Analytical Studies Related to the Maxwell-Boltzmann Equation[†]

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0. Introduction. In spite of all the work based upon HILBERT's and ENSKOG's expansion methods in the kinetic theory of gases, no general existence and uniqueness theory for the Maxwell-Boltzmann equation has yet been established. Only two special cases have been elucidated: (1) for rigid spherical molecules CARLEMAN [4] proved the existence and uniqueness of a class of spatially homogeneous solutions depending only on the speed and demonstrated their uniform convergence in time toward the equilibrium solution, and (2) by use of the monotone character of WILD's [19] iteration process I proved [17] the existence and uniqueness of spatially homogeneous solutions for Maxwellian molecules. I venture to express doubt that assertions of equal extent can hold for the general case, as long as the theory is based upon the classical Maxwell-Boltzmann equation.

On the other hand, in the "derivations" of the Maxwell-Boltzmann equation the choice of approximations and terms neglected is to a certain degree arbitrary. In §5.1 I propose a new fundamental equation, differing from the old in the introduction of a mollifier and reducing to it in the spatially homogeneous case. This new equation follows from the usual physical argument with at least as much justice as does the classical one. While it does not generally imply the field equations of continuum mechanics, it yields the corresponding conservation laws in the large and the H -theorem.

For this equation I prove that within the class of non-negative densities having finite mass, there exists for all time a unique solution to the general initial value problem subject to boundary conditions appropriate to three cases: (1) unlimited space, (2) periodic space, (3) stationary vessel with perfectly reflecting walls. In fact, in Part I similar theorems are proved for differential-functional equations of a far more general type. These results may prove useful in connection with numerous open problems in the kinetic theory. It should be mentioned that the same process permits treatment of measures, which are most easily regarded as elements of the dual space of continuous functions. The estimates remain the same; but this seems of no importance for further investigations since the entropy is infinite.

For the application of these general theorems to the kinetic theory, an assumption of boundedness concerning the collision cross-section is required. This assumption is satisfied by molecules obeying an inverse n^{th} power law if and only if $n = 5$. In particular, for this reason the molecular model of rigid spheres is not included as a special case.

Part III concerns the trend toward equilibrium in the spatially homogeneous case. By means of a fairly general compactness theorem, stated as the main lemma, I show that as $t \rightarrow \infty$ the solution tends weakly toward a Maxwellian density which has the same mass, momentum and energy as does the initial density. It is hoped that the analysis here may be helpful in suggesting further investigation of the trend toward equilibrium.

I. Theory of a certain general differential-functional equation

1. Existence and uniqueness for a finite time. We consider a measure space Ω with points ω , the Banach space L^1 of real integrable functions on Ω with the norm $\|f\| = \int |f(\omega)| d\omega$ and a bounded bilinear operator $\mathfrak{A}(t)$ from $L^1 \times L^1$ into L^1 , depending continuously on t :

$$\begin{aligned} \mathfrak{A}(c_1 f_1 + c_2 f_2, g) &= c_1 \mathfrak{A}(f_1, g) + c_2 \mathfrak{A}(f_2, g) \\ (1.1) \quad \mathfrak{A}(f, c_1 g_1 + c_2 g_2) &= c_1 \mathfrak{A}(f, g_1) + c_2 \mathfrak{A}(f, g_2) \\ \| \mathfrak{A}(f, g) \| &\leq a \|f\| \|g\|. \end{aligned}$$

Take

$$(1.2) \quad t_0 < \frac{1}{4a}.$$

Theorem 1. *For the differential-functional equation*

$$(1.3) \quad \frac{\partial f(t, \omega)}{\partial t} = \mathfrak{A}(t; f(t), f(t)),$$

where the derivative is to be understood in the L^1 sense, corresponding to any initial value f_0 with $\|f_0\| \leq 1$ there exists in $0 \leq t \leq t_0$ a unique continuous solution $f(t, \cdot) \in L^1$. This solution, which depends continuously on f_0 , may be obtained by the iterative scheme

$$(1.4) \quad f_0(t) = f_0, \quad f_{n+1}(t) = f_0 + \int_0^t \mathfrak{A}(\tau; f_n(\tau), f_n(\tau)) d\tau.$$

Proof. We consider the operation

$$(1.5) \quad \phi(F)(t) = f_0 + \int_0^t \mathfrak{A}(\tau; F(\tau), F(\tau)) d\tau$$

and prove: (1) ϕ maps the set

$$(1.6) \quad \|F(t)\| \leq 2, \quad 0 \leq t \leq t_0,$$

into itself. For (1.5) implies

$$(1.7) \quad \|\phi(F)(t)\| \leq 1 + \int_0^t \|\mathfrak{A}(\tau; F(\tau), F(\tau))\| d\tau \leq 1 + 4at_0 < 2.$$

(2) ϕ is contracting. For different entries F_1, F_2 we obtain

$$\begin{aligned}
 (1.8) \quad \|\phi(F_1) - \phi(F_2)\| &\leq \int_0^t \|\mathfrak{A}(\tau; F_1, F_1) - \mathfrak{A}(\tau; F_2, F_2)\| d\tau \\
 &\leq \int_0^t [\|\mathfrak{A}(F_1, F_1, -F_2)\| + \|\mathfrak{A}(F_1 - F_2, F_2)\|] d\tau \\
 &\leq 4a \int_0^t \|F_1 - F_2\| d\tau \leq q \max_{0 \leq t \leq t_0} \|F_1(t) - F_2(t)\|
 \end{aligned}$$

with $q < 1$. Application of the fundamental theorem on contracting operators [1], [2], [18] to the space of L^1 -valued continuous functions on $0 \leq t \leq t_0$ completes the proof of the theorem.

It is possible to calculate the solution by means of another iteration process which corresponds to the expansion method proposed for the Maxwell-Boltzmann equation by JAFFÉ [11] and studied by KELLER [12]. To this end, we introduce into the equation (1.5)* a complex-valued parameter z :

$$(1.9) \quad \frac{\partial f(t, \omega)}{\partial t} = z \mathfrak{A}(t; f(t), f(t)).$$

Then the n^{th} iterate of the iteration process (1.4) becomes a polynomial of degree $2^n - 1$ in z . But all the estimates used in the proof hold uniformly for $|z| \leq 1 + \delta$ (we have to take $a^* = a(1 + \delta)$ and $t_0^* = t_0/(1 + \delta)$), and so we get a uniformly convergent sequence of analytic functions whose values are elements of the Banach space of L^1 -valued continuous functions on $0 \leq t \leq t_0^*$ [10]. By the Weierstrass theorem, the limit function is analytic. Therefore the solution of (1.9) is of the form

$$(1.10) \quad f(t, \omega; z) = \sum_{\lambda=0}^{\infty} \varphi_{\lambda}(t, \omega) z^{\lambda}, \quad |z| \leq 1 + \delta.$$

Thus follows

Theorem 1*. *For any initial value f_0 such that $\|f_0\| \leq 1$, the solution of (1.3) in $0 \leq t \leq t_0^*$ may be calculated from the iterative scheme*

$$\begin{aligned}
 (1.11) \quad \frac{\partial \varphi_0(t, \omega)}{\partial t} &= 0, \quad \varphi_0(0, \omega) = f_0(\omega), \\
 \frac{\partial \varphi_{\lambda}(t, \omega)}{\partial t} &= \sum_{\mu+\nu=\lambda-1} \mathfrak{A}(t; \varphi_{\mu}(t), \varphi_{\nu}(t)), \quad \varphi_{\lambda}(0, \omega) = 0,
 \end{aligned}$$

the series $f(t, \omega) = \sum_{\lambda=0}^{\infty} \varphi_{\lambda}(t, \omega)$ being uniformly convergent in the L^1 -sense for all $0 \leq t \leq t_0^*$.

Proof. We may insert (1.10) into (1.9) and equate like powers of z , then set $z = 1$.

* For the arguments here we admit complex-valued functions and complex Banach spaces.

2. Another iteration process and proof that non-negativeness is preserved.

We consider now a special type of equation described in Theorem 1:

$$(2.1) \quad \frac{\partial f(t, \omega)}{\partial t} = -f(t, \omega)\mathfrak{K}(t; f(t))(\omega) + \mathfrak{M}(t; f(t), f(t)(\omega),$$

where

$\mathfrak{K}$ is a linear non-negative operator from L^1 into L^∞ , bounded by m , and depending continuously on t as a mapping from the L^1 -strong to the L^∞ -weak-star topology, and

$\mathfrak{M}$ is a bilinear non-negative operator from $L^1 \times L^1$ into L^1 , bounded by m , and depending continuously on t .

We are interested in non-negative solutions corresponding to non-negative initial values.

It is possible to obtain the solution of this type of equation by an iteration process which remains within the class of non-negative functions. Such a process, which was applied in his case of the Maxwell-Boltzmann equation by CARLEMAN, is defined by

$$(2.2) \quad \frac{\partial f_{n+1}(t, \omega)}{\partial t} = -f_{n+1}(t, \omega)\mathfrak{K}(t; f_n(t))(\omega) + \mathfrak{M}(t; f_n(t), f_n(t))$$

with $f_0(t, \omega) = f_0(\omega)$ and the initial conditions $f_n(0, \omega) = f_0(\omega) \geq 0$, $\|f_0\| \leq 1$.

We investigate the operator $G = \Psi(F)$ defined by

$$G(0, \omega) = f_0(\omega),$$

$$(2.3) \quad \frac{\partial G(t, \omega)}{\partial t} = -G(t, \omega)\mathfrak{K}(t; F(t))(\omega) + \mathfrak{M}(t; F(t), F(t))(\omega)$$

or

$$(2.4) \quad G(t, \omega) = f_0(\omega) \exp\left(-\int_0^t \nu(\tau) d\tau\right) + \int_0^t \mu(\tau) \exp\left(-\int_\tau^t \nu(\sigma) d\sigma\right) d\tau$$

with

$$(2.5) \quad \nu(t) = \mathfrak{K}(t; F(t)), \quad \mu(t) = \mathfrak{M}(t; F(t), F(t)).$$

The set

$$(2.6) \quad \|F(t)\| \leq 2 \quad (0 \leq t \leq t_0), \quad F(t, \omega) \geq 0$$

is mapped by Ψ into itself, because (2.4) and $\nu \geq 0$ imply $\|\mu\| \leq 4m$,

$$(2.7) \quad \|G(t)\| \leq 1 + 4mt < 2.$$

For different entries F_1, F_2 with $\|F_i(t)\| \leq 2$ and $F_i(t, \omega) \geq 0$,

$$\|F_1(t) - F_2(t)\| < \epsilon,$$

we get for the difference

$$(2.8) \quad \begin{aligned} |\nu_1 - \nu_2| &\leq m\epsilon, & \left| \int_{\tau}^t (\nu_1(\sigma) - \nu_2(\sigma)) d\sigma \right| &\leq m t \epsilon, & \|\mu_1 - \mu_2\| &\leq 4m\epsilon, \\ \left| \exp \left(- \int_{\tau}^t \nu_1(\sigma) d\sigma \right) - \exp \left(- \int_{\tau}^t \nu_2(\sigma) d\sigma \right) \right| &\leq m t \epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} \|G_1(t) - G_2(t)\| &\leq m t \epsilon + \int_0^t (\mu_1(\tau) - \mu_2(\tau)) \exp \left(- \int_{\tau}^t \nu_1(\sigma) d\sigma \right) d\tau \\ &\quad + \int_0^t \mu_2(\tau) \left[\exp \left(- \int_{\tau}^t \nu_1(\sigma) d\sigma \right) - \exp \left(- \int_{\tau}^t \nu_2(\sigma) d\sigma \right) \right] d\tau \\ &\leq m t_0 \epsilon + 4m t_0 \epsilon + 4m^2 t_0^2 \epsilon \\ &\leq \epsilon(5t_0 m + 4t_0^2 m^2) \leq \epsilon q. \end{aligned}$$

If $q < 1$, Ψ is contracting in the set (2.6). Hence we select t_0 such that

$$5t_0 m + 4t_0^2 m^2 < 1$$

and obtain

Theorem 2. *For the equation (2.1), corresponding to any initial value $f_0(\omega) \geq 0$ with $\|f_0\| \leq 1$ there exists in $0 \leq t \leq t_0$ a unique non-negative continuous solution $f(t, \cdot) \in L^1$. This solution, which depends continuously on f_0 , may be obtained by any of the iterative processes (2.2), (1.4), (1.11).*

The last alternatives are a consequence of Theorems 1 and 1*.

Remark. Theorems 1, 1* and 2 remain valid for $\|f_0\| \leq C$ with properly adjusted t_0 .

3. Existence and uniqueness for all time.

Theorem 3 (Main theorem). *If the equation (2.1) has the further property that the integral of its right hand side with respect to ω vanishes for all $f \in L^1$, then for all t the initial value problem for a non-negative initial value has a unique non-negative solution, depending continuously on the initial value. The solution for a small time interval may be obtained by any of the iteration processes (1.4), (1.11) or (2.2).*

Proof. By Theorem 2, there is a non-negative solution for the time interval $0 \leq t \leq t_0$, where t_0 depends on f_0 only through $\|f_0\|$. Our new condition implies that $\int f(t, \omega) d\omega$ is independent of t . Hence

$$\|f(t_0)\| = \int |f(t_0, \omega)| d\omega = \int f(t_0, \omega) d\omega = \int f(0, \omega) d\omega = \|f_0\|.$$

Thus $f(t_0)$ may be taken as a new initial value in Theorem 2, which yields a solution in the interval $t_0 \leq t \leq 2t_0$, etc. Q.E.D.

II. Application to the Maxwell-Boltzmann equation

4. Maxwell-Boltzmann equation over phase space. Let Ω be phase space and let D/Dt denote differentiation along the trajectories of the continuous measure-preserving phase motion T^t . Conservation of density $F(t, \omega)$ would imply

$$DF(t, \omega)/Dt = 0.$$

I now call generalized Maxwell-Boltzmann equation over this phase space an equation describing the deviation from this process in the form

$$(4.1) \quad \frac{D}{Dt} F(t, \omega) = -F(t, \omega)\mathfrak{R}(F(t))(\omega) + \mathfrak{M}(F(t), F(t))(\omega),$$

with $\mathfrak{R}$ and $\mathfrak{M}$ satisfying the conditions of the main theorem of Part I. To show that we can apply the results of Part I, where we had true partial derivatives, we transform this equation by a measure preserving—and therefore L^1 -norm preserving—transformation depending on t , introducing

$$(4.2) \quad G(t, \omega) = F(t, T^t \omega) \quad \text{or} \quad F(t, \omega) = G(t, T^{-t} \omega),$$

which reduces the phase trajectories to rest. Then our equation falls under the type (4.1) but with true partial derivatives:

$$(4.3) \quad \frac{\partial}{\partial t} G(t, \omega) = -G(t, \omega)K(t; G(t))(\omega) + M(t; G(t), G(t))(\omega),$$

and the main theorem applies to it, giving the general existence and uniqueness theorem for non-negative solutions and the three iterative schemes for small time intervals. In the following everything will be formulated in terms of F .

5. Boundary value problems for a modified Maxwell-Boltzmann equation.

The modified equation. The ordinary Maxwell-Boltzmann equation, the central equation in the kinetic theory of gases, is an equation for the molecular density $F(t, x, \xi)$ depending on the time t , the position x in 3-space and the velocity ξ :

$$(5.1) \quad \begin{aligned} \frac{D}{Dt} F &\equiv \frac{\partial}{\partial t} F + \xi \operatorname{grad}_x F + g(x) \operatorname{grad}_\xi F \\ &= \iint B(p, \theta) [F(t, x, \eta) F(t, x, \eta') - F(t, x, \xi) F(t, x, \xi')] d\xi' d\alpha. \end{aligned}$$

Here the variables η, η' depend on ξ, ξ' through

$$(5.2) \quad \begin{aligned} \eta &= \xi + \alpha(\alpha(\xi' - \xi)) \\ \eta' &= \xi' - \alpha(\alpha(\xi' - \xi)) \end{aligned} \quad \begin{aligned} \alpha &= \text{unit vector} \\ \theta &= \angle(\alpha, \xi' - \xi) \\ p &= |\xi - \xi'|. \end{aligned}$$

For fixed α these formulæ represent a unitary transformation in the 6-space ξ, ξ' , so giving $\partial(\eta, \eta')/\partial(\xi, \xi') = 1$. $B \geq 0$ is a function of p and θ , calculated from the specific interaction law of collision; for Maxwellian molecules B is independent of p .

I propose to modify the collision integral so as to obtain

$$(5.3) \quad \frac{D}{Dt} F = \iiint B(p, \theta) [F(t, x, \eta) F(t, y, \eta') - F(t, x, \xi) F(t, y, \xi')] \cdot h(x, y) dy d\xi' d\alpha \equiv C_h(F)$$

where the continuous bounded mollifier h satisfies the conditions

$$(5.4) \quad h(x, y) = h(y, x) \geq 0, \quad h(x, x) > 0.$$

If $\int h(x, y) dy = 1$ then for spatially homogeneous solutions this equation reduces to the old one. In the general case (5.3) reduces formally to (5.1) if h is a delta-function. However, the restriction to bounded h is essential for proof of the theorems that follow, and thus except in the spatially homogeneous case our results will not apply to the classical Maxwell-Boltzmann equation.

The usual “derivations” [3], [8], [14] of the Maxwell-Boltzmann equation deliver at an intermediate stage of the argument a collision term containing $F(t, y, \xi')$ and $F(t, y, \eta')$, where y is the position of the oncoming molecule. It is customary to assume F “slowly varying” with respect to position so as to justify replacement of $F(t, y, \xi')$ by $F(t, x, \xi')$, yielding the classical equation (5.1). The assumption of “slowly varying” F certainly lends at least equal justice to replacement of $F(t, y, \xi')$ by values smoothed with some factor h over a region about x , thus yielding the new equation (5.3). The results proved below are unaffected by the functional form of h .

Conservation laws. We modify the usual argument by integrating over x as well as ξ . We interchange ξ, ξ' with η, η' ; then interchange ξ, η, x with ξ, η', y , thus obtaining

$$(5.5) \quad \begin{aligned} C_h[\phi] &\equiv \iint \phi(x, \xi) C_h(F) dx d\xi \\ &= \frac{1}{4} \iint [\phi(x, \xi) + \phi(y, \xi') - \phi(x, \eta) - \phi(y, \eta')] C_h(F) dx d\xi. \end{aligned}$$

Therefore $C_h[\phi] = 0$ when ϕ yields a summational invariant: $\phi = 1, \xi, \xi^2$. These choices of ϕ yield expressions of the conservation of total mass, momentum,

and energy. They are of the same forms as may be obtained by integrating the field equations of continuum mechanics over x -space. While the field equations themselves follow from the classical Maxwell-Boltzmann equation (5.1), in general from the new equation (5.3) they do not*. An exception is the continuity equation, $\rho_{,t} + (\rho \dot{x}_i)_{,i} = 0$, which can easily be derived by integration of (5.3) over ξ only.

H-theorem. Taking $\phi = \log F$ we obtain

$$(5.6) \quad C_h[\log F] = \frac{1}{4} \iiint\!\!\!\int B \log \frac{F(t, x, \xi)F(t, y, \xi')}{F(t, x, \eta)F(t, y, \eta')} \\ \cdot [F(t, x, \eta)F(t, y, \eta') - F(t, x, \xi)F(t, y, \xi')] h(x, y) dx dy d\alpha d\xi d\xi' \leq 0$$

because of the definiteness of the integrand. Therefore, as in the classical theory, if we set

$$(5.2) \quad H(t) = \iint F(t, x, \xi) \log F(t, x, \xi) dx d\xi$$

and assume the flux term vanishes, we get

$$(5.8) \quad \frac{dH(t)}{dt} \leq 0.$$

For $dH/dt = 0$ it is necessary that

$$(5.9) \quad F(x, \eta)F(y, \eta') = F(x, \xi)F(y, \xi')$$

not only for $x = y$ but also for x, y such that $h(x, y) \neq 0$. Therefore the only stationary solutions are

$$(5.10) \quad F = a \exp [-b(\xi - \xi_0)^2]$$

with constant a, b and ξ_0 . These results hold for all three boundary value problems stated below in the existence theorem.

Theorem of existence and uniqueness. Take Ω as the phase space x, ξ of a single molecule; write

$$(5.11) \quad \mathfrak{R}(f)(x, \xi) = \iiint B(p, \theta) h(x, y) f(y, \xi') dy d\xi' d\alpha, \\ \mathfrak{M}(f, g)(x, \xi) = \iiint B(p, \theta) h(x, y) f(x, \eta) g(y, \eta') dy d\xi' d\alpha;$$

* I express the opinion that they need not be implied by a kinetic theory, since they involve the interior pressure and heat flux, hidden quantities never susceptible of measurement.

and assume $\int B d\alpha < B_0$, a constant. Then our equation (5.3) assumes a form included in (2.1), since

$$(5.12) \quad \begin{aligned} \| \mathfrak{R}(f) \| = \text{ess. sup } | \mathfrak{R}(f)(x, \xi) | &\leq \text{Max } h \int B d\alpha \\ &\cdot \int | f(y, \xi') | dy d\xi' \leq \text{Max } h B_0 \| f \| \end{aligned}$$

and

$$\| \mathfrak{M}(f, g) \| \leq \text{Max } h \| g \| \| f \| B_0.$$

The integral of the right hand side is zero since $C_k[1] = 0$. Therefore we may apply Theorem 3 and obtain:

General theorem of existence and uniqueness. *In the modified Maxwell-Boltzmann equation (5.3), assume $\int B d\alpha$ bounded and impose one of the following boundary conditions:*

- (a) *unlimited space*
- (b) *periodic space*
- (c) *at a boundary point x of a region with continuously varying tangent plane $F(t, x, \xi^\perp) = F(t, x, \xi)$, where $\xi^\perp$ is obtained from ξ by specular reflection.*

Then corresponding to any non-negative initial density $F_0(x, \xi)$ yielding finite total mass there exists a unique solution for all time. This solution depends continuously on F_0 .

To justify the assertion in case (c) observe that the usual theorem on conservation of volume in phase space for continuous motion can be extended to specular reflection, thus permitting reduction to rest by (4.2) as in cases (a) and (b).

Remark. For inverse n^{th} power molecules, $\int B d\alpha$ is bounded if and only if $n = 5$. Whether there exist simple molecular models other than the Maxwellian for which $\int B d\alpha$ is bounded is unknown.

Remark concerning generalized solutions of the ordinary Maxwell-Boltzmann equation. It would be interesting to consider the limit case when h becomes a delta function. If the initial density possesses besides finite mass and energy a finite entropy and if the H -theorem could be proved to hold in this generality, then the compactness theorem (the main lemma to be proved in Part III) would state the existence at any t of at least one weak limit function which could be interpreted as a generalized solution of the classical Maxwell-Boltzmann equation. The uniqueness of this generalized solution and whether under a suitable condition of regularity it is a true solution remain open problems.

The existence of moments. For later use and to justify the theorem of conservation of energy we notice that the existence of the energy at $t = 0$ implies the existence of energy for some $0 \leq t \leq t_0$. For

$$\frac{1 + \xi^2}{(1 + \eta^2)(1 + \eta'^2)} \leq \frac{1 + \xi^2}{1 + \eta^2 + \eta'^2} = \frac{1 + \xi^2}{1 + \xi^2 + \xi'^2} \leq 1.$$

Hence in the iteration process of §1 we may replace the norm $\|f\|$ by the norm

$$\|f\|^* = \iint |f(x, \xi)| (1 + \xi^2) dx d\xi.$$

For the fourth moments there is a similar argument using

$$\|f\|^{**} = \iint |f(x, \xi)| (1 + \xi^2)^2 dx d\xi.$$

III. Trend toward equilibrium in the spatially homogeneous case

6. Outline of the proof. We wish to prove the

Main theorem on asymptotic behavior. *For any initial density $F_0(\xi)$ with finite mass $1 = \int F_0(\xi) d\xi$, energy $E = \int F_0(\xi)\xi^2 d\xi$, fourth moment $\int F_0(\xi)\xi^4 d\xi$ and $H = \int F_0(\xi) \log F_0(\xi) d\xi$ (essentially the negative of the entropy), and for certain B -functions, the corresponding spatially homogeneous solution of the Maxwell-Boltzmann equation when the extrinsic force g is zero tends weakly as $t \rightarrow \infty$ to the unique Maxwellian density F_M with the same mass, momentum and energy.*

Remarks. (1) The weak convergence means: Given a finite number of bounded functions $\varphi_\nu(\xi)$ and an $\epsilon > 0$, then there exists a $t(\epsilon)$ such that

$$(6.1) \quad \left| \int (F(t, \xi) - F_M(\xi)) \varphi_\nu(\xi) d\xi \right| < \epsilon \quad \text{for } t \geq t(\epsilon).$$

The weak convergence asserted by the theorem will hold even for the broader class of functions satisfying

$$(6.2) \quad |\varphi(\xi)| \leq C(1 + \xi^2).$$

The weak convergence is appropriate to the kinetic theory, since all physical quantities are expressed as integrals over the density, which in itself cannot be measured.

(2) The class of B -functions consists in those which have bounded $\int B d\alpha$, yield the boundedness of the fourth moments (§9) and admit the Gauss transformation (§10).

We give here the outline of the proof; the details are provided in the sections which follow.

In the first place it will be essential for the proof that the initial values, and for some B -functions, also the solution to the initial value problem, have not only constant mass and energy, but also bounded fourth moments (§9).

Another property is that the Gauss transforms

$$(6.3) \quad G(t, \zeta) = \int e^{-\pi(\xi-\zeta)^2} F(t, \xi) d\xi,$$

where the factor in the exponential is given the special value π only for simplicity, shall for some B -functions satisfy a Maxwell-Boltzmann equation with the same B -function; this is proved in §10. It would be sufficient if they yielded another B -function B^* with bounded $\int B^* d\alpha$.

But these Gauss transforms G of functions F which have mass 1 and energy E and bounded fourth moments $\leq \Gamma$ are shown in §10 to have sufficient regularity

that the usual argument [5] indicating the monotone decrease of H becomes a proof, since all integrals involved converge uniformly. These estimates are demonstrated in §11. Hence follows the decrease of H of the solution F (§12). By an argument similar to that used by CARLEMAN [4] it is also possible to prove for these Gauss transforms the trend toward the Maxwellian equilibrium solution (§13).

Thus the solution $F(t, \xi)$ to our initial value problem may be considered as a family of non-negative functions with

$$(6.4) \quad \int F(\xi) d\xi = 1, \quad \int F(\xi) \xi^2 d\xi = E, \quad \int F(\xi) \log F(\xi) d\xi \leq H.$$

It is proved as the main lemma in §7 that for such a family, any sequence contains a subsequence converging weakly to some $f^* \in L^1$. Hence there exists for any sequence $t_\nu \rightarrow \infty$ at least one subsequence $t_{\nu_i} \rightarrow \infty$ for which $F(t_{\nu_i}, \xi) \rightharpoonup F^*(\xi)$. The proof is completed if we can show that this limit function is a Maxwellian density $a \exp [-b(\xi - \xi_0)^2]$ whose parameters are independent of the sequence uniquely determined by the mass, momentum and energy of the initial density.

For this purpose let us again consider the Gauss transforms. Since the Gauss transformation converts any weakly convergent sequence into a strongly convergent one, uniformly convergent in any bounded set, and since we already established the convergence of $G(t, \xi)$ toward a Maxwellian density, $F^*(\xi)$ has to be Maxwellian as a consequence of the one-to-one property of the Gauss transformation (§8). The parameters of F^* are uniquely determined by the mass, momentum and energy, which are preserved even for the weak limit function, as a consequence of the weak convergence and the boundedness of the fourth moment:

(1) The 4th moment of a weak limit function is also $\leq \Gamma$. For

$$(6.5) \quad \int_{|\xi| \leq R} F^*(\xi) \xi^4 d\xi = \lim_{t_{\nu_i} \rightarrow \infty} \int_{|\xi| \leq R} F(t, \xi) \xi^4 d\xi \leq \Gamma \quad \text{for all } R.$$

(2) Let $\varphi = 1, \xi, \xi^2$. Then

$$(6.6) \quad \int F(t, \xi) \varphi(\xi) d\xi = \int_{|\xi| \leq R} F(t, \xi) \varphi(\xi) d\xi + \int_{|\xi| \geq R} F(t, \xi) \varphi(\xi) d\xi,$$

and the last term is in all three cases

$$(6.7) \quad \left| \int_{|\xi| \geq R} F(\xi) \varphi(\xi) d\xi \right| \leq \frac{1}{R^2} \left| \int_{|\xi| \geq R} F(\xi) \xi^4 d\xi \right| \leq \frac{\Gamma}{R^2} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

while the first term converges as $t \rightarrow \infty$ as implied by the weak convergence. Q.E.D.

7. The main lemma. Here we prove the fundamental compactness theorem.

Preliminary remark concerning integrals. The integrals $\int \cdots d\xi$, if not absolutely convergent, are to be interpreted as

$$(7.1) \quad \lim_{R \rightarrow \infty} \int_{|\xi| \leq R} \cdots d\xi.$$

Sublemma. *Assumptions:*

- (a) $f = f(x, \xi) \geq 0$, x -region bounded.
- (b) $M(f) = \iint f dx d\xi$,
- (c) $E(f) = \iint f \xi^2 dx d\xi$,
- (d) $M(f) \leq kE(f)$, $E(f) \leq E^*$.

Conclusion: There exists a C depending only on k and E^* such that

$$(7.3) \quad H(f) \equiv \iint f(x, \xi) \log f(x, \xi) dx d\xi \geq C$$

and $C \rightarrow 0$ if $k \rightarrow 0$, E^* fixed.

Proof. Because of the convexity of the function $s \log s$ it is sufficient to consider functions independent of x and depending on $|\xi|$ only, since taking

$$(7.4) \quad \tilde{f}(|\xi|) = \frac{\iint f(x, |\xi| \alpha) dx d\alpha}{\iint 1 dx d\alpha} \quad (\alpha = \text{unit vector})$$

does not change the assumptions but decreases $H(f) \geq H(\tilde{f})$. Then we apply the

Lemma of Gibbs [7] (cf. [4] §7, p. 133–143). If

$$f = f(\xi) \geq 0, \quad \int f d\xi = A, \quad \int f \xi^2 d\xi \leq B,$$

then

$$H(f) \equiv \int f(\xi) \log f(\xi) d\xi \geq C(A, B),$$

and the infimum is achieved only for $f = a \exp(-b\xi^2)$, where a and b are to be determined so as to give $\int f d\xi = A$, $\int f \xi^2 d\xi = B$.

Therefore we need only consider functions of the type

$$(7.5) \quad f(\xi) = \beta \exp(-\alpha^2 \xi^2) \quad (\alpha, \beta \geq 0).$$

They give*

$$(7.6) \quad \begin{aligned} M &= \beta \int_0^\infty \rho^2 \exp(-\alpha^2 \rho^2) d\rho = \frac{\beta}{\alpha^3} M_0 \quad (\rho = |\xi|), \\ E &= \beta \int_0^\infty \rho^4 \exp(-\alpha^2 \rho^2) d\rho = \frac{\beta}{\alpha^5} E_0, \end{aligned}$$

and the assumptions state that

$$(7.7) \quad \frac{\beta}{\alpha^3} \leq k_0 \frac{\beta}{\alpha^5}, \quad \frac{\beta}{\alpha^5} \leq E_0^* \quad \left(E_0^* = \frac{E^*}{E_0}, k_0 = \frac{E_0}{M_0} k \right),$$

which implies $\alpha \leq \sqrt{k_0}, \beta \leq \sqrt{k_0^5} E'$. Hence we obtain

$$(7.8) \quad \begin{aligned} H(f) &= \beta \int_0^\infty \rho^2 \exp(-\alpha^2 \rho^2) (-\alpha^2 \rho^2 + \log \beta) d\rho = \frac{\beta}{\alpha^3} E_0 + \frac{\beta \log \beta}{\alpha^3} M_0 \\ &\geq -k_1 E^* + \left(\frac{\beta}{\alpha^5} \right)^{\frac{1}{2}} \beta^{\frac{1}{2}} \log \beta M_0 \geq -k_1 E^* - E'^{\frac{1}{2}} M_0 b(k_0), \end{aligned}$$

since β is bounded and $\beta^{\frac{1}{2}} \log \beta$ tends to zero as $\beta \rightarrow 0$. It is plain that this lower bound tends to zero as $k \rightarrow 0$.

Corollary. For densities with $\iint f dx d\xi = 1, \iint f \xi^2 dx d\xi = E$, the sublemma asserts the existence of $\iint f \log f dx d\xi$ in the sense of the preliminary remark, provided $\iint_{|\xi| \leq R} f \log f dx d\xi$ stays bounded for all R .

Proof of the corollary. For ease of writing, set

$$(7.9) \quad a(\rho) = \iint f(x, \alpha \rho) \log f(x, \alpha \rho) dx d\alpha \rho^2.$$

Assume $\int_0^R a(\rho) d\rho \leq C; \int_0^R a(\rho) d\rho \geq -D$ by (7.3). We wish to conclude the existence of $\lim_{R \rightarrow \infty} \int_0^R a(\rho) d\rho$. If this limit did not exist we should have a sequence $R_1, R_2, \dots$ with

$$(7.10) \quad \left| \int_{R_\nu}^{R_{\nu+1}} a(\rho) d\rho \right| \geq \epsilon > 0.$$

Hence either

$$\int_{R_\nu}^{R_{\nu+1}} a(\rho) d\rho \geq \epsilon \quad \text{or} \quad \leq -\epsilon \quad \text{infinitely often.}$$

* Here we take the number of dimensions as 3, although all the theorems stated hold in fact in Euclidean n -space.

In the first case we take the complements of the intervals R_ν , $R_{\nu+1}$ and get in any case a sequence of non-overlapping intervals I_ν with

$$\int_{I_\nu} a(\rho) d\rho \leq -\epsilon.$$

Therefore

$$\int_{\cup I_\nu} a(\rho) d\rho = -\infty,$$

in contradiction to the sublemma, applied to the function

$$(7.11) \quad f^*(\xi) = \begin{cases} f(\xi) & \text{if } |\xi| \in \cup I_\nu \\ 0. & \end{cases}$$

Main lemma. Any infinite set of functions such that

$$f = f(x, \xi) \geq 0 \quad (x - \text{region bounded})$$

$$\iint f dx d\xi = M,$$

$$(7.12) \quad \iint f \xi^2 dx d\xi = E,$$

$$\iint f \log f dx d\xi \leq H,$$

contains a subsequence converging weakly to an element in L^1 .

Remark. Convergence holds even for functions such that

$$|\varphi(x, \xi)| \leq C\sqrt{1 + \xi^2}:$$

$$(7.13) \quad \iint (f_n - f^*)\varphi(x, \xi) dx d\xi \rightarrow 0.$$

Proof of the main lemma. In order to prove this compactness property, we have to show that $\|f\|_{|\xi| \geq R} \rightarrow 0$ uniformly as $R \rightarrow \infty$ and prove the uniform integrability of f in any bounded set [6].

The former property follows easily from

$$(7.14) \quad \|f\|_{|\xi| \geq R} \equiv \iint_{|\xi| \geq R} f dx d\xi \leq \frac{1}{R^2} \iint_{|\xi| \geq R} \xi^2 f dx d\xi \leq \frac{E}{R^2}.$$

Application of the sublemma yields then

$$(7.15) \quad \iint_{|\xi| \geq R} f \log f dx d\xi \geq C(R).$$

Therefore

$$(7.16) \quad \iint_{|\xi| \leq R} f \log f \, dx \, d\xi \leq H - C(R),$$

uniformly bounded for fixed R . Because of $s \log^+ s \leq s \log s + 1/e$ (7.16) implies furthermore that

$$(7.17) \quad \iint_{|\xi| \leq R} f \log^+ f \, dx \, d\xi \leq H - C(R) + \frac{1}{e} \iint_{|\xi| \leq R} dx \, d\xi \leq C_0(R).$$

Hence by NAGUMO's criterion [16] the uniform integrability follows.

8. Some lemmas.

A geometrical lemma. *Given two similar rectangles $P_1 P_2 P_3 P_4$ and $Q_1 Q_2 Q_3 Q_4$ with order-preserving similarity in a plane, we have*

$$(8.1) \quad \sum_{\nu=1}^4 (-1)^\nu \overline{P_\nu Q_\nu}^2 = 0.$$

This may be proved by elementary calculations.

Modified Gibbs lemma. *If $f = f(\xi) \geq 0$,*

$$(8.2) \quad \int f \, d\xi = A, \quad \int f \xi \, d\xi = I, \quad \int f \xi^2 \, d\xi \leq E,$$

then

$$H(f) \equiv \int f(\xi) \log f(\xi) \, d\xi \geq C(A, I, E)$$

and the infimum is achieved only for $f_0(\xi) = a \exp(-b(\xi - \xi_0)^2)$, where a , b and ξ_0 are to be determined so that

$$\int f_0 \, d\xi = A, \quad \int f_0 \xi \, d\xi = I, \quad \int f_0 \xi^2 \, d\xi = E.$$

Proof. We write

$$(8.3) \quad f(\xi) = \varphi(\xi) f_0(\xi), \quad H_0 = H(f_0).$$

By the assumption, we have

$$(8.4) \quad \int \varphi f_0 \, d\xi = A = \int f_0 \, d\xi,$$

$$\int \varphi(\xi - \xi_0)^2 f_0 \, d\xi \leq E - 2\xi_0 I + \xi_0^2 \int \varphi f_0 \, d\xi = \int (\xi - \xi_0)^2 f_0 \, d\xi.$$

Therefore we get

$$H(f) - H(f_0) = \int f_0 [\phi \log \phi + (\phi - 1) \log a - (\phi - 1)b(\xi - \xi_0)^2] d\xi \geq 0$$

because $\phi \log \phi - \phi + 1 \geq 0$, and equality only if $\phi = 1$, i.e. $f = f_0$ a.e. Q.E.D.

One-to-one mapping by the Gauss transformation. If $f \in L^1$ and

$$(8.5) \quad f_\delta(\xi) = \mathfrak{G}_\delta(f) = \sqrt{\frac{\delta^3}{\pi}} \int e^{-\delta(\xi-\zeta)^2} f(\zeta) d\zeta = 0 \quad \text{a.e.,}$$

then $f(\zeta) = 0$ a.e.

Proof. $f_\delta(\xi)$ is an analytic function of both variables $1/\delta$ and ξ . It is zero for some $1/\delta$; then it is zero for all greater $1/\delta$ (i.e. smaller δ) by the classical semi-group property of the Gauss transformation:

$$(8.6) \quad \mathfrak{G}_\delta \mathfrak{G}_\epsilon = \mathfrak{G}_{\delta\epsilon/\delta+\epsilon}.$$

Being analytic, it is therefore zero for all $\delta > 0$; but, as usually proved in connection with the initial value problem of the heat equation for unlimited space, $f_\delta \rightarrow f$ in L^1 as $\delta \rightarrow \infty$; therefore $f = 0$ a.e. Q.E.D.

9. Boundedness of the fourth moment. Since $\int F(t, \xi) d\xi$ and $\int F(t, \xi) \xi^2 d\xi$ are constant for any spatially homogeneous solution, we derive the differential equation

$$(9.1) \quad \frac{dF_4(t)}{dt} = -A_1 F_4 + A_2 \quad (A_i = \text{const.}, A_1 > 0),$$

and from it we infer the boundedness of the 4th moment $F_4(t) = \int F(t, \xi) \xi^4 d\xi$ for all times if it exists initially.

In order to prove (9.1) I assume B to be constant, say $B = 1$. We have

$$\begin{aligned} \frac{dF_4}{dt} &= \int \frac{\partial F(t, \xi)}{\partial t} \xi^4 d\xi = \frac{1}{4} \iiint (\xi^4 + \xi'^4 - \eta^4 - \eta'^4) \\ &\quad (F(\eta)F(\eta') - F(\xi)F(\xi')) d\xi d\xi' d\alpha \\ (9.2) \quad &= \frac{1}{2} \iiint [\eta^2(\alpha\eta')^2 - \eta^2(\alpha\eta)^2 - \eta'^2(\alpha\eta')^2 + \eta'^2(\alpha\eta)^2 + (\alpha\eta')^4 \\ &\quad - 2(\alpha\eta')^2(\alpha\eta)^2 + (\alpha\eta)^4] F(\eta)F(\eta') d\eta d\eta' d\alpha \\ &= \text{constant terms (involving mass and energy)} \\ &\quad - \iiint [\eta^2(\alpha\eta)^2 - (\alpha\eta)^4] F(\eta) d\eta d\alpha. \end{aligned}$$

Because of

$$(9.3) \quad \int [\eta^2(\alpha\eta)^2 - (\alpha\eta)^4] d\alpha = C\eta^4 \quad (C > 0),$$

we obtain (9.1).

The same result holds when $B = B(\theta)$, both in three dimensions (Maxwellian molecules) and in the plane, because the collision term for any polynomial of degree m can be expressed as a polynomial in the moments of degree $\leq m$ [15]; for reasons of symmetry in the present case the form (9.1) results for dF_4/dt .

10. The Gauss transforms. In this section we prove that for certain B -functions the Gauss transform

$$(10.1) \quad F_\delta(t, \zeta) = \sqrt{\frac{\delta^3}{\pi}} \int e^{-\delta(\xi-\zeta)^2} F(t, \xi) d\xi$$

of the solution of the Maxwell-Boltzmann equation satisfies itself a Maxwell-Boltzmann equation. For simplicity we take $\delta = \pi$ and write G instead of F_π . Assuming B to be constant, say $B = 1$, we multiply the equation

$$\frac{\partial F(t, \xi)}{\partial t} = \iint [F(t, \eta)F(t, \eta') - F(t, \xi)F(t, \xi')] d\xi' d\alpha$$

by

$$(10.2) \quad \exp(-\pi(\xi - \zeta)^2 - \pi(\xi' - \zeta')^2)$$

and integrate with respect to ξ, ξ' . If χ, χ' are related to ζ, ζ' as η, η' to ξ, ξ' , this transformation being unitary, we get

$$(10.3) \quad (\eta - \chi)^2 + (\eta' - \chi')^2 = (\xi - \zeta)^2 + (\xi' - \zeta')^2$$

for each value of α . Therefore the result of this integration is

$$(10.4) \quad \frac{\partial G(t, \zeta)}{\partial t} = \iint [G(t, \chi)G(t, \chi') - G(t, \zeta)G(t, \zeta')] d\zeta' d\alpha.$$

For $B = B(\theta)$ this same result can be obtained in the plane case. To see this, we consider two transformations $\xi, \xi' \rightarrow \eta, \eta'$ and $\zeta, \zeta' \rightarrow \chi, \chi'$, having the same θ . Since the two rectangles ξ, η, ξ', η' and $\zeta, \chi, \zeta', \chi'$ are similar, the geometrical lemma (§8.1) yields the essential equation (10.3). I conjecture that a similar result but with different B^* holds in three dimensions for Maxwellian molecules.

Properties of the Gauss transforms. Regarding the Gauss transforms of functions $F \geq 0$ with $\int F d\xi = 1$ and $\int F\xi^2 d\xi = E$, we observe the following properties:

(a) G is non-negative, infinitely differentiable, and satisfies

$$\int G(\zeta) d\zeta = 1.$$

- (b) $\int G(\zeta) \zeta d\zeta = \int F(\xi) \xi d\xi$.
- (c) *Second moment relation:* $\int G(\zeta) \zeta^2 d\zeta = \int F(\xi) \xi^2 d\xi + \int F(\xi) d\xi = E + 1$, and similar results for the higher moments.
- (d) $-\infty < H(G) \equiv \int G(\zeta) \log G(\zeta) d\zeta \leq \int F(\xi) \log F(\xi) d\xi \equiv H(F)$ because of the convexity of $s \log s$, $H(G)$ being bounded from below by the Gibbs lemma.
- (e) All G are uniformly bounded by 1 and are uniformly continuous.
- (f) Any infinite sequence of G possesses a subsequence converging uniformly in any bounded set $|\xi| \leq R$ (this follows from Arzela's theorem by iterated selection).
- (g) Weakly convergent sequences are transformed into strongly convergent sequences, converging uniformly in any bounded set.
- (h) If $|\xi| \geq R$, where R is sufficiently large, then $\exp(-C\xi^2) \leq G(\xi) \leq 1$, and therefore $-C\xi^2 \leq \log G(\xi) \leq 0$.

Proof. The right inequality was stated in (e), the left

$$\int_{|\xi| \leq R_1} F(\xi) d\xi = 1 - \int_{|\xi| \geq R_1} F(\xi) d\xi \geq 1 - \frac{1}{R_1^2} \int_{|\xi| \geq R_1} \xi^2 F(\xi) d\xi \geq \frac{1}{2}$$

for proper R_1 . Then for

$$\begin{aligned} G(\zeta) &\geq \int_{|\xi| \leq R_1} \exp[-\pi(\zeta - \xi)^2] F(\xi) d\xi \geq \exp[-\pi(|\zeta| + R_1)^2] \int_{|\xi| \leq R_1} F(\xi) d\xi \\ &\geq \frac{1}{2} \exp(-\pi R_1^2) \exp(-\pi \zeta^2). \end{aligned}$$

- (i) In $|\xi| \leq R$ we have

$$0 < c \leq G(\xi) \leq C.$$

As in the proof of (h) follows

$$G(\xi) \geq \frac{1}{2} \exp[-\pi(R_1 + R)^2].$$

11. Uniform estimates for the H and H' -integrals. (a) *The H -integral.* We have

$$\begin{aligned} (11.1) \quad \int_{|\xi| \geq R} |G(\xi) \log G(\xi)| d\xi &\leq \int_{|\xi| \geq R} G(\xi) C \xi^2 d\xi \leq \frac{C}{R^2} \int_{|\xi| \geq R} G(\xi) \xi^4 d\xi \\ &\leq \frac{C}{R^2} \Gamma \rightarrow \sigma \text{ as } R \rightarrow \infty \quad (\Gamma \text{ a bound for } \int G(\xi) \xi^4 d\xi). \end{aligned}$$

Since $s \log s$ is uniformly continuous in $0 < c \leq s \leq C$, which is by property (i) the range of the function values of G for $|\xi| \leq R$, the continuity with respect to uniform convergence in any bounded set $|\xi| \leq R$ follows.

(b) *The H' -integral.* Here we have

$$(11.2) \quad H'(G) \equiv \iiint B \log G(\xi)(G(\eta)G(\eta') - G(\xi)G(\xi')) d\xi d\xi' d\alpha.$$

The absolute convergence of the second term, which is equal to

$$H(G) \int G(\xi) d\xi \int B d\alpha,$$

follows from (a).

For the first term we have

$$(11.3) \quad \begin{aligned} & \iiint_{|\xi| \geq R} |B \log G(\xi)G(\eta)G(\eta')| d\xi d\xi' d\alpha \\ & \leq \iiint_{|\xi| \geq R} BC\xi^2 G(\eta)G(\eta') d\xi d\xi' d\alpha \\ & \leq \iiint_{\xi^2 + \xi'^2 \geq R^2} BC(\xi^2 + \xi'^2)G(\eta)G(\eta') d\xi' d\xi d\alpha \\ & = \iiint_{\eta^2 + \eta'^2 \geq R^2} BC(\eta^2 + \eta'^2)G(\eta)G(\eta') d\eta d\eta' d\alpha \\ & \leq \frac{1}{R^2} \int BC d\alpha \iint_{\eta^2 + \eta'^2 \geq R^2} (\eta^2 + \eta'^2) G(\eta)G(\eta') d\eta d\eta' \\ & \leq \frac{2B_0}{R^2} (E^2 + \Gamma) \rightarrow \sigma \text{ as } R \rightarrow \infty. \end{aligned}$$

12. The H -theorem. For functions which are Gauss transforms, the H -theorem follows directly from the negativeness of H' (§5.6) and the foregoing estimates for the H and the H' -integrals.

In order to prove that

$$(12.1) \quad H(F(t)) \leq H(F(0)) \equiv H_0,$$

we must assume B such as to yield bounded fourth moments for the solution and admitting the Gauss transformation. We then have

$$(12.2) \quad H(F_\delta(t)) \leq H(F_\delta(0)) \leq H(F(0)) = H_0 \text{ for all } \delta.$$

Since by the sublemma

$$(12.3) \quad \int_{|\xi| \geq R} F_\delta \log F_\delta d\xi \geq -\epsilon$$

if R is sufficiently large, we obtain

$$\int_{|\xi| \leq R} F_\delta \log F_\delta d\xi \leq H_0 + \epsilon.$$

Now, as $\delta \rightarrow \infty$, F_δ converges to F in L^1 ; therefore a subsequence F_{δ_s} converges almost everywhere. Since $s \log s \geq -1/e$, we can apply the lemma of FÉLIX in any region $|\xi| \leq R$, yielding

$$(12.4) \quad \int_{|\xi| \leq R} F \log F d\xi \leq H_0 + \epsilon.$$

Since R may be taken arbitrarily large, this proves our assertion.

13. Proof of the limit behavior for the Gauss transforms. We assume here that $G(t, \xi)$ is a Gauss transform of $F(t, \xi) \geq 0$ with

$$\int F d\xi = 1, \quad \int F \xi^2 d\xi = E, \quad \int F \xi^4 d\xi \leq \Gamma,$$

and satisfies the Maxwell-Boltzmann equation. We prove that $G(t, \xi)$ tends toward a Maxwellian density as $t \rightarrow \infty$.

The H -theorem together with the lower bound for the values of H implies by the sublemma (§7) that

$$(13.1) \quad H(t) \equiv H(G(t)) \equiv \int G(t, \xi) \log G(t, \xi) d\xi \searrow H_0.$$

Since the derivative $H'(t)$ exists, we have at least one sequence $t_\nu^* \rightarrow \infty$ for which

$$(13.2) \quad \lim H'(t_\nu^*) = 0.$$

For a suitable subsequence $t_{\nu_i}^*$ (§10) we have $G(t_{\nu_i}^*, \xi) \rightarrow G^*(\xi)$ uniformly in any bounded set. Therefore by the continuity of the H' -integral we have

$$(13.3) \quad H'(G^*) = 0,$$

which, by the usual theorems on summational invariants [13], applicable since G^* is infinitely many times differentiable, implies that $G^*(\xi)$ is Maxwellian.

From the continuity of H also $H(G^*) = H_0$. Given any other sequence $t_\nu \rightarrow \infty$ we chose a subsequence $t_{\nu_i} \rightarrow \infty$ for which $G(t_{\nu_i}, \xi) \rightarrow G^{**}(\xi)$.

Because of the continuity of the H -integral, we have $H(G^{**}) = H_0$. But by the modified Gibbs lemma (§8), the quantity H for the Maxwellian density is minimum among all densities with given mass, momentum and energy, and the minimum is achieved only for the Maxwellian density. Hence $G^{**} = G^*$ and $G(t, \xi) \rightarrow$ a Maxwellian density, as $t \rightarrow \infty$. Q.E.D.

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