot{\omega} = i\xi(\ddot{Z} + (\dot{\xi}/\xi)\dot{Z}),$$

where we have set $\xi = e^{-i\alpha}/(1 + \frac{1}{4}|Z|^2)$, for convenience. Using these expressions in equation (5) we obtain:

$$\sqrt{2} \left(\mathbf{u}_0 - \frac{1}{4}Z^2 \bar{\mathbf{u}}_0 - \frac{Z}{\sqrt{2}} \mathbf{C}_0 \right) \cdot \mathbf{T} = iI_T \left(\ddot{Z} - i\rho s \dot{Z} - \frac{1}{2}\dot{Z}^2 \frac{\bar{Z}}{1 + \frac{1}{4}|Z|^2} \right) \quad (35)$$

This and

$$\mathbf{T} \cdot \mathbf{C} = I\dot{s}$$

are exact equations of motion, and, suppressing $\dot{Z}^2$ and Z^2 , we obtain the approximate equations of motion.

In the case in which $\mathbf{T} = 0$, (the free body) we obtain

$$\ddot{Z} - i\rho s \dot{Z} - \frac{1}{2}\dot{Z}^2 \frac{\bar{Z}}{1 + \frac{1}{4}|Z|^2} = 0 \quad (36)$$

$$\dot{s} = 0.$$

By assuming that $Z = ae^{i\rho_a s t}$ where ρ_a is real, we obtain as a special solution to (36)

$$Z = ae^{i\rho_a s t},$$

where

$$\rho_a = \rho \left[\frac{1 + \frac{1}{4}|a|^2}{1 - \frac{1}{4}|a|^2} \right].$$

By observing that any other fixed axes might have been used for $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, and using equation (34) with $\alpha_2 = 0$, $Z_2 = B$, $Z_1 = ae^{i\rho_a s t}$ we obtain as a more general solution,

$$Z = \frac{B + ae^{i\rho_a s t}}{1 - \frac{1}{4}\bar{B}ae^{i\rho_a s t}}. \quad (37)$$

Since the solution contains two constants, A and B , it is sufficient to match arbitrary values of $Z(0)$ and $\dot{Z}(0)$, and hence is a completely general solution. When $|a|^2$ and $|\bar{B}a|$ can be ignored compared to unity, the solution reduces to that previously obtained. It will be noted that according to (37), Z travels in a circle, but not quite at uniform speed, unless the center of the circle coincides with the point $Z = 0$. Further, the time taken to trace out the circle is a second-order function of $|a|$.

Thus, we see that the approximate solution for the free-spinning body is a good first-order approximation to the exact solution, and we are led to suppose that such is the case in the more complicated solutions derived in the paper from the linear equation.