*Solution, for all Values of the Time, of Initial Value Problems for the Wave Equation and for a System of Equations in Acoustics**

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Communicated by M. H. MARTIN

1. Introduction. If in Euler's equations of motion and continuity for two dimensional flow:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y},$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,$$

one makes the *acoustic approximation*

$$p = \text{const.} + c^2 \rho,$$

* This work was sponsored by the U. S. Atomic Energy Commission.

where c is a positive constant, and if one also neglects the quadratic terms, one obtains the following linear system of three first order partial differential equations with constant coefficients:

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \lambda}{\partial x}, \quad \frac{\partial v}{\partial t} = -c^2 \frac{\partial \lambda}{\partial y}, \quad \frac{\partial \lambda}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (\text{S})$$

where $\lambda \equiv \log \rho$. The purpose of the present paper is to solve the Cauchy problem for the system (S), Cauchy data (*i.e.*, the values of u, v, λ) being prescribed on the plane $t = 0$ (see equations (31), (32) of section 3). An explicit formula for the solution, in terms of the given Cauchy data, is obtained in section 3. No direct appeal is made to the general theory of systems of partial differential equations with constant coefficients, as developed by BAREISS [7], F. BUREAU [5], K. O. FRIEDRICH [10], and J. LERAY [12]. The simple fact that the time derivative of each function appearing in (S) is a solution of the wave equation,

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0, \quad (1)$$

forms the basis of the present paper. Section 2 deals with a *past, present, and future* form of the Cauchy problem for the wave equation (1), in that a solution is sought for *all* times t , which satisfies the partial differential equation *even on* the plane $t = 0$, the carrier of the Cauchy data. (Questions of a related nature, with Cauchy data, however, given on the time-like manifold, $x = 0$, rather than on the space-like manifold $t = 0$ considered here, are studied by F. JOHN [11].) The hyperbolic equation (1) is in sharp contrast with the elliptic equation which results from it by replacing $-c^2$ by $+c^2$. For this elliptic equation (which is essentially Laplace's equation) no such global result as theorem 2 of the present paper is possible, for either the Cauchy or the Dirichlet problem with data given on $t = 0$. Section 3 deals with the Cauchy problem for the time derivative of the wave equation (1), and finally section 4 is concerned with the above-mentioned Cauchy problem for the system (S).

2. Vibrating membrane. Consider the determination of the deflection of an infinite vibrating membrane, once its initial deflection and initial velocity are specified at time $t = 0$. Analytically, this consists in the solution of the Cauchy problem:

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0, \quad (1)$$

$$w(x, y, 0) = f(x, y); \quad \frac{\partial w}{\partial t}(x, y, 0) = g(x, y), \quad (2)$$

where the functions f and g are given functions, and c is a positive constant. The classical solution of this problem, which VOLTERRA [1, p. 215] has traced back to POISSON and PARSEVAL, is given by

$$w(x, y, t) = \pm \frac{1}{2\pi c} \iint_{\rho^2 < c^2 t^2} \frac{g(\xi, \eta)}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta \quad (3)$$

$$\pm \frac{1}{2\pi c} \frac{\partial}{\partial t} \left[\iint_{\rho^2 < c^2 t^2} \frac{f(\xi, \eta)}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta \right],$$

for $t \geq 0$, where $\rho^2 = (\xi - x)^2 + (\eta - y)^2$, and $(\)^{\frac{1}{2}}$ denotes the positive square root.

For the present purposes, it is necessary to have a precise result stating under what circumstances the formula (3) actually furnishes a solution of the Cauchy problem (1), (2). It appears that in most discussions of the Cauchy problem (1), (2) it is only shown that if a solution exists, then the solution is given by (3), but no attention is given to the question of whether (3), actually furnishes a solution of (1), (2). A significant exception is the discussion of a similar Cauchy problem for the wave equation in three space dimensions:

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = 0, \quad (4)$$

$$w(x, y, z, 0) = w_0(x, y, z); \quad \frac{\partial w}{\partial t}(x, y, z, 0) = w_1(x, y, z), \quad (5)$$

which is given by HADAMARD [4, pp. 60–69]. Although the solution of (1), (2) can be obtained from that of (4), (5) by HADAMARD'S *method of descent* [4, pp. 69–71], the information thus obtained about (3) is insufficient for the problem treated in the present paper.

To simplify the statement of the results the following terminology will be used throughout. A real-valued function $f(x, y)$ defined for all real (x, y) will be said to be of class $C^n(x, y)$ if it is continuous together with all its partial derivatives of all orders up to and including n , for all (x, y) . A similar interpretation applies to the statement that " $w(x, y, t)$ is in $C^n(x, y, t)$ ".

Theorem 1. *If f is in $C^3(x, y)$ and g is in $C^2(x, y)$, then the Cauchy problem (1), (2) possesses one and only one solution $w(x, y, t)$ in $C^2(x, y, t)$.*

Proof. The uniqueness of the solution, under the hypotheses made on the Cauchy data f and g , follows from the fact that if there is a solution then (see e.g. VOLTERRA [1, p. 215]) it must be given by formula (3). Alternatively, one may use a uniqueness argument of ZAREMBA [2] (see also [4, pp. 476–478]). It only remains to show that the function $w(x, y, t)$ defined by (3), or by an equivalent formula, is in $C^3(x, y, t)$ and that it satisfies the partial differential equation (1) for all (x, y, t) , even on the plane $t = 0$, which is the bearer of the Cauchy data, and also fulfills the Cauchy conditions (2).

In order to do this, formula (3) for the solution w of (1), (2), which is usually considered only for positive t , will be rewritten in a form which is valid for all real t . Writing

$$\xi - x = \rho \cos \theta, \quad \eta - y = \rho \sin \theta; \quad (6)$$

making the change of variable $\rho = ctR$ (which renders constant the limits of integration in (3)); and then carrying out the indicated differentiation with respect to t , one obtains

$$w(x, y, t) = \frac{t}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{g(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR \\ + \frac{\partial}{\partial t} \left[\frac{t}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{f(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR \right], \quad (7)$$

and

$$w(x, y, t) = \frac{t}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{g(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR \\ + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{f(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR \quad (8) \\ + \frac{t}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{\frac{\partial f}{\partial t}(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR,$$

where, in the last integrand, $\partial f / \partial t$ stands for $(\partial f / \partial x)ctR \cos \theta + (\partial f / \partial y)ctR \sin \theta$. Either of the expressions on the right hand sides of (7) or (8) serves to define w for all (x, y, t) , once f and g are given. It is clear from either of these two formulas that if f is in $C^3(x, y)$ and g is in $C^2(x, y)$ then w is in $C^2(x, y, t)$. (As a matter of fact, if f is in $C^{n+1}(x, y)$, and g is in $C^n(x, y)$, then w is in $C^n(x, y, t)$). It is also clear from the structure of (7) (compare *Stokes' rule* in RAYLEIGH [6, vol. I, pp. 128–129; vol. II, p. 99]) that to complete the proof of the theorem it suffices to consider only the special case when f is replaced by zero. In this special case one has

$$w(x, y, t) = \frac{t}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{g(x + ctR \cos \theta, y + ctR \sin \theta)}{(1 - R^2)^{\frac{1}{2}}} R dR; \quad (9)$$

and, after an integration by parts (cf. MARTIN [9, p. 244]), that

$$\frac{\partial w}{\partial t}(x, y, t) = g(x, y) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{1}{(1 - R^2)^{\frac{1}{2}}} \frac{\partial g}{\partial R} dR, \quad (10)$$

so that $w(x, y, t)$, given by (9), indeed satisfies the initial conditions of the special problem in question. That the function w of (9) also satisfies the partial differential equation (2) for all (x, y, t) follows from the continuity of $(\partial^2 w / \partial t^2) - c^2[(\partial^2 w / \partial x^2) + (\partial^2 w / \partial y^2)]$ and the readily verified fact (see (10)) that for $t \neq 0$ one has

$$\begin{aligned}
& \left[\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right] (x, y, t) \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{1}{(1-R^2)^{\frac{1}{2}}} \left[\frac{\partial^2 g}{\partial t \partial R} - Rtc^2 \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) \right] dR \quad (11) \\
&= -\frac{1}{2\pi t} \int_{R=0}^1 dR \int_{\theta=0}^{2\pi} \frac{1}{(1-R^2)^{\frac{1}{2}}} R \frac{\partial^2 g}{\partial \theta^2} d\theta = 0,
\end{aligned}$$

concluding the proof of theorem 1.

Formulas (7) and (8) define w for all (x, y, t) , whereas (3) represents w only for $t \neq 0$. For $t < 0$, $w(x, y, t)$ is given by the *minus* sign on the right hand side of (3), as may be seen by putting $\rho = -ctR$ in (7) and recalling that $(\)^{\frac{1}{2}}$ stands for the positive square root. There are other representations for the solution w , which are sometimes useful. For example, FRIEDRICHS & LEWY [3, pp. 135–136] (see also BERNSTEIN [8, pp. 156–157]) write

$$\begin{aligned}
& w(x, y, t) \\
&= \frac{1}{2\pi ct} \iint_{\rho^2 < c^2 t^2} \frac{[f(\xi, \eta) + \frac{\partial f}{\partial x}(\xi, \eta) \cdot (\xi - x) + \frac{\partial f}{\partial y}(\xi, \eta) \cdot (\eta - y) + tg(\xi, \eta)]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta, \quad (12)
\end{aligned}$$

which is valid for $t > 0$, as follows from (8). For $t < 0$, minus the right hand side of (12) must be taken. Further, (8) may be put in still another form, which is again valid for all (x, y, t) , upon integrating by parts the integral involving f . The result is (use being made of the fact that $t(\partial f/\partial t) = R(\partial f/\partial R)$)

$$w(x, y, t) = f(x, y) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{\left[\frac{\partial f}{\partial R} + Rtg \right]}{(1-R^2)^{\frac{1}{2}}} dR, \quad (13)$$

where the arguments of the functions $\partial f/\partial R$ and g in the integrand are understood to be $(x + ctR \cos \theta, y + ctR \sin \theta)$. For $t > 0$, equation (13), upon putting $\rho = ctR$, and using $\partial f/\partial R = ct(\partial f/\partial \rho)$, may be written

$$\begin{aligned}
w(x, y, t) &= f(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \frac{\left[\frac{ct}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{c} g \right]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta = f(x, y) + \\
&\frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \frac{\left[\frac{ct}{\rho^2} \left\{ \frac{\partial f}{\partial x}(\xi, \eta) \cdot (\xi - x) + \frac{\partial f}{\partial y}(\xi, \eta) \cdot (\eta - y) \right\} + \frac{1}{c} g(\xi, \eta) \right]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta, \quad (14)
\end{aligned}$$

which, for $c = 1$, agrees with equation (20) of MARTIN [9, p. 244]. For $t < 0$, the plus sign in front of the integrals in (14) is to be replaced by a minus sign.

3. Cauchy problem for a third order partial differential equation. Consider the Cauchy problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \frac{\partial W}{\partial t} = 0, \quad (15)$$

$$W(x, y, 0) = F(x, y); \quad \frac{\partial W}{\partial t}(x, y, 0) = G(x, y); \quad (16)$$

$$\frac{\partial^2 W}{\partial t^2}(x, y, 0) = H(x, y),$$

where the functions F, G, H are given functions. Since the differential operator in (15) is just $\partial/\partial t$ times the wave operator of (2), it is easy to obtain, from theorem 1, a corresponding existence and uniqueness theorem for the Cauchy problem (15), (16).

Theorem 2. (a) Uniqueness. *If G is in $C^3(x, y)$, and H is in $C^2(x, y)$, then the Cauchy problem (15), (16) possesses at most one solution $W(x, y, t)$ in $C^3(x, y, t)$.* (b) Existence. *If G is in $C^4(x, y)$, and F and H are in $C^3(x, y)$, then the Cauchy problem (15), (16) possesses a solution $W(x, y, t)$ in $C^3(x, y, t)$.*

Proof. Consider uniqueness first. Suppose there is a function W in $C^3(x, y, t)$ which solves (15), (16). Then $\partial W/\partial t$ is in $C^2(x, y, t)$ and solves (1), (2), with f, g of (2) replaced by G, H of (16), respectively. Therefore $\partial W/\partial t$ can be obtained from (8) or (13). Using (8), the result is

$$\begin{aligned} \frac{\partial W}{\partial t}(x, y, \tau) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{\left[\tau H + G + c\tau R \left(\frac{\partial G}{\partial x} \cos \theta + \frac{\partial G}{\partial y} \sin \theta \right) \right]}{(1 - R^2)^{\frac{1}{2}}} R dR \end{aligned} \quad (17)$$

where the arguments of H, G , and $\partial G/\partial t$ in the integrand are $(x + c\tau R \cos \theta, y + c\tau R \sin \theta)$; while using (13) one gets

$$\frac{\partial W}{\partial t}(x, y, \tau) = G(x, y) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{\left[\frac{\partial G}{\partial R} + R\tau H \right]}{(1 - R^2)^{\frac{1}{2}}} dR. \quad (18)$$

Thus $W(x, y, t)$ may be obtained from either (17) or (18) by integrating with respect to τ from 0 to t (see (16)), so that W is indeed uniquely determined. From (17),

$$\begin{aligned} W(x, y, t) &= F(x, y) \\ &+ \frac{1}{2\pi} \int_0^t d\tau \int_0^{2\pi} d\theta \int_{R=0}^1 \frac{\left[\tau H + G + c\tau R \left(\frac{\partial G}{\partial x} \cos \theta + \frac{\partial G}{\partial y} \sin \theta \right) \right]}{(1 - R^2)^{\frac{1}{2}}} R dR, \end{aligned} \quad (19)$$

while from (18)

$$W(x, y, t) = F(x, y) + tG(x, y) + \frac{1}{2\pi} \int_0^t d\tau \int_0^{2\pi} d\theta \int_{R=0}^1 \left[\frac{\partial G}{\partial R} + R\tau H \right] \frac{dR}{(1-R^2)^{\frac{1}{2}}}, \quad (20)$$

where $\partial G/\partial R$ stands for $(\partial G/\partial x)c\tau \cos \theta + (\partial G/\partial y)c\tau \sin \theta$.

Next, consider the existence of a solution of the Cauchy problem (15), (16), under the hypotheses made about F , G and H in (b). From either (19) or (20) it follows that if $W(x, y, t)$ is defined by the right hand sides of these equations then W is in $C^3(x, y, t)$. Further, $W(x, y, 0) = F(x, y)$, and $\partial W/\partial t$ is given by either (17) or (18), which implies, by section 1, (see (8) or (13)) that $\partial W/\partial t$ is a solution in $C^2(x, y, t)$ of the wave equation (2), with

$$\frac{\partial W}{\partial t}(x, y, 0) = G(x, y) \quad \text{and} \quad \frac{\partial^2 W}{\partial t^2}(x, y, 0) = H(x, y),$$

completing the proof.

Formulas (19) and (20) for the solution may be simplified somewhat by reducing the triple integrals appearing there to double integrals. In (19), put $\rho = c\tau R$, when $t > 0$, $\tau > 0$, and note that then

$$\frac{R dR}{(1-R^2)^{\frac{1}{2}}} = \frac{1}{c\tau} \frac{\rho d\rho}{(c^2\tau^2 - \rho^2)^{\frac{1}{2}}}. \quad (21)$$

Inverting the order of the integrations with respect to τ and ρ ; making use of Dirichlet's formula for an arbitrary function $\alpha(\tau, \rho)$:

$$\int_{\tau=0}^t d\tau \int_{\rho=0}^{c\tau} \alpha(\tau, \rho) d\rho = \int_{\rho=0}^{ct} d\rho \int_{\tau=\rho/c}^t \alpha(\tau, \rho) d\tau; \quad (22)$$

and also using the definite integrals

$$\begin{aligned} \int_{\tau=\rho/c}^t \frac{1}{(c^2\tau^2 - \rho^2)^{\frac{1}{2}}} d\tau &= \frac{1}{c} \cosh^{-1} \frac{tc}{\rho}, \\ \int_{\tau=\rho/c}^t \frac{1}{\tau} \frac{1}{(c^2\tau^2 - \rho^2)^{\frac{1}{2}}} d\tau &= \frac{1}{\rho} \cos^{-1} \frac{\rho}{tc}, \end{aligned} \quad (23)$$

one obtains from (19) the following formula:

$$\begin{aligned} W(x, y, t) &= F(x, y) + \frac{1}{2\pi c} \int_{\theta=0}^{2\pi} d\theta \int_{\rho=0}^{ct} \left[\left(\frac{1}{c} \cosh^{-1} \frac{tc}{\rho} \right) H \right. \\ &\quad \left. + \left(\frac{1}{\rho} \cos^{-1} \frac{\rho}{tc} \right) \left\{ G + \rho \frac{\partial G}{\partial x} \cos \theta + \rho \frac{\partial G}{\partial y} \sin \theta \right\} \right] \rho d\rho. \end{aligned} \quad (24)$$

This last formula may be rewritten

$$W(x, y, t) = F(x, y) + \frac{1}{2\pi c} \iint_{\rho^2 < c^2 t^2} \left[\left(\frac{1}{c} \cosh^{-1} \frac{tc}{\rho} \right) H \right. \\ \left. + \left(\frac{1}{\rho} \cosh^{-1} \frac{\rho}{tc} \right) \left\{ G + \frac{\partial G}{\partial x} \cdot (\xi - x) + \frac{\partial G}{\partial y} \cdot (\eta - y) \right\} \right] d\xi d\eta, \quad (25)$$

where the arguments of H and G in the integrand are (ξ, η) . Both (24) and (25) are valid for $t > 0$.

Another way of arriving at formula (25) is to start with $(\partial W / \partial t)(x, y, \tau)$ as given by (12), for $\tau > 0$, and then integrating with respect to τ from 0 to $t > 0$. Then

$$W(x, y, t) = F(x, y) + \int_{\tau=0}^t d\tau \left\{ \frac{1}{2\pi c \tau} \iint_{\rho^2 < c^2 \tau^2} \frac{1}{(c^2 \tau^2 - \rho^2)^{\frac{1}{2}}} \right. \\ \left. \cdot \left[G + \frac{\partial G}{\partial x} \cdot (\xi - x) + \frac{\partial G}{\partial y} \cdot (\eta - y) + \tau H \right] d\xi d\eta \right\}, \quad (26)$$

and interchanging the order of the integrations with respect to τ and ρ yields (25) anew.

A similar procedure serves to reduce the triple integral in (20) to a double integral. In (20), put $\rho = c\tau R$, when $t > 0$, $\tau > 0$, and note that (21) holds. Inverting the order of the integrations with respect to τ and ρ ; making use of Dirichlet's formula (22); and also using the definite integrals

$$\int_{\tau=\rho/c}^t \frac{\tau}{(c^2 \tau^2 - \rho^2)^{\frac{1}{2}}} d\tau = \frac{1}{c^2} (c^2 t^2 - \rho^2)^{\frac{1}{2}}, \\ \int_{\tau=\rho/c}^t \frac{1}{(c^2 \tau^2 - \rho^2)^{\frac{1}{2}}} d\tau = \frac{1}{c} \cosh^{-1} \frac{tc}{\rho}, \quad (27)$$

one obtains from (20) the following formula:

$$W(x, y, t) = F(x, y) + tG(x, y) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{\rho=0}^{ct} \left[\left(\frac{\rho}{c^2} \cosh^{-1} \frac{tc}{\rho} \right) H \right. \\ \left. + \frac{1}{c} (c^2 t^2 - \rho^2)^{\frac{1}{2}} \left\{ \frac{\partial G}{\partial x} \cos \theta + \frac{\partial G}{\partial y} \sin \theta \right\} \right] d\rho. \quad (28)$$

This last formula may be rewritten

$$W(x, y, t) = F(x, y) + tG(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \left[\left(\frac{1}{c^2} \cosh^{-1} \frac{tc}{\rho} \right) H \right. \\ \left. + \frac{1}{c\rho^2} (c^2 t^2 - \rho^2)^{\frac{1}{2}} \left\{ \frac{\partial G}{\partial x} \cdot (\xi - x) + \frac{\partial G}{\partial y} \cdot (\eta - y) \right\} \right] d\xi d\eta, \quad (29)$$

where the arguments of H and G in the integrand are (ξ, η) . Both (28) and (29) are valid for $t > 0$.

Another way of arriving at formula (29) is to start with $(\partial W/\partial t)(x, y, \tau)$ as given by (14), for $\tau > 0$, and then integrating with respect to τ from 0 to $t > 0$. Then

$$W(x, y, t) = F(x, y) + tG(x, y) + \int_{\tau=0}^t d\tau \left\{ \iint_{\rho^2 < c^2 \tau^2} \frac{1}{(c^2 \tau^2 - \rho^2)^{\frac{1}{2}}} \left[\frac{1}{c} H + \frac{c\tau}{\rho^2} \left\{ \frac{\partial G}{\partial x} \cdot (\xi - x) + \frac{\partial G}{\partial y} \cdot (\eta - y) \right\} \right] \right\}, \quad (30)$$

and interchanging the order of the integrations with respect to τ and ρ yields (29) anew.

4. Cauchy problem for a system of equations. The Cauchy problem in question is the following:

$$\frac{\partial u}{\partial t} + c^2 \frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + c^2 \frac{\partial \lambda}{\partial y} = 0, \quad \frac{\partial \lambda}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (31)$$

$$u(x, y, 0) = U(x, y); \quad v(x, y, 0) = V(x, y); \quad \lambda(x, y, 0) = \Lambda(x, y), \quad (32)$$

where U, V, Λ are given functions, and c is a positive constant. The system (31) may be written in matrix form thus

$$\begin{pmatrix} \frac{\partial}{\partial t} & 0 & c^2 \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial t} & c^2 \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (33)$$

Proceeding formally, one sees, upon multiplying (on the left) the matrix equation (33) by the transpose of matrix of cofactors of the 3 by 3 matrix appearing in (33), that each of the functions u, v , and λ is a solution of equation (15), and this shows the connection between the system (31) and the third order equation (15).

Without striving for the most general result, one can easily demonstrate the following theorem, in the course of whose proof an explicit representation of u, v, λ in terms of the Cauchy data U, V, Λ will be obtained.

Theorem 3. (a) Uniqueness. *If U, V, Λ are in $C^4(x, y)$ then the Cauchy problem (31), (32) possesses at most one solution u, v, λ in $C^3(x, y, t)$.* (b) Existence. *If U, V, Λ are in $C^6(x, y)$ then the Cauchy problem (31), (32) possesses a solution u, v, λ in $C^3(x, y, t)$.*

Proof. Suppose u, v, λ are in $C^3(x, y, t)$ and constitute a solution of (31), (32). Then it follows from (31) and (32) that u is a solution of the Cauchy problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \frac{\partial u}{\partial t} = 0, \quad (34)$$

$$u(x, y, 0) = U(x, y),$$

$$\frac{\partial u}{\partial t}(x, y, 0) = -c^2 \frac{\partial \Lambda}{\partial x}(x, y), \quad (35)$$

$$\frac{\partial^2 u}{\partial t^2}(x, y, 0) = c^2 \left(\frac{\partial^2 U}{\partial x^2}(x, y) + \frac{\partial^2 V}{\partial x \partial y}(x, y) \right);$$

and that v is a solution of the Cauchy problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \frac{\partial v}{\partial t} = 0, \quad (36)$$

$$v(x, y, 0) = V(x, y),$$

$$\frac{\partial v}{\partial t}(x, y, 0) = -c^2 \frac{\partial \Lambda}{\partial y}(x, y), \quad (37)$$

$$\frac{\partial^2 v}{\partial t^2}(x, y, 0) = c^2 \left(\frac{\partial^2 U}{\partial x \partial y}(x, y) + \frac{\partial^2 V}{\partial y^2}(x, y) \right);$$

while λ is a solution of the Cauchy problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \frac{\partial \lambda}{\partial t} = 0, \quad (38)$$

$$\lambda(x, y, 0) = \Lambda(x, y),$$

$$\frac{\partial \lambda}{\partial t}(x, y, 0) = - \left(\frac{\partial U}{\partial x}(x, y) + \frac{\partial V}{\partial y}(x, y) \right), \quad (39)$$

$$\frac{\partial^2 \lambda}{\partial t^2}(x, y, 0) = c^2 \left(\frac{\partial^2 \Lambda}{\partial x^2}(x, y) + \frac{\partial^2 \Lambda}{\partial y^2}(x, y) \right).$$

As a matter of fact, from (38) and the last equation in (39) it follows that λ is indeed a solution of the second order problem

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \lambda = 0, \quad (40)$$

$$\lambda(x, y, 0) = \Lambda(x, y); \quad \frac{\partial \lambda}{\partial t}(x, y, 0) = - \left(\frac{\partial U}{\partial x}(x, y) + \frac{\partial V}{\partial y}(x, y) \right). \quad (41)$$

From the hypotheses concerning U , V , Λ , plus theorem 2, it follows that u , v , λ are indeed uniquely determined. For example, for $t > 0$, from (25), (34), (35), (36), (37), follow the explicit formulas

$$u(x, y, t) = U(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \left[\left(\cosh^{-1} \frac{tc}{\rho} \right) \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial x \partial y} \right) - \left(\frac{1}{\rho} \cos^{-1} \frac{\rho}{tc} \right) \left\{ \frac{\partial \Lambda}{\partial x} + \frac{\partial^2 \Lambda}{\partial x^2} \cdot (\xi - x) + \frac{\partial^2 \Lambda}{\partial x \partial y} \cdot (\eta - y) \right\} \right] d\xi d\eta, \quad (42)$$

and

$$v(x, y, t) = V(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \left[\left(\cosh^{-1} \frac{tc}{\rho} \right) \left(\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} \right) - \left(\frac{1}{\rho} \cos^{-1} \frac{\rho}{tc} \right) \left\{ \frac{\partial \Lambda}{\partial y} + \frac{\partial^2 \Lambda}{\partial x \partial y} \cdot (\xi - x) + \frac{\partial^2 \Lambda}{\partial y^2} \cdot (\eta - y) \right\} \right] d\xi d\eta; \quad (43)$$

while, again for $t > 0$, from (12), (40), (41) follows that

$$\lambda(x, y, t) = \frac{1}{2\pi ct} \iint_{\rho^2 < c^2 t^2} \left[\frac{\Lambda + \frac{\partial \Lambda}{\partial x} \cdot (\xi - x) + \frac{\partial \Lambda}{\partial y} \cdot (\eta - y) - t \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right)}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} \right] d\xi d\eta. \quad (44)$$

Turning now to the proof of part (b) of the theorem, suppose U , V , Λ are in $C^6(x, y)$, and define u , v , λ for all x , y , t by means of the right hand sides of either (19), (20), using in each case the appropriate functions F , G , H , which are given in (35), (37) and (39) respectively. Then u , v , λ so defined are in $C^4(x, y, t)$, and they solve by theorem 2 the three Cauchy problems (34)–(39), so that the Cauchy data (32) are indeed fulfilled by u , v , λ . It remains to verify that the system (31) is satisfied. But this is easily seen. Consider, for example, the function $(\partial u / \partial t) + c^2(\partial \lambda / \partial x)$, which is in $C^3(x, y, t)$. This function, as can be checked immediately by using (35), (37) and (39), is a solution of the Cauchy problem (15), (16), with the F , G , H of (16) being identically zero. Hence by part (a) of theorem 2, the function $(\partial u / \partial t) + c^2(\partial \lambda / \partial x)$ vanishes for all (x, y, t) , and thus the first equation of the system (31) holds. A similar reasoning, using $(\partial v / \partial t) + c^2(\partial \lambda / \partial y)$ and $(\partial \lambda / \partial t) + \partial u / \partial x + (\partial v / \partial y)$, shows that the last two equations of (31) are also satisfied.

The formula (42) for $u(x, y, t)$ may be put in a form more convenient for applications, since it involves only first partial derivatives of the Cauchy data U , V , Λ . For, in the integrand of (42), one may write

$$\left(\frac{\xi - x}{\rho} \right) \frac{\partial}{\partial \rho} \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) = (\cos \theta) \frac{\partial}{\partial \rho} \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right)$$

instead of

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial x \partial y},$$

thence by integration by parts obtaining

$$\begin{aligned} \int_0^{2\pi} d\theta \int_{\rho=0}^{ct} \left(\cosh^{-1} \frac{tc}{\rho} \right) \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right] \rho d\rho \\ = \int_0^{2\pi} \left(\frac{\xi - x}{\rho} \right) d\theta \int_{\rho=0}^{ct} \frac{ct}{\rho(c^2 t^2 - \rho^2)^{\frac{1}{2}}} \left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right] d\rho. \end{aligned} \quad (45)$$

Similarly, since

$$\left\{ \frac{1}{\rho} \frac{\partial \Lambda}{\partial x} + \frac{\partial^2 \Lambda}{\partial x^2} \frac{(\xi - x)}{\rho} + \frac{\partial^2 \Lambda}{\partial x \partial y} \frac{(\eta - y)}{\rho} \right\} \rho = \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Lambda}{\partial x} \right),$$

another integration by parts yields

$$\int_0^{2\pi} d\theta \int_{\rho=0}^{ct} \left(-c \cos^{-1} \frac{\rho}{tc} \right) \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Lambda}{\partial x} \right) d\rho = \int_0^{2\pi} d\theta \int_{\rho=0}^{ct} \frac{\rho \frac{\partial \Lambda}{\partial x}}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\rho. \quad (46)$$

The desired formula for $u(x, y, t)$ now follows from (42), (45), and (46), for $t > 0$:

$$u(x, y, t) = U(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \frac{\left[ct \left(\frac{\xi - x}{\rho} \right) \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + \frac{\partial \Lambda}{\partial x} \right]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta. \quad (47)$$

Similar integrations by parts yield a companion formula for $v(x, y, t)$, starting from (43), again for $t > 0$:

$$v(x, y, t) = V(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \frac{\left[ct \left(\frac{\eta - y}{\rho} \right) \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + \frac{\partial \Lambda}{\partial y} \right]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta. \quad (48)$$

It is of interest to note that a formal Fourier transform approach to the Cauchy problem (31), (32) leads, after many vicissitudes, to formulas (47) and (48) for the solution.

It is also interesting to notice that formula (14) of section 2 leads to an expression for $\lambda(x, y, t)$ (see (40), (41)), which is of the same general structure as (47) and (48), namely

$$\begin{aligned} \lambda(x, y, t) = \Lambda(x, y) + \frac{1}{2\pi} \iint_{\rho^2 < c^2 t^2} \\ \cdot \frac{\left[\frac{ct}{\rho^2} \left\{ \frac{\partial \Lambda}{\partial x} \cdot (\xi - x) + \frac{\partial \Lambda}{\partial y} \cdot (\eta - y) \right\} - \frac{1}{c} \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right]}{(c^2 t^2 - \rho^2)^{\frac{1}{2}}} d\xi d\eta. \end{aligned} \quad (49)$$

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