Schlicht Functions and Linear Differential Equations of Second Order

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1. Introduction. In two recent papers [1], [2] Z. NEHARI obtained some interesting conditions for the schlichtness of analytic functions $f(z)$ in a circle. His method is to show that no solution of a certain linear differential equation of second order whose coefficient is determined by the given analytic function $f(z)$ has more than one zero in the circle. To do this he employed various inequalities relating to the *Green's transform* of the associated differential equation or to its solutions expressed in terms of the Green's function of certain operator. Green's transform plays a fundamental role in HILLE's investigations of zero-free regions of the solutions of differential equations [3], [4], [5].

In this paper we shall give an extension to the complex domain of a criterion used by A. WINTNER [6] in the investigations of the real zeros of real solutions of linear differential equations of second order. Some sufficient conditions for the schlichtness of functions can be obtained easily from this criterion. We also use the *Sturmian comparison method* [7, Chap. X] based on a theorem extending the classical Sturm's comparison theorem and obtained recently by the author [8, Theorem 2]. NEHARI's results follow easily by comparing one differential equation with another whose distribution of zeros of the solutions is known. In view of its historical background, this method seems natural and makes the oscillation theorems in the real domain available for the determination of the schlichtness of analytic functions.

2. Preliminary results. In the sequel let $f(z)$ be an analytic function in the domain D under consideration and suppose that the Schwarzian derivative of $f(z)$ with respect to z

$$[f, z] = (f''/f')' - \frac{1}{2}(f''/f')^2$$

is also analytic in D . Let

$$Q(z) = [f, z]/2$$

and consider the differential equation

$$(2.1) \quad W'' + Q(z)W = 0.$$

The following lemma relates the schlichtness of $f(z)$ to the zeros of the solutions of (2.1) (see [2]).

Lemma 1. *The analytic function $f(z)$ is schlicht in D if and only if no solution of (2.1) has more than one zero in D .*

Proof. We have $f = W_1/W_2$, where W_1 and W_2 are a certain pair of linearly independent solutions of (2.1), because by direct substitution one can verify that W_1/W_2 is a solution of the differential equation $[f, z] = 2Q(z)$ and every solution can be expressed in such form. If $f(z_1) = f(z_2) = a$, then $W_1 - aW_2 = 0$ at z_1 and z_2 ; and if $aW_1 + bW_2 = 0$, $a \neq 0$ at z_1 and z_2 , then $f(z_1) = f(z_2)$. This proves the lemma.

In order to obtain information about the schlicht properties of $f(z)$, we now turn our attention to the associated differential equation (2.1).

In our discussion a solution $W(z)$ of (2.1) is always assumed different from the trivial solution ($\equiv 0$). Write

$$z = x + iy, \quad W = u + iv, \quad Q = q_1 + iq_2,$$

where x, y, u, v, q_1 and q_2 are real. Denote the real and imaginary part of A by $\Re A$ and $\Im A$ respectively. A^* is the complex conjugate of A . Let

$$(2.2) \quad L(x, y) = j\Re(W^*W') - k\Im(W^*W') - mWW^*,$$

where j and k are two real constants, $m = m(x, y)$ is a real-valued function of class C^1 . By use of equation (2.1) it is not difficult to verify that

$$(2.3) \quad L_x = j \left(\left(u_x - \frac{mu}{j} \right)^2 + \left(v_x - \frac{mv}{j} \right)^2 \right) - \left(m_x + \frac{m^2}{j} + jq_1 - kq_2 \right) (u^2 + v^2),$$

$$(2.4) \quad L_y = k \left(\left(u_y - \frac{mu}{k} \right)^2 + \left(v_y - \frac{mv}{k} \right)^2 \right) - \left(m_y + \frac{m^2}{k} - kq_1 - jq_2 \right) (u^2 + v^2),$$

where $L_x, u_y, \dots$ denote partial derivatives with respect to the variable indicated.

Let C be the curve

$$(2.5) \quad x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

where $x'(t)$ and $y'(t)$ are piecewise continuous in $a \leq t \leq b$. It is also assumed that C has no double points and that its end-points are distinct.

Theorem 1. *Suppose that the following conditions are satisfied:*

1. *C is a curve of the type (2.5) in a domain D ,*
2. *j and k are two real constants, $jk \neq 0$,*
3. *$j \frac{dx}{dt}$ and $k \frac{dy}{dt}$ do not change sign and do not have different signs on C ,*
4. *$m(x, y)$ is a real-valued function of class C^1 on C ,*
5. *$j \left(m_x + \frac{m^2}{j} + jq_1 - kq_2 \right) \leq 0, \quad k \left(m_y + \frac{m^2}{k} - kq_1 - jq_2 \right) \leq 0$ on C .*

Then every solution of (2.1) has at most one zero on C . Furthermore there is a solution of (2.1) which does not vanish on C .

Proof. Suppose that the theorem is not true, and let z_1 and z_2 be two zeros of a solution $W(z)$ of (2.1) on C , $z_\alpha = x_\alpha + iy_\alpha$ corresponding to t_α , $\alpha = 1, 2$. Clearly

$$(2.6) \quad L(x_2, y_2) - L(x_1, y_1) = \int_{t_1}^{t_2} \left(L_x \frac{dx}{dt} + L_y \frac{dy}{dt} \right) dt,$$

the integral being taken along C . At (x_1, y_1) , $u = v = 0$ and

$$u_x^2 + v_x^2 = u_y^2 + v_y^2 > 0,$$

otherwise both W and W' would vanish at (x_1, y_1) and $W \equiv 0$ in D , which contradicts our assumption that $W \not\equiv 0$. It is now easy to verify that $L_x \frac{dx}{dt} + L_y \frac{dy}{dt}$ does not change sign on C and does not vanish in the neighborhood of (x_1, y_1) ; hence the integral on the right-hand side of (2.6) is different from zero. On the other hand L vanishes at (x_1, y_1) and (x_2, y_2) ; hence the left-hand side of (2.6) is zero. The contradiction proves the first part of the theorem.

In (2.6) replace t_1 by a and t_2 by t , $a < t \leq b$. If the integrand in (2.6) is non-negative, then

$$(2.7) \quad L(x(t), y(t)) \geq L(x(a), y(a)).$$

Let $W(z)$ be a solution with $W = 1$, $jW' = m(x(a), y(a)) + 1$ at $x(a) + iy(a)$. Then $L(x(a), y(a)) = 1$ and $L(x(t), y(t))$, and so $W(z)$ does not vanish on C . If the integrand in (2.6) is non-positive, a similar method shows that the solution $W(z)$ with $W = 1$, $jW' = m(x(a), y(a)) - 1$ at $x(a) + iy(a)$ does not vanish on C . This completes the proof of Theorem 1.

This theorem extends further part of a criterion used by A. WINTNER in his paper [6] (see also [8]). We observe also that (2.6) becomes the "Green's transform" of HILLE [3] when $m = 0$, $j = 1$, $k = 0$ and $m = 0$, $j = 0$, $k = 1$.

3. Some sufficient conditions of schlichtness of $f(z)$. The results in this section are obtained from Theorem 1 by constructing various functions $m(x, y)$ similar to those used by A. WINTNER [6] in the real domain. We shall also need the

following result (Theorem 2) which is due to HILLE [5] and is proved again here, for the sake of completeness, by a slightly different method based on the function $L(x, y)$.

Theorem 2. *Let D be a convex domain in the complex plane and suppose that*

1. $\theta_1 \geq \arg Q(z) \geq \theta_2$, $\pi \geq \theta_1 - \theta_2 \geq 0$, for z in D ,
2. z_1 is a zero of a solution $W(z)$ of (2.1) in D .

Then $W(z)$ does not vanish again in the two sectors of D defined by

$$(3.1) \quad p\pi - \frac{1}{2}\theta_2 < \arg(z - z_1) < (p+1)\pi - \frac{1}{2}\theta_1, \quad p = 0, 1.$$

Proof. Let z_2 be any point in D and $z_2 - z_1 = r_0 \exp(i\theta_0)$, $z_h = x_h + iy_h$, $h = 1, 2$, and $\theta = \arg Q(z)$. In (2.2) set $m = 0$, $j = \sin(\theta_0 + \theta_i)$, $k = \cos(\theta_0 + \theta_i)$. A little computation shows that the integral (2.6) taken along a line segment from z_1 to z_2 is

$$(3.2) \quad L(x_2, y_2) - L(x_1, y_1) = \int_0^{r_0} [|W'|^2 \sin(2\theta_0 + \theta_i) + |QW^2| \sin(\theta - \theta_i)] dr$$

where $r = |z - z_1|$ for z lying on the line segment. If $\theta_1 = \theta_2$, then $Q(z)$ is a constant and the theorem follows by solving (2.1). Suppose then $\theta_1 > \theta_2$ and let z_2 be another zero of $W(z)$ in D . If $(2p+1)\pi < 2\theta_0 + \theta_1 < 2(p+1)\pi$, $p = 0, 1$, then (3.2) with $\theta_i = \theta_1$ leads to contradiction as the left-hand side is zero while the right-hand side is different from zero. If $2p\pi < 2\theta_0 + \theta_2 < (2p+1)\pi$, $p = 0, 1$, we have again contradiction in (3.2) with $\theta_i = \theta_2$. Clearly these inequalities for θ_0 define the same region as (3.1).

Theorem 3. *Let D be a convex domain and suppose that*

1. $q_2 < 0$, $|q_1| \leq -q_2$ in D ,
2. the following inequality holds in D :

$$4 \left| \int_{x_0, y_0}^{x, y} (q_1 - q_2) dx - (q_1 + q_2) dy \right|^2 \leq -|q_1| - q_2,$$

where (x_0, y_0) is a fixed point of D and the path of integration lies in D .

Then $f(z)$ is schlicht in D .

Proof. The line integral

$$m(x, y) = -2 \int_{x_0, y_0}^{x, y} (q_1 - q_2) dx - (q_1 + q_2) dy$$

is independent of the path of integration in D and has partial derivatives

$$m_x = -2(q_1 - q_2), \quad m_y = 2(q_1 + q_2).$$

Clearly the inequalities

$$\begin{aligned} m_x + m^2 + q_1 - q_2 &= m^2 - (q_1 - q_2) \leq 0, \\ m_y + m^2 - (q_1 + q_2) &= m^2 + q_1 + q_2 \leq 0 \end{aligned}$$

hold in D . Let $W(z)$ be a solution of (2.1) vanishing at a point z_1 of D . Denote by D_1 the intersection of D and

$$p\pi \leq \arg(z - z_1) \leq (p + \tfrac{1}{2})\pi, \quad p = 0, 1.$$

To prove that $W(z)$ does not vanish again in D_1 , take an arbitrary point z_2 in D_1 . Letting $j = k = 1$, Theorem 1 shows that $W(z)$ does not vanish again on the line segment $z_1 z_2$, and hence in D_1 . On the other hand, condition (1) implies

$$-\frac{3}{4}\pi \leq \arg Q(z) \leq -\frac{\pi}{4}$$

in D and hence, according to Theorem 2, $W(z)$ does not vanish again in the intersection D_2 of D and

$$p\pi + \frac{3\pi}{8} < \arg(z - z_1) < (p + 1)\pi + \frac{\pi}{8}, \quad p = 0, 1.$$

Clearly the sum of D_1 , D_2 and z_1 is D . The theorem then follows from Lemma 1.

Theorem 4. Let D be a convex domain and suppose that

1. $q_2 > 0$, $q_2 \geq |q_1|$ in D ,
2. the following inequality holds in D

$$4 \left| \int_{x_0, y_0}^{x, y} (q_1 + q_2) dx + (q_1 - q_2) dy \right|^2 \leq q_2 - |q_1|,$$

where (x_0, y_0) is a fixed point of D and the path of integration lies in D .

Then $f(z)$ is schlicht in D .

Proof. The proof is similar to that of Theorem 3, applying Theorems 1 and 2. In applying Theorem 1, take $j = 1$, $k = 1$, and

$$m(x, y) = -2 \int_{x_0, y_0}^{x, y} (q_1 + q_2) dx + (q_1 - q_2) dy.$$

Condition (2) implies that a solution $W(z)$ of (2.1) vanishing at z_1 does not vanish again in the common part D_1 of D and

$$(p + \tfrac{1}{2})\pi \leq \arg(z - z_1) \leq (p + 1)\pi, \quad p = 0, 1,$$

while condition (1) guarantees that $W(z)$ does not vanish again in $D - D_1$.

Theorem 5. Let D be the interior of the square with vertices at $-\frac{1}{2}, -i/2, \frac{1}{2}$ and $i/2$.

Suppose that

1. $q_2 \leq 0$, $Q \neq 0$ in D ,
2. the following inequalities

$$A - 4 \leq \int_{0,0}^{x,y} (q_1 - q_2) dx - (q_1 + q_2) dy \leq A$$

hold in D for some real constant A .

Then $f(z)$ is schlicht in D .

Proof. In D let

$$m(x, y) = A + m_0(x, y) - \int_{0,0}^{x,y} (q_1 - q_2) dx - (q_1 + q_2) dy,$$

where

$$\begin{aligned} m_0(x, y) &= (x + y + \tfrac{1}{2})^{-1} - 4 \quad \text{if} \quad -\tfrac{1}{2} < x + y \leq 0, \\ &= (x + y - \tfrac{1}{2})^{-1} \quad \text{if} \quad 0 < x + y < \tfrac{1}{2}. \end{aligned}$$

Clearly m is of class C^1 in D . Now it is not difficult to verify that

$$\begin{aligned} m_x + m^2 + q_1 - q_2 &\leq 0, \\ m_y + m^2 - q_1 - q_2 &\leq 0 \end{aligned}$$

in D . Let $W(z)$ be a solution of (2.1) having a zero z_1 in D . Taking $j = k = 1$, Theorem 1 guarantees that $W(z)$ will not vanish again on any line segment $z_1 z_2$, z_2 being in D and

$$p\pi \leq \arg(z_2 - z_1) \leq (p + \tfrac{1}{2})\pi, \quad p = 0, 1.$$

On the other hand $W(z)$ will not vanish again in the common part of D and

$$(p - \tfrac{1}{2})\pi < \arg(z - z_1) < p\pi, \quad p = 0, 1,$$

in view of the fact that $\pi \leq \arg Q(z) \leq 2\pi$ in D and Theorem 2. This proves that $W(z)$ has at most one zero in D and the theorem follows from Lemma 1.

If $Q(z)$ is analytic in a domain containing the closure of D , then the integral in condition 2 of Theorem 5, being a harmonic function, assumes its maximum and minimum values on the boundary of D . It follows that in order that the inequalities in condition 2 are satisfied in D it is sufficient they are satisfied on the boundary of D , i.e., the difference of the maximum and minimum values of the integral in condition 2 along the boundary of D is not greater than 4.

The proof of Theorem 5 remains valid if the inequalities in condition 2 are replaced by

$$(3.3) \quad A - 4 \leq I(x, y) \leq A - 4 + 2(x + y + \tfrac{1}{2})^{-1}$$

whenever (x, y) is in D and $-\tfrac{1}{2} < x + y \leq 0$ and

$$(3.4) \quad A + 2(x + y - \tfrac{1}{2})^{-1} \leq I(x, y) \leq A$$

whenever (x, y) is in D and $0 < x + y < \tfrac{1}{2}$, $I(x, y)$ being the integral in condition 2.

Theorem 6. Let D be the interior of the square with vertices at $-\frac{1}{2}$, $-i/2$, $\frac{1}{2}$, $i/2$. Suppose that

1. $q_2 \geq 0$, $Q \neq 0$ in D ,
2. the following inequalities

$$A - 4 \leq \int_{0,0}^{x,y} (q_1 + q_2) dx + (q_1 - q_2) dy \leq A$$

hold in D for some real constant A .

Then $f(z)$ is schlicht in D .

Proof. The proof is similar to that of Theorem 5, applying Theorems 1 and 2. In applying Theorem 1, take $j = 1$, $k = -1$ and

$$m(x, y) = A + m_0(x, y) - \int_{0,0}^{x,y} (q_1 + q_2) dx + (q_1 - q_2) dy,$$

where

$$\begin{aligned} m_0(x, y) &= (x - y + \tfrac{1}{2})^{-1} - 4 \quad \text{if} \quad -\tfrac{1}{2} < x - y \leq 0, \\ &= (x - y - \tfrac{1}{2})^{-1} \quad \text{if} \quad 0 < x - y < \tfrac{1}{2}. \end{aligned}$$

Condition 2 implies that a solution $W(z)$ of (2.1) vanishing at z_1 of D does not vanish again in the common part D_1 of D and

$$(p + \tfrac{1}{2})\pi \leq \arg(z - z_1) \leq (p + 1)\pi, \quad p = 0, 1,$$

while condition 1 guarantees that $W(z)$ does not vanish again in $D - D_1$.

Theorem 7. Let D be the interior of the square with vertices at $-\frac{1}{2}$, $-i/2$, $\frac{1}{2}$, $i/2$. Suppose that the following inequalities

$$(3.5) \quad A - 4 \leq \int_{0,0}^{x,y} (q_1 - q_2) dx - (q_1 + q_2) dy \leq A,$$

$$(3.6) \quad B - 4 \leq \int_{0,0}^{x,y} (q_1 + q_2) dx + (q_1 - q_2) dy \leq B$$

hold in D for some constants A and B .

Then $f(z)$ is schlicht in D .

Proof. The theorem follows by combining the first halves of the proofs of Theorems 5 and 6.

In Theorem 7, we may replace the inequalities (3.5) by the pair of inequalities (3.3) and (3.4) and (3.6) by another pair similar to (3.3) and (3.4) which are not difficult to obtain if the reader completes the details of the proof of Theorem 6.

Let S be the sector in the right half-plane bounded by the lines $L_1 x - y = 0$ and $L_2 x + y = 0$, $L_1 > 0$, $L_2 > 0$, the two lines being excluded. In the next

theorem and its corollary, we assume that the integral $\int_z^\infty Q(t) dt$, its path of integration being taken parallel to the real axis and z lying in S , converges uniformly in any bounded region in S so that it is an analytic function in S .

Theorem 8. *Let S be defined as above and suppose that the following inequalities hold in S :*

$$(3.7) \quad -\frac{3L_1}{4(L_1 x - y)} \leq \int_x^\infty (q_1 + L_1 q_2) dx \leq \frac{L_1}{4(L_1 x - y)},$$

$$(3.8) \quad -\frac{3L_2}{4(L_2 x + y)} \leq \int_x^\infty (q_1 - L_2 q_2) dx \leq \frac{L_2}{4(L_2 x + y)}.$$

Then $f(z)$ is schlicht in S .

Proof. Set

$$m(x, y) = \int_x^\infty (q_1 - L_2 q_2) dx + \frac{L_2}{4(L_2 x + y)},$$

where $q_1 = q_1(x, y)$, $q_2 = q_2(x, y)$. Then we have

$$m_x = -(q_1 - L_2 q_2) - \frac{L_2^2}{4(L_2 x + y)^2},$$

$$m_y = (L_2 q_1 + q_2) - \frac{L_2}{4(L_2 x + y)^2}.$$

Using (3.8), one can verify that the inequalities

$$m_x + m^2 + q_1 - L_2 q_2 \leq 0,$$

$$m_y + m^2/L_2 - L_2 q_1 - q_2 \leq 0$$

hold in S . Let $W(z)$ be a solution of (2.1) vanishing at z_1 in S . Taking $j = 1$, $k = L_2$, Theorem 1 guarantees that $W(z)$ does not vanish again on any line segment $z_1 z_2, z_2$ lying in the common part S_1 of S and

$$p\pi \leq \arg(z - z_1) \leq (p + \tfrac{1}{2})\pi, \quad p = 0, 1.$$

On the other hand for

$$m(x, y) = \int_x^\infty (q_1 + L_1 q_2) dx + \frac{L_1}{4(L_1 x - y)}$$

and $j = 1$, $k = -L_1$, similar argument shows that (3.7) implies that $W(z)$ does not vanish again on any line segment $z_1 z_2, z_2$ lying in $S - S_1$. Hence $W(z)$ has at most one zero in S and $f(z)$ is schlicht in S by Lemma 1.

In applications the following corollary is more convenient.

Corollary 8.1. *Let $0 < L_1 \leq L_2$ and suppose that the inequality*

$$(3.9) \quad \int_x^\infty (|q_1| + L_2 |q_2|) dx \leq \frac{L_1}{4(L_2 x + |y|)}$$

holds in S .

Then $f(z)$ is schlicht in S .

Proof. Clearly (3.9) implies (3.7) and (3.8).

4. Comparison method. In this section we shall use a *comparison method* based on a comparison theorem (Lemma 2) to study the distribution of zeros of the solutions of (2.1). The results collected here are either the same as those of NEHARI [1], [2] or slight generalizations of his. In view of its historical background, the comparison method seems an interesting and natural approach to our problem. Lemma 2 makes it possible to apply oscillation theorems in the real domain to related problems in the complex domain.

Let $h(t)$, $h_1(t)$ and $h_2(t)$ be real-valued continuous functions of the real variable t and $H(t) = h_1(t) + ih_2(t)$. The following comparison theorem compares the distribution of zeros of solutions of the two differential equations.

$$(4.1) \quad s'' + h(t)s = 0,$$

$$(4.2) \quad S'' + H(t)S = 0$$

on an interval I .

Lemma 2. *Let I be an interval of the t -axis which is either closed or open, and if open need not be bounded. Suppose that*

1. *no solution of (4.1) has more than one zero on I ,*
2. *for some real constant k , the inequality $h_1 + kh_2 \leq h$ holds on I .*

Then no solution of (4.2) has more than one zero on I .

For a proof of this lemma, see [8, Theorem 2].

To investigate the distribution of zeros of the solutions of (2.1) on the unit circle $|z| = 1$, let $z = \exp(it)$, $0 \leq t < 2\pi$, and the equation (2.1) is transformed to

$$(4.3) \quad W'' - iW' - e^{2it} Q(e^{it})W = 0.$$

Setting $W = U \exp(it/2)$, we have

$$(4.4) \quad U'' + G(t)U = 0,$$

where

$$(4.5) \quad G(t) = \frac{1}{4} - e^{2it} Q(e^{it}) = \frac{1}{4} - \frac{z^2}{2} [f, z], \quad z = e^{it}.$$

The derivatives in (4.3) and (4.4) are taken with respect to t . Clearly W and U have the same zeros. Write

$$G(t) = g_1(t) + ig_2(t), \quad g(t, k) = g_1(t) + kg_2(t),$$

where g_1 and g_2 are real, and k a real constant. Denote $\max(g, 0)$ by g^+ .

Theorem 9. *If there exists a real constant k_0 such that*

$$(4.6) \quad \int_0^{2\pi} g^+(t, k_0) dt \leq \frac{8}{\pi},$$

then $f(z)$ is schlicht in $|z| \leq 1$.

Proof. First we show that when (4.6) is satisfied no solution $W(t)$ of (4.3) has more than one zero on $0 \leq t < 2\pi$. Suppose that a solution $W(t)$ has two zeros t_1 and t_2 , $0 \leq t_1 < t_2 < 2\pi$, then $W(t)$, and so $U(t)$, has two zeros in each of the two closed intervals $[t_1, t_2]$ and $[t_2, t_1 + 2\pi]$. According to Lemma 2, some solution of the equation

$$(4.7) \quad V'' + g^+(t, k_0)V = 0$$

has two zeros on $[t_1, t_2]$, and some solution has two zeros on $[t_2, t_1 + 2\pi]$. According to an oscillation criterion of LIAPOUNOFF (see, for example, [6]), we have

$$(4.8) \quad \int_{t_1}^{t_2} g^+(t, k_0) dt > \frac{4}{t_2 - t_1}$$

and

$$(4.9) \quad \int_{t_2}^{t_1+2\pi} g^+(t, k_0) dt > \frac{4}{2\pi - (t_2 - t_1)}.$$

Since $g^+(t, k_0)$ has a period of 2π and the sum of the right-hand members of (4.8) and (4.9) has a minimum value $8/\pi$, (4.8) and (4.9) together contradict the assumption (4.6). Since $f(z)$ can be expressed as $f(z) = W_1(z)/W_2(z)$, where $W_1(z)$ and $W_2(z)$ are two linearly independent solutions of (2.1), and no solution of (2.1) has more than one zero on $|z| = 1$, $f(z)$ does not take any value more than once on $|z| = 1$. Hence $f(z)$ is schlicht in $|z| \leq 1$ (see [9, p. 201] and [10, p. 154]).

As a corollary we have

Corollary 9.1. *If one of the following conditions*

$$(4.10) \quad \int_0^{2\pi} |g_1(t)| dt \leq \frac{8}{\pi},$$

$$(4.11) \quad \int_0^{2\pi} \left| \frac{1}{2} - z^2[f, z] \right| dt \leq \frac{16}{\pi}, \quad z = e^{it},$$

are satisfied, then $f(z)$ is schlicht in $|z| \leq 1$.

Proof. Either one of (4.10) and (4.11) implies (4.6) for $k_0 = 0$.

Theorem 10. *If one of the following conditions*

$$(4.12) \quad g(t, k_0) \leq 1 \text{ for some } k_0,$$

$$(4.13) \quad \Re(z^2[f, z]) \geq -\frac{3}{2}, \quad z = e^{it},$$

$$(4.14) \quad \Im(z^2[f, z]) \geq 0 \quad \text{or} \quad \Im(z^2[f, z]) \leq 0, \quad z = e^{it},$$

holds on $0 \leq t < 2\pi$, with equality satisfied on at most a set of measure zero, then $f(z)$ is schlicht in $|z| \leq 1$.

Proof. Let (4.12) be satisfied. As in the proof of Theorem 9, it is sufficient to prove that no solution of the equation

$$(4.15) \quad V'' + g(t, k_0)V = 0$$

has two zeros t_1 and t_2 , $0 < t_2 - t_1 \leq \pi$; this follows immediately by applying STURM's comparison theorem [7, p. 225] to (4.15) and $V'' + V = 0$. The inequality (4.13) is equivalent to (4.12) for $k_0 = 0$. If $\Im(z^2[f, z]) = -2g_2(t) > 0$ (or < 0), except at a set of measure 0, clearly we can find a k_0 such that (4.6) holds. This completes the proof.

Theorem 11. *If either one of the following conditions*

$$(4.16) \quad |[f, z]| \leq 2(1 - |z|^2)^{-2},$$

$$(4.17) \quad |[f, z]| \leq \pi^2/2$$

holds in $|z| < 1$, then $f(z)$ is schlicht in $|z| < 1$.

Proof. First we prove that $f(z)$ is schlicht in $|z| < 1$ when (4.16) is satisfied. The inequality (4.16) does not change under the linear transformation

$$(4.18) \quad z = k(a - z_1)/(1 - a^*z_1), \quad |a| < 1, \quad |k| = 1,$$

which maps the unit circle onto itself. To see this, we observe that (4.18) can be built up from the transformations

$$(4.19) \quad z = kz_1, \quad z = (b - z_1)/(1 - bz_1), \quad |k| = 1, \quad b \text{ real},$$

and hence it suffices to show that (4.16) does not change under these transformations. For the first transformation of (4.19) we have $[f, z_1] = k^2[f, z]$, while for the second transformation

$$[f, z_1] = (1 - b^2)^2(1 - bz_1)^{-4}[f, z]$$

and

$$|[f, z_1]| \leq 2 \left[\frac{1 - b^2}{|1 - bz_1|^2(1 - |z|^2)} \right]^2 = \frac{2}{(1 - |z_1|^2)^2},$$

hence they do not affect (4.16). On the other hand the transformation (4.18) does not affect the schlichtness of $f(z)$. Since by a proper transformation of the

type (4.18) any two points in $|z| < 1$ can be transformed into two points in $|z| < 1$ symmetrical with respect to the origin, if $f(z)$ takes some value twice in $|z| < 1$, we may assume that that value occurs at z_1 and z_2 and $-z_1 = z_2 = r$, $0 < r < 1$. This implies that some solution W of (2.1), and hence of

$$(4.20) \quad \frac{d^2 W}{dx^2} + Q(x)W = 0,$$

has at least two zeros on the closed interval $[-r, r]$ of the real axis. According to Lemma 2, some solution of the differential equation

$$(4.21) \quad \frac{d^2 s}{dx^2} + \frac{1}{(1-x^2)^2} s = 0$$

must have at least two zeros on $[-r, r]$. This is impossible, as $(1-x^2)^{\frac{1}{2}}$ is a solution of (4.21) vanishing nowhere on $[-r, r]$ and hence no solution of (4.21) has more than one zero on $[-r, r]$ (STURM's separation theorem [7, p. 224]). This proves that $f(z)$ is schlicht in $|z| < 1$ if (4.16) is satisfied.

Suppose that (4.17) is satisfied and that $f(z)$ takes some value twice in $|z| < 1$, say, at z_1 and z_2 . Let $|z_2 - z_1| = r_0$ ($0 < r_0 < 2$), $\theta_0 = \arg(z_2 - z_1)$ and $z = z_1 + r \exp(i\theta_0)$ for real r . Along the line segment $z_1 z_2$, equation (2.1) becomes

$$(4.22) \quad \frac{d^2 W}{dr^2} + e^{2i\theta_0} Q(z_1 + re^{i\theta_0})W = 0.$$

The assumption that $f(z)$ takes the same value at z_1 and z_2 implies that there is a solution of (2.1) vanishing at z_1 and z_2 and hence there is a solution of (4.22) vanishing twice on the closed interval $[0, r_0]$. According to Lemma 2, some solution of

$$(4.23) \quad \frac{d^2 s}{dr^2} + \frac{\pi^2}{4} s = 0$$

has at least two zeros on $[0, r_0]$. This is impossible since every solution of (4.23) has at most one zero on $[0, r_0]$. The contradiction proves that $f(z)$ is schlicht in $|z| < 1$ if (4.17) is satisfied.

The inequalities (4.11), (4.13), (4.16) and (4.17) are the same as those obtained by NEHARI in [1] and [2].

5. Another result. Let S be the sector $-\pi/4 < \arg z < \pi/4$ of the complex plane, and suppose that

$$(5.1) \quad P(z) = z \int_z^\infty Q(t) dt$$

is well defined in S when the integral is taken along a line parallel to the real axis. In a recent paper [11, Theorem 13], E. HILLE proved that there is a solution of

(2.1) which has no zero in S provided $|P(z)| \leq k_0 < \frac{1}{4}$ for all z in S and observed that in some simple cases the value zero is of low frequency in S for every solution of (2.1) (HILLE considered a more general sector S). In this section we shall prove that if another condition is added to that of HILLE, namely, that the real part of $Q(z)$ is non-negative in S , then no solution of (2.1) has more than one zero in S , and hence $f(z)$ is schlicht in S .

Theorem 12. *Let S be defined as above and suppose that*

$$(5.2) \quad \left| z \int_z^\infty Q(t) dt \right| \leq \frac{1}{4} \quad \text{in } S,$$

$$(5.3) \quad Q(z) \neq 0, \quad \Re Q(z) \geq 0 \quad \text{in } S,$$

where the integral is well defined in S when it is taken along a line parallel to the real axis.

Then $f(z)$ is schlicht in S .

Proof. Consider a solution $W(z)$ of (2.1) and suppose that it vanishes at z_1 in S . Let S_1 be that part of S such that $(k - \frac{1}{4})\pi \leq \arg(z - z_1) \leq (k + \frac{1}{4})\pi$, $k = 0, 1$ and $S_2 = S - S_1 - z_1$. Let z_0 be the origin or a point on the ray $\arg z = \pi/4$ or $\arg z = -\pi/4$, $\theta = \arg(z_1 - z_0)$, such that the half-line $L_\theta: z_0 + r \exp(i\theta)$, $0 < r < \infty$, lies in $S_1 + z_1$. Along L_θ $W(z)$ satisfies the differential equation

$$(5.4) \quad \frac{d^2 W}{dr^2} + H(r)W = 0, \quad H(r) = e^{2i\theta} Q(z_0 + re^{i\theta}).$$

According to Lemma 2, $W(z)$ has no zero on L_θ other than z_1 if no solution of the differential equation

$$(5.5) \quad \frac{d^2 s}{dr^2} + h_1(r)s = 0, \quad h_1(r) = \Re H(r),$$

has more than one zero on $0 < r < \infty$. According to an oscillation criterion of

KNESER (see, for example, [6]), if $\int_r^\infty h_1(u) du$ exists and

$$(5.6) \quad r \left| \int_r^\infty h_1(u) du \right| \leq \frac{1}{4}, \quad 0 < r < \infty,$$

then no solution of (5.5) can vanish more than once on $0 < r < \infty$. We shall prove that (5.2) implies (5.6). Since $P(z)/z$ is analytic and has a derivative $-Q(z)$ in S , we have

$$(5.7) \quad \int_z^{z_2} Q(t) dt - \int_z^{z_3} Q(t) dt = P(z_3)/z_3 - P(z_2)/z_2,$$

where z is a point on L_θ and the first integral is taken along a line parallel to the real axis and the second along L_θ . In view of the boundedness of $P(z)$ in S

the second integral in (5.7) is convergent and equal to the first integral as both z_2 and z_3 tend to infinity independently. Hence for any point $z = z_0 + r \exp(i\theta)$ on L_θ , we have $r \leq |z|$ and

$$(5.8) \quad r \left| \int_r^\infty h_1(u) du \right| \leq r \left| \int_r^\infty H(u) du \right| \leq |P(z)| \leq \frac{1}{4}$$

on $0 < r < \infty$. This proves that $W(z)$ has no zero on L_θ other than z_1 . By varying θ from $-\pi/4$ to $\pi/4$, it is clear that $W(z)$ has no zero in S_1 . That $W(z)$ has no zero in S_2 follows immediately from Theorem 2. This completes the proof.

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