Stress-Deformation Relations for Isotropic Materials

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1. Introduction. It has been shown (see C. TRUESDELL (1952) for a comprehensive review of this subject) that the mechanics of a homogeneous isotropic ideally elastic material may be developed on the basis of a description of the relevant elastic properties of the material in terms of a strain-energy function W which is a single-valued function of three scalar invariants of the deformation, I_1 , I_2 and I_3 . These invariants are independent scalar invariants of a symmetric tensor C_{ij} of second order, the components of which, in any rectangular Cartesian coordinate system x_i , are explicit functions of the displacement gradients only. C_{ij} is defined by

$$C_{ij} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k}, \quad (1.1)$$

where x_i is the position in the deformed state of a particle of the body which is at X_i in the undeformed state. I_1 , I_2 and I_3 are defined in terms of C_{ij} by

$$I_1 = C_{ii}, \quad I_2 = \frac{1}{2}[(C_{ii})^2 - C_{ij}C_{ij}] \quad \text{and} \quad I_3 = \det C_{ij}. \quad (1.2)$$

It has been shown on the basis of this theory that the stress components t_{ij} in the coordinate system x_i are given by

$$t_{ij} = \frac{2}{I_3^{\frac{1}{2}}} \left[C_{ij} \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) - C_{ik} C_{kj} \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \delta_{ij} \right]. \quad (1.3)$$

These relations form a natural extension of the stress-strain relations of classical elasticity theory, valid even though the displacement gradients are not infinitesimally small.

Corresponding extensions of the Navier-Stokes equations for a viscous fluid have been proposed by REINER (1945) and RIVLIN (1948). Essentially these theories involve an assumption that the stress components t_{ij} , in the coordinate system x_i , at a point of the fluid, depend only on the velocity gradients at that point (see §39). It is shown that the stress components t_{ij} may be expressed in terms of the velocity-strain tensor $A_{ij}^{(1)}$ by

$$t_{ij} = \varphi_1 A_{ij}^{(1)} + \varphi_2 A_{ik}^{(1)} A_{kj}^{(1)} + \varphi_3 \delta_{ij}, \quad (1.4)$$

where $A_{ij}^{(1)}$ is defined in terms of the velocity $v_i^{(1)}$ of the fluid by

$$A_{ij}^{(1)} = \frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i}. \quad (1.5)$$

φ_1 , φ_2 and φ_3 are single-valued functions of the scalar invariants I_1 , I_2 , I_3 of the tensor $A_{ij}^{(1)}$ defined by

$$I_1 = A_{ii}^{(1)}, \quad I_2 = \frac{1}{2}[(A_{ii}^{(1)})^2 - A_{ij}^{(1)} A_{ij}^{(1)}] \quad \text{and} \quad I_3 = \det A_{ij}^{(1)}. \quad (1.6)$$

The object of the present paper is to derive expressions for the components of the stress tensor in terms of the deformation, for materials with more complex mechanical properties than those discussed in the previous theories. We assume that the materials are homogeneous and isotropic and that the components of stress depend only on the gradients of displacement, velocity, acceleration, second acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration or on the gradients of velocity, acceleration, second acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration. It is implicitly assumed that temperature effects may be neglected.

In chapter I the necessary kinematics of displacement, velocity, acceleration, second acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration fields is discussed.

In chapter II it is proven that if the stress components are assumed to depend on the gradients of displacement, velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration and the stress-deformation relations are invariant under a transformation from one rectangular Cartesian coordinate system to another, having any relative motion with respect to the first, then the stress components must be expressible in terms of these gradients through the components of $(n+1)$ symmetric tensors C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$, which are defined in terms of the gradients. The tensors C_{ij} and $A_{ij}^{(1)}$ are defined in terms of the displacement gradients and the velocity gradients respectively by (1.1) and (1.5), while the tensor $A_{ij}^{(r)}$ is defined in terms of the gradients of velocity, acceleration, $\dots$, $(r-1)^{\text{th}}$ acceleration. This set of tensors is equivalent to that employed by OLDROYD (1950).

It is also shown in chapter II (§§16–18) that if certain relations between the tensors C_{ij} and $A_{ij}^{(1)}$ or $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ (*e.g.* the existence of a common principal direction) exist throughout a finite time interval, then corresponding relations between these tensors and subsequent ones in the sequence C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n-1)}$ are implied.

The remainder of the paper is devoted to the reduction of the expressions for the stress tensor to a closed form analogous to (1.3) and (1.4). The matrix notation has been found to lend itself more naturally to this purpose than the Cartesian tensor notation used in chapters I and II.

In chapters III and IV, a number of results in the algebra of matrices are derived which are necessary for this purpose. In chapter V these results are applied to show that if the stress components are assumed to depend on the gradients of displacement, velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration, then the stress matrix must be expressible as a linear combination of simple matrix polynomials in the matrices $\mathbf{C}(= \| c_{ij} \|)$, $\mathbf{A}_1(= \| A_{ij}^{(1)} \|)$ and $\mathbf{A}_2(= \| A_{ij}^{(2)} \|)$, with coefficients which are scalar invariants of the matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$, where $\mathbf{A}_r = \| A_{ij}^{(r)} \|$ ($r = 1, 2, \dots, n$). It is also shown that if the stress components are assumed to depend on the gradients of velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration, then the stress matrix must be expressible as a linear combination of simple matrix polynomials in the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$, with coefficients which are scalar invariants of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$.

In the expressions for the stress matrix, obtained in chapter V, an indeter-

minacy in the values of the stress components may arise when certain relations exist between the matrices $\mathbf{C}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ or $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$. However, it is shown from the results obtained in §§16–18 that if these relations persist over a finite interval of time, then the indeterminacy can be removed. Consequently, the expressions for the stress matrix enable us to determine the stress components throughout a body, from a knowledge of the deformation undergone by the body, except perhaps at isolated instants of time. In these expressions, the particular mechanical properties of the material enter through coefficients which depend on the scalar invariants of the set of kinematic tensors $C_{ij}, A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$. It is apparent that as a result of this, expressions for the forces associated with any completely specified deformation of a body of the material could be obtained in a manner analogous to that adopted in the calculations (RIVLIN 1948, 1949a, 1949b) based on equations (1.3) and (1.4). It is desirable, before carrying out such calculations, to transform the stress-deformation relations obtained in the present paper to a general curvilinear coordinate system, *i.e.* to express them in a general tensor notation. This will be done in a subsequent paper.

CHAPTER I. KINEMATICS OF DEFORMATION

2. Description of the deformation of a body in a fixed co-ordinate system.

Let us consider a body of some material to be undergoing deformation. Let us specify the position of the body at time t by its co-ordinates x_i ($i = 1, 2, 3$) in a fixed rectangular Cartesian co-ordinate system x_i , so that for each material particle of the body x_i is a function of t . Suppose further that at some time $t = T$, x_i is at X_i in the same co-ordinate system. We can identify each material particle by its co-ordinates X_i at time T and consider x_i to be a function of X_j and t . Hence,

$$x_i = x_i(X_j, t). \quad (2.1)$$

Thus, if the three functional relationships (2.1) are known, the deformation throughout the body is completely specified.

The components, in the co-ordinate system x_i , of the velocity $v_i^{(1)}$ at time t , of the material particle which was at x_i at time T , can be obtained from (2.1) as

$$v_i^{(1)} = Dx_i/Dt, \quad (2.2)$$

where D/Dt denotes differentiation with respect to time keeping X_j constant.

The velocity components $v_i^{(1)}$ obtained by differentiating (2.1) with respect to time are obtained as functions of X_j and t . They may equally be considered as functions of x_j and t . If $v_i^{(1)}$ be so regarded, then we obtain for the acceleration components $v_i^{(2)}$

$$v_i^{(2)} = \frac{Dv_i^{(1)}}{Dt} = \frac{\partial v_i^{(1)}}{\partial t} + v_l^{(1)} \frac{\partial v_i^{(1)}}{\partial x_l}, \quad (2.3)$$

where $\partial/\partial t$ denotes differentiation with respect to time keeping x_j constant.

More generally, the $(n - 1)^{\text{th}}$ acceleration components $v_i^{(n)}$ may be expressed as

$$\begin{aligned} v_i^{(n)} &= \left(\frac{\partial}{\partial t} + v_l^{(1)} \frac{\partial}{\partial x_l} \right) v_i^{(n-1)} = \dots = \left(\frac{\partial}{\partial t} + v_l^{(1)} \frac{\partial}{\partial x_l} \right)^r v_i^{(n-r)} \\ &= \dots = \left(\frac{\partial}{\partial t} + v_l^{(1)} \frac{\partial}{\partial x_l} \right)^{n-1} v_i^{(1)}. \end{aligned} \quad (2.4)$$

3. Description of the deformation of a body in a moving co-ordinate system.

The co-ordinates y_i of a point in a moving rectangular Cartesian co-ordinate system are given in terms of the co-ordinates x_i of the same point in the fixed system introduced earlier by

$$y_i = a_{ij}(x_j - b_j), \quad (3.1)$$

where a_{ij} and b_j are functions of t such that

$$a_{ik} a_{jk} = a_{ki} a_{kj} = \delta_{ij}. \quad (3.2)$$

We consider only proper transformations in this paper, so the quantities a_{ij} must also satisfy $\det \|a_{ij}\| = 1$.

The velocity $u_i^{(1)}$ relative to the moving co-ordinate system y_i of the material particle, which at time t is at x_i in the fixed co-ordinate system, is given, from (3.1) and (2.2), as

$$u_i^{(1)} = \frac{Dy_i}{Dt} = \frac{Da_{ij}}{Dt} (x_j - b_j) + a_{ij} \left(v_j^{(1)} - \frac{Db_j}{Dt} \right). \quad (3.3)$$

The acceleration $u_i^{(2)}$ relative to the co-ordinate system y_i is given by $Du_i^{(1)}/Dt$ and more generally the $(n-1)^{\text{th}}$ acceleration $u_i^{(n)}$ is given by

$$u_i^{(n)} = \frac{D^n y_i}{Dt^n} = \frac{D^n a_{ij}}{Dt^n} (x_j - b_j) + \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ij}}{Dt^{n-r}} \left(v_j^{(r)} - \frac{D^r b_j}{Dt^r} \right). \quad (3.4)$$

The velocity $u_i^{(1)}$ and accelerations $u_i^{(n)}$ of the particle considered relative to the moving co-ordinate system y_i has components $w_i^{(n)}$ parallel to the axes of the fixed co-ordinate system x_i which are given by

$$w_i^{(n)} = a_{ji} u_j^{(n)} = a_{ji} \frac{D^n a_{jk}}{Dt^n} (x_k - b_k) + \sum_{r=1}^n \binom{n}{r} a_{ji} \frac{D^{n-r} a_{jk}}{Dt^{n-r}} \left(v_k^{(r)} - \frac{D^r b_k}{Dt^r} \right). \quad (3.5)$$

We can obtain the relations between the accelerations of various order relative to the fixed and moving co-ordinate systems x_i and y_i in inverse form in the following manner. From equations (3.1) and (3.2), we have

$$x_i = b_i + a_{ji} y_j. \quad (3.6)$$

Whence,

$$\begin{aligned} v_i^{(n)} &= \frac{D^n x_i}{Dt^n} = \frac{D^n b_i}{Dt^n} + \sum_{r=0}^n \binom{n}{r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} \frac{D^r y_j}{Dt^r} \\ &= \frac{D^n b_i}{Dt^n} + \frac{D^n a_{ji}}{Dt^n} y_j + \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} u_j^{(r)}. \end{aligned} \quad (3.7)$$

Employing the relation (3.1) to substitute for y_j , we obtain

$$v_i^{(n)} = \frac{D^n b_i}{Dt^n} + a_{jk} \frac{D^n a_{ji}}{Dt^n} (x_k - b_k) + \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} u_j^{(r)}. \quad (3.8)$$

The term $D^n b_i/Dt^n$ on the right-hand side of equation (3.8) represents the $(n-1)^{\text{th}}$ acceleration of the origin of the co-ordinate system y_i in the co-ordi-

nate system x_i . The term $a_{jk}(x_k - b_k)D^n a_{ji}/Dt^n = W_i^{(n)}$ (say) represents the additional $(n - 1)^{\text{th}}$ acceleration which the material particle would have in the co-ordinate system x_i if it were stationary at y_i relative to the co-ordinate system y_i . The remaining terms, which we shall denote by $U_i^{(n)}$, represent the $(n - 1)^{\text{th}}$ acceleration of the material particle relative to the co-ordinate system x_i due to its motion relative to the co-ordinate system y_i .

4. Discussion of the results of the previous section. It has been seen in the previous section that the $(n - 1)^{\text{th}}$ acceleration $v_i^{(n)}$ of a material particle relative to the fixed co-ordinate system x_i can be expressed by equation (3.8) in terms of its velocity, acceleration, second acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration relative to a moving co-ordinate system y_i together with quantities determined by the motion of the co-ordinate system y_i relative to the co-ordinate system x_i . We have

$$v_i^{(n)} = \frac{D^n b_i}{Dt^n} + W_i^{(n)} + U_i^{(n)}, \quad (4.1)$$

where

$$W_i^{(n)} = a_{jk} \frac{D^n a_{ji}}{Dt^n} (x_k - b_k)$$

and

$$U_i^{(n)} = \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} u_j^{(r)}. \quad (4.2)$$

Let us first consider the velocity $v_i^{(1)}$. This is given, from (4.1), by

$$v_i^{(1)} = \frac{Db_i}{Dt} + W_i^{(1)} + U_i^{(1)}, \quad (4.3)$$

where, from (4.2),

$$W_i^{(1)} = a_{jk} \frac{Da_{ji}}{Dt} (x_k - b_k) \text{ and } U_i^{(1)} = a_{ji} u_j^{(1)}. \quad (4.4)$$

The term Db_i/Dt on the right-hand side of equation (4.3) represents the velocity, in the co-ordinate system x_i , of the origin of the co-ordinate system y_i . The term $U_i^{(1)}$ represents the components in the co-ordinate system x_i of the velocity of the particle considered relative to the co-ordinate system y_i , when the latter is treated as stationary. The term $W_i^{(1)}$ represents the velocity of the material particle in the co-ordinate system x_i resulting from the angular velocity of the co-ordinate system y_i , the material particle being regarded as stationary at y_i relative to the system y_i . Let $\Omega_i^{(1)}$ be the components of this angular velocity in the co-ordinate system x_i . We have

$$(\Omega_1^{(1)}, \Omega_2^{(1)}, \Omega_3^{(1)}) = \left(a_{j2} \frac{Da_{j3}}{Dt}, a_{j3} \frac{Da_{j1}}{Dt}, a_{j1} \frac{Da_{j2}}{Dt} \right). \quad (4.5)$$

We note that, since $a_{jk} a_{ji} = \delta_{ki}$,

$$a_{jk} \frac{Da_{ji}}{Dt} + a_{ji} \frac{Da_{jk}}{Dt} = 0. \quad (4.6)$$

Defining $\omega_{ik}^{(1)}$ by

$$\omega_{ik}^{(1)} = \frac{1}{2} \left(a_{jk} \frac{Da_{ji}}{Dt} - a_{ji} \frac{Da_{jk}}{Dt} \right) = a_{jk} \frac{Da_{ji}}{Dt}, \quad (4.7)$$

so that $\omega_{ik}^{(1)} = -\omega_{ki}^{(1)}$, we see from (4.5) and (4.7) that

$$\Omega_1^{(1)} = \omega_{32}^{(1)}, \quad \Omega_2^{(1)} = \omega_{13}^{(1)} \quad \text{and} \quad \Omega_3^{(1)} = \omega_{21}^{(1)}. \quad (4.8)$$

Now let us consider the first acceleration $v_i^{(2)}$. We have

$$v_i^{(2)} = \frac{D^2 b_i}{Dt^2} + W_i^{(2)} + U_i^{(2)}, \quad (4.9)$$

where

$$W_i^{(2)} = a_{jk} \frac{D^2 a_{ji}}{Dt^2} (x_k - b_k)$$

and

$$U_i^{(2)} = 2 \frac{Da_{ji}}{Dt} u_j^{(1)} + a_{ji} u_j^{(2)}. \quad (4.10)$$

The term $D^2 b_i/Dt^2$ represents the acceleration of the origin of the co-ordinate system y_i in the system x_i . In the expression (4.10) for $U_i^{(2)}$, the term $2u_j^{(1)} Da_{ji}/Dt$ represents the Coriolis acceleration of the moving particle and the term $a_{ji} u_j^{(2)}$ represents the acceleration of the particle in the co-ordinate system x_i due to its acceleration relative to the system y_i . The term $W_i^{(2)}$ in (4.9) represents the acceleration of the particle in the co-ordinate system x_i due to the rotation of the co-ordinate system y_i , if the particle were stationary at y_i in the latter system. The term $W_i^{(1)}$ corresponding to this in the expression (4.3) for $v_i^{(1)}$ was seen to arise as a result of the non-zero angular velocity of the co-ordinate system y_i . The term $W_i^{(2)}$ does not, however, arise entirely as a result of the non-zero angular acceleration of the co-ordinate system y_i . We can express $W_i^{(2)}$ in the form

$$\begin{aligned} W_i^{(2)} = & \left[\frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} + a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right) \right. \\ & \left. + \frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} - a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right) \right] (x_k - b_k). \end{aligned} \quad (4.11)$$

The term $\frac{1}{2}(a_{jk} D^2 a_{ji}/Dt^2 + a_{ji} D^2 a_{jk}/Dt^2) (x_k - b_k)$ represents the centrifugal acceleration in the co-ordinate system x_i of a particle stationary at y_i , in the co-ordinate system y_i . This may be expressed in a more conventional form in the following way. Differentiating (4.6) with respect to t , we have

$$\begin{aligned} \frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} + a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right) &= -\frac{Da_{jk}}{Dt} \frac{Da_{ji}}{Dt} \\ &= -\delta_{il} \frac{Da_{jk}}{Dt} \frac{Da_{jl}}{Dt} = -a_{mi} a_{ml} \frac{Da_{jk}}{Dt} \frac{Da_{jl}}{Dt}. \end{aligned} \quad (4.12)$$

From equation (3.2), we have $a_{ml} Da_{jl}/Dt = -a_{jl} Da_{ml}/Dt$, so that

$$\frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} + a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right) = a_{mi} \frac{Da_{ml}}{Dt} a_{jl} \frac{Da_{jk}}{Dt}. \quad (4.13)$$

It is seen from (4.7) and (4.13) that

$$\frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} + a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right) = \omega_{kl}^{(1)} \omega_{li}^{(1)}. \quad (4.14)$$

The term $\frac{1}{2}(a_{jk} D^2 a_{ji}/Dt^2 - a_{ji} D^2 a_{jk}/Dt^2) (x_k - b_k)$ represents the acceleration of a particle, stationary with respect to y_i , resulting from the angular acceleration $\Omega_i^{(2)}$, defined as $D\Omega_i^{(1)}/Dt$, of the co-ordinate system y_i . For, we readily see, differentiating equation (4.7) with respect to t , that

$$\frac{D\omega_{ik}^{(1)}}{Dt} = \frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} - a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right). \quad (4.15)$$

Thus, in view of the relations (4.8), we have

$$\Omega_1^{(2)} = \omega_{32}^{(2)}, \quad \Omega_2^{(2)} = \omega_{13}^{(2)} \quad \text{and} \quad \Omega_3^{(2)} = \omega_{21}^{(2)}, \quad (4.16)$$

where

$$\omega_{ik}^{(2)} = -\omega_{ki}^{(2)} = \frac{1}{2} \left(a_{jk} \frac{D^2 a_{ji}}{Dt^2} - a_{ji} \frac{D^2 a_{jk}}{Dt^2} \right). \quad (4.17)$$

We may now consider the second acceleration $v_i^{(3)}$, which is given from equation (4.1) as

$$v_i^{(3)} = \frac{D^3 b_i}{Dt^3} + W_i^{(3)} + U_i^{(3)}, \quad (4.18)$$

where

$$W_i^{(3)} = a_{jk} \frac{D^3 a_{ji}}{Dt^3} (x_k - b_k)$$

and

$$U_i^{(3)} = 3 \frac{D^2 a_{ji}}{Dt^2} u_j^{(1)} + 3 \frac{Da_{ji}}{Dt} u_j^{(2)} + a_{ji} u_j^{(3)}. \quad (4.19)$$

The term $D^3 b_i/Dt^3$ in (4.18) represents the second acceleration of the origin of the co-ordinate system y_i relative to x_i . The term $a_{ji} u_j^{(3)}$ in the expression (4.19) for $U_i^{(3)}$ represents the components in the co-ordinate system x_i of the second acceleration of the material particle relative to y_i , if the co-ordinate system y_i is treated as stationary. The terms $3u_j^{(1)} D^2 a_{ji}/Dt^2$ and $3u_j^{(2)} Da_{ji}/Dt$ are interaction terms of the Coriolis type between the motion of the particle relative to the co-ordinate system y_i and the rotation of the co-ordinate system y_i relative to x_i . The term $W_i^{(3)}$ given by (4.19) represents the second acceleration in the co-ordinate system x_i , resulting from the rotation of the co-ordinate system y_i , if the particle were fixed at y_i relative to the co-ordinate system y_i . We may express $W_i^{(3)}$ in the form

$$W_i^{(3)} = \left[\frac{1}{2} \left(a_{jk} \frac{D^3 a_{ji}}{Dt^3} + a_{ji} \frac{D^3 a_{jk}}{Dt^3} \right) + \frac{1}{2} \left(a_{jk} \frac{D^3 a_{ji}}{Dt^3} - a_{ji} \frac{D^3 a_{jk}}{Dt^3} \right) \right] (x_k - b_k). \quad (4.20)$$

The term $\frac{1}{2}(a_{jk} D^3 a_{ji}/Dt^3 + a_{ji} D^3 a_{jk}/Dt^3)(x_k - b_k)$ represents a centrifugal second acceleration. The term $\frac{1}{2}(a_{jk} D^3 a_{ji}/Dt^3 - a_{ji} D^3 a_{jk}/Dt^3)(x_k - b_k)$ does not, however, arise entirely from the angular second acceleration $\Omega_i^{(3)}$, which is defined by

$$\Omega_i^{(3)} = \frac{D\Omega_i^{(2)}}{Dt} = \frac{D^2 \Omega_i^{(1)}}{Dt^2}. \quad (4.21)$$

Employing the notation

$$\Omega_1^{(3)} = \omega_{32}^{(3)}, \quad \Omega_2^{(3)} = \omega_{13}^{(3)} \quad \text{and} \quad \Omega_3^{(3)} = \omega_{21}^{(3)}, \quad (4.22)$$

with $\omega_{ik}^{(3)} = -\omega_{ki}^{(3)}$, we have, from (4.21), (4.16) and (4.15),

$$\omega_{ik}^{(3)} = \frac{1}{2} \left[a_{jk} \frac{D^3 a_{ji}}{Dt^3} - a_{ji} \frac{D^3 a_{jk}}{Dt^3} + \frac{Da_{jk}}{Dt} \frac{D^2 a_{ji}}{Dt^2} - \frac{Da_{ji}}{Dt} \frac{D^2 a_{jk}}{Dt^2} \right]. \quad (4.23)$$

We thus see that

$$\begin{aligned} & \frac{1}{2} \left(a_{jk} \frac{D^3 a_{ji}}{Dt^3} - a_{ji} \frac{D^3 a_{jk}}{Dt^3} \right) (x_k - b_k) \\ &= \left[\omega_{ik}^{(3)} - \frac{1}{2} \left(\frac{Da_{jk}}{Dt} \frac{D^2 a_{ji}}{Dt^2} - \frac{Da_{ji}}{Dt} \frac{D^2 a_{jk}}{Dt^2} \right) \right] (x_k - b_k) \\ &= \left[\omega_{ik}^{(3)} - \frac{1}{2} \left(a_{mk} a_{ml} \frac{Da_{jl}}{Dt} \frac{D^2 a_{ji}}{Dt^2} - a_{mi} a_{ml} \frac{Da_{jl}}{Dt} \frac{D^2 a_{jk}}{Dt^2} \right) \right] (x_k - b_k). \end{aligned} \quad (4.24)$$

From (4.7), we have

$$a_{mi} \frac{D a_{jl}}{Dt} = \omega_{lp}^{(1)} a_{mi} a_{jp}. \quad (4.25)$$

Substituting from (4.25) in (4.24), we obtain

$$\frac{1}{2} \left(a_{jk} \frac{D^3 a_{ji}}{Dt^3} - a_{ji} \frac{D^3 a_{jk}}{Dt^3} \right) = \omega_{ik}^{(3)} - \frac{1}{2} \left(a_{jp} \frac{D^2 a_{ji}}{Dt^2} \omega_{kp}^{(1)} - a_{jp} \frac{D^2 a_{jk}}{Dt^2} \omega_{ip}^{(1)} \right). \quad (4.26)$$

Now, from (4.14) and (4.17) we have

$$a_{jp} \frac{D^2 a_{ji}}{Dt^2} = \omega_{ip}^{(2)} + \omega_{ij}^{(1)} \omega_{jp}^{(1)}. \quad (4.27)$$

Substituting from (4.27) in (4.26) we obtain

$$\frac{1}{2} \left(a_{jk} \frac{D^3 a_{ji}}{Dt^3} - a_{ji} \frac{D^3 a_{jk}}{Dt^3} \right) = \omega_{ik}^{(3)} - \frac{1}{2} (\omega_{ip}^{(2)} \omega_{kp}^{(1)} - \omega_{kp}^{(2)} \omega_{ip}^{(1)}) - \omega_{ij}^{(1)} \omega_{jp}^{(1)} \omega_{kp}^{(1)}. \quad (4.28)$$

It is seen from (4.28) that the term $\frac{1}{2}(a_{jk} D^3 a_{ji}/Dt^3 - a_{ji} D^3 a_{jk}/Dt^3)(x_k - b_k)$ in the expression (4.20) for $W_i^{(3)}$ consists of a term $\omega_{ik}^{(3)}(x_k - b_k)$, which represents the second acceleration, relative to the fixed co-ordinate system x_i , of a point fixed in the moving co-ordinate system y_i , due to the angular second acceleration of the co-ordinate system y_i , together with terms which result from the angular velocity and angular acceleration of the co-ordinate system y_i .

We shall now discuss the more general case of the $(n - 1)^{\text{th}}$ acceleration $v_i^{(n)}$, given by equation (4.1). The term $D^n b_i/Dt^n$ represents the $(n - 1)^{\text{th}}$ acceleration of the origin of the co-ordinate system y_i relative to the co-ordinate system x_i . We can re-write the expression (4.2) for $U_i^{(n)}$ thus

$$U_i^{(n)} = \sum_{r=1}^{n-1} \binom{n}{r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} u_j^{(r)} + a_{ji} u_j^{(n)}. \quad (4.29)$$

The quantities $a_{ji} u_j^{(n)}$ in the expression (4.29) for $U_i^{(n)}$ are the components in the co-ordinate system x_i of the $(n - 1)^{\text{th}}$ acceleration of the material particle relative to y_i , if the co-ordinate system y_i is treated as stationary. The terms in the sum $\sum_{r=1}^{n-1} \binom{n}{r} u_j^{(r)} D^{n-r} a_{ji}/Dt^{n-r}$ are interaction terms of the Coriolis type between the motion of the particle relative to the co-ordinate system y_i and the rotation of the co-ordinate system y_i relative to x_i . The term $W_i^{(n)}$ in (4.1) may be expressed in the form

$$W_i^{(n)} = \left[\frac{1}{2} \left(a_{jk} \frac{D^n a_{ji}}{Dt^n} + a_{ji} \frac{D^n a_{jk}}{Dt^n} \right) + \frac{1}{2} \left(a_{jk} \frac{D^n a_{ji}}{Dt^n} - a_{ji} \frac{D^n a_{jk}}{Dt^n} \right) \right] (x_k - b_k). \quad (4.30)$$

The term $\frac{1}{2}(a_{jk} D^n a_{ji}/Dt^n + a_{ji} D^n a_{jk}/Dt^n)(x_k - b_k)$ represents a centrifugal $(n - 1)^{\text{th}}$ acceleration. As in the case when $n = 3$, the term $\frac{1}{2}(a_{jk} D^n a_{ji}/Dt^n -$

$a_{ji} D^n a_{jk}/Dt^n)(x_k - b_k)$ does not, in general, arise entirely from the angular $(n - 1)^{\text{th}}$ acceleration of the co-ordinate system y_i with respect to the co-ordinate system x_i . We define the angular $(n - 1)^{\text{th}}$ acceleration $\Omega_i^{(n)}$ by

$$\Omega_i^{(n)} = \frac{D^{n-1} \Omega_i^{(1)}}{Dt^{n-1}}. \quad (4.31)$$

Employing the notation

$$\Omega_1^{(n)} = \omega_{32}^{(n)}, \quad \Omega_2^{(n)} = \omega_{13}^{(n)} \quad \text{and} \quad \Omega_3^{(n)} = \omega_{21}^{(n)}, \quad (4.32)$$

with $\omega_{ik}^{(n)} = -\omega_{ki}^{(n)}$, we have from (4.31), (4.7) and (4.8),

$$\begin{aligned} \omega_{ik}^{(n)} &= \frac{1}{2} \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{jk}}{Dt^r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} - \frac{D^r a_{ji}}{Dt^r} \frac{D^{n-r} a_{jk}}{Dt^{n-r}} \right) \\ &= \frac{1}{2} \left(a_{jk} \frac{D^n a_{ji}}{Dt^n} - a_{ji} \frac{D^n a_{jk}}{Dt^n} \right) \\ &\quad + \frac{1}{2} \sum_{r=1}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{jk}}{Dt^r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} - \frac{D^r a_{ji}}{Dt^r} \frac{D^{n-r} a_{jk}}{Dt^{n-r}} \right). \end{aligned} \quad (4.33)$$

Whence we have

$$\begin{aligned} &\frac{1}{2} \left(a_{jk} \frac{D^n a_{ji}}{Dt^n} - a_{ji} \frac{D^n a_{jk}}{Dt^n} \right) (x_k - b_k) \\ &= \left[\omega_{ik}^{(n)} - \frac{1}{2} \sum_{r=1}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{jk}}{Dt^r} \frac{D^{n-r} a_{ji}}{Dt^{n-r}} - \frac{D^r a_{ji}}{Dt^r} \frac{D^{n-r} a_{jk}}{Dt^{n-r}} \right) \right] (x_k - b_k). \end{aligned} \quad (4.34)$$

We shall see in the next section that $D^r a_{jk}/Dt^r$ may be expressed in terms of $\omega_{pq}^{(1)}, \omega_{pq}^{(2)}, \dots, \omega_{pq}^{(r)}$. With this result it is seen that the term $\frac{1}{2}(a_{jk} D^n a_{ji}/Dt^n - a_{ji} D^n a_{jk}/Dt^n)(x_k - b_k)$ in the expression (4.30) for $W_i^{(n)}$ consists of a term $\omega_{ik}^{(n)}(x_k - b_k)$, which represents the $(n - 1)^{\text{th}}$ acceleration, relative to the fixed co-ordinate system x_i , of a point fixed in the moving co-ordinate system y_i , due to the angular $(n - 1)^{\text{th}}$ acceleration of the co-ordinate system y_i , together with terms which result from the angular velocity, angular acceleration, angular second acceleration, $\dots$, angular $(n - 2)^{\text{th}}$ acceleration of the co-ordinate system y_i .

5. A particular choice of the moving co-ordinate system. In this section it will be shown that the moving co-ordinate system y_i may be chosen in such a way that instantaneously at time t' its orientation, angular velocity, angular acceleration, $\dots$, angular $(n - 1)^{\text{th}}$ acceleration relative to the co-ordinate system x_i have any specified values, while at some other time T , its orientation has any specified value. Expressed mathematically this means that a set of a 's, which are analytic functions of time t , satisfying the relations (3.2) at all instants of time, may be chosen in such a way that instantaneously at time t' ,

$$a_{ik} = a'_{ik} \quad (\text{say}) \quad (5.1)$$

and

$$\left[\frac{D^{r-1}}{Dt^{r-1}} \left(a_{jk} \frac{Da_{ji}}{Dt} \right) \right]_{t=t'} = \omega_{ik}^{(r)}, \quad r = 1, \dots, n, \quad (5.2)$$

where the quantities a'_{ik} may be arbitrarily chosen subject to the condition that

$$a'_{jk} a'_{ji} = a'_{kj} a'_{ij} = \delta_{ki} \quad (5.3)$$

and the quantities $\omega_{ik}^{(r)}$ may be arbitrarily chosen subject to the condition that

$$\omega_{ik}^{(r)} = -\omega_{ki}^{(r)}, \quad (5.4)$$

while, at time T ,

$$a_{ik} = A_{ik} \quad (\text{say}) \quad (5.5)$$

where the quantities A_{ij} may be arbitrarily chosen subject to the condition

$$A_{jk} A_{ji} = A_{kj} A_{ij} = \delta_{ik}. \quad (5.6)$$

Let us suppose that the a 's are functions of t such that

$$a_{jk} \frac{Da_{ji}}{Dt} = \sum_{r=1}^n \frac{1}{(r-1)!} \omega_{ik}^{(r)} (t-t')^{r-1}. \quad (5.7)$$

Then, with (5.4), we have

$$\frac{D}{Dt} (a_{jk} a_{ji}) = a_{jk} \frac{Da_{ji}}{Dt} + a_{ji} \frac{Da_{jk}}{Dt} = 0. \quad (5.8)$$

Thus,

$$a_{jk} a_{ji} = \alpha_{ik}, \quad (5.9)$$

where the α 's are arbitrary constants. Let us choose $\alpha_{ik} = \delta_{ik}$. Then, from (5.9), we have

$$a_{jk} a_{ji} = \delta_{ik}, \quad \text{whence} \quad a_{kj} a_{ij} = \delta_{ik}. \quad (5.10)$$

With (5.10) we may re-write (5.7) in the form

$$\frac{Da_{ji}}{Dt} = \sum_{r=1}^n \frac{1}{(r-1)!} \omega_{ik}^{(r)} a_{jk} (t-t')^{r-1}. \quad (5.11)$$

Equation (5.11) represents nine linear differential equations in the nine dependent variables a_{ij} and the single independent variable t . Since the expressions on the right-hand side of these equations are analytic in the ten variables a_{ij} and t , the equations have a unique solution for a_{ij} as functions of t , analytic over some

finite range of values of t about t' and satisfying the condition (5.1) when $t = t'$ (see, for example, E. KAMKE (1947)). We note that, from (5.3), the condition (5.1) is in accord with our choosing $\alpha_{ik} = \delta_{ik}$.

It can readily be seen that if a_{ij} is a solution of equations (5.11) satisfying the condition (5.3) at $t = t'$, then it satisfies the condition (5.10) for all values of t in the range containing $t = t'$ for which the solution exists. For, from (5.11) and the relation $\omega_{ik}^{(r)} = -\omega_{ki}^{(r)}$, we have $D(a_{ki} a_{ji})/Dt = 0$. With (5.3) we obtain (5.10). Since the relation (5.10) holds it is apparent that the relation (5.7) follows from (5.11). We thus see that if a_{ij} is a solution of (5.11) satisfying (5.3) then it is also a solution of (5.7).

This solution is readily seen to satisfy the conditions (5.2). Since it is analytic in the neighborhood of $t = t'$ we can differentiate (5.7) with respect to t any number of times. Introducing $t = t'$ in equation (5.7) and in the equations obtained by differentiating (5.7) with respect to t once, twice, $\dots$, $n - 1$ times, we obtain the relations (5.2).

From (5.10), we have $a_{ij} a_{ij} = 3$, so that a_{ij} is bounded in the range of values of t for which the solution a_{ij} of equation (5.11) exists. Consequently, the range of values of t in which a solution of (5.11) satisfying the boundary condition (5.1) exists may be taken as large as we please; in particular, it may be taken to include the time interval from T to t' .

We thus see that the moving rectangular Cartesian co-ordinate system y_i may be chosen in such a way that instantaneously at time t' its orientation, angular velocity, $\dots$, angular $(n - 1)^{\text{th}}$ acceleration relative to the co-ordinate system x_i have any specified values. It remains to be shown that, in addition, its orientation at time T may be arbitrarily chosen.

In order to do this, we shall first show that we can choose a moving rectangular Cartesian co-ordinate system z_i in such a way that it coincides with the co-ordinate system x_i at time t' and has zero angular velocity, angular acceleration, $\dots$, angular $(n - 1)^{\text{th}}$ acceleration at this time, while at some other instant of time T , its orientation with respect to the co-ordinate system x_i is arbitrarily specified. Mathematically, this means that we must show that there exists an orthogonal transformation s_{ij} which depends on time in such a way that

$$[s_{ij}]_{t=t'} = \delta_{ij}, \quad (5.12)$$

$$[s_{ij}]_{t=T} = S_{ij}, \quad (5.13)$$

where S_{ij} are the prescribed direction-cosines of the axis z_i in the co-ordinate system x_i at time T , and

$$\left[\frac{D^{p-1}}{Dt^{p-1}} \left(s_{ki} \frac{Ds_{kj}}{Dt} \right) \right]_{t=t'} = 0 \quad (p = 1, 2, \dots, n). \quad (5.14)$$

If the orthogonal transformation s_{ij} can be chosen in this manner, then we may, of course, take

$$z_i = s_{ij} x_j. \quad (5.15)$$

Now, it is well-known that any orthogonal transformation s_{ij} , for which $\det s_{ij} = 1$, can be expressed in the form

$$s_{ij} = \sigma_{ik} \vartheta_{kl} \sigma_{jl}, \quad (5.16)$$

where σ_{ij} represents an orthogonal transformation and ϑ_{kl} has the form

$$\|\vartheta_{kl}\| = \begin{vmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}. \quad (5.17)$$

Let φ depend on time in accordance with the relation

$$\varphi = \left(\frac{t - t'}{T - t'} \right)^{n+1} \varphi_0, \quad (5.18)$$

where φ_0 is constant, such that the quantities σ_{ij} in equation (5.16) may be chosen to satisfy the relation $s_{ij} = S_{ij}$ at $t = T$. It can readily be seen that if σ_{ij} are chosen in this manner at $t = T$ and are independent of time, then s_{ij} , given by equation (5.16) satisfies the relations (5.12) and (5.14). When $t = t'$, we see from (5.18) and (5.17) that $\vartheta_{kl} = \delta_{kl}$, so that, from (5.16), $s_{ij} = \sigma_{ik} \sigma_{jk} = \delta_{ij}$, *i.e.* (5.12) is satisfied. Furthermore, it is seen from (5.16) that, since σ_{ij} is independent of time and φ is given by (5.18), equations (5.14) are satisfied.

Now, we can choose a moving rectangular Cartesian co-ordinate system y_i , defined relative to the co-ordinate system z_i by the relation

$$y_i = e_{ij} z_j, \quad (5.19)$$

where

$$[e_{ij}]_{t=t'} = a'_{ij} \quad (5.20)$$

and the angular velocity $(\omega_{32}^{(1)}, \omega_{13}^{(1)}, \omega_{21}^{(1)})$, angular acceleration $(\omega_{32}^{(2)}, \omega_{13}^{(2)}, \omega_{21}^{(2)})$, $\dots$, angular $(n - 1)^{\text{th}}$ acceleration $(\omega_{32}^{(n)}, \omega_{13}^{(n)}, \omega_{21}^{(n)})$ with respect to the co-ordinate system z_i have specified values at time t' . Let the value of e_{ij} at time T for this co-ordinate system be given by

$$[e_{ij}]_{t=T} = E_{ij}. \quad (5.21)$$

Let

$$y_i = a_{ij} x_j \quad \text{or} \quad x_i = a_{ji} y_j \quad (5.22)$$

define the relation of the co-ordinate system y_i to the co-ordinate system x_i . From (5.15), (5.19) and (5.22), we have

$$a_{ij} = e_{ik} s_{kj}. \quad (5.23)$$

It is readily seen, from (5.23), (5.12) and (5.20) that

$$[a_{ij}]_{t=t'} = a'_{ij}; \quad (5.24)$$

i.e. equation (5.1) is satisfied.

Since at time t' , the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system z_i with respect to x_i are zero, we readily see that the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of y_i with respect to x_i have the same values as those with respect to z_i . Thus, equations (5.2) are satisfied.

Now, if we choose S_{ij} in equation (5.16), defining the orientation of the co-ordinate system z_i at time T , in accordance with the relation

$$S_{ij} = E_{ki} A_{kj}, \quad (5.25)$$

we have

$$A_{ij} = E_{ik} S_{kj}, \quad (5.26)$$

so that, from (5.23), we obtain

$$[a_{ij}]_{t=T} = A_{ij}; \quad (5.27)$$

i.e. the relation (5.5) is satisfied.

We thus see that we can choose a moving rectangular Cartesian co-ordinate system y_i so that it has any specified orientations with respect to the co-ordinate system x_i at times t' and T and any specified values of angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration with respect to the co-ordinate system x_i at time t' .

It may be remarked that if the orientation, angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i with respect to x_i are specified at time t' , then the values of a_{ij} , Da_{ij}/Dt , $\dots$, $D^n a_{ij}/Dt^n$ at time t' are uniquely determined. The value of a_{ij} is, of course, directly given by the specification of the orientation. From (5.2), with $r = 1$, we have

$$\left[a_{jk} \frac{Da_{ji}}{Dt} \right]_{t=t'} = \omega_{ik}^{(1)}, \quad (5.28)$$

whence

$$\left[\frac{Da_{ji}}{Dt} \right]_{t=t'} = a'_{jk} \omega_{ik}^{(1)}. \quad (5.29)$$

Again, with $r = 2$, we obtain, from (5.2),

$$\left[a_{jk} \frac{D^2 a_{ji}}{Dt^2} + \frac{Da_{jk}}{Dt} \frac{Da_{ji}}{Dt} \right]_{t=t'} = \omega_{ik}^{(2)}. \quad (5.30)$$

From (5.29) and (5.30), we have

$$\left[\frac{D^2 a_{ji}}{Dt^2} \right]_{t=t'} = a'_{jk} (\omega_{ik}^{(2)} - \omega_{kl}^{(1)} \omega_{il}^{(1)}). \quad (5.31)$$

It will now be shown that if $[D^s a_{ji}/Dt^s]_{t=t'}$ is assumed to be expressible as a polynomial in $a'_{pq}, \omega_{pq}^{(1)}, \omega_{pq}^{(2)}, \dots, \omega_{pq}^{(s)}$ for $s = 1, 2, \dots, r$, then $[D^{r+1} a_{ji}/Dt^{r+1}]_{t=t'}$ is expressible as a polynomial in $a'_{pq}, \omega_{pq}^{(1)}, \omega_{pq}^{(2)}, \dots, \omega_{pq}^{(r+1)}$. We have, from (5.2),

$$\begin{aligned} \left[\frac{D^r}{Dt^r} \left(a_{jk} \frac{Da_{ji}}{Dt} \right) \right]_{t=t'} &= \sum_{s=0}^r \binom{r}{s} \left[\frac{D^s a_{jk}}{Dt^s} \frac{D^{r+1-s} a_{ji}}{Dt^{r+1-s}} \right]_{t=t'} \\ &= \omega_{ik}^{(r+1)}. \end{aligned} \quad (5.32)$$

This relation may be re-written as

$$\sum_{s=1}^r \binom{r}{s} \left[\frac{D^s a_{jk}}{Dt^s} \frac{D^{r+1-s} a_{ji}}{Dt^{r+1-s}} \right]_{t=t'} + \left[a_{jk} \frac{D^{r+1} a_{ji}}{Dt^{r+1}} \right]_{t=t'} = \omega_{ik}^{(r+1)}. \quad (5.33)$$

Whence, we obtain

$$\left[\frac{D^{r+1} a_{ji}}{Dt^{r+1}} \right]_{t=t'} = a'_{jk} \omega_{ik}^{(r+1)} - \sum_{s=1}^r \binom{r}{s} \left[a_{jk} \frac{D^s a_{mk}}{Dt^s} \frac{D^{r+1-s} a_{mi}}{Dt^{r+1-s}} \right]_{t=t'}. \quad (5.34)$$

We note that equation (5.34) expresses $[D^{r+1} a_{ji}/Dt^{r+1}]$ as a polynomial in $\omega_{pq}^{(r+1)}, a'_{pq}$ and $[D^s a_{pq}/Dt^s]_{t=t'}$ ($s = 1, 2, \dots, r$). Since we have assumed that $[D^s a_{pq}/Dt^s]_{t=t'}$ is expressible as a polynomial in $a'_{mn}, \omega_{mn}^{(1)}, \omega_{mn}^{(2)}, \dots, \omega_{mn}^{(s)}$ for $s = 1, 2, \dots, r$, we see that $[D^{r+1} a_{ji}/Dt^{r+1}]_{t=t'}$ is expressible as a polynomial in $a'_{pq}, \omega_{pq}^{(1)}, \omega_{pq}^{(2)}, \dots, \omega_{pq}^{(r+1)}$. Now, our assumption that $[D^s a_{pq}/Dt^s]_{t=t'}$ is expressible as a polynomial in $a'_{mn}, \omega_{mn}^{(1)}, \omega_{mn}^{(2)}, \dots, \omega_{mn}^{(s)}$ is seen, from (5.29) and (5.31) to be valid for $s = 1$ and 2 respectively. It is therefore true generally.

6. The deformation fields. It has been seen in §2 that if the deformation undergone by a body is described by specifying the co-ordinates of each material particle of the body in a fixed rectangular Cartesian co-ordinate system x_i as a function of time t and the co-ordinates of the material particle at some earlier time T , then we can determine the displacement, velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration for each material particle at any time t . If we consider the displacements which have been undergone at time t by all the material particles of the body, they constitute a vector field. This vector field will be called the *displacement field* relative to the co-ordinate system x_i . Again, the velocities of all the material particles at time t constitute a vector field. We shall call this the *velocity field* relative to the co-ordinate system x_i . In a similar manner, the accelerations, second accelerations, $\dots$, $(n-1)^{\text{th}}$ accelerations of all the material particles of the body, at time t , constitute vector fields, which we describe as the *acceleration field*, *second acceleration field*, $\dots$, $(n-1)^{\text{th}}$ *acceleration field* respectively relative to the co-ordinate system x_i .

It has been further seen in §3 that, from the same description of the deformation undergone by the body, we can determine the displacements, velocities,

accelerations, $\dots$, $(n - 1)^{\text{th}}$ accelerations at time t of each material particle of the body relative to a co-ordinate system y_i moving in any specified manner. From these we may define, in an analogous manner, displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration fields relative to the co-ordinate system y_i .

In the following three sections (§§7 to 10) we shall analyse these deformation fields. It will be shown that, in the neighborhood of any material particle, each of the deformation fields relative to the fixed co-ordinate system x_i may be resolved into a rotational and irrotational portion. Thus, in the case of the displacement field, we see that we can consider the displacements undergone at time t by the particles contained in an element of volume to be the resultants of displacements undergone by each of the particles as a result of a rigid body motion (*i.e.* bodily displacement and bodily rotation) followed by a pure homogeneous (*i.e.* irrotational) deformation. This result is, of course, already well-known. As a consequence of this, we can choose a moving rectangular Cartesian co-ordinate system y_i in such a way that the displacement field relative to y_i in the element of volume considered is irrotational throughout any desired time interval.

Again, in the case of the velocity field relative to x_i , we obtain the well-known result that the velocities of the particles contained in an element of volume are the resultants of the velocities due to a rigid body motion of the element and an irrotational velocity field. It is then possible to choose a moving rectangular co-ordinate system y_i in such a way that the velocity field relative to y_i in the element of volume considered is irrotational throughout any desired time interval.

In a similar manner, we find that the r^{th} accelerations of the particles contained in an element of volume are the resultants of the r^{th} accelerations of the particles due to a rotational r^{th} acceleration field and an irrotational r^{th} acceleration field. As a result of this we could choose a moving rectangular Cartesian co-ordinate system y_i in such a manner that the r^{th} acceleration field relative to y_i in the element of volume considered is irrotational throughout a finite time interval. If y_i were chosen in such a manner, the displacement, velocity, acceleration, $\dots$, $(r - 1)^{\text{th}}$ acceleration fields relative to y_i would not, in general, be irrotational. It is shown, however, in §10, that we may choose a co-ordinate system y_i so that, instantaneously at time t , the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration fields relative to y_i are irrotational.

7. Analysis of the displacement field. Consider two neighboring material particles P and P' which are at X_i and $X_i + dX_i$ at time T and at x_i and $x_i + dx_i$ at time t . We have

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j \quad (7.1)$$

and

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j. \quad (7.2)$$

If the distance between the particles is dS at time T and ds at time t , we have

$$(dS)^2 = dX_i dX_i = \frac{\partial X_i}{\partial x_j} \frac{\partial X_i}{\partial x_k} dx_j dx_k \quad (7.3)$$

and

$$(ds)^2 = dx_i dx_i = \frac{\partial x_i}{\partial X_j} \frac{\partial x_i}{\partial X_k} dX_j dX_k. \quad (7.4)$$

If we consider all material particles P' which at time T lie on an elementary sphere of radius dS with its center at X_i , then at time t such points lie on an elementary ellipsoid with its center at x_i and having the equation (7.3) in a rectangular Cartesian co-ordinate system with origin at x_i and axes parallel to the axes of the co-ordinate system x_i . dx_i denotes the current co-ordinates of the particles P' in this co-ordinate system at time t . The ellipsoid described by (7.3) with dS constant is usually called the strain ellipsoid. It has the property that the length of its radius in any direction gives the value of ds for a linear element of the material which, at time t , is at x_i and lies in a direction coinciding with that of the radius and, at time T , has length dS .

Let us consider the material lying within the elementary sphere of radius dS at time T . We can show that each particle of this can be taken to its position in the elementary ellipsoid, into which the sphere is deformed at time t , by subjecting the sphere successively to a rigid body motion, in which only its position and orientation in space are changed, and a pure homogeneous deformation. Consider a rigid body motion such that the material particles P and P' move to points x'_i ($= x_i$) and $x'_i + dx'_i$ respectively in the co-ordinate system x_i , where dx'_i is given by

$$dx'_i = \bar{a}_{ij} dX_j. \quad (7.5)$$

Since the deformation described by equation (7.5) is assumed to be rigid, we must have

$$\bar{a}_{ij} \bar{a}_{ik} = \bar{a}_{ji} \bar{a}_{ki} = \delta_{jk}, \quad (7.6)$$

so that each element of length is unchanged (*i.e.* $dx'_i dx'_i = dX_i dX_i$) in the deformation described by equation (7.5). Let this rigid body motion be followed by a pure homogeneous deformation in which the material particle P' moves to $x_i + dx_i$, where

$$dx_i = c_{ij} dx'_j. \quad (7.7)$$

If equation (7.7) is to describe a pure homogeneous deformation, we must have

$$c_{ij} = c_{ji}. \quad (7.8)$$

From (7.5) and (7.7), we obtain

$$dx_i = c_{ij} \bar{a}_{jk} dX_k. \quad (7.9)$$

If the deformation described by equation (7.9) is to be that described by equation (7.1), we must have

$$\frac{\partial x_i}{\partial X_j} = c_{ik} \bar{a}_{kj}. \quad (7.10)$$

From equations (7.6), (7.8) and (7.10), c_{ij} and $\bar{a}_{ij}$ can be determined. It follows, from (7.10) and (7.6), that

$$\frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} = c_{il} c_{jm} \bar{a}_{lk} \bar{a}_{mk} = c_{il} c_{jm} \delta_{lm} = c_{ik} c_{jk}. \quad (7.11)$$

From (7.8), it is seen that $c_{ik} c_{jk} = c_{ik} c_{kj}$, so that the quantities c_{ij} form a symmetric matrix $\mathbf{c}$ the square of which is the matrix formed from the elements on the left-hand side of equation (7.11). It will be shown later (§13) that a set of real quantities c_{ij} satisfying (7.8) and (7.11) exists.

From equation (7.10), we obtain

$$\bar{a}_{kj} = \frac{\partial x_i}{\partial X_j} (c^{-1})_{ki}, \quad (7.12)$$

where $(c^{-1})_{ki}$ is the element in the k^{th} row and i^{th} column of the inverse matrix of $\mathbf{c}$.

Thus all particles of an element of volume located at X_i at time T could be taken to the positions which they occupy at time t as a result of the deformation described by equation (2.1) by first subjecting the element to a rigid body motion described by equation (7.5), in which $\bar{a}_{ij}$ is given by (7.12), and then subjecting the volume element to a pure homogeneous deformation described by equation (7.7), in which c_{ij} satisfies equation (7.11).

In the moving co-ordinate system y_i defined by equation (3.1), the material particle P' has co-ordinates $y_i + dy_i$ at time t given by

$$y_i + dy_i = a_{ij}(x_j + dx_j - b_j) \quad (7.13)$$

and co-ordinates $Y_i + dY_i$ at time T given by

$$Y_i + dY_i = A_{ij}(X_j + dX_j - B_j), \quad (7.14)$$

where $A_{ij} = [a_{ij}]_{t=T}$ and $B_j = [b_j]_{t=T}$. Let us choose the origin of the co-ordinate system y_i to coincide with the material particle P at each instant of time. Then, $b_j = x_j$, $y_i = 0$ and $B_j = X_j$, $Y_i = 0$. The co-ordinates of the material particle

P' in the co-ordinate system y_i are therefore given by dy_i at time t and by dY_i at time T , where

$$dy_i = a_{ij} dx_j \quad (7.15)$$

and

$$dY_i = A_{ij} dX_j. \quad (7.16)$$

From (7.15) and (7.9), we have

$$dy_i = a_{ij} c_{jk} \bar{a}_{kl} dX_l. \quad (7.17)$$

Since, from (3.2), $A_{ln} A_{lm} = \delta_{nm}$, we have from (7.16)

$$dX_l = A_{nl} dY_n. \quad (7.18)$$

Introducing (7.18) into (7.17), we obtain

$$\begin{aligned} dy_i &= a_{ij} c_{jk} \bar{a}_{kl} A_{nl} dY_n \\ &= a_{ij} c_{jk} a_{pk} a_{pq} \bar{a}_{ql} A_{nl} dY_n, \end{aligned} \quad (7.19)$$

since $a_{pk} a_{pq} = \delta_{kq}$.

Now, suppose we choose the co-ordinate system y_i so that

$$a_{pq} \bar{a}_{ql} A_{nl} = \delta_{pn}. \quad (7.20)$$

Equation (7.19) then yields

$$dy_i = a_{ij} c_{jk} a_{pk} dY_p. \quad (7.21)$$

Equation (7.20) can be expressed in the alternative forms

$$\bar{a}_{ij} = a_{ki} A_{kj} \quad (7.22)$$

and

$$a_{ij} = \bar{a}_{jk} A_{ik}. \quad (7.23)$$

Noting that $a_{ki} A_{kj}$ represents the cosine of the angle between the direction of the axis y_i at time t and that of y_j at time T , we see that condition (7.20) or (7.22) is equivalent to the specification that, however the axes y_i are oriented with respect to the axes x_i at time T , their orientations at time t are related to their orientations at time T as though they had been rotated with the volume element located at X_i at time T in its rigid body motion described by (7.5). If the co-ordinate system y_i is moved in this way then the deformation of the element of volume relative to this co-ordinate system is described by equation (7.21). It is readily seen that this is a pure homogeneous deformation, for, since $c_{jk} = c_{kj}$,

$$a_{ij} c_{jk} a_{pk} = a_{pj} c_{jk} a_{ik}, \quad (7.24)$$

so that the matrix having components $a_{ij} c_{jk} a_{pk}$ is symmetric.

8. Analysis of the velocity field (*cf.* C. TRUESDELL (1954)). We now consider the relative velocities of the two neighboring material particles P and P' at the instant of time t . In the co-ordinate system x_i , their velocities have components $v_i^{(1)}$ and $v_i^{(1)} + dv_i^{(1)}$. $v_i^{(1)}$ is given by (2.2) and may be regarded either as a function of X_j and t or as a function of x_j and t . We obtain immediately

$$dv_i^{(1)} = \frac{\partial v_i^{(1)}}{\partial X_j} dX_j = \frac{\partial v_i^{(1)}}{\partial x_j} dx_j. \quad (8.1)$$

$dv_i^{(1)}$ is, of course, the velocity of P' relative to P and may be written as

$$dv_i^{(1)} = d_{ij}^{(1)} dx_j + \omega_{ij}^{(1)} dx_j, \quad (8.2)$$

where

$$d_{ij}^{(1)} = \frac{1}{2} \left(\frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i} \right) \quad \text{and} \quad \omega_{ij}^{(1)} = \frac{1}{2} \left(\frac{\partial v_i^{(1)}}{\partial x_j} - \frac{\partial v_j^{(1)}}{\partial x_i} \right). \quad (8.3)$$

We note that

$$d_{ij}^{(1)} = d_{ji}^{(1)} \quad \text{and} \quad \omega_{ij}^{(1)} = -\omega_{ji}^{(1)}. \quad (8.4)$$

Now, the term $\omega_{ij}^{(1)} dx_j$ in (8.2) would be the velocity of P' relative to P if the element of volume of the material containing both P and P' were rotating as a rigid body with angular velocity having components $(\omega_{32}^{(1)}, \omega_{31}^{(1)}, \omega_{21}^{(1)})$, instantaneously at the time t , in the axial system x_i . Again, the term $d_{ij}^{(1)} dx_j$ in (8.2) would be the velocity of P' relative to P if the element of volume of the material containing both P and P' were undergoing, at the instant of time t , an appropriate homogeneous irrotational flow.

If $v_i^{(1)}$ be regarded as a function of X_j and t , rather than as a function of x_j and t , we may express $d_{ij}^{(1)}$ and $\omega_{ij}^{(1)}$ in the alternative forms

$$d_{ij}^{(1)} = \frac{1}{2\tau} \left[\frac{\partial \tau}{\partial (\partial x_j / \partial X_k)} \frac{\partial v_i^{(1)}}{\partial X_k} + \frac{\partial \tau}{\partial (\partial x_i / \partial X_k)} \frac{\partial v_j^{(1)}}{\partial X_k} \right]$$

and

$$\omega_{ij}^{(1)} = \frac{1}{2\tau} \left[\frac{\partial \tau}{\partial (\partial x_j / \partial X_k)} \frac{\partial v_i^{(1)}}{\partial X_k} - \frac{\partial \tau}{\partial (\partial x_i / \partial X_k)} \frac{\partial v_j^{(1)}}{\partial X_k} \right], \quad (8.5)$$

where

$$\tau = \det (\partial x_i / \partial X_j). \quad (8.6)$$

We now consider the velocities $u_i^{(1)}$ and $u_i^{(1)} + du_i^{(1)}$ at time t of the material particles P and P' relative to the moving co-ordinate system y_i defined by equation (3.1). $u_i^{(1)}$ is given by equation (3.3) and since it may be regarded either as a function of X_j and t or as a function of x_j and t , $du_i^{(1)}$ is given by

$$du_i^{(1)} = \frac{\partial u_i^{(1)}}{\partial X_j} dX_j = \frac{\partial u_i^{(1)}}{\partial x_j} dx_j. \quad (8.7)$$

Now, suppose that the moving co-ordinate system y_i is chosen in such a way that its relation to the fixed co-ordinate system x_i is defined by equation (3.1), while its origin coincides with the position of the material particle P at all instants of time, so that

$$x_j = b_j \quad \text{and} \quad v_j^{(1)} = Db_j/Dt. \quad (8.8)$$

We then have $u_i^{(1)} = 0$ and $du_i^{(1)}$, the component parallel to y_i of the velocity of the material particle P' relative to the coordinate system y_i , is given from (3.3) and (8.7) in the alternative forms

$$du_i^{(1)} = \left(\frac{Da_{ik}}{Dt} \frac{\partial x_k}{\partial X_j} + a_{ik} \frac{\partial v_k^{(1)}}{\partial X_j} \right) dX_j \quad (8.9)$$

and

$$du_i^{(1)} = \left(\frac{Da_{ij}}{Dt} + a_{ik} \frac{\partial v_k^{(1)}}{\partial x_j} \right) dx_j. \quad (8.10)$$

Equation (8.10) may be re-written, with (3.6), as

$$\begin{aligned} du_i^{(1)} &= \left(\frac{Da_{ij}}{Dt} + a_{ik} \frac{\partial v_k^{(1)}}{\partial x_j} \right) a_{lj} dy_l \\ &= \frac{1}{2} \left[a_{lj} \frac{Da_{ij}}{Dt} + a_{ij} \frac{Da_{lj}}{Dt} + (a_{ik} a_{lj} + a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} \right] dy_l \\ &\quad + \frac{1}{2} \left[a_{lj} \frac{Da_{ij}}{Dt} - a_{ij} \frac{Da_{lj}}{Dt} + (a_{ik} a_{lj} - a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} \right] dy_l. \end{aligned} \quad (8.11)$$

From (3.2) we have $D(a_{lj} a_{ij})/Dt = 0$, which with (8.11) yields

$$\begin{aligned} du_i^{(1)} &= \frac{1}{2} (a_{ik} a_{lj} + a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} dy_l \\ &\quad + \frac{1}{2} \left[\left(a_{lj} \frac{Da_{ij}}{Dt} - a_{ij} \frac{Da_{lj}}{Dt} \right) + (a_{ik} a_{lj} - a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} \right] dy_l. \end{aligned} \quad (8.12)$$

We note that the expression $\frac{1}{2}(a_{ik} a_{lj} + a_{lk} a_{ij}) \partial v_k^{(1)} / \partial x_j$ is symmetric with respect to interchange of the indices i and l , while the expression

$$\frac{1}{2} \left[\left(a_{lj} \frac{Da_{ij}}{Dt} - a_{ij} \frac{Da_{lj}}{Dt} \right) + (a_{ik} a_{lj} - a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} \right] \quad (8.13)$$

is skew-symmetric.

It can readily be seen that the instantaneous value at time t of the angular

velocity of the co-ordinate system y_i relative to the co-ordinate system x_i may be chosen in such a manner that the skew-symmetric term (8.13) vanishes. If this is the case, we must have

$$\frac{1}{2} \left(a_{lj} \frac{Da_{ij}}{Dt} - a_{ij} \frac{Da_{lj}}{Dt} \right) = a_{lj} \frac{Da_{ij}}{Dt} = \frac{1}{2} (a_{lk} a_{ij} - a_{ik} a_{lj}) \frac{\partial v_k^{(1)}}{\partial x_j}. \quad (8.14)$$

From (8.14) and (3.2), we have

$$\frac{Da_{im}}{Dt} = a_{lm} a_{lj} \frac{Da_{ij}}{Dt} = \frac{1}{2} a_{lm} (a_{lk} a_{ij} - a_{ik} a_{lj}) \frac{\partial v_k^{(1)}}{\partial x_j}. \quad (8.15)$$

Whence,

$$\begin{aligned} a_{in} \frac{Da_{im}}{Dt} &= \frac{1}{2} a_{in} a_{lm} (a_{lk} a_{ij} - a_{ik} a_{lj}) \frac{\partial v_k^{(1)}}{\partial x_j} \\ &= \frac{1}{2} (\delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn}) \frac{\partial v_k^{(1)}}{\partial x_j} \\ &= \frac{1}{2} \left(\frac{\partial v_m^{(1)}}{\partial x_n} - \frac{\partial v_n^{(1)}}{\partial x_m} \right) = \omega_{mn}^{(1)}. \end{aligned} \quad (8.16)$$

Now, it was seen in §4 that $\left(a_{i2} \frac{Da_{i3}}{Dt}, a_{i3} \frac{Da_{i1}}{Dt}, a_{i1} \frac{Da_{i2}}{Dt} \right)$ is the angular velocity of the co-ordinate system y_i relative to x_i at the instant of time t . It therefore follows that if at the instant of time t the co-ordinate system y_i has an angular velocity $(\omega_{32}^{(1)}, \omega_{13}^{(1)}, \omega_{21}^{(1)})$, given by (8.3), relative to the system x_i , the expression (8.13) vanishes and $du_i^{(1)}$ is given by

$$du_i^{(1)} = \frac{1}{2} (a_{ik} a_{lj} + a_{lk} a_{ij}) \frac{\partial v_k^{(1)}}{\partial x_j} dy_l. \quad (8.17)$$

Since $\frac{1}{2} (a_{ik} a_{lj} + a_{lk} a_{ij}) \partial v_k^{(1)} / \partial x_j$ is symmetric with respect to interchange of the indices i and l , we may say that in the neighborhood of the point considered, the velocity field is irrotational relative to a co-ordinate system y_i rotating with angular velocity $(\omega_{32}^{(1)}, \omega_{13}^{(1)}, \omega_{21}^{(1)})$ relative to the co-ordinate system x_i .

9. Analysis of the acceleration fields. Each of the acceleration fields can be analysed in a similar manner to that adopted in analysing the velocity field in the previous section. Let us consider the $(n-1)^{\text{th}}$ acceleration. In the fixed axial system x_i the $(n-1)^{\text{th}}$ acceleration, at time t , of the material particle P is $v_i^{(n)}$ and that of the material particle P' is $v_i^{(n)} + dv_i^{(n)}$. $v_i^{(n)}$ is given by equation (2.4) and $dv_i^{(n)}$ by

$$dv_i^{(n)} = \frac{\partial v_i^{(n)}}{\partial X_j} dX_j = \frac{\partial v_i^{(n)}}{\partial x_j} dx_j. \quad (9.1)$$

$dv_i^{(n)}$ is the $(n - 1)^{\text{th}}$ acceleration of P' relative to P and may be written as

$$dv_i^{(n)} = d_{ij}^{(n)} dx_j + \bar{\omega}_{ij}^{(n)} dx_j$$

where

$$d_{ij}^{(n)} = \frac{1}{2} \left(\frac{\partial v_i^{(n)}}{\partial x_j} + \frac{\partial v_j^{(n)}}{\partial x_i} \right) \quad \text{and} \quad \bar{\omega}_{ij}^{(n)} = \frac{1}{2} \left(\frac{\partial v_i^{(n)}}{\partial x_j} - \frac{\partial v_j^{(n)}}{\partial x_i} \right). \quad (9.2)$$

We note that, as in the case when $n = 1$,

$$d_{ij}^{(n)} = d_{ji}^{(n)} \quad \text{and} \quad \bar{\omega}_{ij}^{(n)} = -\bar{\omega}_{ji}^{(n)}. \quad (9.3)$$

We now consider the $(n - 1)^{\text{th}}$ accelerations $u_i^{(n)}$ and $u_i^{(n)} + du_i^{(n)}$ at time t of the material particles P and P' relative to the moving co-ordinate system y_i defined by equation (3.1). $u_i^{(n)}$ is given by equation (3.4) and, again, as in the case when $n = 1$, $du_i^{(n)}$ is given by

$$du_i^{(n)} = \frac{\partial u_i^{(n)}}{\partial X_j} dX_j = \frac{\partial u_i^{(n)}}{\partial x_j} dx_j. \quad (9.4)$$

Suppose now that we choose the moving co-ordinate system y_i so that its origin coincides with the material particle P at all instants of time. Then, we have

$$x_j = b_j \quad \text{and} \quad v_j^{(r)} = D^r b_j / Dt^r \quad (r = 1, 2, \dots, n). \quad (9.5)$$

Thus, from (3.4), $u_i^{(n)} = 0$ and $du_i^{(n)}$, the component parallel to y_i of the $(n - 1)^{\text{th}}$ acceleration of the material particle P' relative to the co-ordinate system y_i , is given, from (3.4) and (9.4), in the alternative forms

$$du_i^{(n)} = \left[\frac{D^n a_{ik}}{Dt^n} \frac{\partial x_k}{\partial X_j} + \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ik}}{Dt^{n-r}} \frac{\partial v_k^{(r)}}{\partial X_j} \right] dX_j \quad (9.6)$$

and

$$du_i^{(n)} = \left[\frac{D^n a_{ij}}{Dt^n} + \sum_{r=1}^n \binom{n}{r} \frac{D^{n-r} a_{ik}}{Dt^{n-r}} \frac{\partial v_k^{(r)}}{\partial x_j} \right] dx_j. \quad (9.7)$$

Since $u_i^{(n)}$ may be considered as a function of y_i and t , we may write

$$du_i^{(n)} = \frac{\partial u_i^{(n)}}{\partial y_j} dy_j = \frac{1}{2} \left(\frac{\partial u_i^{(n)}}{\partial y_j} + \frac{\partial u_j^{(n)}}{\partial y_i} \right) dy_j + \frac{1}{2} \left(\frac{\partial u_i^{(n)}}{\partial y_j} - \frac{\partial u_j^{(n)}}{\partial y_i} \right) dy_j. \quad (9.8)$$

From the relations (3.4) and (3.6), we have

$$\begin{aligned} \frac{\partial u_i^{(n)}}{\partial y_j} + \frac{\partial u_j^{(n)}}{\partial y_i} &= \frac{D^n a_{il}}{Dt^n} a_{jl} + \frac{D^n a_{jl}}{Dt^n} a_{il} \\ &+ \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{il}}{Dt^{n-r}} a_{jk} + \frac{D^{n-r} a_{jl}}{Dt^{n-r}} a_{ik} \right) \frac{\partial v_i^{(r)}}{\partial x_k} \end{aligned} \quad (9.9)$$

and

$$\begin{aligned} \frac{\partial u_i^{(n)}}{\partial y_j} - \frac{\partial u_j^{(n)}}{\partial y_i} = & \frac{D^n a_{il}}{Dt^n} a_{jl} - \frac{D^n a_{jl}}{Dt^n} a_{il} \\ & + \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{il}}{Dt^{n-r}} a_{jk} - \frac{D^{n-r} a_{jl}}{Dt^{n-r}} a_{ik} \right) \frac{\partial v_i^{(r)}}{\partial x_k}. \end{aligned} \quad (9.10)$$

We shall employ the notation

$$d'_{ij}^{(n)} = \frac{1}{2} \left(\frac{\partial u_i^{(n)}}{\partial y_j} + \frac{\partial u_j^{(n)}}{\partial y_i} \right) \quad \text{and} \quad \bar{\omega}'_{ij}^{(n)} = \frac{1}{2} \left(\frac{\partial u_i^{(n)}}{\partial y_j} - \frac{\partial u_j^{(n)}}{\partial y_i} \right). \quad (9.11)$$

We note that $d'_{ij}^{(n)}$ is symmetric and $\bar{\omega}'_{ij}^{(n)}$ is skew-symmetric with respect to interchange of the indices i and j .

Thus, if the co-ordinate system y_i is chosen in such a way that $\bar{\omega}'_{ij}^{(n)} = 0$, then in the neighborhood of the point considered, the $(n-1)^{\text{th}}$ acceleration field relative to the co-ordinate system y_i is irrotational. We have already seen in §8 that in the particular case when $n=1$, the condition $\bar{\omega}'_{ij}^{(1)} = 0$ is equivalent to a particular choice of the angular velocity of the co-ordinate system y_i relative to x_i . In general, however, there is a measure of ambiguity in the choice of the co-ordinate system y_i which leads to $\bar{\omega}'_{ij}^{(n)} = 0$. For example, we might assume that the angular velocity, angular acceleration, $\dots$, angular $(n-2)^{\text{th}}$ acceleration of the co-ordinate system y_i relative to x_i are zero and determine the angular $(n-1)^{\text{th}}$ acceleration which would lead to the condition $\bar{\omega}'_{ij}^{(n)} = 0$ being satisfied. This would lead to a description of the deformation in the neighborhood of the point considered which is irrotational as far as the $(n-1)^{\text{th}}$ acceleration field is concerned, but not necessarily so as far as the velocity, acceleration, $\dots$, $(n-2)^{\text{th}}$ acceleration fields are concerned.

In the next section it will be shown that by a suitable choice of the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i at an instant of time, the deformation at that instant of time, in the neighborhood of a point, may be described in the co-ordinate system y_i by irrotational velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration fields.

10. The description of the deformation in terms of irrotational deformation fields. We shall show that by a suitable choice of the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i relative to x_i at time t , we can arrange that $\bar{\omega}'_{ij}^{(s)} = 0$ for $s=1, 2, \dots, n$, where $\bar{\omega}'_{ij}^{(s)}$ is defined by (9.11). This is conveniently done by the method of induction. We have already seen in §8 that if the co-ordinate system y_i has an angular velocity $(\omega_{32}^{(1)}, \omega_{13}^{(1)}, \omega_{21}^{(1)})$, given by (8.3), relative to the co-ordinate system x_i at the instant of time t , then $\bar{\omega}'_{ij}^{(1)} = 0$.

Let us assume that we can choose the angular velocity, angular acceleration, $\dots$, angular $(n-2)^{\text{th}}$ acceleration of the co-ordinate system y_i at time t in such a way that $\bar{\omega}'_{ij}^{(s)} = 0$ for $s=1, 2, \dots, n-1$. We shall denote these by $(\omega_{32}^{(s)}, \omega_{13}^{(s)}, \omega_{21}^{(s)})$ with $s=1, 2, \dots, n-1$ respectively.

It has been shown in §5 that if $(\omega_{32}^{(s)}, \omega_{13}^{(s)}, \omega_{21}^{(s)})$, with $s = 1, 2, \dots, n-1$, are specified, then we can obtain expressions for $D^s a_{ij}/Dt^s$ at time t , with $s = 1, 2, \dots, n-1$. Thus, $D^s a_{ij}/Dt^s$ may be expressed as a polynomial in a_{pq} and the quantities $\omega_{pq}^{(r)}$ ($r = 1, 2, \dots, s$). From the expression for $\bar{\omega}_{ij}^{(n)}$, given by (9.10) and (9.11), it is seen that if $\bar{\omega}_{ij}^{(n)} = 0$,

$$\frac{D^n a_{il}}{Dt^n} a_{jl} - \frac{D^n a_{jl}}{Dt^n} a_{il} = - \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{il}}{Dt^{n-r}} a_{jk} - \frac{D^{n-r} a_{jl}}{Dt^{n-r}} a_{ik} \right) \frac{\partial v_i^{(r)}}{\partial x_k}. \quad (10.1)$$

Now, if the angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i relative to the system x_i is $(\omega_{32}^{(n)}, \omega_{13}^{(n)}, \omega_{21}^{(n)})$, we have, from (5.34),

$$\frac{D^n a_{ij}}{Dt^n} = a_{ik} \omega_{jk}^{(n)} - a_{ik} \sum_{r=1}^{n-1} \binom{n-1}{r} \frac{D^r a_{mk}}{Dt^r} \frac{D^{n-r} a_{mj}}{Dt^{n-r}}. \quad (10.2)$$

Introducing this result into (10.1), we obtain

$$\begin{aligned} (a_{lj} a_{ik} - a_{ij} a_{lk}) \omega_{jk}^{(n)} &= (a_{lj} a_{ik} - a_{ij} a_{lk}) \sum_{r=1}^{n-1} \binom{n-1}{r} \frac{D^r a_{mk}}{Dt^r} \frac{D^{n-r} a_{mj}}{Dt^{n-r}} \\ &\quad - \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{ij}}{Dt^{n-r}} a_{lk} - \frac{D^{n-r} a_{lj}}{Dt^{n-r}} a_{ik} \right) \frac{\partial v_j^{(r)}}{\partial x_k}, \end{aligned} \quad (10.3)$$

whence, using the relation $\omega_{ij}^{(n)} = -\omega_{ji}^{(n)}$, we obtain

$$\begin{aligned} 2\omega_{pq}^{(n)} &= \sum_{r=1}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{mq}}{Dt^r} \frac{D^{n-r} a_{mp}}{Dt^{n-r}} - \frac{D^r a_{mp}}{Dt^r} \frac{D^{n-r} a_{mq}}{Dt^{n-r}} \right) \\ &\quad - \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{ij}}{Dt^{n-r}} a_{iq} \frac{\partial v_j^{(r)}}{\partial x_p} - \frac{D^{n-r} a_{lj}}{Dt^{n-r}} a_{lp} \frac{\partial v_j^{(r)}}{\partial x_q} \right) \\ &= \sum_{r=1}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{mq}}{Dt^r} \frac{D^{n-r} a_{mp}}{Dt^{n-r}} - \frac{D^r a_{mp}}{Dt^r} \frac{D^{n-r} a_{mq}}{Dt^{n-r}} \right) \\ &\quad - \sum_{r=1}^{n-1} \binom{n}{r} \left(\frac{D^{n-r} a_{ij}}{Dt^{n-r}} a_{iq} \frac{\partial v_j^{(r)}}{\partial x_p} - \frac{D^{n-r} a_{lj}}{Dt^{n-r}} a_{lp} \frac{\partial v_j^{(r)}}{\partial x_q} \right) \\ &\quad + \left(\frac{\partial v_p^{(n)}}{\partial x_q} - \frac{\partial v_q^{(n)}}{\partial x_p} \right) \\ &= \sum_{r=1}^{n-1} \binom{n-1}{r} \left(\frac{D^r a_{mq}}{Dt^r} \frac{D^{n-r} a_{mp}}{Dt^{n-r}} - \frac{D^r a_{mp}}{Dt^r} \frac{D^{n-r} a_{mq}}{Dt^{n-r}} \right) \\ &\quad - \sum_{r=1}^{n-1} \binom{n}{r} \left(\frac{D^{n-r} a_{ij}}{Dt^{n-r}} a_{iq} d_{jp}^{(r)} - \frac{D^{n-r} a_{lj}}{Dt^{n-r}} a_{lp} d_{jq}^{(r)} \right) \\ &\quad - \sum_{r=1}^{n-1} \binom{n}{r} \left(\frac{D^{n-r} a_{ij}}{Dt^{n-r}} a_{iq} \bar{\omega}_{jp}^{(r)} - \frac{D^{n-r} a_{lj}}{Dt^{n-r}} a_{lp} \bar{\omega}_{jq}^{(r)} \right) + 2\bar{\omega}_{pq}^{(n)}, \end{aligned} \quad (10.4)$$

where $d_{kp}^{(r)}$ and $\tilde{\omega}_{kp}^{(r)}$ are defined by equations (9.2). Since $D^r a_{ij}/Dt^r$ may be expressed as a polynomial in the quantities a_{pq} and $\omega_{pq}^{(s)}$ ($s = 1, 2, \dots, r$), we see that by means of equation (10.4), $\omega_{pq}^{(n)}$ may be expressed as a polynomial in the quantities a_{ij} , $\omega_{ij}^{(s)}$, $\tilde{\omega}_{ij}^{(s)}$, $d_{ij}^{(s)}$ ($s = 1, 2, \dots, n-1$) and $\tilde{\omega}_{ij}^{(n)}$. By repeated substitution for $\omega_{pq}^{(n-1)}$ by means of such a formula and noting that $\omega_{ij}^{(1)} = \tilde{\omega}_{ij}^{(1)}$, it is seen that $\omega_{pq}^{(n)}$ may be expressed as a polynomial in the quantities a_{ij} , $\tilde{\omega}_{ij}^{(s)}$, $d_{ij}^{(s)}$ ($s = 1, 2, \dots, n-1$) and $\tilde{\omega}_{ij}^{(n)}$. It is thus seen that we may choose instantaneous values at time t of the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i with respect to the system x_i in such a way that $\tilde{\omega}_{ij}^{(r)} = 0$ for $r = 1, 2, \dots, n$; *i.e.* the velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration fields relative to the co-ordinate system y_i are irrotational in the neighborhood of the point considered at the instant of time t .

Let $\omega_{pq}^{\prime(r)}$ be the value of $\omega_{pq}^{(r)}$ ($r = 1, 2, \dots, n$) for which $\tilde{\omega}_{ij}^{\prime(r)} = 0$ ($r = 1, 2, \dots, n$). From (9.9), (9.11), (9.2) and the relation

$$\frac{D^n}{Dt^n} (a_{il} a_{jl}) = \sum_{r=0}^n \binom{n}{r} \frac{D^r a_{il}}{Dt^r} \frac{D^{n-r} a_{jl}}{Dt^{n-r}} = 0, \quad (10.5)$$

we obtain

$$\begin{aligned} 2d_{ij}^{\prime(n)} = & - \sum_{r=1}^{n-1} \binom{n}{r} \frac{D^r a_{il}}{Dt^r} \frac{D^{n-r} a_{jl}}{Dt^{n-r}} \\ & + \sum_{r=1}^n \binom{n}{r} \left(\frac{D^{n-r} a_{il}}{Dt^{n-r}} a_{jk} + \frac{D^{n-r} a_{jl}}{Dt^{n-r}} a_{ik} \right) (d_{kl}^{(r)} + \tilde{\omega}_{kl}^{(r)}). \end{aligned} \quad (10.6)$$

By putting $\omega_{pq}^{(r)} = \omega_{pq}^{\prime(r)}$ ($r = 1, 2, \dots, n$) in (5.34), we could in principle obtain an expression for $D^r a_{ij}/Dt^r$ as a polynomial in a_{pq} and $\omega_{pq}^{\prime(s)}$ ($s = 1, 2, \dots, r$). The expression for $D^r a_{ij}/Dt^r$ so obtained could be introduced into (10.6) to yield an expression for $d_{ij}^{\prime(n)}$ as a polynomial in a_{pq} , $\omega_{pq}^{\prime(r)}$ ($r = 1, 2, \dots, n-1$), $d_{pq}^{(r)}$ and $\tilde{\omega}_{pq}^{(r)}$ ($r = 1, 2, \dots, n$). Since $\omega_{pq}^{\prime(r)}$ is, in principle, expressible as a polynomial in a_{kl} , $\tilde{\omega}_{kl}^{(s)}$, $d_{kl}^{(s)}$ ($s = 1, 2, \dots, r-1$), and $\tilde{\omega}_{kl}^{(r)}$ we could, in principle, express $d_{ij}^{\prime(n)}$ as a polynomial in terms of a_{pq} , $d_{pq}^{(r)}$ and $\tilde{\omega}_{pq}^{(r)}$ ($r = 1, 2, \dots, n$).

We may, however, obtain such an expression for $d_{ij}^{\prime(n)}$ more readily in a somewhat different manner. If we consider two neighboring material particles which are located at x_i and $x_i + dx_i$ in the co-ordinate system x_i at time t , the length ds of the linear element joining them is given by

$$(ds)^2 = dx_i dx_i = dy_i dy_i. \quad (10.7)$$

Following DUPONT (1933) and OLDROYD (1950), we can readily obtain the n^{th} time derivative of $(ds)^2$ as

$$\frac{D^n (ds)^2}{Dt^n} = \sum_{r=0}^n \binom{n}{r} \frac{D^{n-r} (dx_i)}{Dt^{n-r}} \frac{D^r (dx_i)}{Dt^r} = \sum_{r=0}^n \binom{n}{r} \frac{D^{n-r} (dy_i)}{Dt^{n-r}} \frac{D^r (dy_i)}{Dt^r}. \quad (10.8)$$

Now,

$$\frac{D^r(dx_i)}{Dt^r} = \frac{\partial v_i^{(r)}}{\partial x_j} dx_j \quad (r \geq 1) \quad (10.9)$$

and

$$\frac{D^r(dy_i)}{Dt^r} = \frac{\partial u_i^{(r)}}{\partial y_j} dy_j \quad (r \geq 1). \quad (10.10)$$

Hence,

$$\frac{D^n(ds)^2}{Dt^n} = A_{jk}^{(n)} dx_j dx_k = A'_{jk}{}^{(n)} dy_j dy_k, \quad (10.11)$$

where $A_{jk}^{(n)}$ and $A'_{jk}{}^{(n)}$ are defined by

$$A_{jk}^{(n)} = 2d_{jk}^{(n)} + \sum_{r=1}^{n-1} \binom{n}{r} \frac{\partial v_i^{(n-r)}}{\partial x_j} \frac{\partial v_i^{(r)}}{\partial x_k} \quad (10.12)$$

and

$$A'_{jk}{}^{(n)} = 2d'_{jk}{}^{(n)} + \sum_{r=1}^{n-1} \binom{n}{r} \frac{\partial u_i^{(n-r)}}{\partial y_j} \frac{\partial u_i^{(r)}}{\partial y_k}. \quad (10.13)$$

From (10.11) and (3.6), we have

$$\begin{aligned} A'_{jk}{}^{(n)} dy_j dy_k &= A_{jk}^{(n)} a_{lj} a_{mk} dy_l dy_m \\ &= A_{lm}^{(n)} a_{jl} a_{km} dy_j dy_k, \end{aligned} \quad (10.14)$$

whence

$$A'_{jk}{}^{(n)} = A_{lm}^{(n)} a_{jl} a_{km}. \quad (10.15)$$

From (10.13) and (10.15), we obtain

$$2d'_{jk}{}^{(n)} = A_{lm}^{(n)} a_{jl} a_{km} - \sum_{r=1}^{n-1} \binom{n}{r} \frac{\partial u_i^{(n-r)}}{\partial y_j} \frac{\partial u_i^{(r)}}{\partial y_k}. \quad (10.16)$$

It has been shown earlier that the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration at time t of the co-ordinate system y_i relative to the co-ordinate system x_i may be chosen in such a way that

$$\bar{\omega}'_{jk}{}^{(r)} = \frac{1}{2} \left(\frac{\partial u_j^{(r)}}{\partial y_k} - \frac{\partial u_k^{(r)}}{\partial y_j} \right) = 0 \quad (r = 1, 2, \dots, n). \quad (10.17)$$

If the co-ordinate system y_i is chosen in this way, then

$$d'_{jk}{}^{(r)} = \frac{\partial u_j^{(r)}}{\partial y_k} \quad (r = 1, 2, \dots, n) \quad (10.18)$$

and equation (10.16) becomes

$$2d'_{jk}{}^{(n)} = A_{lm}^{(n)} a_{jl} a_{km} - \sum_{r=1}^{n-1} \binom{n}{r} d'_{ij}{}^{(n-r)} d'_{ik}{}^{(r)}. \quad (10.19)$$

Equation (10.19) is a recurrence formula by means of which $d'_{jk}{}^{(n)}$ may be expressed in terms of a_{lm} , $A_{lm}^{(n)}$ and $d'_{lm}{}^{(r)}$ ($r = 1, 2, \dots, n-1$). By successive application of this formula, together with the relations

$$2d'_{jk}{}^{(1)} = A'_{jk}{}^{(1)} = A_{lm}^{(1)} a_{jl} a_{km}, \quad (10.20)$$

obtained from (10.13) and (10.15) by putting $n = 1$, we see that $d'_{jk}{}^{(n)}$ may be obtained as a polynomial in the quantities a_{lm} and $A_{lm}^{(r)}$ ($r = 1, 2, \dots, n$). For example, if $n = 2$, we have

$$\begin{aligned} 2d'_{jk}{}^{(2)} &= A_{lm}^{(2)} a_{jl} a_{km} - 2d'_{ij}{}^{(1)} d'_{ik}{}^{(1)} \\ &= (A_{lm}^{(2)} - \tfrac{1}{2} A_{li}^{(1)} A_{im}^{(1)}) a_{jl} a_{km}. \end{aligned} \quad (10.21)$$

Again, if $n = 3$, we have

$$\begin{aligned} 2d'_{jk}{}^{(3)} &= A_{lm}^{(3)} a_{jl} a_{km} - 3d'_{ij}{}^{(1)} d'_{ik}{}^{(2)} - 3d'_{ij}{}^{(2)} d'_{ik}{}^{(1)} \\ &= [A_{lm}^{(3)} - \tfrac{3}{4} (A_{pi}^{(1)} A_{pm}^{(2)} + A_{pm}^{(1)} A_{pi}^{(2)} - A_{lp}^{(1)} A_{pr}^{(1)} A_{rm}^{(1)})] a_{km} a_{jl}. \end{aligned} \quad (10.22)$$

CHAPTER II. THE STRESS IN A BODY OF ISOTROPIC MATERIAL

11. Introduction. In this chapter, we shall consider how the stress at a point of a body of isotropic material* may be expressed in terms of quantities defining the deformation undergone by the body. In order to do this, we make certain rather broad assumptions regarding the physical nature of the material of the body. We assume the material to be such that at any instant each component of the stress at a point of the body is determined uniquely by the gradients of displacement, velocity, acceleration, . . . , $(n - 1)^{\text{th}}$ acceleration at that point at that instant of time. Underlying this assumption is the conception that the gradients of displacement, velocity, acceleration, . . . , $(n - 1)^{\text{th}}$ acceleration define adequately for our purpose the physical state of the material in an element of volume surrounding the point and that the stress at the point is determined uniquely by this state. By implication, we are considering all other parameters defining the state of the material to be constant scalars. Thus, we consider the conditions in the body under consideration to be isothermal, since we have omitted to introduce temperature as an independent parameter on which the stress depends.

We note too that the assumption that the stress at a point of the body depends only on the gradients of displacement, velocity, acceleration, . . . , $(n - 1)^{\text{th}}$ acceleration at that point implies that the material of the body is isotropic when all of these gradients are zero. If the material were anisotropic under these conditions, then we should have to assume that the components of stress are also dependent on the orientation of the co-ordinate system, in which the kinematically defined gradients are measured, to certain directions defined intrinsically in the material. Again, the fact that we do not assume any explicit dependence of the stress on the position of the point considered implies that the material of the body is homogeneous in the undeformed state, *i.e.* when the gradients of the displacement, velocity, acceleration, . . . , $(n - 1)^{\text{th}}$ acceleration are all zero.

From the assumption that the stress components at a point of the body are dependent only on the gradients of displacement, velocity, acceleration, . . . , $(n - 1)^{\text{th}}$ acceleration at that point, together with the assumption that the functional form of this dependence is independent of the orientation or motion of the co-ordinate system in which the stress components and kinematic gradients are measured, it follows that certain limitations are imposed on the form of this dependence.

*Those who prefer not to regard "isotropic material" as a primitive concept may think of this as being just a name for the types of ideal materials discussed here.

In §13, we discuss the case when the stress components in a rectangular Cartesian co-ordinate system depend only on the nine displacement gradients measured in that co-ordinate system. We then find that each component of the stress depends only on six quantities defined in terms of these nine displacement gradients. These six quantities may be considered as the components of a symmetric Cartesian tensor, which we may call the strain tensor, and it is shown that the form of dependence of the stress on the components of this tensor must be such that the six components of stress may be transformed from one rectangular Cartesian co-ordinate system to another, moving in any way with respect to the first, according to the rules for the transformation of symmetric Cartesian tensors. We may express this result by saying that the stress tensor must be a symmetric tensor function of the strain tensor.

In §14, we discuss the analogous problem for the case when the components of stress depend only on the velocity gradients. In this case we find that the stress must be a symmetric tensor function of a symmetric tensor, defined in terms of the velocity gradients, which we may describe as the velocity-strain tensor.

Then, in §15, we discuss the general problem when the components of stress depend on the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration. In this case, we find that the stress must be a symmetric tensor function of the strain tensor, velocity-strain tensor and $n - 1$ further symmetric tensors the components of which are defined in terms of the gradients of velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration. Expressed otherwise, this means that the six components of the stress in any rectangular Cartesian co-ordinate system must be expressible in terms of the components of $n + 1$ kinematically defined symmetric tensors and must be such functions of these that they transform from one rectangular Cartesian co-ordinate system to another, moving in any manner with respect to it, according to the rules for the transformation of symmetric Cartesian tensors.

12. The transformation of the stress. Let us assume that the stress at time t at any point of a body undergoing deformation is determined by the gradients at time t of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration in the material at the point considered. Expressed mathematically, we thus have, for any component t_{ij} of the stress in the axial system x_i ,

$$t_{ij} = f_{ij} \left(\frac{\partial x_k}{\partial X_l}, \frac{\partial v_k^{(1)}}{\partial x_l}, \frac{\partial v_k^{(2)}}{\partial x_l}, \dots, \frac{\partial v_k^{(n)}}{\partial x_l} \right). \quad (12.1)$$

It can, of course, be readily shown in a manner similar to that adopted in the classical theories of elasticity and hydrodynamics that $t_{ij} = t_{ji}$, if only body forces and surface tractions are applied to the body under consideration. Now suppose we consider the deformation of the material to be defined relative to a moving co-ordinate system y_i defined by equations (3.1) and (3.2). The displacement gradients relative to the co-ordinate system y_i are $\partial y_k / \partial Y_l = \delta_{kl}$,

the velocity gradients are $\partial u_k^{(1)}/\partial y_l$, the acceleration gradients are $\partial u_k^{(2)}/\partial y_l$ and the $(n - 1)^{\text{th}}$ acceleration gradients are $\partial u_k^{(n)}/\partial y_l$. Now,

$$\frac{\partial y_k}{\partial Y_l} = \frac{\partial y_k}{\partial x_j} \frac{\partial x_j}{\partial X_i} \frac{\partial X_i}{\partial Y_l}. \quad (12.2)$$

With (3.1) and (3.6), equation (12.2) yields

$$\frac{\partial y_k}{\partial Y_l} = a_{kj} A_{li} \frac{\partial x_j}{\partial X_i}. \quad (12.3)$$

Also,

$$\frac{\partial u_k^{(s)}}{\partial y_l} = \frac{\partial u_k^{(s)}}{\partial x_i} \frac{\partial x_i}{\partial y_l}. \quad (12.4)$$

From equation (3.4), we have

$$\frac{\partial u_k^{(s)}}{\partial x_i} = \frac{D^s a_{ki}}{Dt^s} + \sum_{r=1}^s \binom{s}{r} \frac{D^{s-r} a_{kj}}{Dt^{s-r}} \frac{\partial v_j^{(r)}}{\partial x_i}. \quad (12.5)$$

Employing the relations (12.5) and (3.6), equation (12.4) becomes

$$\frac{\partial u_k^{(s)}}{\partial y_l} = a_{li} \left[\frac{D^s a_{ki}}{Dt^s} + \sum_{r=1}^s \binom{s}{r} \frac{D^{s-r} a_{kj}}{Dt^{s-r}} \frac{\partial v_j^{(r)}}{\partial x_i} \right]. \quad (12.6)$$

Let t'_{ij} be the stress components in the co-ordinate system y_i . Then,

$$t_{pq} = t'_{ij} a_{ip} a_{jq}. \quad (12.7)$$

Now, t'_{ij} must be the same function of the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration measured in the co-ordinate system y_i as t_{ij} is of those measured in the co-ordinate system x_i . Expressed mathematically we have

$$t'_{ij} = f_{ij} \left(\frac{\partial y_k}{\partial Y_l}, \frac{\partial u_k^{(1)}}{\partial y_l}, \frac{\partial u_k^{(2)}}{\partial y_l}, \dots, \frac{\partial u_k^{(n)}}{\partial y_l} \right). \quad (12.8)$$

From equations (12.7), (12.8) and (12.1), we obtain

$$\begin{aligned} f_{pq} \left(\frac{\partial x_k}{\partial X_l}, \frac{\partial v_k^{(1)}}{\partial x_l}, \frac{\partial v_k^{(2)}}{\partial x_l}, \dots, \frac{\partial v_k^{(n)}}{\partial x_l} \right) \\ = f_{ij} \left(\frac{\partial y_k}{\partial Y_l}, \frac{\partial u_k^{(1)}}{\partial y_l}, \frac{\partial u_k^{(2)}}{\partial y_l}, \dots, \frac{\partial u_k^{(n)}}{\partial y_l} \right) a_{ip} a_{jq}. \end{aligned} \quad (12.9)$$

This relation imposes certain conditions on the manner in which the stress components can depend on the gradients of the displacement, velocity, $\dots$, $(n - 1)^{\text{th}}$ acceleration for isotropic materials. Before obtaining these conditions generally, we shall first illustrate the manner in which these conditions can be

obtained in the cases when the stress components are assumed to depend only on the displacement gradients and when they are assumed to depend only on the velocity gradients.

13. Stress dependent on displacement gradients. If the stress components at any point of the material depend only on the displacement gradients at the point at the instant of time considered, then we have

$$t_{ij} = f_{ij} \left(\frac{\partial x_k}{\partial X_l} \right). \quad (13.1)$$

Equation (12.9) takes the form

$$f_{pq} \left(\frac{\partial x_k}{\partial X_l} \right) = f_{ij} \left(\frac{\partial y_k}{\partial Y_l} \right) a_{ip} a_{jq}. \quad (13.2)$$

This equation must be valid for any choice of the co-ordinate system y_i , *i.e.* for any choice of the a 's, subject only to the condition that

$$a_{ij} a_{kj} = a_{ji} a_{jk} = \delta_{ik} \quad (13.3)$$

at all instants of time.

Now, we saw in §7, that if we consider the deformation undergone, in the time interval from T to t , by an element of volume of the material containing the material particle P , which is at X_i at time T and at x_i at time t in the fixed co-ordinate system x_i , then we can choose a moving co-ordinate system y_i in such a way that the displacement field relative to y_i is irrotational for the element of volume. It was shown that this is the case if the origin of the co-ordinate system y_i is chosen to coincide with the positions of the material particle P at times T and t while its orientation changes with time in such a way that

$$a_{ij} = \bar{a}_{jk} A_{ik}, \quad (13.4)$$

where A_{ik} denotes $[a_{ik}]_{t \leftarrow T}$ and $\bar{a}_{jk}$ is given by equation (7.12) as

$$\bar{a}_{jk} = \frac{\partial x_i}{\partial X_k} (c^{-1})_{ji}, \quad (13.5)$$

and $\mathbf{c}$ is a symmetric matrix such that

$$\mathbf{c}^2 = \left\| \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \right\|. \quad (13.6)$$

A_{ik} may be chosen arbitrarily provided only the conditions $A_{ij} A_{kj} = A_{ji} A_{jk} = \delta_{ik}$ for mutual perpendicularity of the co-ordinate axes y_i at time T are satisfied. It was shown further, in §7, that if the co-ordinate system y_i is chosen in the manner described by equations (13.4), (13.5) and (13.6), then the

co-ordinates at time t of a material particle, which is at dY_i in the co-ordinate system y_i at time T , are dy_i , given by

$$dy_i = a_{ij} c_{jk} a_{pk} dY_p. \quad (13.7)$$

Thus, if the co-ordinate system y_i is chosen in this manner, we have in equation (13.2)

$$\frac{\partial y_k}{\partial Y_l} = a_{kj} c_{ji} a_{li}, \quad (13.8)$$

so that

$$f_{pq} \left(\frac{\partial x_k}{\partial X_l} \right) = f_{ij} (a_{kn} c_{nm} a_{lm}) a_{ip} a_{jq}. \quad (13.9)$$

This relation must be valid for arbitrary choice of the a 's subject to the conditions (13.3). In particular it must be valid if we choose the co-ordinate system y_i in such a way that it coincides with the co-ordinate system x_i at time t . Then, $a_{ij} = \delta_{ij}$ and equation (13.9) becomes

$$f_{pq} \left(\frac{\partial x_k}{\partial X_l} \right) = f_{ij} (\delta_{kn} \delta_{lm} c_{nm}) \delta_{ip} \delta_{jq} = f_{pq} (c_{kl}). \quad (13.10)$$

It is thus seen that if equation (13.2) is valid, t_{pq} must be expressible as a function of the six quantities c_{kl} . Equation (13.2) must therefore take the form

$$f_{pq}(c_{kl}) = f_{ij}(c'_{kl}) a_{ip} a_{jq}, \quad (13.11)$$

where c'_{kl} are the elements of a matrix $\mathbf{c}'$ such that

$$\mathbf{c}'^2 = \left\| \frac{\partial y_i}{\partial Y_k} \frac{\partial y_j}{\partial Y_k} \right\|. \quad (13.12)$$

The relation (13.11) must, of course, be valid for any choice of the a 's subject to the condition (13.3).

From equation (13.12), we have

$$c'_{ik} c'_{kj} = \frac{\partial y_i}{\partial Y_k} \frac{\partial y_j}{\partial Y_k}. \quad (13.13)$$

With the relations (3.1), (3.6) and (13.8), we obtain

$$c'_{ik} c'_{kj} = a_{il} a_{jm} A_{kr} A_{ks} \frac{\partial x_l}{\partial X_r} \frac{\partial x_m}{\partial X_s} = a_{il} a_{jm} \frac{\partial x_l}{\partial X_r} \frac{\partial x_m}{\partial X_r} = a_{il} a_{jm} c_{lr} c_{rm}. \quad (13.14)$$

Thus, $c_{ik} c_{kj}$ transforms as a tensor.

We note that c_{ij} (*i.e.* $\mathbf{c}$) is not uniquely determined by equation (13.6) and correspondingly $\bar{a}_{jk}$ is not uniquely determined by (13.5). However, it can be

shown that the three eigen-values of $\mathbf{c}^2$, defined by (13.6), are real and positive, the principal directions of $\mathbf{c}$ coincide with the principal directions of $\mathbf{c}^2$ and the eigen-values of $\mathbf{c}$ are equal to the square-roots of the eigen-values of $\mathbf{c}^2$. The ambiguity in $\mathbf{c}$ thus arises from the fact that in choosing the eigen-values for $\mathbf{c}$ which satisfy equation (13.6), we can attach positive or negative signs to the square roots of the eigen-values of $\mathbf{c}^2$. It can, in fact, be shown that if the deformation described by equation (3.1) represents a deformation which is possible in a real material, we must have $\det(\partial x_i / \partial X_j) > 0$ and from this and the fact that $\det \bar{a}_{ij} = 1$ (see §3), we can show that the product of the three eigen-values of $\mathbf{c}$ must be positive; *i.e.* either all of them must be positive or two of them must be negative and one positive. If we take the eigen-values of $\mathbf{c}$ to be positive, c_{ij} is uniquely determined and hence $\bar{a}_{jk}$ is uniquely determined through equation (13.5). Similarly, $\mathbf{c}'$ is uniquely defined by equation (13.12) if we take all its eigen-values to be positive. If $\mathbf{c}$ and $\mathbf{c}'$ are both chosen with all their eigen-values positive, then it is seen from (13.14) that

$$c'_{ij} = a_{il} a_{jm} c_{lm}; \quad (13.15)$$

i.e. c_{ij} transforms as a tensor. Introducing (13.15) into (13.11), we obtain

$$f_{pq}(c_{kl}) = f_{ij}(a_{km} a_{ln} c_{mn}) a_{ip} a_{jq}. \quad (13.16)$$

The relation (13.16) must be valid for any choice of the co-ordinate system y_i . This will be the case if and only if t_{pq} is a tensor function of c_{kl} . t_{pq} must, of course, be a symmetric tensor function of c_{kl} , since $t_{pq} = t_{qp}$. Since, with the choice that the eigen-values of c_{ij} are all positive, c_{ij} is uniquely determined by $c_{ik} c_{kl}$, we may re-write equation (13.16) as

$$f_{pq}(c_{ks} c_{sl}) = f_{ij}(a_{km} a_{ln} c_{ms} c_{sn}) a_{ip} a_{jq}. \quad (13.17)$$

The actual forms of the functional dependencies on $c_{ks} c_{sl}$ and $a_{km} a_{ln} c_{ms} c_{sn}$ indicated in (13.17) are, of course, different from those in (13.16). We thus have the result that, if for an isotropic material the stress components depend on the displacement gradients only, then the components of the stress must be the components of a tensor which is expressible as a symmetric tensor function of $c_{ij} c_{jk}$ defined by (13.6).

14. Stress dependent on velocity gradients. If the stress components at any point of the material depend only on the velocity gradients at the point at the instant of time considered, then we have

$$t_{ij} = f_{ij} \left(\frac{\partial v_k^{(1)}}{\partial x_l} \right). \quad (14.1)$$

Equation (12.9) then yields

$$f_{pq} \left(\frac{\partial v_k^{(1)}}{\partial x_l} \right) = f_{ij} \left(\frac{\partial u_k^{(1)}}{\partial y_l} \right) a_{ip} a_{jq}. \quad (14.2)$$

Equation (14.2) must be valid for any choice of the co-ordinate system y_i , *i.e.* for any choice of the a 's subject only to the condition that

$$a_{ij} a_{kj} = a_{ji} a_{jk} = \delta_{ik} \quad (14.3)$$

at all instants of time. Now, from (3.3) and (3.6), we have

$$\frac{\partial u_k^{(1)}}{\partial y_l} = \frac{Da_{km}}{Dt} a_{lm} + \frac{\partial v_m^{(1)}}{\partial x_n} a_{km} a_{ln}. \quad (14.4)$$

Let us choose the co-ordinate system y_i so that instantaneously at time t it satisfies the following conditions:

- (i) it has an angular velocity with components $(\omega_{32}^{(1)}, \omega_{13}^{(1)}, \omega_{21}^{(1)})$ in the co-ordinate system x_i , given by equation (8.3) and
- (ii) its directions coincide with those of the co-ordinate system x_i .

From (8.16), it is seen that condition (i) leads to the relations

$$a_{mk} \frac{Da_{ml}}{Dt} = \omega_{lk}^{(1)} = \frac{1}{2} \left(\frac{\partial v_l^{(1)}}{\partial x_k} - \frac{\partial v_k^{(1)}}{\partial x_l} \right). \quad (14.5)$$

From (14.5), we have

$$\frac{Da_{km}}{Dt} = a_{kn} \omega_{mn}^{(1)}. \quad (14.6)$$

Introducing this result into (14.4), we obtain, with (14.5),

$$\frac{\partial u_k^{(1)}}{\partial y_l} = a_{kn} a_{lm} \omega_{mn}^{(1)} + \frac{\partial v_m^{(1)}}{\partial x_n} a_{km} a_{ln} = a_{km} a_{ln} \left(\omega_{nm}^{(1)} + \frac{\partial v_m^{(1)}}{\partial x_n} \right) = a_{km} a_{ln} d_{mn}^{(1)}, \quad (14.7)$$

$d_{mn}^{(1)}$ being given by equation (8.3).

Now, condition (ii) implies that at time t

$$a_{ij} = \delta_{ij}. \quad (14.8)$$

Introducing this relation into (14.7), we obtain

$$\frac{\partial u_k^{(1)}}{\partial y_l} = d_{kl}^{(1)}. \quad (14.9)$$

Thus, if (14.2) is to be valid for the particular choice of the co-ordinate system y_i made, we must have

$$f_{pq} \left(\frac{\partial v_k^{(1)}}{\partial x_l} \right) = f_{ij} (d_{kl}^{(1)}) \delta_{ip} \delta_{jq} = f_{pq} (d_{kl}^{(1)}); \quad (14.10)$$

i.e. t_{pq} must be expressible as a function of the six quantities $d_{kl}^{(1)}$ defined by equation (8.3). It follows that equation (14.2) will be valid for any choice of the co-ordinate system y_i , *i.e.* for any choice of the a 's subject only to condition $a_{ij} a_{ik} = a_{ji} a_{ki} = \delta_{jk}$, if and only if

$$f_{pq}(d_{kl}^{(1)}) = f_{ij}(d_{kl}^{(1)})a_{ip}a_{jq}, \quad (14.11)$$

where $d_{kl}^{(1)}$ is defined by

$$d_{kl}^{(1)} = \frac{1}{2} \left(\frac{\partial u_k^{(1)}}{\partial y_l} + \frac{\partial u_l^{(1)}}{\partial y_k} \right). \quad (14.12)$$

We see from (14.4), that

$$d_{kl}^{(1)} = d_{mn}^{(1)} a_{km} a_{ln}, \quad (14.13)$$

$$\text{since } a_{lm} \frac{Da_{km}}{Dt} + a_{km} \frac{Da_{lm}}{Dt} = 0.$$

Thus, equation (14.2) will be valid for any choice of the co-ordinate system y_i if and only if

$$f_{pq}(d_{kl}^{(1)}) = f_{ij}(d_{mn}^{(1)} a_{km} a_{ln}) a_{ip} a_{jq}; \quad (14.14)$$

i.e. if and only if t_{pq} is a symmetric tensor function of $d_{kl}^{(1)}$.

15. The general case. Let us assume that the stress components are unique functions of the gradients of displacement, velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration, so that equations (12.1) and (12.9) are applicable. It has been shown in §10 that we may choose the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration, at the instant of time t , of the co-ordinate system y_i relative to the system x_i in such a way that the velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration fields relative to the co-ordinate system y_i , in the immediate neighborhood of the material particle considered, are irrotational at the instant of time t . It has been seen in §5 that we can choose the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of a moving rectangular Cartesian co-ordinate system y_i relative to a fixed system x_i arbitrarily at time t and its orientation arbitrarily at time t and a previous instant of time T , subject only to the condition that the axes y_i are mutually perpendicular at times t and T , while, in the interval of time from T to t , the axial system y_i moves as a rigid body.

In particular we may choose the co-ordinate system y_i in the following manner:

- (i) Instantaneously at time t , the directions of the axes of the co-ordinate system y_i are parallel to those of the co-ordinate system x_i , so that

$$a_{ij} = \delta_{ij}. \quad (15.1)$$

- (ii) Instantaneously at time t , the angular velocity, angular acceleration, $\dots$, angular $(n-1)^{\text{th}}$ acceleration of the co-ordinate system y_i relative to the system x_i are such that the velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration fields relative to the co-ordinate system y_i , in the immediate neighborhood of the material particle considered, are irrotational at the

instant of time t ; *i.e.* they are given by $(\omega'_{32}{}^{(r)}, \omega'_{13}{}^{(r)}, \omega'_{21}{}^{(r)})$ for $r = 1, 2, \dots, n$, defined in §10.

- (iii) Instantaneously at time T , the axes of the co-ordinate system y_i have directions defined by the direction-cosines A_{ij} , which are related to the a 's, given by equation (15.1), by equations (7.23) and (7.12); *i.e.* the A 's are such that

$$\delta_{ij} = \bar{a}_{jk} A_{ik}, \quad (15.2)$$

where

$$\bar{a}_{jk} = \frac{\partial x_i}{\partial X_k} (c^{-1})_{ji}. \quad (15.3)$$

Thus,

$$A_{ij} = \bar{a}_{ij} = \frac{\partial x_k}{\partial X_j} (c^{-1})_{ik}. \quad (15.4)$$

$\mathbf{c}$ ($= \parallel c_{ij} \parallel$) is defined by equation (7.11) and is uniquely defined by this equation provided we choose it to have all its eigen-values positive (see §13). Then A_{ij} is uniquely defined by equation (15.4). We shall consider $\mathbf{c}$ to be that value of $\mathbf{c}$ defined by equation (7.11) which has positive eigen-values.

It is seen in §10, that if the co-ordinate system y_i is chosen in accordance with (ii), $d'_{ij}{}^{(r)}$, defined by (10.18), must be expressible as a polynomial in the quantities a_{pq} and $A_{pq}^{(s)}$ ($s = 1, 2, \dots, r$) defined by equation (10.12). Since

$$d'_{ij}{}^{(r)} = \frac{\partial u_i^{(r)}}{\partial y_j} \quad (r = 1, 2, \dots, n), \quad (15.5)$$

$\partial u_i^{(r)}/\partial y_j$ must be expressible as a polynomial in the quantities a_{pq} and $A_{pq}^{(s)}$ ($s = 1, 2, \dots, r$). The expressions for $\partial u_i^{(r)}/\partial y_j$ are obtained by means of the recurrence formula, given in equation (10.19), which may be re-written

$$2 \frac{\partial u_i^{(r)}}{\partial y_j} = A_{im}^{(r)} a_{il} a_{jm} - \sum_{s=1}^{r-1} \binom{r}{s} \frac{\partial u_k^{(r-s)}}{\partial y_i} \frac{\partial u_k^{(s)}}{\partial y_j} \quad (r = 1, 2, \dots, n). \quad (15.6)$$

Now, if further the co-ordinate system y_i is chosen in accordance with (i), we have, introducing (15.1) into (15.6),

$$2 \frac{\partial u_i^{(r)}}{\partial y_j} = A_{ij}^{(r)} - \sum_{s=1}^{r-1} \binom{r}{s} \frac{\partial u_k^{(r-s)}}{\partial y_i} \frac{\partial u_k^{(s)}}{\partial y_j} \quad (r = 1, 2, \dots, n), \quad (15.7)$$

which is a recurrence formula by means of which $\partial u_i^{(r)}/\partial y_j$ can be expressed as a polynomial in the quantities $A_{pq}^{(s)}$ ($s = 1, 2, \dots, r$) only.

It is seen from equation (7.21) that if the co-ordinate system y_i is chosen in accordance with (i) and (iii), then

$$\frac{\partial y_i}{\partial Y_j} = c_{ij}, \quad (15.8)$$

where c_{ij} are the elements of a matrix $\mathbf{c}$ such that

$$\mathbf{c}^2 = \left\| \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \right\|. \quad (15.9)$$

Thus, if the co-ordinate system y_i is chosen in accordance with (i), (ii) and (iii), $\partial y_i / \partial Y_j = c_{ij}$ and $\partial u_i^{(r)} / \partial y_j$ ($r = 1, 2, \dots, n$) is expressible as a polynomial in the quantities $A_{pq}^{(s)}$ ($s = 1, 2, \dots, r$) defined by equation (10.12).

Now, since the relation (12.9) must be valid for any choice of the co-ordinate system y_i , it must be valid for a choice in accordance with the limitations imposed by (i), (ii) and (iii). If the co-ordinate system y_i is chosen in this way, equation (12.1) yields that the components of stress t_{pq} in the co-ordinate system x_i must be expressible as a function of c_{ij} and $A_{ij}^{(r)}$ ($r = 1, 2, \dots, n$) defined, from (10.12), by

$$A_{ij}^{(r)} = \frac{\partial v_i^{(r)}}{\partial x_j} + \frac{\partial v_j^{(r)}}{\partial x_i} + \sum_{s=1}^{r-1} \binom{r}{s} \frac{\partial v_k^{(r-s)}}{\partial x_i} \frac{\partial v_k^{(s)}}{\partial x_j}. \quad (15.10)$$

This may be expressed by the equation

$$t_{ij} = f_{ij}(c_{kl}, A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}). \quad (15.11)$$

The form of the dependence of t_{ij} on c_{kl} , $A_{kl}^{(1)}$, $\dots$, $A_{kl}^{(n)}$ is, of course, not, in general, the same as the form of its dependence on $\partial x_k / \partial X_l$, $\partial v_k^{(1)} / \partial x_l$, $\dots$, $\partial v_k^{(n)} / \partial x_l$. The relation (12.9) now becomes

$$f_{pq}(c_{kl}, A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}) = f_{ij}(c'_{kl}, A'^{(1)}_{kl}, A'^{(2)}_{kl}, \dots, A'^{(n)}_{kl}) a_{ip} a_{jq}, \quad (15.12)$$

where c'_{kl} is defined by equation (13.12) and $A'^{(r)}_{kl}$ ($r = 1, 2, \dots, n$) is defined by equation (10.13); *i.e.* c'_{kl} is defined in terms of y_i and Y_i in the same way as c_{kl} is defined in terms of x_i and X_i and $A'^{(r)}_{kl}$ is defined in terms of $u_i^{(s)}$ and y_i in the same way as $A_{kl}^{(r)}$ is defined in terms of $v_i^{(s)}$ and x_i .

The relation (15.12) must be valid for any choice of the co-ordinate system y_i . We have already seen in equations (10.15) and (13.15) that, for any choice of the co-ordinate system y_i ,

$$A'^{(r)}_{kl} = A_{mn}^{(r)} a_{km} a_{ln} \quad \text{and} \quad c'_{kl} = c_{mn} a_{km} a_{ln}; \quad (15.13)$$

i.e. $A_{mn}^{(r)}$ and c_{kl} transform as tensors from the co-ordinate system x_i to the co-ordinate system y_i moving in any manner. With (15.13), equation (15.12) becomes

$$f_{pq}(c_{kl}, A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}) = f_{ij}(c_{mn} a_{km} a_{ln}, A_{mn}^{(1)} a_{km} a_{ln}, A_{mn}^{(2)} a_{km} a_{ln}, \dots, A_{mn}^{(n)} a_{km} a_{ln}) a_{ip} a_{jq}. \quad (15.14)$$

A necessary and sufficient condition for this equation to be valid is that t_{pq} shall be a tensor function of the tensors c_{kl} , $A_{kl}^{(1)}$, $\dots$, $A_{kl}^{(n)}$. We thus see that, for an isotropic material, the condition that the stress components shall be dependent only on the gradients of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration implies that the stress must be expressible as a symmetric tensor function of the tensors c_{kl} , $A_{kl}^{(1)}$, $\dots$, $A_{kl}^{(n)}$. Since c_{kl} can be expressed uniquely in terms of $c_{pj} c_{iq}$, defined by (7.11), we see that the stress must be expressible as a symmetric tensor function of the tensors $c_{kj} c_{il}$, $A_{kl}^{(1)}$, $\dots$, $A_{kl}^{(n)}$.

It is readily seen from the above discussion that if the stress components are assumed to depend only on the gradients of the velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration, then the stress must be expressible as a symmetric tensor function of the tensors $A_{kl}^{(1)}$, $A_{kl}^{(2)}$, $\dots$, $A_{kl}^{(n)}$.

It is also apparent from the above discussion that if the stress components are assumed to be expressible as polynomials in the gradients of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration, they must be expressible as polynomials in the components of the tensors C_{kl} , $A_{kl}^{(1)}$, $A_{kl}^{(2)}$, $\dots$, $A_{kl}^{(n)}$. Again, if the stress components are expressible as polynomials in the gradients of the velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration only, they must be expressible as polynomials in the components of the tensors $A_{kl}^{(1)}$, $A_{kl}^{(2)}$, $\dots$, $A_{kl}^{(n)}$ only.

16. Relations between the kinematic tensors. Although the kinematic tensors C_{ij} , (defined as $c_{ik} c_{kj}$) $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ may be independently chosen instantaneously, the fact that their elements are defined as time and spatial derivatives of the displacement components imposes certain relationships on them. In particular, we shall discuss in the following sections (§§17 and 18) the effect of restrictions, considered to apply over a finite time interval, imposed on the tensors C_{ij} , $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ for a specified volume element of the material, on the succeeding tensors in the sequence C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$. We shall first (in §17) consider the implications of certain restrictions on the velocity-strain and acceleration-strain tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$.

Our point of departure will be the relation

$$A_{ij}^{(r+1)} = \frac{DA_{ij}^{(r)}}{Dt} + A_{mj}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_i} + A_{im}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_j}, \quad (16.1)$$

which may be readily obtained from equations (10.11). We have from (10.11), writing $n = r$ and differentiating with respect to time,

$$\frac{D^{r+1}(ds)^2}{Dt^{r+1}} = \frac{DA_{jk}^{(r)}}{Dt} dx_j dx_k + A_{jk}^{(r)} (dv_j^{(1)} dx_k + dv_k^{(1)} dx_j). \quad (16.2)$$

Substituting the relation $dv_j^{(1)} = (\partial v_j^{(1)} / \partial x_l) dx_l$ in (16.2), we have

$$\frac{D^{r+1}(ds)^2}{Dt^{r+1}} = \left(\frac{DA_{jk}^{(r)}}{Dt} + A_{mk}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_j} + A_{jm}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_k} \right) dx_j dx_k. \quad (16.3)$$

Now, directly from (10.11), we have

$$\frac{D^{r+1}(ds)^2}{Dt^{r+1}} = A_{jk}^{(r+1)} dx_j dx_k. \quad (16.4)$$

Comparing equations (16.3) and (16.4), we obtain the relation (16.1) which is valid in any rectangular Cartesian co-ordinate system x_i , fixed or moving in any manner.

In §18, we consider the effect of certain restrictions on the strain and velocity-strain tensors C_{ij} and $A_{ij}^{(1)}$. Our point of departure here is the relation

$$\frac{DC_{ij}}{Dt} = C_{kj} \frac{\partial v_i^{(1)}}{\partial x_k} + C_{ki} \frac{\partial v_j^{(1)}}{\partial x_k}, \quad (16.5)$$

together with the relation (16.1). The relation (16.5) may be obtained directly from the definition of C_{ij} as

$$C_{ij} = c_{ik} c_{kj} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k}. \quad (16.6)$$

From (16.6), we have

$$\frac{DC_{ij}}{Dt} = \frac{\partial v_i^{(1)}}{\partial X_k} \frac{\partial x_j}{\partial X_k} + \frac{\partial v_j^{(1)}}{\partial X_k} \frac{\partial x_i}{\partial X_k} = \frac{\partial v_i^{(1)}}{\partial x_l} \frac{\partial x_l}{\partial X_k} \frac{\partial x_j}{\partial X_k} + \frac{\partial v_j^{(1)}}{\partial x_l} \frac{\partial x_l}{\partial X_k} \frac{\partial x_i}{\partial X_k}, \quad (16.7)$$

which with (16.6) leads to (16.5). Again, it can readily be seen that the relation (16.5) is valid in any rectangular Cartesian co-ordinate system x_i , fixed or moving in any manner. It may be re-written in the form

$$\frac{DC_{ij}}{Dt} - C_{kj} \omega_{ik}^{(1)} - C_{ki} \omega_{jk}^{(1)} = \frac{1}{2} C_{kj} A_{ik}^{(1)} + \frac{1}{2} C_{ki} A_{jk}^{(1)}, \quad (16.8)$$

where

$$\omega_{ik}^{(1)} = \frac{1}{2} \left(\frac{\partial v_i^{(1)}}{\partial x_k} - \frac{\partial v_k^{(1)}}{\partial x_i} \right).$$

17. The effect of restrictions on the velocity-strain and acceleration-strain tensors.

(i) *The tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a principal direction in common.*

Let us assume that no two eigen-values of the tensor $A_{ij}^{(1)}$ are equal.

We choose the co-ordinate system x_i to move in such a manner that at each instant of time its axes coincide in direction with the principal axes of $A_{ij}^{(1)}$. Then, in this co-ordinate system we have

$$A_{23}^{(1)} = A_{31}^{(1)} = A_{12}^{(1)} = 0. \quad (17.1)$$

The components $A_{11}^{(1)}$, $A_{22}^{(1)}$, $A_{33}^{(1)}$ of $A_{ij}^{(1)}$ are then the eigen-values of the tensor $A_{ij}^{(1)}$. If the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a principal direction in common, say

the direction corresponding to the eigen-value $A_{11}^{(1)}$, then in this co-ordinate system we have

$$A_{12}^{(2)} = A_{31}^{(2)} = 0. \quad (17.2)$$

If the relations (17.2) are valid over a finite interval of time, then it can be shown that

$$A_{12}^{(r)} = A_{31}^{(r)} = 0 \quad (r = 2, 3, \dots, n). \quad (17.3)$$

This may be done readily by an inductive method. We first observe from (16.1) that

$$A_{ij}^{(2)} = \frac{DA_{ij}^{(1)}}{Dt} + A_{mj}^{(1)} \frac{\partial v_m^{(1)}}{\partial x_i} + A_{im}^{(1)} \frac{\partial v_m^{(1)}}{\partial x_j}. \quad (17.4)$$

Since the relation (17.1) is valid over a finite interval of time, we have

$$\frac{DA_{ij}^{(1)}}{Dt} = 0 \quad (i \neq j). \quad (17.5)$$

Introducing the relations (17.1) and (17.5) into (17.4), we obtain

$$\begin{aligned} A_{23}^{(2)} &= A_{22}^{(1)} \frac{\partial v_2^{(1)}}{\partial x_3} + A_{33}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_2}, \\ A_{31}^{(2)} &= A_{33}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_1} + A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_3}, \end{aligned}$$

and

$$A_{12}^{(2)} = A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_2} + A_{22}^{(1)} \frac{\partial v_2^{(1)}}{\partial x_1}. \quad (17.6)$$

Since, from (10.12)

$$A_{ij}^{(1)} = \frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i}, \quad (17.7)$$

equations (17.1) yield

$$\frac{\partial v_i^{(1)}}{\partial x_j} = - \frac{\partial v_j^{(1)}}{\partial x_i} \quad (i \neq j), \quad (17.8)$$

so that the relations (17.6) become

$$A_{23}^{(2)} = (A_{22}^{(1)} - A_{33}^{(1)}) \frac{\partial v_2^{(1)}}{\partial x_3}, \quad A_{31}^{(2)} = (A_{33}^{(1)} - A_{11}^{(1)}) \frac{\partial v_3^{(1)}}{\partial x_1}$$

and

$$A_{12}^{(2)} = (A_{11}^{(1)} - A_{22}^{(1)}) \frac{\partial v_1^{(1)}}{\partial x_2}. \quad (17.9)$$

From (17.9) and (17.2), bearing in mind that we have assumed $A_{11}^{(1)} \neq A_{22}^{(1)} \neq A_{33}^{(1)} \neq A_{11}^{(1)}$, we obtain

$$\frac{\partial v_1^{(1)}}{\partial x_2} = \frac{\partial v_3^{(1)}}{\partial x_1} = 0. \quad (17.10)$$

We now assume that

$$A_{12}^{(r)} = A_{31}^{(r)} = 0, \quad (17.11)$$

over a finite interval of time and prove that

$$A_{12}^{(r+1)} = A_{31}^{(r+1)} = 0. \quad (17.12)$$

From (16.1), we have

$$A_{12}^{(r+1)} = \frac{DA_{12}^{(r)}}{Dt} + A_{m2}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_1} + A_{1m}^{(r)} \frac{\partial v_m^{(1)}}{\partial x_2}. \quad (17.13)$$

With (17.11), this yields

$$A_{12}^{(r+1)} = A_{22}^{(r)} \frac{\partial v_2^{(1)}}{\partial x_1} + A_{32}^{(r)} \frac{\partial v_3^{(1)}}{\partial x_1} + A_{11}^{(r)} \frac{\partial v_1^{(1)}}{\partial x_2}. \quad (17.14)$$

Introducing (17.10) and (17.8) into (17.14), we obtain $A_{12}^{(r+1)} = 0$. In a similar manner we can show that $A_{31}^{(r+1)} = 0$, thus obtaining (17.12). Since the relation (17.11) is assumed for $r = 2$, it is seen to be true generally. Thus, if $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a common principal direction throughout a finite time interval and no two eigen-values of $A_{ij}^{(1)}$ are equal, then $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have a common principal direction.

It follows immediately that if throughout a finite time interval no two eigen-values of $A_{ij}^{(1)}$ are equal and all the principal directions of $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ coincide, then all the principal directions of $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ coincide.

(ii) *Each of the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ has a pair of equal eigen-values.*

If $A_{ij}^{(1)}$ has a pair of equal eigen-values, but its third eigen-value unequal to these, then a principal plane is defined for the tensor, such that any pair of perpendicular directions lying in this plane may be considered as principal directions for the tensor. Again, if $A_{ij}^{(2)}$ has a pair of equal eigen-values, but a third eigen-value unequal to these then a principal plane is similarly defined for $A_{ij}^{(2)}$ having the property that any pair of perpendicular directions lying in this plane may be regarded as principal directions for $A_{ij}^{(2)}$. If these principal planes for $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ coincide, then all three principal directions of $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ coincide and, by a suitable choice of the co-ordinate system x_i , we can arrange that $A_{ij}^{(1)} = A_{ij}^{(2)} = 0$ ($i \neq j$).

If the principal planes of the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ defined by the pairs of

equal eigen-values do not coincide, then the line in which they intersect may be taken as a common principal direction for the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$. If the co-ordinate system x_i is chosen with one of its axes x_1 coinciding with this direction while x_2 lies in the principal plane of $A_{ij}^{(1)}$, then

$$A_{ij}^{(1)} = 0 \quad (i \neq j) \quad \text{and} \quad A_{11}^{(1)} = A_{22}^{(1)}. \quad (17.15)$$

In the same coordinate system $A_{ij}^{(2)}$ will have a principal direction in that of the axis x_1 , so that

$$A_{12}^{(2)} = A_{31}^{(2)} = 0. \quad (17.16)$$

The eigen-value equation of $A_{ij}^{(2)}$ is

$$\det (A_{ij}^{(2)} - \lambda \delta_{ij}) = 0, \quad (17.17)$$

which in view of (17.16) may be written

$$(A_{11}^{(2)} - \lambda)[(A_{22}^{(2)} - \lambda)(A_{33}^{(2)} - \lambda) - (A_{23}^{(2)})^2] = 0. \quad (17.18)$$

Now, from (17.4) and (17.15), we have

$$A_{11}^{(2)} = A_{22}^{(2)} = \frac{DA_{11}^{(1)}}{Dt} + 2A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_1}, \quad (17.19)$$

since from (17.15) and the definitions of $A_{11}^{(1)}$ and $A_{22}^{(1)}$, we have

$$\frac{\partial v_1^{(1)}}{\partial x_1} = \frac{\partial v_2^{(1)}}{\partial x_2}. \quad (17.20)$$

Also, from (17.4) and (17.15), we have

$$A_{33}^{(2)} = \frac{DA_{33}^{(1)}}{Dt} + 2A_{33}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_3}, \quad A_{23}^{(2)} = A_{22}^{(1)} \frac{\partial v_2^{(1)}}{\partial x_3} + A_{33}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_2},$$

$$A_{31}^{(2)} = A_{33}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_1} + A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_3}$$

and

$$A_{12}^{(2)} = A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_2} + A_{22}^{(1)} \frac{\partial v_2^{(1)}}{\partial x_1}. \quad (17.21)$$

Now, from (17.15) and the definitions of $A_{23}^{(1)}$, $A_{31}^{(1)}$ and $A_{12}^{(1)}$, we have

$$\frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i} = 0 \quad (i \neq j). \quad (17.22)$$

Introducing (17.22) into the expressions (17.21) for $A_{23}^{(2)}$, $A_{31}^{(2)}$ and $A_{12}^{(2)}$, we

obtain

$$A_{23}^{(2)} = (A_{22}^{(1)} - A_{33}^{(1)}) \frac{\partial v_2^{(1)}}{\partial x_3}, \quad A_{31}^{(2)} = (A_{33}^{(1)} - A_{11}^{(1)}) \frac{\partial v_3^{(1)}}{\partial x_1}$$

and

$$A_{12}^{(2)} = (A_{11}^{(1)} - A_{22}^{(1)}) \frac{\partial v_1^{(1)}}{\partial x_2}. \quad (17.23)$$

Since $A_{31}^{(2)} = 0$ and $A_{11}^{(1)} \neq A_{33}^{(1)}$, we have from (17.23)

$$\frac{\partial v_3^{(1)}}{\partial x_1} = 0. \quad (17.24)$$

The eigen-value equation (17.18) yields, with (17.19),

$$\lambda = A_{11}^{(2)} \quad \text{and} \quad \lambda = \frac{1}{2}[(A_{11}^{(2)} + A_{33}^{(2)}) \pm \sqrt{(A_{11}^{(2)} - A_{33}^{(2)})^2 + 4(A_{23}^{(2)})^2}]. \quad (17.25)$$

Since the components of $A_{ij}^{(2)}$ are real, two of the values of λ given by (17.25) are equal if and only if $A_{23}^{(2)} = 0$. We then have $\lambda = A_{11}^{(2)}$, $A_{11}^{(2)}$ and $A_{33}^{(2)}$. Thus, if the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ each have a pair of eigen-values equal over a finite interval of time, their principal planes defined by the pairs of equal eigen-values coincide and hence their principal directions coincide; *i.e.* in the co-ordinate system x_i chosen in the manner described above, we have

$$A_{11}^{(2)} = A_{22}^{(2)} \quad \text{and} \quad A_{ij}^{(2)} = 0 \quad (i \neq j). \quad (17.26)$$

Since we have assumed $A_{22}^{(1)} \neq A_{33}^{(1)}$, we obtain from (17.26) and (17.23),

$$\frac{\partial v_2^{(1)}}{\partial x_3} = 0. \quad (17.27)$$

We can now show by an inductive method that each of the tensors $A_{ij}^{(3)}$, $A_{ij}^{(4)}$, $\dots$, $A_{ij}^{(n)}$ has a pair of equal eigen-values and that the planes defined by these pairs of equal eigen-values coincide with each other and also with those defined by the pairs of equal eigen-values of the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$. In mathematical language, we shall show that if the tensors $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ are all referred to the co-ordinate system x_i defined above, we have

$$\begin{aligned} A_{11}^{(r)} &= A_{22}^{(r)} & (r = 1, 2, \dots, n) \\ \text{and} \\ A_{ij}^{(r)} &= 0 & (i \neq j, r = 1, 2, \dots, n). \end{aligned} \quad (17.28)$$

We assume (17.28) to be valid for an arbitrary value of r , over a finite interval of time. Then, from (16.1) and $A_{ij}^{(r)} = 0$ ($i \neq j$), we have

$$\begin{aligned}
A_{11}^{(r+1)} &= \frac{DA_{11}^{(r)}}{Dt} + 2A_{11}^{(r)} \frac{\partial v_1^{(1)}}{\partial x_1}, & A_{22}^{(r+1)} &= \frac{DA_{22}^{(r)}}{Dt} + 2A_{22}^{(r)} \frac{\partial v_2^{(1)}}{\partial x_2}, \\
A_{33}^{(r+1)} &= \frac{DA_{33}^{(r)}}{Dt} + 2A_{33}^{(r)} \frac{\partial v_3^{(1)}}{\partial x_3}, & A_{23}^{(r+1)} &= A_{22}^{(r)} \frac{\partial v_2^{(1)}}{\partial x_3} + A_{33}^{(r)} \frac{\partial v_3^{(1)}}{\partial x_2}, \\
A_{31}^{(r+1)} &= A_{33}^{(r)} \frac{\partial v_3^{(1)}}{\partial x_1} + A_{11}^{(r)} \frac{\partial v_1^{(1)}}{\partial x_3}, & A_{12}^{(r)} &= A_{11}^{(r)} \frac{\partial v_1^{(1)}}{\partial x_2} + A_{22}^{(r)} \frac{\partial v_2^{(1)}}{\partial x_1}.
\end{aligned} \quad (17.29)$$

Since $A_{11}^{(r)} = A_{22}^{(r)}$, over a finite interval of time, we obtain from (17.29)

$$A_{11}^{(r+1)} = A_{22}^{(r+1)}.$$

From (17.29), (17.22), (17.24) and (17.27), we have

$$A_{23}^{(r+1)} = A_{31}^{(r+1)} = 0 \quad \text{and} \quad A_{12}^{(r+1)} = (A_{11}^{(r)} - A_{22}^{(r)}) \frac{\partial v_1^{(1)}}{\partial x_2}. \quad (17.30)$$

Since $A_{11}^{(r)} = A_{22}^{(r)}$, (17.30) yields

$$A_{23}^{(r+1)} = A_{31}^{(r+1)} = A_{12}^{(r+1)} = 0. \quad (17.31)$$

It is thus seen that since (17.28) is valid for $r = 1, 2$, it is valid for $r = 3, 4, \dots, n$.

(iii) *The tensor $A_{ij}^{(1)}$ has three equal eigen-values.*

If $A_{ij}^{(1)}$ has three equal eigen-values during a finite time interval then, for any choice of the rectangular Cartesian co-ordinate system x_i , we have

$$A_{11}^{(1)} = A_{22}^{(1)} = A_{33}^{(1)} \quad \text{and} \quad A_{ij}^{(1)} = 0 \quad (i \neq j), \quad (17.32)$$

throughout that time interval. We see immediately from the definition of the tensor $A_{ij}^{(1)}$ and (17.32) that

$$\frac{\partial v_1^{(1)}}{\partial x_1} = \frac{\partial v_2^{(1)}}{\partial x_2} = \frac{\partial v_3^{(1)}}{\partial x_3}$$

and

$$\frac{\partial v_i^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_i} = 0. \quad (i \neq j). \quad (17.33)$$

From (16.1), (17.32) and (17.33), we have

$$A_{11}^{(2)} = A_{22}^{(2)} = A_{33}^{(2)} \quad \text{and} \quad A_{ij}^{(2)} = 0 \quad (i \neq j). \quad (17.34)$$

By repeated use of the relation (16.1) it can be seen generally that

$$A_{11}^{(r)} = A_{22}^{(r)} = A_{33}^{(r)} \quad \text{and} \quad A_{ij}^{(r)} = 0 \quad (i \neq j), \quad (17.35)$$

for $r = 1, 2, \dots, n$.

(iv) *The tensors $A_{ij}^{(1)}$, $A_{ij}^{(2)}$ and $A_{ij}^{(3)}$ have a principal direction in common.*

It has been seen in §17(i) that if, throughout a finite time interval, the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a common principal direction and no two eigen-values of $A_{ij}^{(1)}$ are equal, then $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have a common principal direction. It will be shown now that if, throughout a finite interval of time, $A_{ij}^{(1)}$, $A_{ij}^{(2)}$ and $A_{ij}^{(3)}$ have a common principal direction, then $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have this principal direction in common, even in the case when $A_{ij}^{(1)}$ has a pair of equal eigen-values. We choose the co-ordinate system x_i to move in such a way that at each instant of time its axes coincide in direction with the principal directions of $A_{ij}^{(1)}$, the axis x_i coinciding with the common principal direction of $A_{ij}^{(1)}$, $A_{ij}^{(2)}$ and $A_{ij}^{(3)}$. In such a co-ordinate system we have

$$A_{12}^{(1)} = A_{31}^{(1)} = A_{23}^{(1)} = A_{12}^{(2)} = A_{31}^{(2)} = A_{12}^{(3)} = A_{31}^{(3)} = 0. \quad (17.36)$$

As in §17(i), we have (equations (17.9))

$$A_{23}^{(2)} = (A_{22}^{(1)} - A_{33}^{(1)}) \frac{\partial v_2^{(1)}}{\partial x_3}, \quad A_{31}^{(2)} = (A_{33}^{(1)} - A_{11}^{(1)}) \frac{\partial v_3^{(1)}}{\partial x_1}$$

and

$$A_{12}^{(2)} = (A_{11}^{(1)} - A_{22}^{(1)}) \frac{\partial v_1^{(1)}}{\partial x_2}. \quad (17.37)$$

From (17.36) and (17.37), we have

$$(A_{33}^{(1)} - A_{11}^{(1)}) \frac{\partial v_3^{(1)}}{\partial x_1} = 0 \quad \text{and} \quad (A_{11}^{(1)} - A_{22}^{(1)}) \frac{\partial v_1^{(1)}}{\partial x_2} = 0. \quad (17.38)$$

From the first of these two equations, we have

$$A_{33}^{(1)} = A_{11}^{(1)} \quad \text{or} \quad \partial v_3^{(1)} / \partial x_1 = 0$$

and, from the second, we have

$$A_{11}^{(1)} = A_{22}^{(1)} \quad \text{or} \quad \partial v_1^{(1)} / \partial x_2 = 0.$$

If $\partial v_1^{(1)} / \partial x_2 = \partial v_3^{(1)} / \partial x_1 = 0$, we can proceed as in §17(i) to establish the result that $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have a principal direction in common. If $A_{11}^{(1)} = A_{22}^{(1)} = A_{33}^{(1)}$, we have the case discussed in §17(iii) and the desired result follows. It remains, then, to discuss the two cases

$$A_{11}^{(1)} = A_{22}^{(1)} \quad \text{and} \quad \partial v_3^{(1)} / \partial x_1 = 0 \quad (17.39)$$

and

$$A_{33}^{(1)} = A_{11}^{(1)} \quad \text{and} \quad \partial v_1^{(1)} / \partial x_2 = 0. \quad (17.40)$$

If the equations (17.39) apply, we have, from (16.1), (17.36) (17.39) and the definition of $A_{ij}^{(1)}$,

$$A_{11}^{(2)} = A_{22}^{(2)} = \frac{DA_{11}^{(1)}}{Dt} + 2A_{11}^{(1)} \frac{\partial v_1^{(1)}}{\partial x_1}. \quad (17.41)$$

From (16.1) and (17.36), we have

$$A_{31}^{(3)} = A_{23}^{(2)} \frac{\partial v_2^{(1)}}{\partial x_1} + (A_{33}^{(2)} - A_{11}^{(2)}) \frac{\partial v_3^{(1)}}{\partial x_1} = 0$$

and

$$A_{12}^{(3)} = A_{23}^{(2)} \frac{\partial v_3^{(1)}}{\partial x_1} + (A_{11}^{(2)} - A_{22}^{(2)}) \frac{\partial v_1^{(1)}}{\partial x_2} = 0. \quad (17.42)$$

From (17.42) and (17.39), we obtain

$$A_{23}^{(2)} = 0 \quad \text{or} \quad \partial v_2^{(1)} / \partial x_1 = 0.$$

If $\partial v_2^{(1)} / \partial x_1 = 0$, then, with $\partial v_3^{(1)} / \partial x_1 = 0$, we can carry through the argument of §17(i) to establish the desired result. If $A_{23}^{(2)} = 0$, then the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have all their principal directions in common. It can then be seen, from the analysis of §17(i), that either two of the three quantities $\partial v_1^{(1)} / \partial x_2$, $\partial v_2^{(1)} / \partial x_3$ and $\partial v_3^{(1)} / \partial x_1$ are zero, or $A_{11}^{(1)} = A_{22}^{(1)} = A_{33}^{(1)}$. In the former case we can, of course, carry through the proof that $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have a principal direction in common. In the latter, we see, as in §17(iii), that each of the tensors $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ has all its eigen-values equal and therefore all their principal directions are common.

It is readily seen that an analogous argument can be carried through if equation (17.40) applies instead of (17.39). We thus conclude that if the tensors $A_{ij}^{(1)}$, $A_{ij}^{(2)}$ and $A_{ij}^{(3)}$ have a principal direction in common over a finite time interval, then this principal direction is also a principal direction for $A_{ij}^{(4)}$, $A_{ij}^{(5)}$, $\dots$, $A_{ij}^{(n)}$.

(v) $A_{ij}^{(1)}$ has a single pair of equal eigen-values and $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a principal direction in common perpendicular to the plane defined by the pair of equal eigen-values of $A_{ij}^{(1)}$.

We refer the tensor $A_{ij}^{(1)}$ to a co-ordinate system x_i having its axis x_3 perpendicular to the plane defined by the pair of equal eigen-values of $A_{ij}^{(1)}$ and its axes x_1 and x_2 in this plane. We then have

$$A_{11}^{(1)} = A_{22}^{(1)}, \quad A_{ij}^{(1)} = 0, \quad (i \neq j)$$

and

$$A_{23}^{(2)} = A_{31}^{(2)} = 0. \quad (17.43)$$

From (16.1), (17.43) and the definition of $A_{11}^{(1)}$ and $A_{22}^{(1)}$, we see that, if the conditions (17.43) persist throughout a finite interval of time, we have

$$A_{11}^{(2)} = A_{22}^{(2)}. \quad (17.44)$$

Also, from (16.1), (17.43), (17.44) and the definition of $A_{12}^{(1)}$, we obtain

$$A_{12}^{(2)} = 0. \quad (17.45)$$

Thus, the tensor $A_{ij}^{(2)}$ has

$$A_{11}^{(2)} = A_{22}^{(2)} \quad \text{and} \quad A_{ij}^{(2)} = 0 \quad (i \neq j); \quad (17.46)$$

i.e. it has a pair of equal eigen-values and the plane defined by these coincides with that defined by the equal eigen-values of $A_{ij}^{(1)}$. It follows from §17(ii) that each of the tensors $A_{ij}^{(3)}, A_{ij}^{(4)}, \dots, A_{ij}^{(n)}$ has a pair of equal eigen-values and that the planes defined by these pairs of equal eigen-values coincide with each other and also with those defined by the pairs of equal eigen-values of the tensors $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$.

18. The effect of restrictions on the strain and velocity-strain tensors.

(i) *The tensors C_{ij} and $A_{ij}^{(1)}$ have a principal direction in common.*

If the tensors C_{ij} and $A_{ij}^{(1)}$ have a principal direction in common, we can readily show that the tensor $C_{ik} A_{kj}^{(1)} + C_{jk} A_{ki}^{(1)}$ also has this direction as a principal direction.

Let us choose the rectangular Cartesian co-ordinate system x_i with its axes in the principal directions of C_{ij} , so that

$$C_{ij} = 0 \quad (i \neq j). \quad (18.1)$$

Then, let us suppose, for example, that the common principal direction of C_{ij} and $A_{ij}^{(1)}$ is that of the x_1 axis. We then have

$$A_{12}^{(1)} = A_{31}^{(1)} = 0. \quad (18.2)$$

From (18.1) and (18.2), we see that the 12 and 13 components of $C_{ik} A_{kj}^{(1)} + C_{jk} A_{ki}^{(1)}$ are zero, so that this tensor and the tensors C_{ij} and $A_{ij}^{(1)}$ have a principal direction in common.

From (16.8), we have, bearing in mind that the relations (18.1) apply over a finite time interval and $\omega_{ij}^{(1)} = -\omega_{ji}^{(1)}$,

$$(C_{11} - C_{22})\omega_{12}^{(1)} = 0 \quad \text{and} \quad (C_{33} - C_{11})\omega_{31}^{(1)} = 0. \quad (18.3)$$

If we assume that no two of the eigen-values of C_{ij} are equal, *i.e.*

$$C_{11} \neq C_{22} \neq C_{33} \neq C_{11},$$

then (18.3) yields

$$\omega_{12}^{(1)} = \omega_{31}^{(1)} = 0. \quad (18.4)$$

Now, from (18.2) and the definition of $A_{ij}^{(1)}$, we have

$$\frac{\partial v_1^{(1)}}{\partial x_2} + \frac{\partial v_2^{(1)}}{\partial x_1} = \frac{\partial v_3^{(1)}}{\partial x_1} + \frac{\partial v_1^{(1)}}{\partial x_3} = 0. \quad (18.5)$$

From (18.4) and (18.5), we obtain

$$\frac{\partial v_1^{(1)}}{\partial x_2} = \frac{\partial v_2^{(1)}}{\partial x_1} = \frac{\partial v_3^{(1)}}{\partial x_1} = \frac{\partial v_1^{(1)}}{\partial x_3} = 0. \quad (18.6)$$

Using the relations (18.6), we can employ an inductive method in a manner similar to that adopted in §17(i) to show that if, throughout a finite interval of time, C_{ij} has all its eigen-values unequal and C_{ij} and $A_{ij}^{(1)}$ have a principal direction in common, then $A_{ij}^{(2)}, A_{ij}^{(3)}, \dots, A_{ij}^{(n)}$ also have this direction as a principal direction.

It is readily seen in a similar manner that if throughout a finite time interval no two eigen-values of C_{ij} are equal and all the principal directions of C_{ij} and $A_{ij}^{(1)}$ coincide, then all the principal directions of $C_{ij}, A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$ coincide.

(ii) *Each of the tensors C_{ij} and $A_{ij}^{(1)}$ has a pair of equal eigen-values.*

By an argument analogous to that of §17(ii), it can be seen that if C_{ij} and $A_{ij}^{(1)}$ each has a pair of equal eigen-values, they must have a common principal direction. We shall assume first that they have only one common principal direction. Then, as in §17(ii), it can be seen that this lies in the principal planes of C_{ij} and $A_{ij}^{(1)}$ corresponding to their pairs of equal eigen-values. Choosing this common principal direction as the axis x_1 of the rectangular Cartesian co-ordinate system x_i in which the tensors are defined and the axis x_2 in the principal plane of C_{ij} corresponding to its pair of equal eigen-values, we have

$$C_{ij} = 0 \quad (i \neq j) \quad \text{and} \quad C_{11} = C_{22}. \quad (18.7)$$

We shall assume that $C_{11} \neq C_{33}$.

In the same co-ordinate system, since x_1 is a principal direction of $A_{ij}^{(1)}$, we must have

$$A_{12}^{(1)} = A_{31}^{(1)} = 0. \quad (18.8)$$

In view of the relations (16.8), we have, as in §18(i), equation (18.3), which now leads to

$$\omega_{31}^{(1)} = 0. \quad (18.9)$$

From (18.9) and (18.8), we have

$$\frac{\partial v_1^{(1)}}{\partial x_3} = \frac{\partial v_3^{(1)}}{\partial x_1} = 0. \quad (18.10)$$

From (16.8), we have, since $\omega_{11}^{(1)} = \omega_{22}^{(1)} = \omega_{33}^{(1)} = 0$,

$$\frac{DC_{11}}{Dt} = C_{11} A_{11}^{(1)} \quad \text{and} \quad \frac{DC_{22}}{Dt} = C_{22} A_{22}^{(1)}. \quad (18.11)$$

With (18.7), this leads to

$$C_{11}(A_{11}^{(1)} - A_{22}^{(1)}) = 0. \quad (18.12)$$

Provided $C_{11} \neq 0$, a condition which is always satisfied physically, we have

$$A_{11}^{(1)} = A_{22}^{(1)}. \quad (18.13)$$

The eigen-value equation of $A_{ij}^{(1)}$ yields, with (18.8) and (18.13), the values

$$\lambda = A_{11}^{(1)} \quad \text{and} \quad \lambda = \frac{1}{2}[(A_{11}^{(1)} + A_{33}^{(1)}) \pm \sqrt{(A_{11}^{(1)} - A_{33}^{(1)})^2 + 4(A_{23}^{(1)})^2}] \quad (18.14)$$

for the eigen-values λ of $A_{ij}^{(1)}$. Since the components of $A_{ij}^{(1)}$ are real, $A_{ij}^{(1)}$ will have a pair of equal eigen-values if and only if $A_{23}^{(1)} = 0$. Then, $\lambda = A_{11}^{(1)}$, $A_{11}^{(1)}$ and $A_{33}^{(1)}$. Writing $i, j = 2, 3$ in (16.8), we have, with (18.7),

$$(C_{22} - C_{33})\omega_{32}^{(1)} = 0. \quad (18.15)$$

Whence, since $C_{22} \neq C_{33}$, $\omega_{32}^{(1)} = 0$. With the relation $A_{23}^{(1)} = 0$, we have

$$\frac{\partial v_2^{(1)}}{\partial x_3} = \frac{\partial v_3^{(1)}}{\partial x_2} = 0. \quad (18.16)$$

Summarizing our conclusions regarding the tensor $A_{ij}^{(1)}$, we have

$$A_{11}^{(1)} = A_{22}^{(1)} \quad \text{and} \quad A_{ij}^{(1)} = 0 \quad (i \neq j). \quad (18.17)$$

We also have the relations (18.10) and (18.16). These relations can be used to show, by an inductive method similar to that employed in §17(ii), that

$$A_{11}^{(r)} = A_{22}^{(r)} \quad (r = 2, 3, \dots, n)$$

and

$$A_{ij}^{(r)} = 0 \quad (i \neq j, r = 2, 3, \dots, n). \quad (18.18)$$

Thus, if C_{ij} and $A_{ij}^{(1)}$ each has a pair of equal eigen-values, throughout a finite interval of time, each of $A_{ij}^{(2)}$, $A_{ij}^{(3)}$, $\dots$, $A_{ij}^{(n)}$ has a pair of equal eigen-values and the principal planes of C_{ij} , $A_{ij}^{(1)}$, $\dots$, $A_{ij}^{(n)}$ corresponding to these pairs of equal eigen-values coincide.

In deriving this result, we have assumed that the three eigen-values of C_{ij} are not all equal. If they are all equal over a finite time interval, then it can be readily shown that each of the tensors $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ has all three of its eigen-values equal.

(iii) *The tensors C_{ij} , $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a principal direction in common.*

It has been seen in §18(i) that if, throughout a finite time interval, the tensors C_{ij} and $A_{ij}^{(1)}$ have a common principal direction and no two eigen-values of C_{ij} are equal, then C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have a common principal direction. It will be shown now that if, throughout a finite interval of time, C_{ij} , $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a common principal direction, then C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$ have this principal direction in common, even in the case when C_{ij} has a pair of equal

eigen-values. We choose the co-ordinate system x_i to move in such a way that at each instant of time its axes coincide with the principal directions of C_{ij} , the axis x_1 coinciding with the common principal direction of C_{ij} , $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$. In such a co-ordinate system we have

$$C_{12} = C_{31} = C_{23} = A_{12}^{(1)} = A_{31}^{(1)} = A_{12}^{(2)} = A_{31}^{(2)} = 0. \quad (18.19)$$

As in §18(i), we obtain from equations (16.8) and (18.9)

$$(C_{11} - C_{22})\omega_{12}^{(1)} = 0 \quad \text{and} \quad (C_{33} - C_{11})\omega_{31}^{(1)} = 0. \quad (18.20)$$

From the first of these equations, we obtain

$$C_{11} = C_{22} \quad \text{or} \quad \omega_{12}^{(1)} = 0$$

and, from the second,

$$C_{11} = C_{33} \quad \text{or} \quad \omega_{31}^{(1)} = 0.$$

If $\omega_{12}^{(1)} = \omega_{31}^{(1)} = 0$, we can proceed, as in §18(i), to obtain the desired result. If $C_{11} = C_{22} = C_{33}$, then all three eigen-values of $C_{ij}^{(1)}$ are equal and we can readily show that each of the tensors $A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$ has all three of its eigen-values equal. It remains then to consider the two cases

$$C_{11} = C_{22} \quad \text{and} \quad \omega_{31}^{(1)} = 0 \quad (18.21)$$

and

$$C_{11} = C_{33} \quad \text{and} \quad \omega_{12}^{(1)} = 0. \quad (18.22)$$

Let us consider the case when (18.21) applies. Since, from (18.19), $A_{31}^{(1)} = 0$, we have, with (18.21),

$$\partial v_3^{(1)} / \partial x_1 = \partial v_1^{(1)} / \partial x_3 = 0. \quad (18.23)$$

From (16.8), (18.19) and (18.21), we have, bearing in mind that C_{11} and C_{22} cannot be zero,

$$A_{11}^{(1)} = A_{22}^{(1)} = \frac{1}{C_{11}} \frac{DC_{11}}{Dt}. \quad (18.24)$$

From (16.1) and (18.19), we have

$$A_{12}^{(2)} = (A_{11}^{(1)} - A_{22}^{(1)}) \frac{\partial v_1^{(1)}}{\partial x_2} + A_{23}^{(1)} \frac{\partial v_3^{(1)}}{\partial x_1} = 0.$$

and

$$A_{31}^{(2)} = (A_{33}^{(1)} - A_{11}^{(1)}) \frac{\partial v_3^{(1)}}{\partial x_1} + A_{23}^{(1)} \frac{\partial v_2^{(1)}}{\partial x_1} = 0. \quad (18.25)$$

With (18.23) and (18.24), equations (18.25) yield

$$A_{23}^{(1)} = 0 \quad \text{or} \quad \partial v_2^{(1)} / \partial x_1 = 0. \quad (18.26)$$

If $\partial v_2^{(1)} / \partial x_1 = 0$, then, with (18.23), we can carry through the argument of §18(i) to show that $C_{ij}, A_{ij}^{(1)}, \dots, A_{ij}^{(n)}$ have a principal direction in common. If $A_{23}^{(1)} = 0$, then the tensors C_{ij} and $A_{ij}^{(1)}$ have all their principal directions in common. It can then be seen, from the analysis of §18(i), that either two of the three quantities $\partial v_1^{(1)} / \partial x_2, \partial v_2^{(1)} / \partial x_3$ and $\partial v_3^{(1)} / \partial x_1$ are zero, or $C_{11} = C_{22} = C_{33}$. In the former case we can, of course, carry through the proof that $C_{ij}, A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$ have a principal direction in common. In the latter, we have all the eigen-values of C_{ij} and hence of each of $A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$ are equal.

A similar argument to that given above can be carried through if equation (18.22) applies instead of (18.21). We thus see that if the tensors $C_{ij}, A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ have a principal direction in common throughout a finite time interval, then this direction is also a principal direction for $A_{ij}^{(3)}, A_{ij}^{(4)}, \dots, A_{ij}^{(n)}$.

(iv) C_{ij} has a single pair of equal eigen-values and C_{ij} and $A_{ij}^{(1)}$ have a principal direction in common perpendicular to the plane defined by the pair of equal eigen-values of C_{ij} .

We refer the tensor C_{ij} to a co-ordinate system x_i having the axis x_3 perpendicular to the plane defined by the pair of equal eigen-values of C_{ij} and its axes x_1 and x_2 in this plane. We then have

$$C_{11} = C_{22}, \quad C_{ij} = 0 \quad (i \neq j)$$

and

$$A_{23}^{(1)} = A_{31}^{(1)} = 0. \quad (18.27)$$

From (16.5), (18.27) and the definitions of $A_{11}^{(1)}$ and $A_{22}^{(1)}$, we obtain

$$A_{11}^{(1)} = A_{22}^{(1)}. \quad (18.28)$$

Also, from (16.5), (18.27), (18.28) and the definition of $A_{12}^{(1)}$, we obtain

$$A_{12}^{(1)} = 0. \quad (18.29)$$

Thus the tensor $A_{ij}^{(1)}$ has

$$A_{11}^{(1)} = A_{22}^{(1)} \quad \text{and} \quad A_{ij}^{(1)} = 0 \quad (i \neq j); \quad (18.30)$$

i.e. it has a pair of equal eigen-values and the plane defined by these coincides with that defined by the equal eigen-values of C_{ij} . It follows from §18(ii) that each of the tensors $C_{ij}, A_{ij}^{(1)}, A_{ij}^{(2)}, \dots, A_{ij}^{(n)}$ has a pair of equal eigen-values and the principal planes defined by these pairs of equal eigen-values coincide.

CHAPTER III. SOME RESULTS IN THE ALGEBRA OF MATRICES

19. Introduction. It has been seen in §15 that if we assume that the components of stress in some rectangular Cartesian co-ordinate system, at a point of a body which is undergoing deformation, depend only on the gradients of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration at that point, then it follows that the stress is a symmetric tensor the components of which in any rectangular Cartesian co-ordinate system are functions of the components of $n + 1$ kinematic tensors in that co-ordinate system, the functional dependence being independent of the co-ordinate system chosen. In any rectangular Cartesian co-ordinate system, fixed or moving, the components of these kinematic tensors are specific functions of the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration.

We may consider the components of each of these kinematic tensors to be the components of a symmetric 3×3 matrix. Transformation of the kinematic tensor from one rectangular Cartesian co-ordinate system to another then corresponds to the transformation of the corresponding kinematic matrix by means of the proper orthogonal transformation relating the two rectangular Cartesian co-ordinate systems.

In the same way the stress components in any rectangular Cartesian co-ordinate system may also be regarded as the components of a symmetric 3×3 matrix and transformation of the stress tensor from one rectangular Cartesian co-ordinate system to another corresponds to the transformation of the stress matrix by means of the proper orthogonal transformation relating the two rectangular Cartesian co-ordinate systems.

The elements of the stress matrix in any rectangular Cartesian co-ordinate system are then functions of the elements of the kinematic matrices in the same co-ordinate system. The forms of these functional relationships are independent of the particular co-ordinate system in which they are expressed.

In this chapter a number of results will be obtained concerning the expression of a symmetric matrix (the dependent matrix) as a linear combination of a number of other matrices. It will be considered that the elements of the dependent matrix are functions of the elements of a number of other matrices (the independent matrices) and that the form of this functional dependence is independent of the rectangular Cartesian co-ordinate system in which it is expressed. It will be shown that a linear relation can be obtained between the dependent matrix and a number of other matrices formed from the elements of the independent matrices, the coefficients in this linear relation being scalar invariants of the independent matrices and the form of the linear relation being independent of the rectangular Cartesian co-ordinate system in which it is expressed. An infinite number of such linear relations can be obtained by making different

choices of the matrices formed from the elements of the independent matrices. For each such choice there exist certain conditions on the independent matrices for which the coefficient of the dependent matrix in the linear relation is zero and for which the linear relation does not therefore provide an expression for the dependent matrix.

In Chapter V we shall identify the dependent matrix with the stress matrix and the independent matrices with the kinematic matrices on the elements of which the elements of the stress matrix depend. It has been seen in §§16 to 18 that if certain restrictions on the behaviour of certain of the kinematic matrices (the strain matrix, the velocity-strain matrix, the acceleration-strain matrix, the second acceleration-strain matrix) exist this imposes a restriction on the behaviour of the remaining kinematic matrices. In the present chapter it is shown that when restrictions of this type are imposed on the independent matrices, then we can obtain a set of matrices formed from the elements of the independent matrices such that the dependent matrix can be expressed as a linear combination of these, in which the coefficients are scalar invariants of the independent matrices and the coefficient of the dependent matrix is not zero.

20. Linear dependence of symmetric matrices. Let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N$, with $N = \frac{1}{2}n(n+1) + 1$, be any $[\frac{1}{2}n(n+1) + 1]$ symmetric $n \times n$ matrices, no $N - 1$ of which are linearly dependent. Then, it can be readily seen that a linear relation exists between them, *i.e.* there exist scalar quantities α_i , none of which is zero, such that

$$\sum_{r=1}^N \alpha_r \mathbf{A}_r = 0. \quad (20.1)$$

Let the elements of the matrix $\mathbf{A}_r$ be $a_{ij}^{(r)}$, with $a_{ij}^{(r)} = a_{ji}^{(r)}$. Then, equation (20.1) represents $N - 1$ homogeneous linear equations

$$\sum_{r=1}^N \alpha_r a_{ij}^{(r)} = 0 \quad (20.2)$$

in the N quantities α_r . These $N - 1$ equations must be linearly independent, since no $N - 1$ of the matrices $\mathbf{A}_r$ are linearly dependent. From equations (20.2), α_r can be determined in terms of the elements of the matrices, apart from an arbitrary non-zero multiplier K (say). We obtain by Cramer's rule

$$\alpha_r = K(-1)^r \det \mathbf{B}_r, \quad (20.3)$$

where the $(N - 1) \times (N - 1)$ matrix $\mathbf{B}_r$ is defined by

$$\mathbf{B}_r = \begin{vmatrix} a_{11}^{(1)} & a_{11}^{(2)} & a_{11}^{(3)} & \dots & a_{11}^{(r-1)} & a_{11}^{(r+1)} & \dots & a_{11}^{(N)} \\ a_{12}^{(1)} & a_{12}^{(2)} & a_{12}^{(3)} & \dots & a_{12}^{(r-1)} & a_{12}^{(r+1)} & \dots & a_{12}^{(N)} \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_{ij}^{(1)} & a_{ij}^{(2)} & a_{ij}^{(3)} & \dots & a_{ij}^{(r-1)} & a_{ij}^{(r+1)} & \dots & a_{ij}^{(N)} \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_{nn}^{(1)} & a_{nn}^{(2)} & a_{nn}^{(3)} & \dots & a_{nn}^{(r-1)} & a_{nn}^{(r+1)} & \dots & a_{nn}^{(N)} \end{vmatrix}. \quad (20.4)$$

It is clear that if $\det \mathbf{B}_r = 0$, then $\alpha_r = 0$, which would contradict the assumption that no $N - 1$ of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N$ are linearly dependent. Thus none of the quantities $\det \mathbf{B}_r$ is zero.

From (20.3), we have

$$\frac{\alpha_r}{\alpha_p} = (-1)^{r+p} \frac{\det \mathbf{B}_r}{\det \mathbf{B}_p} = (-1)^{r+p} \frac{\det \mathbf{B}_r \det \mathbf{B}_p}{(\det \mathbf{B}_p)^2}. \quad (20.5)$$

Let $\mathbf{D}_r$ be the matrix formed from $\mathbf{B}_r$ by multiplying by 2 each of the rows of $\mathbf{B}_r$ consisting of off-diagonal components of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N$. Then, we have

$$\det \mathbf{B}_r = \frac{1}{2^{4n(n-1)}} \det \mathbf{D}_r. \quad (20.6)$$

Denoting by $\mathbf{B}_p^*$ the transpose of $\mathbf{B}_p$, we see that (20.5) yields

$$\frac{\alpha_r}{\alpha_p} = (-1)^{r+p} \frac{\det \mathbf{B}_p^* \mathbf{D}_r}{\det \mathbf{B}_p^* \mathbf{D}_p}. \quad (20.7)$$

We have from (20.4) and the definition of $\mathbf{D}_r$,

$$\mathbf{B}_p^* \mathbf{D}_r = \begin{pmatrix} \alpha_{11}, & \alpha_{12} & \cdots & \alpha_{1,r-1}, & \alpha_{1,r+1} & \cdots & \alpha_{1N} \\ \alpha_{21}, & \alpha_{22} & \cdots & \alpha_{2,r-1}, & \alpha_{2,r+1} & \cdots & \alpha_{2N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{p-1,1}, & \alpha_{p-1,2} & \cdots & \alpha_{p-1,r-1}, & \alpha_{p-1,r+1} & \cdots & \alpha_{p-1,N} \\ \alpha_{p+1,1}, & \alpha_{p+1,2} & \cdots & \alpha_{p+1,r-1}, & \alpha_{p+1,r+1} & \cdots & \alpha_{p+1,N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{N1}, & \alpha_{N2} & \cdots & \alpha_{N,r-1}, & \alpha_{N,r+1} & \cdots & \alpha_{NN} \end{pmatrix}, \quad (20.8)$$

where

$$\alpha_{ij} = \text{tr } \mathbf{A}_i \mathbf{A}_j = \text{tr } \mathbf{A}_j \mathbf{A}_i = \alpha_{ji}. \quad (20.9)$$

We may, of course, re-write (20.7) as

$$\alpha_r = K(-1)^{r+p} \det \mathbf{B}_p^* \mathbf{D}_r, \quad (20.10)$$

where K is an arbitrary non-zero multiplier. We note that (20.10) with (20.8) expresses α_r in terms of the scalar quantities α_{ij} .

The expression (20.10) for α_r may be obtained directly from equation (20.1) in the following manner. We form the $N - 1$ matrix equations

$$\sum_{r=1}^N \alpha_r \mathbf{A}_r \mathbf{A}_s = 0 \quad (s = 1, 2, \dots, p-1, p+1, \dots, N) \quad (20.11)$$

by multiplying equation (20.1) by $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{p-1}, \mathbf{A}_{p+1}, \dots, \mathbf{A}_N$ respectively. Forming the trace of each of the equations (20.11) we have, with the notation of (20.9),

$$\sum_{r=1}^N \alpha_r \mathcal{Q}_{rs} = 0 \quad (s = 1, 2, \dots, p-1, p+1, \dots, N). \quad (20.12)$$

These $N - 1$ scalar equations in $\alpha_1, \alpha_2, \dots, \alpha_N$ may be solved to yield the expressions (20.10) for $\alpha_1, \alpha_2, \dots, \alpha_N$.

Now, suppose that M ($< N$) of the set of matrices considered are linearly dependent. We then have a relation between them of the form (20.1) in which one or more of the α 's are zero. We thus have the result that there exists a linear relation of the form (20.1) between any $N [= \frac{1}{2}n(n+1) + 1]$ symmetric $n \times n$ matrices in which the coefficients α_r are scalars, not all zero.

In the case when M ($< N$) of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_M$ are linearly dependent, but there exists at least one set $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{p-1}, \mathbf{A}_{p+1}, \dots, \mathbf{A}_M$ of $M - 1$ of these matrices which are linearly independent, we can obtain the values of $\alpha_1, \alpha_2, \dots, \alpha_M$ satisfying the relation

$$\sum_{r=1}^M \alpha_r \mathbf{A}_r = 0 \quad (20.13)$$

by setting up the $M - 1$ equations

$$\sum_{r=1}^M \alpha_r \mathcal{Q}_{sr} = 0 \quad (s = 1, 2, \dots, p-1, p+1, \dots, M) \quad (20.14)$$

formed by multiplying (20.13) by $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{p-1}, \mathbf{A}_{p+1}, \dots, \mathbf{A}_M$ respectively and taking the traces of the matrix equations so obtained. These equations may of course, be solved to yield expressions for α_r in terms of the scalar invariants $\mathcal{Q}_{rs}$.

21. Functions of matrices. We shall say that an $n \times n$ matrix $\mathbf{T}$ is a function of the R , $n \times n$ matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ if the elements of $\mathbf{T}$ are functions of the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ and we denote this relationship by

$$\mathbf{T} = \mathbf{F}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R). \quad (21.1)$$

Now, we consider the transformation of this relation by means of an arbitrary orthogonal transformation $\mathbf{S}$. Let $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_R$ and $\bar{\mathbf{T}}$ be the matrices into which $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ and $\mathbf{T}$ respectively are transformed by means of this transformation. We have

$$\bar{\mathbf{A}}_r = \mathbf{S} \mathbf{A}_r \mathbf{S}^{-1} \quad (r = 1, 2, \dots, R)$$

and

$$\bar{\mathbf{T}} = \mathbf{S} \mathbf{T} \mathbf{S}^{-1}. \quad (21.2)$$

If $\bar{\mathbf{T}}$ is the same function of $\bar{\mathbf{A}}_r$ ($r = 1, 2, \dots, R$) as $\mathbf{T}$ is of $\mathbf{A}_r$ ($r = 1, 2, \dots, R$), then $\mathbf{T}$ will be described as an *isotropic* function of the matrices $\mathbf{A}_r$.

It was shown in §20 that any symmetric $n \times n$ matrix may be expressed as a linear combination of any $\frac{1}{2}n(n+1)$ linearly independent symmetric $n \times n$

matrices. Therefore, if $\mathbf{T}$ is a symmetric matrix and if we form, from the elements of the matrices $\mathbf{A}_r$ ($r = 1, 2, \dots, R$), $\frac{1}{2}n(n+1)$ linearly independent symmetric $n \times n$ matrices $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_{N-1}$ ($N = \frac{1}{2}n(n+1) + 1$), then $\mathbf{T}$ may be expressed as a linear combination of the matrices $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_{N-1}$, *i.e.* we have

$$\alpha \mathbf{T} + \sum_{r=1}^{\frac{1}{2}n(n+1)} \alpha_r \mathbf{B}_r = 0, \quad (21.3)$$

where the α 's are scalars formed from the elements of $\mathbf{T}$ and $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_N$.

If $\alpha \neq 0$, as will be the case if $\mathbf{B}_r$ ($r = 1, 2, \dots, \frac{1}{2}n(n+1)$) are linearly independent, then we shall describe an expression for $\mathbf{T}$ of this form as a *representation* of $\mathbf{T}$ in terms of $\mathbf{B}_r$ ($r = 1, 2, \dots, \frac{1}{2}n(n+1)$). More generally, if we can establish a linear relation between $\mathbf{T}$ and any number of matrices, not necessarily linearly independent, such that the coefficient of $\mathbf{T}$ is non-zero, then we shall refer to this expression as a representation of $\mathbf{T}$ in terms of the remaining matrices.

If $\mathbf{T}$ is an isotropic function of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, then the form of equation (21.3) considered as an expression for $\mathbf{T}$ in terms of the elements of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ must be unaltered by a simultaneous orthogonal transformation of the matrices $\mathbf{T}$ and $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. If $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_N$ are isotropic functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, this implies that the α 's are expressible in terms of the elements of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ in a form which is unaltered by an orthogonal transformation.

22. A representation of a symmetric matrix function of R symmetric 2×2 matrices. It is seen from §21 that any symmetric 2×2 matrix $\mathbf{T}$ ($= \|t_{ij}\|$), which is a function of R symmetric 2×2 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, may be expressed as a linear combination of any three linearly independent symmetric matrix functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. In particular, we may express $\mathbf{T}$ as a linear combination of $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{I}$ (the unit matrix), provided that these are linearly independent. The condition for linear independence of the matrices $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{I}$ is that no relation of the form

$$\alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{I} = 0, \quad (22.1)$$

where α_1, α_2 and α_3 are scalars, not all zero, shall exist. Bearing in mind that the relation (22.1) is equivalent to the three relations

$$\alpha_1 a_{ij}^{(1)} + \alpha_2 a_{ij}^{(2)} + \alpha_3 \delta_{ij} = 0, \quad (22.2)$$

where

$$\mathbf{A}_r = \|a_{ij}^{(r)}\|; \quad a_{ij}^{(r)} = a_{ji}^{(r)} \quad (r = 1, 2, \dots, R), \quad (22.3)$$

we see that the necessary and sufficient condition for linear independence of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{I}$ is

$$\begin{vmatrix} 1, & 1, & 0, \\ a_{11}^{(1)}, & a_{22}^{(1)}, & a_{12}^{(1)}, \\ a_{11}^{(2)}, & a_{22}^{(2)}, & a_{12}^{(2)}, \end{vmatrix} \neq 0; \quad (22.4)$$

that is

$$\begin{vmatrix} 1, & 1, & 0, \\ a_{11}^{(1)}, & a_{22}^{(1)}, & a_{12}^{(1)}, \\ a_{11}^{(2)}, & a_{22}^{(2)}, & a_{12}^{(2)}, \end{vmatrix} \times \begin{vmatrix} 1, & a_{11}^{(1)}, & a_{11}^{(2)} \\ 1, & a_{22}^{(1)}, & a_{22}^{(2)} \\ 0, & 2a_{12}^{(1)}, & 2a_{12}^{(2)} \end{vmatrix} \equiv \begin{vmatrix} 2, & \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2 \\ \text{tr } \mathbf{A}_2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2 \end{vmatrix} \neq 0. \quad (22.5)$$

Now, we may choose an orthogonal transformation which transforms $\mathbf{A}_1$ into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)})$ and $\mathbf{A}_2$ into the symmetric matrix $\bar{\mathbf{A}}_2 (= \|\bar{a}_{ij}^{(2)}\|)$. Then, the condition (22.4) becomes

$$\begin{vmatrix} 1, & 1, & 0 \\ \bar{a}_{11}^{(1)}, & \bar{a}_{22}^{(1)}, & 0 \\ \bar{a}_{11}^{(2)}, & \bar{a}_{22}^{(2)}, & \bar{a}_{12}^{(2)} \end{vmatrix} \neq 0, \quad (22.6)$$

that is

$$\bar{a}_{11}^{(1)} \neq \bar{a}_{22}^{(1)} \quad \text{and} \quad \bar{a}_{12}^{(2)} \neq 0. \quad (22.7)$$

A necessary and sufficient condition for linear independence of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{I}$ is thus seen to be that the principal directions of $\mathbf{A}_1$ and $\mathbf{A}_2$ shall not coincide. This is strictly equivalent to $\bar{a}_{12}^{(2)} \neq 0$. It is immediately apparent, however, that if the eigen-values of $\mathbf{A}_1$ are equal, *i.e.* $\bar{a}_{11}^{(1)} = \bar{a}_{22}^{(1)}$, then the principal directions of $\mathbf{A}_1$ and $\mathbf{A}_2$ may be considered to coincide.

Thus, provided that the principal directions of $\mathbf{A}_1$ and $\mathbf{A}_2$ do not coincide, we may write

$$\alpha \mathbf{T} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{I} = 0, \quad (22.8)$$

with $\alpha \neq 0$, where, from (20.10),

$$\begin{aligned} \alpha = K \begin{vmatrix} \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2, & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_2, & 2 \end{vmatrix}, \quad \alpha_1 = -K \begin{vmatrix} \text{tr } \mathbf{A}_1 \mathbf{T}, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_2 \mathbf{T}, & \text{tr } \mathbf{A}_2^2, & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{T}, & \text{tr } \mathbf{A}_2, & 2 \end{vmatrix}, \\ \alpha_2 = K \begin{vmatrix} \text{tr } \mathbf{A}_1 \mathbf{T}, & \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_2 \mathbf{T}, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{T}, & \text{tr } \mathbf{A}_1, & 2 \end{vmatrix}, \quad \alpha_3 = -K \begin{vmatrix} \text{tr } \mathbf{A}_1 \mathbf{T}, & \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2 \\ \text{tr } \mathbf{A}_2 \mathbf{T}, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2 \\ \text{tr } \mathbf{T}, & \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_2 \end{vmatrix}, \end{aligned} \quad (22.9)$$

K being an arbitrary non-zero multiplier. Now, if $\mathbf{T}$ is an isotropic function of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_R$, since $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{I}$ are also isotropic functions of these, $\text{tr } \mathbf{T}$, $\text{tr } \mathbf{A}_1 \mathbf{T}$ and $\text{tr } \mathbf{A}_2 \mathbf{T}$ must be scalar invariants formed from the elements of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ only.

We note (cf. §20) that

$$\alpha = 2K\Delta^2, \quad \alpha_1 = -2K\Delta\Delta_1, \quad \alpha_2 = 2K\Delta\Delta_2 \quad \text{and} \quad \alpha_3 = -2K\Delta\Delta_3, \quad (22.10)$$

where

$$\Delta = \begin{vmatrix} \bar{a}_{11}^{(1)} & \bar{a}_{11}^{(2)} & 1 \\ \bar{a}_{22}^{(1)} & \bar{a}_{22}^{(2)} & 1 \\ 0 & \bar{a}_{12}^{(2)} & 0 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \bar{l}_{11} & \bar{a}_{11}^{(2)} & 1 \\ \bar{l}_{22} & \bar{a}_{22}^{(2)} & 1 \\ \bar{l}_{12} & \bar{a}_{12}^{(2)} & 0 \end{vmatrix}, \quad (22.11)$$

$$\Delta_2 = \begin{vmatrix} \bar{l}_{11} & \bar{a}_{11}^{(1)} & 1 \\ \bar{l}_{22} & \bar{a}_{22}^{(1)} & 1 \\ \bar{l}_{12} & 0 & 0 \end{vmatrix} \quad \text{and} \quad \Delta_3 = \begin{vmatrix} \bar{l}_{11} & \bar{a}_{11}^{(1)} & \bar{a}_{11}^{(2)} \\ \bar{l}_{22} & \bar{a}_{22}^{(1)} & \bar{a}_{22}^{(2)} \\ \bar{l}_{12} & 0 & \bar{a}_{12}^{(2)} \end{vmatrix}.$$

23. A representation of a symmetric matrix function of R symmetric 2×2 matrices in certain degenerate cases.

If $\mathbf{T}$ and $\mathbf{A}_1$ have common principal directions, then we can set up a linear relation between the matrices $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{I}$ thus,

$$\beta \mathbf{T} + \beta_1 \mathbf{A}_1 + \beta_2 \mathbf{I} = 0. \quad (23.1)$$

We can choose an orthogonal transformation which applied simultaneously to the matrices $\mathbf{T}$ and $\mathbf{A}_1$ transforms them both into diagonal matrices $\bar{\mathbf{T}} = (\bar{l}_{11}, \bar{l}_{22})$ and $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)})$. We may now write, from (23.1),

$$\beta \bar{\mathbf{T}} + \beta_1 \bar{\mathbf{A}}_1 + \beta_2 \mathbf{I} = 0. \quad (23.2)$$

This matrix equation represents the two equations

$$\beta \bar{l}_{11} + \beta_1 \bar{a}_{11}^{(1)} + \beta_2 = 0$$

and

$$\beta \bar{l}_{22} + \beta_1 \bar{a}_{22}^{(1)} + \beta_2 = 0, \quad (23.3)$$

which can be solved for β , β_1 and β_2 , apart from an arbitrary multiplying constant K . We have

$$\beta = K(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)}), \quad \beta_1 = -K(\bar{l}_{11} - \bar{l}_{22}), \quad (23.4)$$

$$\beta_2 = -K(\bar{a}_{11}^{(1)} \bar{l}_{22} - \bar{a}_{22}^{(1)} \bar{l}_{11}).$$

Alternatively, we can form from (23.2) the pair of equations

$$\beta \text{tr } \bar{\mathbf{T}} + \beta_1 \text{tr } \bar{\mathbf{A}}_1 + 2\beta_2 \mathbf{I} = 0$$

and

$$\beta \operatorname{tr} \bar{\mathbf{A}}_1 \bar{\mathbf{T}} + \beta_1 \operatorname{tr} \bar{\mathbf{A}}_1^2 + \beta_2 \operatorname{tr} \bar{\mathbf{A}}_1 = 0, \quad (23.5)$$

to obtain, with the relations

$$\operatorname{tr} \mathbf{T} = \operatorname{tr} \bar{\mathbf{T}}, \quad \operatorname{tr} \mathbf{A}_1 = \operatorname{tr} \bar{\mathbf{A}}_1, \quad \operatorname{tr} \mathbf{A}_1^2 = \operatorname{tr} \bar{\mathbf{A}}_1^2 \quad \text{and} \quad \operatorname{tr} \mathbf{A}_1 \mathbf{T} = \operatorname{tr} \bar{\mathbf{A}}_1 \bar{\mathbf{T}},$$

$$\beta = K[2 \operatorname{tr} \mathbf{A}_1^2 - (\operatorname{tr} \mathbf{A}_1)^2], \quad \beta_1 = K(\operatorname{tr} \mathbf{T} \operatorname{tr} \mathbf{A}_1 - 2 \operatorname{tr} \mathbf{A}_1 \mathbf{T})$$

and

$$\beta_2 = K(\operatorname{tr} \mathbf{A}_1 \operatorname{tr} \mathbf{A}_1 \mathbf{T} - \operatorname{tr} \mathbf{T} \operatorname{tr} \mathbf{A}_1^2). \quad (23.6)$$

We note that

$$2 \operatorname{tr} \mathbf{A}_1^2 - (\operatorname{tr} \mathbf{A}_1)^2 = (\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})^2,$$

$$\operatorname{tr} \mathbf{T} \operatorname{tr} \mathbf{A}_1 - 2 \operatorname{tr} \mathbf{A}_1 \mathbf{T} = -(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{t}_{11} - \bar{t}_{22})$$

and

$$\operatorname{tr} \mathbf{A}_1 \operatorname{tr} \mathbf{A}_1 \mathbf{T} - \operatorname{tr} \mathbf{T} \operatorname{tr} \mathbf{A}_1^2 = -(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{a}_{11}^{(1)} \bar{t}_{22} - \bar{a}_{22}^{(1)} \bar{t}_{11}). \quad (23.7)$$

We see that $\beta = 0$, if and only if $\mathbf{A}_1$ has equal eigen-values, *i.e.* $\bar{a}_{11}^{(1)} = \bar{a}_{22}^{(1)}$. Thus, if $\mathbf{T}$ and $\mathbf{A}_1$ have common principal directions, we can obtain a representation of $\mathbf{T}$ in terms of $\mathbf{A}_1$ and $\mathbf{I}$ provided that $\mathbf{A}_1$ has unequal eigen-values.

Again, if $\mathbf{T}$ has equal eigen-values, we can obtain a representation of $\mathbf{T}$ in terms of the unit matrix $\mathbf{I}$. Thus,

$$\beta' \mathbf{T} + \beta'_1 \mathbf{I} = 0, \quad (23.8)$$

where

$$\beta' = 2K' \quad \text{and} \quad \beta'_1 = -K' \operatorname{tr} \mathbf{T}, \quad (23.9)$$

K' being an arbitrary non-zero constant.

We note that the relation (23.8) may be obtained from (23.1) by introducing $\bar{t}_{11} = \bar{t}_{22}$, expressing the equality of the eigen-values of $\mathbf{T}$, into the expressions (23.4) or (23.6) for β , β_1 and β_2 . Further, it can be shown that the relation (23.1), with β , β_1 and β_2 given by (23.6), may be obtained by introducing $\bar{t}_{12} = 0$, expressing the coincidence of the principal axes of $\mathbf{T}$ and $\mathbf{A}_1$, into the expressions (22.10) for α , α_1 , α_2 and α_3 .

24. A representation of a symmetric function of R symmetric 3×3 matrices.

It is seen from §21 that any symmetric 3×3 matrix $\mathbf{T}$ ($= \| t_{ij} \|$) which is a function of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ may be expressed as a linear combination of any six linearly independent symmetric matrix functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. In particular, we may express $\mathbf{T}$ as a linear combination of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^2, \mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1, \mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2$ and $\mathbf{I}$ (the unit matrix), provided that

these six matrices are linearly independent. From §20, we readily see that a necessary and sufficient condition for linear independence of these six matrices is

$$\det \mathbf{B} \neq 0, \quad (24.1)$$

where

$\mathbf{B} =$

$$\begin{vmatrix} a_{11}^{(1)}, & a_{11}^{(2)}, & a_{1j}^{(1)} a_{j1}^{(1)}, & 2a_{1j}^{(1)} a_{j1}^{(2)}, & 2a_{1j}^{(1)} a_{jk}^{(1)} a_{k1}^{(2)}, & 1 \\ a_{22}^{(1)}, & a_{22}^{(2)}, & a_{2j}^{(1)} a_{j2}^{(1)}, & 2a_{2j}^{(1)} a_{j2}^{(2)}, & 2a_{2j}^{(1)} a_{jk}^{(1)} a_{k2}^{(2)}, & 1 \\ a_{33}^{(1)}, & a_{33}^{(2)}, & a_{3j}^{(1)} a_{j3}^{(1)}, & 2a_{3j}^{(1)} a_{j3}^{(2)}, & 2a_{3j}^{(1)} a_{jk}^{(1)} a_{k3}^{(2)}, & 1 \\ a_{23}^{(1)}, & a_{23}^{(2)}, & a_{2j}^{(1)} a_{j3}^{(1)}, & a_{2j}^{(1)} a_{j3}^{(2)} + a_{3j}^{(1)} a_{j2}^{(2)}, & a_{2j}^{(1)} a_{jk}^{(1)} a_{k3}^{(2)} + a_{2j}^{(2)} a_{jk}^{(1)} a_{k3}^{(1)}, & 0 \\ a_{31}^{(1)}, & a_{31}^{(2)}, & a_{3j}^{(1)} a_{j1}^{(1)}, & a_{3j}^{(1)} a_{j1}^{(2)} + a_{1j}^{(1)} a_{j3}^{(2)}, & a_{3j}^{(1)} a_{jk}^{(1)} a_{k1}^{(2)} + a_{3j}^{(2)} a_{jk}^{(1)} a_{k1}^{(1)}, & 0 \\ a_{12}^{(1)}, & a_{12}^{(2)}, & a_{1j}^{(1)} a_{j2}^{(1)}, & a_{1j}^{(1)} a_{j2}^{(2)} + a_{2j}^{(1)} a_{j1}^{(2)}, & a_{1j}^{(1)} a_{jk}^{(1)} a_{k2}^{(2)} + a_{1j}^{(2)} a_{jk}^{(1)} a_{k2}^{(1)}, & 0 \end{vmatrix}. \quad (24.2)$$

We may re-write condition (24.1) as

$$\det \mathbf{B}^* \mathbf{D} = 8 (\det \mathbf{B})^2 \neq 0, \quad (24.3)$$

where $\mathbf{D}$ is the matrix obtained by multiplying each of the last three rows of $\mathbf{B}$ by 2 and $\mathbf{B}^*$ is the transpose of $\mathbf{B}$. Denoting $\mathbf{B}^* \mathbf{D}$ by $\mathbf{E}$, we have

$$\mathbf{E} = \mathbf{B}^* \mathbf{D} = \begin{vmatrix} \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } \mathbf{A}_1(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2, & \text{tr } \mathbf{A}_1^2 \mathbf{A}_2, & \text{tr } \mathbf{A}_2(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } \mathbf{A}_2(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1^2 \mathbf{A}_2, & \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1^2(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } \mathbf{A}_1^2(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } \mathbf{A}_1^2 \\ \text{tr } \mathbf{A}_1(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } \mathbf{A}_2(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } \mathbf{A}_1^2(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), & \text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1)^2, & \text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1)(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & 2\text{tr } \mathbf{A}_1 \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } \mathbf{A}_2(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } \mathbf{A}_1^2(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1)(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2), & \text{tr } (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2)^2, & 2\text{tr } \mathbf{A}_1^2 \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_2, & \text{tr } \mathbf{A}_1^2, & 2\text{tr } \mathbf{A}_1 \mathbf{A}_2, & 2\text{tr } \mathbf{A}_1^2 \mathbf{A}_2, & 3 \end{vmatrix}. \quad (24.4)$$

We now choose an orthogonal transformation which transforms the matrix $\mathbf{A}_1$ into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$ and the matrix $\mathbf{A}_2$ into the symmetric matrix $\bar{\mathbf{A}}_2 (= \| \bar{a}_{ij}^{(2)} \|)$. Then, the condition (24.1) is equivalent to

$$\begin{vmatrix} 1, \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2 \\ 1, \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2 \\ 1, \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2 \end{vmatrix} \begin{vmatrix} \bar{a}_{23}^{(2)}, & \bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{23}^{(2)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\ \bar{a}_{31}^{(2)}, & \bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{31}^{(2)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\ \bar{a}_{12}^{(2)}, & \bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & \bar{a}_{12}^{(2)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2] \end{vmatrix} \neq 0, \quad (24.5)$$

that is to

$$\bar{a}_{11}^{(1)} \neq \bar{a}_{22}^{(1)} \neq \bar{a}_{33}^{(1)} \neq \bar{a}_{11}^{(1)} \quad \text{and} \quad \bar{a}_{23}^{(2)} \neq 0, \quad \bar{a}_{31}^{(2)} \neq 0, \quad \bar{a}_{12}^{(2)} \neq 0. \quad (24.6)$$

The relation (24.5) is obtained from (24.1) by replacing the elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ in (24.3) by those of $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ respectively.

If the conditions (24.6) are satisfied by $\mathbf{A}_1$ and $\mathbf{A}_2$, then we can obtain a representation of $\mathbf{T}$ of the form

$$\begin{aligned} \alpha \mathbf{T} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2 + \alpha_4 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ + \alpha_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \alpha_6 \mathbf{I} = 0, \end{aligned} \quad (24.7)$$

with $\alpha \neq 0$.

Expressions for $\alpha, \alpha_1, \dots, \alpha_6$ can be obtained apart from an arbitrary non-zero multiplier K , by solving the six scalar equations obtained from (24.7) by multiplying by $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^2, (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1), (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2)$ and $\mathbf{I}$ respectively and forming the traces of the matrix equations so obtained. We then obtain

$$\alpha = K \det \mathbf{E} = 8K (\det \mathbf{B})^2, \quad (24.8)$$

where $\mathbf{E}$ and $\mathbf{B}$ are given by (24.4) and (24.2) respectively, and

$$\alpha_i = -K \det \mathbf{E}_i \quad (i = 1, 2, \dots, 6), \quad (24.9)$$

where $\mathbf{E}_i$ is the matrix obtained by replacing the i^{th} column of $\mathbf{E}$ by the column $\text{tr } \mathbf{A}_1 \mathbf{T}, \text{tr } \mathbf{A}_2 \mathbf{T}, \text{tr } \mathbf{A}_1^2 \mathbf{T}, \text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \mathbf{T}, \text{tr } (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) \mathbf{T}, \text{tr } \mathbf{T}$.

The quantities $\alpha, \alpha_1, \dots, \alpha_6$ are thus obtained as scalar invariants formed from the elements of $\mathbf{T}, \mathbf{A}_1$ and $\mathbf{A}_2$. Hence they may be obtained in terms of scalar invariants formed from the elements of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, if $\mathbf{T}$ is assumed to be an isotropic function of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, since $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^2, \mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1, \mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2$ and $\mathbf{I}$ are also isotropic functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$.

We readily see (*cf.* §20) that

$$\det \mathbf{E}_i = 8 \det \mathbf{B} \cdot \det \mathbf{B}_i \quad (i = 1, 2, \dots, 6), \quad (24.10)$$

where $\mathbf{B}_i$ is the matrix formed from $\mathbf{B}$ by replacing its i^{th} column by the column $t_{11}, t_{22}, t_{33}, t_{23}, t_{31}, t_{12}$. We note that if the conditions (24.6) are not satisfied, so that $\det \mathbf{B} = 0$, then $\alpha, \alpha_1, \alpha_2, \dots, \alpha_6$ are all zero.

25. An alternative representation of a symmetric matrix function of R symmetric 3×3 matrices. In this section, we shall discuss a representation of the symmetric 3×3 matrix function $\mathbf{T}$, in terms of six linearly independent isotropic symmetric matrix functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, slightly different from that discussed in the previous section. We shall examine the conditions under which $\mathbf{T}$ may be expressed as a linear combination of the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1^2, \mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1, \mathbf{A}_2^2$ and $\mathbf{I}$; *i.e.* the conditions under which these six matrices are linearly independent. We readily see that this condition is

$$\det \mathbf{B}' \neq 0, \quad (25.1)$$

where $\mathbf{B}'$ is the matrix formed by replacing the fifth column of $\mathbf{B}$, defined by (24.2), by $a_{1j}^{(2)} a_{j1}^{(2)}, a_{2j}^{(2)} a_{j2}^{(2)}, a_{3j}^{(2)} a_{j3}^{(2)}, a_{2j}^{(2)} a_{j3}^{(2)}, a_{3j}^{(2)} a_{j1}^{(2)}, a_{1j}^{(2)} a_{j2}^{(2)}$. The condition (25.1) is, of course, equivalent to $\det \mathbf{E}' \neq 0$, where $\mathbf{E}'$ is obtained by replacing

the fifth row and fifth column by $\text{tr } \mathbf{A}_1 \mathbf{A}_2^2$, $\text{tr } \mathbf{A}_2^3$, $\text{tr } \mathbf{A}_1^2 \mathbf{A}_2^2$, $\text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \mathbf{A}_2^2$, $\text{tr } \mathbf{A}_2^4$, $\text{tr } \mathbf{A}_2^2$. As in the previous section, we apply an orthogonal transformation to $\mathbf{A}_1$ and $\mathbf{A}_2$ which transforms $\mathbf{A}_1$ into the diagonal matrix $\bar{\mathbf{A}}_1$ and $\mathbf{A}_2$ into the symmetric matrix $\bar{\mathbf{A}}_2$. The condition (25.1) is then seen to be equivalent to

$$\begin{vmatrix} 1, & 1, & 1 \\ \bar{a}_{11}^{(1)}, & \bar{a}_{22}^{(1)}, & \bar{a}_{33}^{(1)} \\ (\bar{a}_{11}^{(1)})^2, & (\bar{a}_{22}^{(1)})^2, & (\bar{a}_{33}^{(1)})^2 \end{vmatrix} \begin{vmatrix} \bar{a}_{23}^{(2)} & \bar{a}_{31}^{(2)} & \bar{a}_{12}^{(2)} \\ \bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}) \\ \bar{a}_{2j}^{(2)}\bar{a}_{j3}^{(2)} & \bar{a}_{3j}^{(2)}\bar{a}_{j1}^{(2)} & \bar{a}_{1j}^{(2)}\bar{a}_{j2}^{(2)} \end{vmatrix} \\ \neq 0. \quad (25.2)$$

This is equivalent to the conditions

$$\bar{a}_{11}^{(1)} \neq \bar{a}_{22}^{(1)} \neq \bar{a}_{33}^{(1)} \neq \bar{a}_{11}^{(1)}$$

and

$$\begin{vmatrix} \bar{a}_{23}^{(2)}, & \bar{a}_{31}^{(2)}, & \bar{a}_{12}^{(2)} \\ \bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}) \\ \bar{a}_{2j}^{(2)}\bar{a}_{j3}^{(2)}, & \bar{a}_{3j}^{(2)}\bar{a}_{j1}^{(2)}, & \bar{a}_{1j}^{(2)}\bar{a}_{j2}^{(2)} \end{vmatrix} \neq 0. \quad (25.3)$$

If the conditions (25.3) are satisfied, then we may write

$$\alpha' \mathbf{T} + \alpha'_1 \mathbf{A}_1 + \alpha'_2 \mathbf{A}_2 + \alpha'_3 \mathbf{A}_1^2 + \alpha'_4 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \alpha'_5 \mathbf{A}_2^2 + \alpha'_6 \mathbf{I} = 0, \quad (25.4)$$

with $\alpha' \neq 0$.

In this relation α' and α'_i ($i = 1, 2, \dots, 6$) are given, apart from the arbitrary non-zero multiplier K' , by

$$\alpha' = K' \det \mathbf{E}' \quad \text{and} \quad \alpha'_i = -K' \det \mathbf{E}'_i, \quad (25.5)$$

where $\mathbf{E}'_i$ is the matrix obtained by replacing the i^{th} column of $\mathbf{E}'$ by the column $\text{tr } \mathbf{A}_1 \mathbf{T}$, $\text{tr } \mathbf{A}_2 \mathbf{T}$, $\text{tr } \mathbf{A}_1^2 \mathbf{T}$, $\text{tr } (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \mathbf{T}$, $\text{tr } \mathbf{A}_2^2 \mathbf{T}$, $\text{tr } \mathbf{T}$.

Again, we see that

$$\det \mathbf{E}' = 8 (\det \mathbf{B}')^2 \quad \text{and} \quad \det \mathbf{E}'_i = 8 \det \mathbf{B}' \cdot \det \mathbf{B}'_i. \quad (25.6)$$

$\mathbf{B}'$ is defined earlier in this section and $\mathbf{B}'_i$ is the matrix formed by replacing the i^{th} column of $\mathbf{B}'$ by the column t_{11} , t_{22} , t_{33} , t_{23} , t_{31} , t_{12} .

26. The removal of a degeneracy in the representation. We have seen in §24 that if and only if the conditions (24.6) are satisfied, we may obtain a representation for $\mathbf{T}$ of the form (24.7) in terms of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_1^2$, $\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1$, $\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2$ and $\mathbf{I}$. The coefficients α , α_i ($i = 1, 2, \dots, 6$) in this representation are then given by equations (24.8) and (24.9). If the conditions (24.6) are not satisfied then $\alpha = \alpha_i = 0$ in this representation.

Again, we have seen in §25 that if and only if the conditions (25.3) are satisfied, we may obtain a representation of $\mathbf{T}$ in terms of the matrices, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_1^2$, $\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1$, $\mathbf{A}_2^2$ and $\mathbf{I}$ of the form (25.4). The coefficients α' , α'_i ($i = 1, 2, \dots, 6$) in this representation are then given by equations (25.5). If the conditions (25.3) are not satisfied, then $\alpha' = \alpha'_i = 0$ in this representation.

We readily see from (24.7) and (25.4) that

$$\begin{aligned} (\alpha + \alpha')\mathbf{T} + (\alpha_1 + \alpha'_1)\mathbf{A}_1 + (\alpha_2 + \alpha'_2)\mathbf{A}_2 + (\alpha_3 + \alpha'_3)\mathbf{A}_1^2 \\ + (\alpha_4 + \alpha'_4)(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \alpha_5(\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) \\ + \alpha'_5 \mathbf{A}_2^2 + (\alpha_6 + \alpha'_6)\mathbf{I} = 0. \end{aligned} \quad (26.1)$$

This expression affords a representation for $\mathbf{T}$ unless $\alpha + \alpha' = 0$. Now we see from the expressions (24.8), (25.5) and (25.6), that, provided we take the arbitrary multipliers K and K' to have the same sign and assume that the elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ are real, $\alpha + \alpha' = 0$ if and only if $\alpha = 0$ and $\alpha' = 0$. Thus $\mathbf{T}$ affords a representation of $\mathbf{T}$ if and only if the conditions (24.6) and (25.3) are simultaneously satisfied. In this representation we have

$$\begin{aligned} \alpha = K \det \mathbf{E}, \quad \alpha' = K' \det \mathbf{E}', \quad \alpha_i = -K \det \mathbf{E}_i \quad \text{and} \quad \alpha'_i = -K' \det \mathbf{E}'_i \\ (i = 1, 2, \dots, 6), \end{aligned} \quad (26.2)$$

where K and K' are arbitrary constants having the same sign. We may without loss of generality take them both positive.

We shall now examine further the conditions under which (26.1) does not provide a representation of $\mathbf{T}$. This occurs if the conditions (24.6) and (25.3) are simultaneously disobeyed; *i.e.* if either of the following two situations arise:

- (i) $\mathbf{A}_1$ has a pair of equal eigen-values; or
- (ii) one or more of the off-diagonal components of $\bar{\mathbf{A}}_2$ is zero and

$$\begin{vmatrix} \bar{a}_{23}^{(2)}, & \bar{a}_{31}^{(2)}, & \bar{a}_{12}^{(2)} \\ \bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}) \\ \bar{a}_{2j}^{(2)}\bar{a}_{j3}^{(2)}, & \bar{a}_{3j}^{(2)}\bar{a}_{j1}^{(2)}, & \bar{a}_{1j}^{(2)}\bar{a}_{j2}^{(2)} \end{vmatrix} = 0. \quad (26.3)$$

Let us assume, for the moment, that no two eigen-values of $\mathbf{A}_1$ are equal and that one of the off-diagonal components of $\bar{\mathbf{A}}_2$, $\bar{a}_{12}^{(2)}$ say, is zero. Then, the relation (26.3) becomes

$$[\bar{a}_{13}^{(2)} \bar{a}_{23}^{(2)}]^2 (\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)}) = 0; \quad (26.4)$$

that is

$$\bar{a}_{23}^{(2)} = 0 \quad \text{or} \quad \bar{a}_{13}^{(2)} = 0. \quad (26.5)$$

If (26.5) is satisfied we see, with the assumption that $\bar{a}_{12}^{(2)} = 0$, that $\bar{\mathbf{A}}_2$ must have at least two of its off-diagonal components zero; *i.e.* $\mathbf{A}_1$ and $\mathbf{A}_2$ must have a principal direction in common.

It is thus seen that (26.1) provides a representation for $\mathbf{T}$ unless $\mathbf{A}_1$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have a principal direction in common.

27. Removal of a further degeneracy in the representation. It has been seen in the previous section that equation (26.1) provides a representation for $\mathbf{T}$ provided that no two eigen-values of $\mathbf{A}_1$ are equal and $\mathbf{A}_1$ and $\mathbf{A}_2$ do not have a principal direction in common. In this section it will be shown how the first of these two restrictions can be replaced by a weaker restriction. Writing

$$\begin{aligned}\beta &= \alpha + \alpha', & \beta_1 &= \alpha_1 + \alpha'_1, & \beta_2 &= \alpha_2 + \alpha'_2, & \beta_3 &= \alpha_3 + \alpha'_3, \\ \beta_4 &= \beta_4 + \beta'_4, & \beta_5 &= \alpha_5, & \beta_6 &= \alpha'_5, & \beta_7 &= \alpha_6 + \alpha'_6,\end{aligned}\quad (27.1)$$

in equation (26.1), we have

$$\begin{aligned}\beta \mathbf{T} + \beta_1 \mathbf{A}_1 + \beta_2 \mathbf{A}_2 + \beta_3 \mathbf{A}_1^2 + \beta_4 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) \\ + \beta_6 \mathbf{A}_2^2 + \beta_7 \mathbf{I} = 0,\end{aligned}\quad (27.2)$$

where

$$\begin{aligned}\beta &= K \det \mathbf{E} + K' \det \mathbf{E}', & \beta_i &= -K \det \mathbf{E}_i - K' \det \mathbf{E}'_i \\ (i &= 1, 2, 3, 4),\end{aligned}\quad (27.3)$$

$$\beta_5 = -K \det \mathbf{E}_5, \quad \beta_6 = -K' \det \mathbf{E}'_5 \quad \text{and} \quad \beta_7 = -K \det \mathbf{E}_6 - K' \det \mathbf{E}'_6.$$

Now, we define the matrices $\mathbf{F}$, $\mathbf{F}_i$, $\mathbf{F}'$ and $\mathbf{F}'_i$ as the matrices obtained by interchanging $\mathbf{A}_1$ and $\mathbf{A}_2$ in the matrices $\mathbf{E}$, $\mathbf{E}_i$, $\mathbf{E}'$ and $\mathbf{E}'_i$ respectively. We can then write by analogy with (27.2)

$$\begin{aligned}\beta' \mathbf{T} + \beta'_1 \mathbf{A}_2 + \beta'_2 \mathbf{A}_1 + \beta'_3 \mathbf{A}_2^2 + \beta'_4 (\mathbf{A}_2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2) + \beta'_5 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) \\ + \beta'_6 \mathbf{A}_1^2 + \beta'_7 \mathbf{I} = 0,\end{aligned}\quad (27.4)$$

where

$$\begin{aligned}\beta' &= L \det \mathbf{F} + L' \det \mathbf{F}', & \beta_i &= -L \det \mathbf{F}_i - L' \det \mathbf{F}'_i \\ (i &= 1, 2, 3, 4),\end{aligned}\quad (27.5)$$

$$\beta_5 = -L \det \mathbf{F}_5, \quad \beta_6 = -L' \det \mathbf{F}'_5 \quad \text{and} \quad \beta_7 = -L \det \mathbf{F}_6 - L' \det \mathbf{F}'_6.$$

L and L' are arbitrary non-zero multipliers, which we shall take as having the same sign as each other and as K and K' . Equation (27.4) provides a representation for $\mathbf{T}$ provided that no two eigen-values of $\mathbf{A}_2$ are equal and $\mathbf{A}_1$ and $\mathbf{A}_2$ do not have a principal direction in common. Adding (27.2) and (27.4), we

obtain

$$\begin{aligned}
 (\beta + \beta')\mathbf{T} + (\beta_1 + \beta'_1)\mathbf{A}_1 + (\beta_2 + \beta'_1)\mathbf{A}_2 + (\beta_3 + \beta'_1)\mathbf{A}_1^2 + (\beta_6 + \beta'_1)\mathbf{A}_2^2 \\
 + (\beta_4 + \beta'_4)(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_5(\mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1^2) \\
 + \beta'_5(\mathbf{A}_2^2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2^2) + (\beta_7 + \beta'_7)\mathbf{I} = 0. \quad (27.6)
 \end{aligned}$$

Equation (27.6) provides a representation for $\mathbf{T}$ unless $\beta + \beta' = 0$. We have already seen that $\det \mathbf{E}$ and $\det \mathbf{E}'$ are positive if the elements of $\mathbf{A}_1$ are real. Analogously $\det \mathbf{F}$ and $\det \mathbf{F}'$ are positive if the elements of $\mathbf{A}_2$ are real. Since K , K' , L and L' all have the same sign, we see that β and β' have the same sign, so that $\beta + \beta' = 0$ if and only if $\beta = \beta' = 0$. Thus (27.6) provides a representation for $\mathbf{T}$ unless each of $\mathbf{A}_1$ and $\mathbf{A}_2$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have a principal direction in common.

28. A further representation. It is readily seen that a linear relation can be established between the seven symmetric 3×3 matrices $\mathbf{T}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_2^2$, $\mathbf{A}_3$, $\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_2$ and $\mathbf{I}$ of the form

$$\gamma\mathbf{T} + \gamma_1\mathbf{A}_1 + \gamma_2\mathbf{A}_2 + \gamma_3\mathbf{A}_2^2 + \gamma_4\mathbf{A}_3 + \gamma_5(\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_2) + \gamma_6\mathbf{I} = 0. \quad (28.1)$$

This provides a representation for $\mathbf{T}$ provided that $\gamma \neq 0$, *i.e.* the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_2^2$, $\mathbf{A}_3$, $\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_2$ and $\mathbf{I}$ are linearly independent. This condition may be written as

$$\det \mathbf{B}'' \neq 0, \quad (28.2)$$

where

$$\mathbf{B}'' = \begin{vmatrix} a_{11}^{(1)}, & a_{11}^{(2)}, & a_{1j}^{(2)} a_{j1}^{(2)}, & a_{11}^{(3)}, & 2a_{1j}^{(2)} a_{j1}^{(3)}, & 1 \\ a_{22}^{(1)}, & a_{22}^{(2)}, & a_{2j}^{(2)} a_{j2}^{(2)}, & a_{22}^{(3)}, & 2a_{2j}^{(2)} a_{j2}^{(3)}, & 1 \\ a_{33}^{(1)}, & a_{33}^{(2)}, & a_{3j}^{(2)} a_{j3}^{(2)}, & a_{33}^{(3)}, & 2a_{3j}^{(2)} a_{j3}^{(3)}, & 1 \\ a_{23}^{(1)}, & a_{23}^{(2)}, & a_{2j}^{(2)} a_{j3}^{(2)}, & a_{23}^{(3)}, & a_{2j}^{(2)} a_{j3}^{(3)} + a_{2j}^{(3)} a_{j3}^{(2)}, & 0 \\ a_{31}^{(1)}, & a_{31}^{(2)}, & a_{3j}^{(2)} a_{j1}^{(2)}, & a_{31}^{(3)}, & a_{3j}^{(2)} a_{j1}^{(3)} + a_{3j}^{(3)} a_{j1}^{(2)}, & 0 \\ a_{12}^{(1)}, & a_{12}^{(2)}, & a_{1j}^{(2)} a_{j2}^{(2)}, & a_{12}^{(3)}, & a_{1j}^{(2)} a_{j2}^{(3)} + a_{1j}^{(3)} a_{j2}^{(2)}, & 0 \end{vmatrix}. \quad (28.3)$$

The coefficients γ, γ_i ($i = 1, 2, \dots, 6$) in (28.1) can be determined in a similar manner to that employed in previous sections as

$$\gamma = K'' \det \mathbf{E}'' \quad \text{and} \quad \gamma_i = -K'' \det \mathbf{E}_i'', \quad (28.4)$$

where K'' is an arbitrary non-zero multiplier and $\mathbf{E}''$ is given by

$$\mathbf{E}'' = \begin{vmatrix} \text{tr } \mathbf{A}_1^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_1 \mathbf{A}_2^2, & \text{tr } \mathbf{A}_1 \mathbf{A}_3, & \text{tr } \mathbf{A}_1 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2, & \text{tr } \mathbf{A}_2^3, & \text{tr } \mathbf{A}_2 \mathbf{A}_3, & \text{tr } \mathbf{A}_2 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_2^2, & \text{tr } \mathbf{A}_2^3, & \text{tr } \mathbf{A}_2^4, & \text{tr } \mathbf{A}_2^2 \mathbf{A}_3, & \text{tr } \mathbf{A}_2^2 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_2^2 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_3, & \text{tr } \mathbf{A}_2 \mathbf{A}_3, & \text{tr } \mathbf{A}_2^2 \mathbf{A}_3, & \text{tr } \mathbf{A}_3^2, & \text{tr } \mathbf{A}_3 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_3 \\ \text{tr } \mathbf{A}_1 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_2 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_2^2 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } \mathbf{A}_3 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & \text{tr } (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2)^2, & \text{tr } (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2) \\ \text{tr } \mathbf{A}_1, & \text{tr } \mathbf{A}_2, & \text{tr } \mathbf{A}_2^2, & \text{tr } \mathbf{A}_3, & \text{tr } (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2), & 3 \end{vmatrix}. \quad (28.5)$$

$\mathbf{E}_i''$ is obtained from $\mathbf{E}''$ by replacing the i^{th} column of $\mathbf{E}''$ by the column $\text{tr } \mathbf{A}_1 \mathbf{T}$, $\text{tr } \mathbf{A}_2 \mathbf{T}$, $\text{tr } \mathbf{A}_2^2 \mathbf{T}$, $\text{tr } \mathbf{A}_3 \mathbf{T}$, $\text{tr}(\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2) \mathbf{T}$, $\text{tr } \mathbf{T}$. It can readily be seen that we have the relations

$$\det \mathbf{E}'' = 8 (\det \mathbf{B}'')^2 \quad \text{and} \quad \det \mathbf{E}_i'' = 8 \det \mathbf{B}'' \det \mathbf{B}_i'', \quad (28.6)$$

where $\mathbf{B}_i''$ is obtained from $\mathbf{B}''$ by replacing the i^{th} column by the column t_{11} , t_{22} , t_{33} , t_{23} , t_{31} , t_{12} .

From equations (28.1) and (27.6), we have

$$\begin{aligned} &(\beta + \beta' + \gamma) \mathbf{T} + (\beta_1 + \beta'_1 + \gamma_1) \mathbf{A}_1 + (\beta_2 + \beta'_2 + \gamma_2) \mathbf{A}_2 + (\beta_3 + \beta'_3) \mathbf{A}_1^2 \\ &\quad + (\beta_6 + \beta'_3 + \gamma_3) \mathbf{A}_2^2 + (\beta_4 + \beta'_4) (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ &\quad + \beta_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \beta'_5 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) \\ &\quad + \gamma_4 \mathbf{A}_3 + \gamma_5 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2) + (\beta_7 + \beta'_7 + \gamma_6) \mathbf{I} = 0, \end{aligned} \quad (28.7)$$

where β , β' , β_i and β'_i are defined by (27.3) and (27.5). If now we take K'' to have the same sign as K , K' , L and L' , then, we see that β , β' and γ all have the same sign provided that the elements of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are real. Thus, $\beta + \beta' + \gamma = 0$ if and only if $\beta = \beta' = \gamma = 0$. We have already seen that $\beta = \beta' = 0$ if and only if $\mathbf{A}_1$ and $\mathbf{A}_2$ each have a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have a common principal direction. The condition $\gamma = 0$ further requires that

$$\det \mathbf{B}'' = 0. \quad (28.8)$$

Now, we can choose an orthogonal transformation which transforms $\mathbf{A}_1$ into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$. Let $\bar{\mathbf{A}}_2 (= \|\bar{a}_{ij}^{(2)}\|)$ and $\bar{\mathbf{A}}_3 (= \|\bar{a}_{ij}^{(3)}\|)$ be the symmetric matrices into which $\mathbf{A}_2$ and $\mathbf{A}_3$ are transformed by this orthogonal transformation. Then, the condition (28.8) can be re-written in terms of the elements of $\bar{\mathbf{A}}_1$, $\bar{\mathbf{A}}_2$ and $\bar{\mathbf{A}}_3$ as

$$\det \bar{\mathbf{B}}'' = 0, \quad (28.9)$$

where $\bar{\mathbf{B}}''$ is the matrix formed from $\mathbf{B}''$, defined in (28.3), by replacing the elements of $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ by those of $\bar{\mathbf{A}}_1$, $\bar{\mathbf{A}}_2$ and $\bar{\mathbf{A}}_3$. Thus, (28.7) provides a representation for $\mathbf{T}$ unless (28.9) is satisfied and either each of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have a common principal direction.

29. A representation of a symmetric matrix function of R symmetric 3×3 matrices in certain degenerate cases.

(i) $\mathbf{T}$, $\mathbf{A}_1 \mathbf{A}_2$ have a principal direction in common.

By a suitable orthogonal transformation, $\mathbf{A}_1$ may be transformed into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$ and $\mathbf{T}$ and $\mathbf{A}_2$ may be transformed into the symmetric matrices $\bar{\mathbf{T}} (= \|\bar{t}_{ij}\|)$ and $\bar{\mathbf{A}}_2 (= \|\bar{a}_{ij}^{(2)}\|)$, in which $\bar{t}_{12} = \bar{t}_{13} = \bar{a}_{12}^{(2)} = \bar{a}_{13}^{(2)} = 0$. We may then establish a linear relation between the matrices $\bar{\mathbf{T}}$, $\bar{\mathbf{A}}_1$, $\bar{\mathbf{A}}_1^2$, $\bar{\mathbf{A}}_2$ and $\mathbf{I}$, thus:

$$\epsilon \bar{\mathbf{T}} + \epsilon_1 \bar{\mathbf{A}}_1 + \epsilon_2 \bar{\mathbf{A}}_1^2 + \epsilon_3 \bar{\mathbf{A}}_2 + \epsilon_4 \mathbf{I} = 0. \quad (29.1)$$

That this is the case may readily be seen when we realize that equation (29.1) represents the four equations

$$\begin{aligned} \epsilon \bar{t}_{11} + \epsilon_1 \bar{a}_{11}^{(1)} + \epsilon_2 (\bar{a}_{11}^{(1)})^2 + \epsilon_3 \bar{a}_{11}^{(2)} + \epsilon_4 &= 0, \\ \epsilon \bar{t}_{22} + \epsilon_1 \bar{a}_{22}^{(1)} + \epsilon_2 (\bar{a}_{22}^{(1)})^2 + \epsilon_3 \bar{a}_{22}^{(2)} + \epsilon_4 &= 0, \\ \epsilon \bar{t}_{33} + \epsilon_1 \bar{a}_{33}^{(1)} + \epsilon_2 (\bar{a}_{33}^{(1)})^2 + \epsilon_3 \bar{a}_{33}^{(2)} + \epsilon_4 &= 0 \end{aligned} \quad (29.2)$$

and

$$\epsilon \bar{t}_{23} + \epsilon_3 \bar{a}_{23}^{(2)} = 0,$$

from which ϵ , ϵ_1 , ϵ_2 , ϵ_3 and ϵ_4 can be calculated apart from an arbitrary non-zero multiplying constant. We obtain

$$\epsilon = K \det \bar{\mathbf{B}} \quad \text{and} \quad \epsilon_i = -K \det \bar{\mathbf{B}}_i \quad (i = 1, 2, 3, 4), \quad (29.3)$$

where

$$\bar{\mathbf{B}} = \begin{vmatrix} \bar{a}_{11}^{(1)} & (\bar{a}_{11}^{(1)})^2 & \bar{a}_{11}^{(2)} & 1 \\ \bar{a}_{22}^{(1)} & (\bar{a}_{22}^{(1)})^2 & \bar{a}_{22}^{(2)} & 1 \\ \bar{a}_{33}^{(1)} & (\bar{a}_{33}^{(1)})^2 & \bar{a}_{33}^{(2)} & 1 \\ 0 & 0 & \bar{a}_{23}^{(2)} & 0 \end{vmatrix}, \quad (29.4)$$

K is an arbitrary non-zero constant and $\bar{\mathbf{B}}_i$ is obtained from $\bar{\mathbf{B}}$ by replacing the i^{th} column of $\bar{\mathbf{B}}$ by $\bar{t}_{11}$, $\bar{t}_{22}$, $\bar{t}_{33}$, $\bar{t}_{23}$.

Alternatively, we may obtain ϵ , ϵ_1 , ϵ_2 , ϵ_3 , ϵ_4 directly in terms of scalar invariants of $\bar{\mathbf{T}}$, $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ in a manner similar to that employed in previous sections. We then have

$$\epsilon = K \det \bar{\mathbf{E}} = 2K (\det \bar{\mathbf{B}})^2$$

and

$$\epsilon_i = -K \det \bar{\mathbf{E}}_i = -2K \det \bar{\mathbf{B}} \det \bar{\mathbf{B}}_i \quad (i = 1, 2, 3, 4), \quad (29.5)$$

where

$$\bar{\mathbf{E}} = \begin{vmatrix} \text{tr } \bar{\mathbf{A}}_1^2 & \text{tr } \bar{\mathbf{A}}_1^3 & \text{tr } \bar{\mathbf{A}}_1 \bar{\mathbf{A}}_2 & \text{tr } \bar{\mathbf{A}}_1 \\ \text{tr } \bar{\mathbf{A}}_1^3 & \text{tr } \bar{\mathbf{A}}_1^4 & \text{tr } \bar{\mathbf{A}}_1^2 \bar{\mathbf{A}}_2 & \text{tr } \bar{\mathbf{A}}_1^2 \\ \text{tr } \bar{\mathbf{A}}_1 \bar{\mathbf{A}}_2 & \text{tr } \bar{\mathbf{A}}_1^2 \bar{\mathbf{A}}_2 & \text{tr } \bar{\mathbf{A}}_2^2 & \text{tr } \bar{\mathbf{A}}_2 \\ \text{tr } \bar{\mathbf{A}}_1 & \text{tr } \bar{\mathbf{A}}_1^2 & \text{tr } \bar{\mathbf{A}}_2 & 3 \end{vmatrix}, \quad (29.6)$$

K is an arbitrary non-zero constant and $\bar{\mathbf{E}}_i$ is obtained from $\bar{\mathbf{E}}$ by replacing the i^{th} column of $\bar{\mathbf{E}}$ by $\text{tr } \bar{\mathbf{A}}_1 \mathbf{T}$, $\text{tr } \bar{\mathbf{A}}_1^2 \mathbf{T}$, $\text{tr } \bar{\mathbf{A}}_2 \mathbf{T}$, $\text{tr } \mathbf{T}$. We readily see, in view of relations of the type $\text{tr } \bar{\mathbf{T}} = \text{tr } \mathbf{T}$, $\text{tr } \bar{\mathbf{A}}_1 = \text{tr } \mathbf{A}_1$, $\text{tr } \bar{\mathbf{A}}_1 \bar{\mathbf{T}} = \text{tr } \mathbf{A}_1 \mathbf{T}$, *etc.*, that

$$\epsilon = K \det \mathbf{E} \quad \text{and} \quad \epsilon_i = -K \det \mathbf{E}_i \quad (i = 1, 2, 3, 4), \quad (29.7)$$

where

$$\mathbf{E} = \begin{vmatrix} \text{tr } \mathbf{A}_1^2 & \text{tr } \mathbf{A}_1^3 & \text{tr } \mathbf{A}_1 \mathbf{A}_2 & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_1^3 & \text{tr } \mathbf{A}_1^4 & \text{tr } \mathbf{A}_1^2 \mathbf{A}_2 & \text{tr } \mathbf{A}_1^2 \\ \text{tr } \mathbf{A}_1 \mathbf{A}_2 & \text{tr } \mathbf{A}_1^2 \mathbf{A}_2 & \text{tr } \mathbf{A}_2^2 & \text{tr } \mathbf{A}_2 \\ \text{tr } \mathbf{A}_1 & \text{tr } \mathbf{A}_1^2 & \text{tr } \mathbf{A}_2 & 3 \end{vmatrix}, \quad (29.8)$$

and $\mathbf{E}_i$ is obtained from $\mathbf{E}$ by replacing the i^{th} column of $\mathbf{E}$ by $\text{tr } \mathbf{A}_1 \mathbf{T}$, $\text{tr } \mathbf{A}_1^2 \mathbf{T}$, $\text{tr } \mathbf{A}_2 \mathbf{T}$, $\text{tr } \mathbf{T}$.

From (29.1) we see that

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_2 \mathbf{A}_1^2 + \epsilon_3 \mathbf{A}_2 + \epsilon_4 \mathbf{I} = 0, \quad (29.9)$$

where ϵ , ϵ_1 , ϵ_2 , ϵ_3 and ϵ_4 are given by (29.7). This provides a representation for $\mathbf{T}$ if and only if $\epsilon \neq 0$, *i.e.* if and only if

$$\det \bar{\mathbf{B}} \neq 0. \quad (29.10)$$

It is readily seen that the condition (29.10) is equivalent to

$$\bar{a}_{11}^{(1)} \neq \bar{a}_{22}^{(1)} \neq \bar{a}_{33}^{(1)} \neq \bar{a}_{11}^{(1)} \quad \text{and} \quad \bar{a}_{23}^{(1)} \neq 0. \quad (29.11)$$

Thus, (29.9) provides a representation for $\mathbf{T}$ if and only if $\mathbf{A}_1$ has unequal eigen-values and $\mathbf{A}_1$ and $\mathbf{A}_2$ have only one principal direction in common.

In a similar manner we can obtain a representation for $\mathbf{T}$ in terms of the matrices $\mathbf{A}_2$, $\mathbf{A}_2^2$, $\mathbf{A}_1$ and $\mathbf{I}$ of the form

$$\epsilon' \mathbf{T} + \epsilon'_1 \mathbf{A}_2 + \epsilon'_2 \mathbf{A}_2^2 + \epsilon'_3 \mathbf{A}_1 + \epsilon'_4 \mathbf{I} = 0, \quad (29.12)$$

with

$$\epsilon = K' \det \mathbf{E}' \quad \text{and} \quad \epsilon'_i = -K' \det \mathbf{E}'_i, \quad (29.13)$$

where $\mathbf{E}'$ and $\mathbf{E}'_i$ are obtained from $\mathbf{E}$ and $\mathbf{E}_i$ respectively, defined in (29.8), by interchanging $\mathbf{A}_1$ and $\mathbf{A}_2$ and K' is an arbitrary non-zero constant, which we will take as having the same sign as K . (29.12) provides a representation for $\mathbf{T}$ unless $\mathbf{A}_2$ has a pair of eigen-values equal or $\mathbf{A}_1$ and $\mathbf{A}_2$ have all their principal directions in common.

Adding (29.9) and (29.12), we obtain

$$(\epsilon + \epsilon') \mathbf{T} + (\epsilon_1 + \epsilon'_1) \mathbf{A}_1 + (\epsilon_3 + \epsilon'_3) \mathbf{A}_2 + \epsilon_2 \mathbf{A}_1^2 + \epsilon'_2 \mathbf{A}_2^2 + (\epsilon_4 + \epsilon'_4) \mathbf{I} = 0, \quad (29.14)$$

which provides a representation for $\mathbf{T}$ unless each of $\mathbf{A}_1$ and $\mathbf{A}_2$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have all their principal directions in common.

(ii) $\mathbf{T}$ and $\mathbf{A}_1$ have all their principal directions in common.

By a suitable orthogonal transformation, $\mathbf{T}$ and $\mathbf{A}_1$ may be simultaneously

transformed into diagonal matrices $\bar{\mathbf{T}} = (\bar{t}_{11}, \bar{t}_{22}, \bar{t}_{33})$ and $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$. We can now establish a linear relation

$$\epsilon \bar{\mathbf{T}} + \epsilon_1 \bar{\mathbf{A}}_1 + \epsilon_2 \bar{\mathbf{A}}_1^2 + \epsilon_4 \mathbf{I} = 0 \quad (29.15)$$

between the matrices $\bar{\mathbf{T}}$, $\bar{\mathbf{A}}_1$, $\bar{\mathbf{A}}_1^2$ and $\mathbf{I}$ and hence the linear relation

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_2 \mathbf{A}_1^2 + \epsilon_4 \mathbf{I} = 0, \quad (29.16)$$

between the matrices $\mathbf{T}$, $\mathbf{A}_1$, $\mathbf{A}_1^2$ and $\mathbf{I}$. We have

$$\epsilon = K \det \mathbf{E} = K(\det \bar{\mathbf{B}})^2$$

and

$$\epsilon_i = -K \det \mathbf{E}_i = -K \det \bar{\mathbf{B}} \det \bar{\mathbf{B}}_i \quad (i = 1, 2, 4), \quad (29.17)$$

where K is an arbitrary non-zero constant and $\mathbf{E}$ and $\bar{\mathbf{B}}$ are defined by

$$\mathbf{E} = \begin{vmatrix} \text{tr } \mathbf{A}_1^2 & \text{tr } \mathbf{A}_1^3 & \text{tr } \mathbf{A}_1 \\ \text{tr } \mathbf{A}_1^3 & \text{tr } \mathbf{A}_1^4 & \text{tr } \mathbf{A}_1^2 \\ \text{tr } \mathbf{A}_1 & \text{tr } \mathbf{A}_1^2 & 3 \end{vmatrix} \quad \text{and} \quad \bar{\mathbf{B}} = \begin{vmatrix} \bar{a}_{11}^{(1)} & (\bar{a}_{11}^{(1)})^2 & 1 \\ \bar{a}_{22}^{(1)} & (\bar{a}_{22}^{(1)})^2 & 1 \\ \bar{a}_{33}^{(1)} & (\bar{a}_{33}^{(1)})^2 & 1 \end{vmatrix}. \quad (29.18)$$

The matrices $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_4$ are defined to be those obtained by replacing the first, second and third columns of $\mathbf{E}$ respectively by $\text{tr } \mathbf{A}_1 \mathbf{T}$, $\text{tr } \mathbf{A}_1^2 \mathbf{T}$, $\text{tr } \mathbf{T}$ and $\bar{\mathbf{B}}_1$, $\bar{\mathbf{B}}_2$, $\bar{\mathbf{B}}_4$ are defined as the matrices obtained by replacing the first, second and third columns respectively of $\bar{\mathbf{B}}$ by $\bar{t}_{11}$, $\bar{t}_{22}$, $\bar{t}_{33}$.

If $\mathbf{T}$ and $\mathbf{A}_1$ have all their principal directions in common, then (29.16) provides a representation for $\mathbf{T}$ unless $\epsilon = 0$, *i.e.* $\det \bar{\mathbf{B}} = 0$. This occurs only if $\mathbf{A}_1$ has a pair of equal eigen-values.

We now consider that $\mathbf{T}$ and $\mathbf{A}_1$ have all their principal directions in common and each has a pair of equal eigen-values, the principal planes corresponding to these pairs of equal eigen-values coinciding for the two matrices. Then, we have $\bar{t}_{11} = \bar{t}_{22}$ and $\bar{a}_{11}^{(1)} = \bar{a}_{22}^{(1)}$. We can now establish a linear relation of the form

$$\epsilon \bar{\mathbf{T}} + \epsilon_1 \bar{\mathbf{A}}_1 + \epsilon_4 \mathbf{I} = 0, \quad (29.19)$$

between the matrices $\bar{\mathbf{T}}$, $\bar{\mathbf{A}}_1$ and $\mathbf{I}$, and hence one of the form

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_4 \mathbf{I} = 0, \quad (29.20)$$

between the matrices $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{I}$, where ϵ , ϵ_1 and ϵ_4 are given by

$$\epsilon = K[3 \text{tr } \mathbf{A}_1^2 - (\text{tr } \mathbf{A}_1)^2], \quad \epsilon_1 = K(\text{tr } \mathbf{A}_1 \text{tr } \mathbf{T} - 3 \text{tr } \mathbf{A}_1 \mathbf{T})$$

and

$$\epsilon_4 = K(\text{tr } \mathbf{A}_1 \text{tr } \mathbf{A}_1 \mathbf{T} - \text{tr } \mathbf{A}_1^2 \text{tr } \mathbf{T}). \quad (29.21)$$

It can readily be shown that (29.20) provides a representation for $\mathbf{T}$ unless all the eigen-values of $\mathbf{A}_1$ are equal.

If all the eigen-values of $\mathbf{T}$ are equal then we obtain a representation for $\mathbf{T}$ in terms of the unit matrix $\mathbf{I}$ only, thus:

$$3\mathbf{T} - (\text{tr } \mathbf{T})\mathbf{I} = 0. \quad (29.22)$$

30. Some properties of symmetric isotropic matrix functions of R symmetric 3×3 matrices. In the previous section, we have considered representations of the symmetric isotropic 3×3 matrix function $\mathbf{T}$ of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ in certain special cases which are expressed as relations between $\mathbf{T}$ and the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. In this section it will be shown how such relations between $\mathbf{T}$ and $\mathbf{A}_1$ and $\mathbf{A}_2$ follow as a result of appropriate relations between the R matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, when $\mathbf{T}$ is an isotropic function of these.

We shall first show that if $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ have a common principal direction, then this principal direction is also a principal direction of $\mathbf{T}$.

Since $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ have a common principal direction, they can be transformed simultaneously, by an orthogonal transformation, into matrices $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_R$ respectively, having the forms

$$\bar{\mathbf{A}}_r = \begin{vmatrix} \bar{a}_{11}^{(r)} & 0 & 0 \\ 0 & \bar{a}_{22}^{(r)} & \bar{a}_{23}^{(r)} \\ 0 & \bar{a}_{23}^{(r)} & \bar{a}_{33}^{(r)} \end{vmatrix} \quad (r = 1, 2, \dots, R). \quad (30.1)$$

Let $\bar{\mathbf{T}} = \|\bar{t}_{ij}\|$, with $\bar{t}_{ij} = \bar{t}_{ji}$, be the matrix into which $\mathbf{T}$ is transformed by this orthogonal transformation.

Now, the orthogonal transformation $\mathbf{S}$ defined by

$$\mathbf{S} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

is readily seen to leave the matrices $\bar{\mathbf{A}}_r$ ($r = 1, 2, \dots, R$) unaltered. From the definition of an isotropic matrix function, it is therefore seen that

$$\bar{\mathbf{T}} = \mathbf{S}\bar{\mathbf{T}}\mathbf{S}^{-1}. \quad (30.2)$$

It is readily seen that (30.2) is satisfied if and only if $\bar{t}_{12} = -\bar{t}_{12}$ and $\bar{t}_{13} = -\bar{t}_{13}$, *i.e.* if and only if $\bar{t}_{12} = \bar{t}_{13} = 0$. Thus, $\mathbf{T}$ must have a common principal direction with $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$.

It follows immediately that if all of the principal directions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ coincide, then the principal directions of $\mathbf{T}$ also coincide with those of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$.

If, in addition, each of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ has a pair of equal eigen-values and the principal plane corresponding to this pair is the same for each of the matrices, then $\mathbf{T}$ also has a pair of equal eigen-values and the principal plane of $\mathbf{T}$ corresponding to this pair of equal eigen-values is the same as that corresponding to the pairs of equal eigen-values of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. We can express the assumptions regarding the equality of eigen-values of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ by writing $\bar{\mathbf{A}}_r = (\bar{a}_{11}^{(r)}, \bar{a}_{11}^{(r)}, \bar{a}_{33}^{(r)})$ ($r = 1, 2, \dots, R$). Since $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ have all their principal directions in common we can write $\bar{\mathbf{T}} = (\bar{t}_{11}, \bar{t}_{22}, \bar{t}_{33})$. Now $\bar{\mathbf{A}}_r$ is unaltered by any orthogonal transformation $\mathbf{S}$ of the form

$$\mathbf{S} = \begin{vmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad (30.3)$$

where

$$s_{11}^2 + s_{12}^2 = s_{21}^2 + s_{22}^2 = 1 \quad \text{and} \quad s_{11}s_{21} + s_{12}s_{22} = 0. \quad (30.4)$$

From the definition of $\mathbf{T}$ as an isotropic matrix function, we must therefore have $\bar{\mathbf{T}} = \mathbf{S}\bar{\mathbf{T}}\mathbf{S}^{-1}$, where $\mathbf{S}$ is given by (30.3). This yields

$$\bar{t}_{11} = s_{11}^2 \bar{t}_{11} + s_{12}^2 \bar{t}_{22}$$

and

$$\bar{t}_{22} = s_{21}^2 \bar{t}_{11} + s_{22}^2 \bar{t}_{22}. \quad (30.5)$$

With (30.4), we obtain

$$\bar{t}_{11} = \bar{t}_{22}; \quad (30.6)$$

i.e. $\mathbf{T}$ has a pair of eigen-values equal and the principal plane corresponding to these is the principal plane corresponding to the pairs of equal eigen-values of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$.

It readily follows that if each of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ has all three of its eigen-values equal, then $\mathbf{T}$ also has all three of its eigen-values equal.

Analogous results to those given above can be derived for 2×2 matrices. Thus, if $\mathbf{T}$ is a symmetric isotropic matrix function of R symmetric 2×2 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ and these matrices have common principal directions, it is readily seen that these principal directions are also principal directions for $\mathbf{T}$. Furthermore, if each of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ has equal eigen-values, then $\mathbf{T}$ also has equal eigen-values.

31. Summary of principal conclusions. In the foregoing sections, we have considered a symmetric 3×3 matrix $\mathbf{T}$ which is a matrix function of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. It has been assumed that all the matrices

have real elements. We shall summarize below the principal results that have been obtained regarding the relations which can be established between the matrices $\mathbf{T}$ and $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$.

(i) A relation of the form

$$\begin{aligned} \alpha \mathbf{T} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2 + \alpha_4 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ + \alpha_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \alpha_6 \mathbf{I} = 0 \end{aligned} \quad (31.1)$$

exists, in which $\alpha, \alpha_1, \alpha_2, \dots, \alpha_6$ are scalars given by equations (24.8) and (24.9). This relation provides a representation for $\mathbf{T}$, *i.e.* $\alpha \neq 0$, provided that no two eigen-values of $\mathbf{A}_1$ are equal and none of the off-diagonal components of $\bar{\mathbf{A}}_2$ is zero, where $\bar{\mathbf{A}}_2$ is the matrix into which $\mathbf{A}_2$ is transformed by an orthogonal transformation which transforms $\mathbf{A}_1$ into a diagonal matrix (see §24).

(ii) A relation of the form

$$\begin{aligned} (\alpha + \alpha') \mathbf{T} + (\alpha_1 + \alpha'_1) \mathbf{A}_1 + (\alpha_2 + \alpha'_2) \mathbf{A}_2 + (\alpha_3 + \alpha'_3) \mathbf{A}_1^2 \\ + (\alpha_4 + \alpha'_4) (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \alpha_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) \\ + \alpha'_5 \mathbf{A}_2^2 + (\alpha_6 + \alpha'_6) \mathbf{I} = 0 \end{aligned} \quad (31.2)$$

exists, in which $\alpha, \alpha_1, \alpha_2, \dots, \alpha_6$ and $\alpha', \alpha'_1, \alpha'_2, \dots, \alpha'_6$ are scalars given by equations (26.2). This relation provides a representation for $\mathbf{T}$, *i.e.* $\alpha + \alpha' \neq 0$, provided that no two eigen-values of $\mathbf{A}_1$ are equal and $\mathbf{A}_1$ and $\mathbf{A}_2$ do not have a principal direction in common (see §26).

(iii) A relation of the form

$$\begin{aligned} (\beta + \beta') \mathbf{T} + (\beta_1 + \beta'_1) \mathbf{A}_1 + (\beta_2 + \beta'_2) \mathbf{A}_2 + (\beta_3 + \beta'_3) \mathbf{A}_1^2 + (\beta_6 + \beta'_6) \mathbf{A}_2^2 \\ + (\beta_4 + \beta'_4) (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) \\ + \beta'_5 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) + (\beta_7 + \beta'_7) \mathbf{I} = 0 \end{aligned} \quad (31.3)$$

exists, in which $\beta, \beta_1, \beta_2, \dots, \beta_7$ and $\beta', \beta'_1, \beta'_2, \dots, \beta'_7$ are scalars given by equations (27.5). This relation provides a representation for $\mathbf{T}$, *i.e.* $\beta + \beta' \neq 0$, provided that $\mathbf{A}_1$ and $\mathbf{A}_2$ do not both have a pair of equal eigen-values and $\mathbf{A}_1$ and $\mathbf{A}_2$ do not have a principal direction in common (see §27).

(iv) A relation of the form

$$\begin{aligned} (\beta + \beta' + \gamma) \mathbf{T} + (\beta_1 + \beta'_1 + \gamma_1) \mathbf{A}_1 + (\beta_2 + \beta'_2 + \gamma_2) \mathbf{A}_2 + (\beta_3 + \beta'_3) \mathbf{A}_1^2 \\ + (\beta_6 + \beta'_6 + \gamma_3) \mathbf{A}_2^2 + (\beta_4 + \beta'_4) (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ + \beta_5 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \beta'_5 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) + \gamma_4 \mathbf{A}_3 \\ + \gamma_5 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2) + (\beta_7 + \beta'_7 + \gamma_6) \mathbf{I} = 0 \end{aligned} \quad (31.4)$$

exists, in which $\beta, \beta_1, \beta_2, \dots, \beta_7$ and $\beta', \beta'_1, \beta'_2, \dots, \beta'_7$ are scalars given, as before, by (27.5) and $\gamma, \gamma_1, \gamma_2, \dots, \gamma_6$ are scalars given by (28.4). This relation provides a representation for $\mathbf{T}$, *i.e.* $\beta + \beta' + \gamma \neq 0$, unless the relation (28.9) is satisfied and either each of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have a common principal direction.

(v) If $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ have a principal direction in common, then a relation of the form

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_2 \mathbf{A}_1^2 + \epsilon_3 \mathbf{A}_2 + \epsilon_4 \mathbf{I} = 0 \quad (31.5)$$

exists, where $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_4 are scalars given by (29.7). This relation provides a representation for $\mathbf{T}$, *i.e.* $\epsilon \neq 0$, provided that $\mathbf{A}_1$ has unequal eigen-values and $\mathbf{A}_1$ and $\mathbf{A}_2$ have only one principal direction in common.

(vi) If $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ have a principal direction in common, then a relation of the form

$$\begin{aligned} (\epsilon + \epsilon') \mathbf{T} + (\epsilon_1 + \epsilon'_1) \mathbf{A}_1 + (\epsilon_3 + \epsilon'_3) \mathbf{A}_2 \\ + \epsilon_2 \mathbf{A}_1^2 + \epsilon'_2 \mathbf{A}_2^2 + (\epsilon_4 + \epsilon'_4) \mathbf{I} = 0 \end{aligned} \quad (31.6)$$

exists, in which $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ are scalars given by (29.7) and $\epsilon', \epsilon'_1, \epsilon'_2, \epsilon'_3, \epsilon'_4$ are scalars given by (29.13). This relation provides a representation for $\mathbf{T}$, *i.e.* $\epsilon + \epsilon' \neq 0$, provided that $\mathbf{A}_1$ and $\mathbf{A}_2$ do not both have a pair of equal eigen-values and $\mathbf{A}_1$ and $\mathbf{A}_2$ have only one principal direction in common.

(vii) If $\mathbf{T}$ and $\mathbf{A}_1$ have all their principal directions in common, then a relation of the form

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_2 \mathbf{A}_1^2 + \epsilon_4 \mathbf{I} = 0 \quad (31.7)$$

exists, where $\epsilon, \epsilon_1, \epsilon_2$ and ϵ_4 are given by (29.17). This relation provides a representation for $\mathbf{T}$, *i.e.* $\epsilon \neq 0$, unless $\mathbf{A}_1$ has a pair of equal eigen-values.

(viii) If $\mathbf{T}$ and $\mathbf{A}_1$ have all their principal directions in common and each has a pair of equal eigen-values, the principal planes corresponding to these pairs of equal eigen-values coinciding for the two matrices, then a relation of the form

$$\epsilon \mathbf{T} + \epsilon_1 \mathbf{A}_1 + \epsilon_4 \mathbf{I} = 0 \quad (31.8)$$

exists, in which ϵ, ϵ_1 and ϵ_4 are scalars given by (29.21). This relation provides a representation for $\mathbf{T}$, *i.e.* $\epsilon \neq 0$, unless all the eigen-values of $\mathbf{A}_1$ are equal.

(ix) If $\mathbf{T}$ has all its eigen-values equal, then we have the relation

$$3\mathbf{T} - (\text{tr } \mathbf{T})\mathbf{I} = 0. \quad (31.9)$$

(x) If the matrix $\mathbf{T}$ is an isotropic matrix function of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, then, in each of the representations for $\mathbf{T}$ considered, the scalar coefficients are expressible as scalar invariants of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ only.

CHAPTER IV. SCALAR INVARIANTS OF SYMMETRIC MATRICES.

32. Introduction. In Chapter III it was shown that if a symmetric 3×3 matrix $\mathbf{T}$ is a symmetric matrix function of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, then it may be expressed as a linear combination of a set of simple symmetric matrix functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, the coefficients in this linear combination being scalar invariants of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. Such an expression for $\mathbf{T}$ provides a representation for $\mathbf{T}$ in terms of the symmetric matrix functions of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ chosen as a basis for the representation provided that the coefficient of $\mathbf{T}$ is not zero. The vanishing of the coefficient of $\mathbf{T}$ implies certain relations between the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ and these relations were investigated in Chapter III for certain choices of the matrix functions employed as a basis for the representation.

In the present chapter it will be shown that, provided no two eigen-values of $\mathbf{A}_1$ are equal and if, when $\mathbf{A}_1$ and $\mathbf{A}_2$ are referred to the principal directions of $\mathbf{A}_1$, none of the off-diagonal components of $\mathbf{A}_2$ is zero, then any scalar invariant of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ may be expressed as a single-valued function of $6R - 2$ scalar invariants.

It would no doubt be possible to obtain a set of scalar invariants of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, greater in number than $6R - 2$, in terms of which an arbitrary scalar invariant of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ could be expressed as a single-valued function, under less restrictive conditions on the set of matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$. However, this will not be done in the present paper.

33. Functionally independent scalar invariants of two symmetric 3×3 matrices. We consider the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}, \mathcal{Q}_{1122}$ formed from the elements of two symmetric 3×3 matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ according to the definitions

$$\begin{aligned} \mathcal{Q}_1 &= \text{tr } \mathbf{A}_1, & \mathcal{Q}_{11} &= \text{tr } \mathbf{A}_1^2, & \mathcal{Q}_{111} &= \text{tr } \mathbf{A}_1^3, & \mathcal{Q}_2 &= \text{tr } \mathbf{A}_2, & \mathcal{Q}_{12} &= \text{tr } \mathbf{A}_1 \mathbf{A}_2, \\ \mathcal{Q}_{112} &= \text{tr } \mathbf{A}_1^2 \mathbf{A}_2, & \mathcal{Q}_{22} &= \text{tr } \mathbf{A}_2^2, & \mathcal{Q}_{122} &= \text{tr } \mathbf{A}_1 \mathbf{A}_2^2 \text{ and} \\ \mathcal{Q}_{1122} &= \text{tr } \mathbf{A}_1^2 \mathbf{A}_2^2. \end{aligned} \tag{33.1}$$

Let $\mathbf{A}_1 = \| a_{ij}^{(1)} \|$ and $\mathbf{A}_2 = \| a_{ij}^{(2)} \|$, with $a_{ij}^{(1)} = a_{ji}^{(1)}$ and $a_{ij}^{(2)} = a_{ji}^{(2)}$. We choose an orthogonal transformation by means of which $\mathbf{A}_1$ can be transformed into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$ say. Let $\mathbf{A}_2$ be transformed by this transformation into $\bar{\mathbf{A}}_2 = \| \bar{a}_{ij}^{(2)} \|$. The scalar invariants defined in (33.1) may now be written as

$$\begin{aligned}
\mathcal{Q}_1 &= \bar{a}_{ii}^{(1)}, & \mathcal{Q}_{11} &= (\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2, & \mathcal{Q}_{111} &= (\bar{a}_{11}^{(1)})^3 \\
&+ (\bar{a}_{22}^{(1)})^3 + (\bar{a}_{33}^{(1)})^3, & \mathcal{Q}_2 &= \bar{a}_{ii}^{(2)}, & \mathcal{Q}_{12} &= \bar{a}_{11}^{(1)} \bar{a}_{11}^{(2)} + \bar{a}_{22}^{(1)} \bar{a}_{22}^{(2)} \\
&+ \bar{a}_{33}^{(1)} \bar{a}_{33}^{(2)}, & \mathcal{Q}_{112} &= (\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(2)} + (\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(2)} + (\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(2)}, \\
\mathcal{Q}_{22} &= \bar{a}_{ij}^{(2)} \bar{a}_{ji}^{(2)}, & \mathcal{Q}_{122} &= \bar{a}_{11}^{(1)} \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(2)} + \bar{a}_{22}^{(1)} \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(2)} + \bar{a}_{33}^{(1)} \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(2)}
\end{aligned} \tag{33.2}$$

and

$$\mathcal{Q}_{1122} = (\bar{a}_{11}^{(1)})^2 \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(2)} + (\bar{a}_{22}^{(1)})^2 \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(2)} + (\bar{a}_{33}^{(1)})^2 \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(2)}.$$

A necessary and sufficient condition that the nine scalar invariants defined by equation (33.1) shall be functionally independent is that their Jacobian matrix **J**, formed by regarding them as functions of the $\bar{a}$'s in accordance with equations (33.2), shall have rank 9.

Now, we may define the matrix **J** as the 9×9 matrix, the element in the p^{th} row and q^{th} column of which is the derivative of the q^{th} element of

$$(\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}, \mathcal{Q}_{1122})$$

with respect to the p^{th} element of

$$(\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)}, \bar{a}_{11}^{(2)}, \bar{a}_{22}^{(2)}, \bar{a}_{33}^{(2)}, \bar{a}_{23}^{(2)}, \bar{a}_{31}^{(2)}, \bar{a}_{12}^{(2)}).$$

Then,

$$\mathbf{J} = \begin{vmatrix}
1, & 2\bar{a}_{11}^{(1)}, & 3(\bar{a}_{11}^{(1)})^2, & 0, & \bar{a}_{11}^{(2)}, & 2\bar{a}_{11}^{(1)}\bar{a}_{11}^{(2)}, & 0, & \bar{a}_{1j}^{(2)}\bar{a}_{j1}^{(2)}, & 2\bar{a}_{11}^{(1)}\bar{a}_{1j}^{(2)}\bar{a}_{j1}^{(2)} \\
1, & 2\bar{a}_{22}^{(1)}, & 3(\bar{a}_{22}^{(1)})^2, & 0, & \bar{a}_{22}^{(2)}, & 2\bar{a}_{22}^{(1)}\bar{a}_{22}^{(2)}, & 0, & \bar{a}_{2j}^{(2)}\bar{a}_{j2}^{(2)}, & 2\bar{a}_{22}^{(1)}\bar{a}_{2j}^{(2)}\bar{a}_{j2}^{(2)} \\
1, & 2\bar{a}_{33}^{(1)}, & 3(\bar{a}_{33}^{(1)})^2, & 0, & \bar{a}_{33}^{(2)}, & 2\bar{a}_{33}^{(1)}\bar{a}_{33}^{(2)}, & 0, & \bar{a}_{3j}^{(2)}\bar{a}_{j3}^{(2)}, & 2\bar{a}_{33}^{(1)}\bar{a}_{3j}^{(2)}\bar{a}_{j3}^{(2)} \\
0, & 0, & 0, & 1, & \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2, & 2\bar{a}_{11}^{(2)}, & 2\bar{a}_{11}^{(1)}\bar{a}_{11}^{(2)}, & 2(\bar{a}_{11}^{(1)})^2\bar{a}_{11}^{(2)} \\
0, & 0, & 0, & 1, & \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2, & 2\bar{a}_{22}^{(2)}, & 2\bar{a}_{22}^{(1)}\bar{a}_{22}^{(2)}, & 2(\bar{a}_{22}^{(1)})^2\bar{a}_{22}^{(2)} \\
0, & 0, & 0, & 1, & \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2, & 2\bar{a}_{33}^{(2)}, & 2\bar{a}_{33}^{(1)}\bar{a}_{33}^{(2)}, & 2(\bar{a}_{33}^{(1)})^2\bar{a}_{33}^{(2)} \\
0, & 0, & 0, & 0, & 0, & 0, & 4\bar{a}_{23}^{(2)}, & 2\bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & 2\bar{a}_{23}^{(2)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\
0, & 0, & 0, & 0, & 0, & 0, & 4\bar{a}_{31}^{(2)}, & 2\bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & 2\bar{a}_{31}^{(2)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\
0, & 0, & 0, & 0, & 0, & 0, & 4\bar{a}_{12}^{(2)}, & 2\bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & 2\bar{a}_{12}^{(2)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2]
\end{vmatrix} \tag{33.3}$$

We readily see that

$$\det \mathbf{J} = \det \mathbf{B}_1 \det \mathbf{B}_2 \det \mathbf{B}_3, \tag{33.4}$$

where

$$\mathbf{B}_2 = \frac{1}{6}\mathbf{B}_1 = \begin{vmatrix} 1, & \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2 \\ 1, & \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2 \\ 1, & \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2 \end{vmatrix}$$

and

$$\mathbf{B}_3 = \begin{vmatrix} 4\bar{a}_{23}^{(2)}, & 2\bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & 2\bar{a}_{23}^{(2)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\ 4\bar{a}_{31}^{(2)}, & 2\bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & 2\bar{a}_{31}^{(2)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\ 4\bar{a}_{12}^{(2)}, & 2\bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & 2\bar{a}_{12}^{(2)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2] \end{vmatrix}. \quad (33.5)$$

We have

$$\det \mathbf{B}_2 = \frac{1}{6} \det \mathbf{B}_1 = (\bar{a}_{22}^{(1)} - \bar{a}_{33}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})$$

and

$$\det \mathbf{B}_3 = 16\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)}(\bar{a}_{22}^{(1)} - \bar{a}_{33}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)}), \quad (33.6)$$

so that, from (33.4),

$$\det \mathbf{J} = 96\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)}[(\bar{a}_{22}^{(1)} - \bar{a}_{33}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})]^3. \quad (33.7)$$

The condition $\det \mathbf{J} \neq 0$ for $\mathbf{J}$ to have rank 9 will be satisfied if and only if

$$\bar{a}_{11}^{(1)} \neq \bar{a}_{22}^{(1)} \neq \bar{a}_{33}^{(1)} \neq \bar{a}_{11}^{(1)} \quad (33.8)$$

and

$$\bar{a}_{23}^{(2)} \neq 0, \quad \bar{a}_{31}^{(2)} \neq 0 \quad \text{and} \quad \bar{a}_{12}^{(2)} \neq 0; \quad (33.9)$$

i.e. if and only if the three eigen-values of the matrix $\mathbf{A}_1$ are all unequal and none of the off-diagonal components of the matrix $\bar{\mathbf{A}}_2$ is zero. These then are necessary and sufficient conditions for the set of nine scalar invariants defined by (33.1) to be functionally independent.*

If the conditions (33.8) and (33.9) are satisfied, then any scalar invariant $\mathcal{J}$ formed from the elements of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ is functionally related to the nine scalar invariants defined by equations (33.1). This can be readily seen by considering $\mathcal{J}$ to be expressed in terms of the nine independent elements $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$, $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$, $\bar{a}_{33}^{(2)}$, $\bar{a}_{23}^{(2)}$, $\bar{a}_{31}^{(2)}$, $\bar{a}_{12}^{(2)}$ of the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$. The condition for

* It will be appreciated that the term functionally independent is used here and throughout this paper in a somewhat unconventional fashion. If the determinant of the Jacobian matrix vanishes, when certain constraints are imposed on the variation of the independent variables, so that it is possible to find a functional relationship between the dependent variables which is valid when these constraints are satisfied, then we consider that the dependent variables are not functionally independent when these constraints are imposed.

functional independence of the ten scalar invariants $\mathcal{J}$, $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ is that the rank of their Jacobian when they are considered as a function of the $\bar{a}$'s shall be 10. Since the Jacobian concerned is a 9×10 matrix its rank cannot be greater than 9 and the ten scalar invariants must therefore be functionally related. Otherwise expressed, since the ten scalar invariants are expressible in terms of only nine independent variables a functional relationship must exist between them.

If condition (33.8) is satisfied, but one of the off-diagonal components of $\bar{\mathbf{A}}_2$, $\bar{a}_{23}^{(2)}$ say, is zero, then it is readily seen that the Jacobian matrix $\mathbf{J}'$ of the set of eight scalar invariants formed from that defined by (33.3) by omitting $\mathcal{Q}_{1122}$ has rank 8, so that the eight invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$ defined by (33.1) form a functionally independent set. It can then be seen that, since any scalar invariant formed from the elements of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ and these eight functionally independent invariants may be regarded as functions of the eight components $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$, $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$, $\bar{a}_{33}^{(2)}$, $\bar{a}_{31}^{(2)}$, $\bar{a}_{12}^{(2)}$ of the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$, a functional relationship must exist between $\mathcal{J}$, $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$.

In a similar way it can be seen that if the condition (33.8) is satisfied, but two of the off-diagonal components of $\bar{\mathbf{A}}_2$ are zero, then the seven scalar invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$ form a functionally independent set and the arbitrary scalar invariant $\mathcal{J}$ is functionally related to these. Again, if the condition (33.8) is satisfied, while $\bar{a}_{23}^{(2)} = \bar{a}_{31}^{(2)} = \bar{a}_{12}^{(2)} = 0$, then the six scalar invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$ form a functionally independent set and the arbitrary scalar invariant $\mathcal{J}$ is functionally related to these.

Combining the results obtained when both conditions (33.8) and (33.9) are satisfied by $\mathbf{A}_1$ and $\mathbf{A}_2$ with those obtained when only condition (33.8) is satisfied, we see that provided the eigen-values of $\mathbf{A}_1$ are all unequal, any scalar invariant formed from the elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ is functionally related to the scalar invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ defined by equations (33.1).

In general, this functional relationship does not allow $\mathcal{J}$ to be expressed as a single-valued function of the nine scalar invariants described by equations (33.1), since the elements of the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ cannot be uniquely determined, apart from an orthogonal transformation applied simultaneously to both of them, from a knowledge of these nine scalar invariants. From the expressions (33.2) for $\mathcal{Q}_2$, $\mathcal{Q}_{12}$ and $\mathcal{Q}_{112}$, we obtain

$$\bar{a}_{11}^{(2)} = - \frac{\mathcal{Q}_{112} - (\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)})\mathcal{Q}_{12} + \bar{a}_{22}^{(1)}\bar{a}_{33}^{(1)}\mathcal{Q}_2}{(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})}. \quad (33.10)$$

$\bar{a}_{22}^{(2)}$ and $\bar{a}_{33}^{(2)}$ are obtained from (33.10) by cyclic permutation of $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$.

From the expressions (33.2) for $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ we obtain

$$(\bar{a}_{23}^{(2)})^2 = - \frac{[\beta\mathcal{Q}_1 - \gamma - \frac{1}{2}\alpha(\mathcal{Q}_1^2 - \mathcal{Q}_{11})] - \beta\bar{a}_{11}^{(1)} + \alpha(\bar{a}_{11}^{(1)})^2}{(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})}, \quad (33.11)$$

where

$$\begin{aligned}\alpha &= \frac{1}{2}[\mathcal{Q}_{22} - (\bar{a}_{11}^{(2)})^2 - (\bar{a}_{22}^{(2)})^2 - (\bar{a}_{33}^{(2)})^2], \\ \beta &= \mathcal{Q}_{122} - \bar{a}_{11}^{(1)}(\bar{a}_{11}^{(2)})^2 - \bar{a}_{22}^{(1)}(\bar{a}_{22}^{(2)})^2 - \bar{a}_{33}^{(1)}(\bar{a}_{33}^{(2)})^2\end{aligned}\quad (33.12)$$

and

$$\gamma = \mathcal{Q}_{1122} - (\bar{a}_{11}^{(1)} \bar{a}_{11}^{(2)})^2 - (\bar{a}_{22}^{(1)} \bar{a}_{22}^{(2)})^2 - (\bar{a}_{33}^{(1)} \bar{a}_{33}^{(2)})^2.$$

$(\bar{a}_{31}^{(2)})^2$ and $(\bar{a}_{12}^{(2)})^2$ can be obtained from (33.11) by cyclic permutation of $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$.

Now, from a knowledge of $\mathcal{Q}_1$, $\mathcal{Q}_{11}$ and $\mathcal{Q}_{111}$, we can find the values of $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$ uniquely apart from the effect of an orthogonal transformation which permutes these in any manner. It will readily be seen from equations (33.10) and (33.11) that a permutation of $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$ effects corresponding permutations of $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$ and $\bar{a}_{33}^{(2)}$ and of $(\bar{a}_{23}^{(2)})^2$, $(\bar{a}_{31}^{(2)})^2$ and $(\bar{a}_{12}^{(2)})^2$. For example, suppose we replace $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$ by $\bar{a}_{11}^{(1)}$, $\bar{a}_{33}^{(1)}$ and $\bar{a}_{22}^{(1)}$ respectively, as we may do by subjecting the matrix $\bar{\mathbf{A}}_1$ to the orthogonal transformation

$$\begin{vmatrix} 1, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & 1, & 0 \end{vmatrix}.$$
(33.13)

From (33.10) and (33.11), it is seen that $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$ and $\bar{a}_{33}^{(2)}$ are changed to $\bar{a}_{11}^{(2)}$, $\bar{a}_{33}^{(2)}$ and $\bar{a}_{22}^{(2)}$ respectively and $(\bar{a}_{23}^{(2)})^2$, $(\bar{a}_{31}^{(2)})^2$ and $(\bar{a}_{12}^{(2)})^2$ are changed to $(\bar{a}_{23}^{(2)})^2$, $(\bar{a}_{12}^{(2)})^2$ and $(\bar{a}_{31}^{(2)})^2$ respectively. We readily see that these are precisely the changes that are made by subjecting the matrix $\bar{\mathbf{A}}_2$ to the orthogonal transformation (33.13). Thus, if the nine scalar invariants defined by equations (33.1) are specified, $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$, $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$, $\bar{a}_{33}^{(2)}$, $(\bar{a}_{23}^{(2)})^2$, $(\bar{a}_{31}^{(2)})^2$ and $(\bar{a}_{12}^{(2)})^2$ are uniquely determined apart from the effect of an orthogonal transformation applied simultaneously to the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$.

For any set of values of the scalar invariants and a particular choice of $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$ in accordance with these values, $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$ and $\bar{a}_{33}^{(2)}$ are uniquely determined by equation (33.10) and two similar equations. $\bar{a}_{23}^{(2)}$, $\bar{a}_{31}^{(2)}$ and $\bar{a}_{12}^{(2)}$ are uniquely determined from equation (33.11) and two similar equations apart from an ambiguity of sign in each of them. Thus, the matrix $\bar{\mathbf{A}}_2$ is uniquely determined apart from an arbitrariness as to sign in each of the off-diagonal components. It is readily seen that a change in the signs of two of the independent off-diagonal components, say $\bar{a}_{12}^{(2)}$ and $\bar{a}_{31}^{(2)}$, is equivalent to transformation of the matrix $\bar{\mathbf{A}}_2$ by the orthogonal transformation

$$\begin{vmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{vmatrix}.$$
(33.14)

This orthogonal transformation leaves the matrix $\bar{\mathbf{A}}_1$ unchanged. Consequently, the fact that the signs of two of the independent off-diagonal components of $\bar{\mathbf{A}}_2$ may be changed simultaneously without affecting the values of the nine scalar invariants defined by equations (33.1) only affects the determination of the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ from the scalar invariants to the extent of an orthogonal transformation applied simultaneously to both of them and consequently the value of any scalar invariant formed from the elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ will be unchanged. However, if the sign of only one of the independent off-diagonal components of $\bar{\mathbf{A}}_2$ is changed, this is not the case. For example, we may consider the scalar invariant $\det \mathbf{A}_2$ which is, of course, given in terms of $\bar{a}_{11}^{(2)}$, $\bar{a}_{22}^{(2)}$, $\bar{a}_{33}^{(2)}$, $\bar{a}_{23}^{(2)}$, $\bar{a}_{31}^{(2)}$, $\bar{a}_{12}^{(2)}$ by

$$\Delta_2 = \det \mathbf{A}_2 = \det \bar{\mathbf{A}}_2 = \bar{a}_{11}^{(2)} \bar{a}_{22}^{(2)} \bar{a}_{33}^{(2)} - \bar{a}_{11}^{(2)} (\bar{a}_{23}^{(2)})^2 - \bar{a}_{22}^{(2)} (\bar{a}_{31}^{(2)})^2 - \bar{a}_{33}^{(2)} (\bar{a}_{12}^{(2)})^2 + 2\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)}. \quad (33.15)$$

A change in sign of one, or three, of the elements $\bar{a}_{23}^{(2)}$, $\bar{a}_{31}^{(2)}$, $\bar{a}_{12}^{(2)}$ will clearly alter the value of Δ_2 . Thus, although Δ_2 must be functionally related to the nine scalar invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$, $\mathcal{Q}_{1122}$, defined by equations (33.1), this functional relationship does not enable us to establish a single-valued dependence of Δ_2 on these nine scalar invariants.

This may be illustrated by means of an example. Suppose the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are given by

$$\bar{\mathbf{A}}_1 = \begin{vmatrix} 1, & 0, & 0 \\ 0, & 2, & 0 \\ 0, & 0, & 3 \end{vmatrix} \quad \text{and} \quad \bar{\mathbf{A}}_2 = \begin{vmatrix} 1, & 1, & -2, \\ 1, & -1, & -1, \\ -2, & -1, & 2, \end{vmatrix}. \quad (33.16)$$

From (33.2) it is seen that, in this case,

$$\begin{aligned} \mathcal{Q}_1 &= 6, & \mathcal{Q}_{11} &= 14, & \mathcal{Q}_{111} &= 36, & \mathcal{Q}_2 &= 2, & \mathcal{Q}_{12} &= 5, \\ \mathcal{Q}_{112} &= 15, & \mathcal{Q}_{22} &= 18, & \mathcal{Q}_{122} &= 39 & \text{and} & \mathcal{Q}_{1122} &= 99. \end{aligned} \quad (33.17)$$

If the values of the nine scalar invariants $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\dots$, $\mathcal{Q}_{1122}$ are specified by (33.17), then we obtain $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$ as having the values 1, 2, 3 in any order. Suppose, we take $\bar{a}_{11}^{(1)}$, $\bar{a}_{22}^{(1)}$, $\bar{a}_{33}^{(1)}$ = 1, 2, 3. Then, from (33.10), we obtain $\bar{a}_{11}^{(2)} = 1$, $\bar{a}_{22}^{(2)} = -1$, and $\bar{a}_{33}^{(2)} = 2$. Again, from equations (33.11), we obtain $(\bar{a}_{23}^{(2)})^2 = 1$, $(\bar{a}_{31}^{(2)})^2 = 4$ and $(\bar{a}_{12}^{(2)})^2 = 1$. If $\bar{a}_{23}^{(2)} = -1$, $\bar{a}_{31}^{(2)} = -2$ and $\bar{a}_{12}^{(2)} = 1$, we have the matrix $\bar{\mathbf{A}}_2$ defined by (33.16) and $\Delta_2 = 3$. If, however, we take $\bar{a}_{23}^{(2)} = 1$, $\bar{a}_{31}^{(2)} = -2$ and $\bar{a}_{12}^{(2)} = 1$, as we can do consistently with (33.17), then $\Delta_2 = -5$.

It can be seen similarly that if Δ_2 is specified instead of $\mathcal{Q}_{1122}$, so that we have

$$\begin{aligned} \mathcal{Q}_1 &= 6, & \mathcal{Q}_{11} &= 14, & \mathcal{Q}_{111} &= 36, & \mathcal{Q}_2 &= 2, & \mathcal{Q}_{12} &= 5, \\ \mathcal{Q}_{112} &= 15, & \mathcal{Q}_{22} &= 18, & \mathcal{Q}_{122} &= 39 & \text{and} & \Delta_2 &= 3, \end{aligned} \quad (33.18)$$

then $\mathcal{Q}_{1122}$ is not uniquely determined. With the values given by (33.18) and taking the choice $\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)} = 1, 2, 3$ for the elements of $\bar{\mathbf{A}}_1$, we obtain from (33.10) $\bar{a}_{11}^{(2)}, \bar{a}_{22}^{(2)}, \bar{a}_{33}^{(2)} = 1, -1, 2$ as before. Now, from the expressions (33.2) for $\mathcal{Q}_{22}$ and $\mathcal{Q}_{122}$, we obtain

$$(\bar{a}_{23}^{(2)})^2 + (\bar{a}_{31}^{(2)})^2 + (\bar{a}_{12}^{(2)})^2 = 6,$$

and

$$5(\bar{a}_{23}^{(2)})^2 + 4(\bar{a}_{31}^{(2)})^2 + 3(\bar{a}_{12}^{(2)})^2 = 24. \quad (33.19)$$

Also, from (33.15), we have

$$-(\bar{a}_{23}^{(2)})^2 + (\bar{a}_{31}^{(2)})^2 - 2(\bar{a}_{12}^{(2)})^2 = 5 - 2\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)}. \quad (33.20)$$

Solving equations (33.19) and (33.20) for $(\bar{a}_{23}^{(2)})^2$, $(\bar{a}_{31}^{(2)})^2$ and $(\bar{a}_{12}^{(2)})^2$, we obtain

$$(\bar{a}_{23}^{(2)})^2 = \frac{1}{5}(1 + 2\mu), \quad (\bar{a}_{31}^{(2)})^2 = \frac{4}{5}(7 - \mu) \quad \text{and} \quad (\bar{a}_{12}^{(2)})^2 = \frac{1}{5}(1 + 2\mu), \quad (33.21)$$

where

$$\mu = \bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)}. \quad (33.22)$$

From (33.21) and (33.22), we have

$$125\mu^2 = 4(1 + 2\mu)^2(7 - \mu);$$

that is

$$(\mu - 2)(16\mu^2 + 61\mu + 14) = 0. \quad (33.23)$$

Whence

$$\mu = 2 \quad \text{or} \quad \mu = \frac{1}{32}(-61 \pm 5\sqrt{113}). \quad (33.24)$$

The solution $\mu = 2$ leads, from (33.21), to

$$(\bar{a}_{23}^{(2)})^2 = 1, \quad (\bar{a}_{31}^{(2)})^2 = 4 \quad \text{and} \quad (\bar{a}_{12}^{(2)})^2 = 1, \quad (33.25)$$

so that, from (33.2),

$$\mathcal{Q}_{1122} = 99. \quad (33.26)$$

It is similarly seen that the solutions $\mu = \frac{1}{32}(-61 \pm 5\sqrt{113})$ lead to

$$\begin{aligned} (\bar{a}_{23}^{(2)})^2 &= \frac{1}{16}(-9 \pm \sqrt{113}), \quad (\bar{a}_{31}^{(2)})^2 = \frac{1}{8}(57 \mp \sqrt{113}), \\ (\bar{a}_{12}^{(2)})^2 &= \frac{1}{16}(-9 \pm \sqrt{113}) \quad \text{and} \quad \mathcal{Q}_{1122} = \frac{1}{8}(817 \mp \sqrt{113}). \end{aligned} \quad (33.27)$$

Thus, if the values of $\mathcal{Q}_1$, $\mathcal{Q}_{11}$, $\mathcal{Q}_{111}$, $\mathcal{Q}_2$, $\mathcal{Q}_{12}$, $\mathcal{Q}_{112}$, $\mathcal{Q}_{22}$, $\mathcal{Q}_{122}$ and Δ_2 are given by (33.18), there are three possible corresponding values of the scalar invariant $\mathcal{Q}_{1122}$. Moreover, we note that the matrices $\bar{\mathbf{A}}_2$ corresponding to two of these

(those determined by $\mu = 2$ and $\mu = \frac{1}{32}(-61 + 5\sqrt{113})$) have all their elements real.

There are, of course, other sets of nine functionally independent scalar invariants, for example the symmetric set $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{22}, \mathcal{Q}_{222}$ ($= \text{tr } \mathbf{A}_2^3$), $\mathcal{Q}_{12} = \mathcal{Q}_{21}, \mathcal{Q}_{112}$ and $\mathcal{Q}_{122}$. It is not known whether any such set determines $\mathbf{A}_1$ and $\mathbf{A}_2$ to within an orthogonal transformation.

34. Single-valued functional relationships between scalar invariants of two symmetric 3×3 matrices. It has been seen in the previous section that the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$, formed from the elements of two symmetric 3×3 matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ according to equations (33.1) are, in general, functionally independent and that any arbitrary scalar invariant $\mathcal{J}$ formed from the elements of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ is functionally related to these nine scalar invariants. It has further been shown that even when the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ are not functionally independent, provided no two eigen-values of $\mathbf{A}_1$ are equal, a functional relation exists between the arbitrary scalar invariant $\mathcal{J}$ and a sub-set of the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \dots, \mathcal{Q}_{1122}$.

This functional relationship does not necessarily yield a single-valued expression for $\mathcal{J}$ in terms of the nine scalar invariants defined by (33.1). It can be shown, however, that, provided no two eigen-values of $\mathbf{A}_1$ are equal, $\mathcal{J}$ can be expressed as a single-valued function of the ten scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}, \mathcal{Q}_{1122}$ and Δ_2 defined by equations (33.1) and (33.15).

It has also been seen in the previous section that if the first nine of these scalar invariants are specified, then, apart from the effects of an orthogonal transformation applied simultaneously to $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ and the ambiguity in the signs of the off-diagonal components of $\bar{\mathbf{A}}_2$, the elements of the matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are determined. It has been seen that a change in the signs of two of the independent off-diagonal components of $\bar{\mathbf{A}}_2$ corresponds to an orthogonal transformation of the matrix $\bar{\mathbf{A}}_2$ which does not affect the matrix $\bar{\mathbf{A}}_1$. A change in sign of one, or three, of the off-diagonal components of $\bar{\mathbf{A}}_2$ alters the value of Δ_2 and consequently if Δ_2 is specified, then this type of ambiguity is removed. Thus, the ten scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}, \mathcal{Q}_{1122}$ and Δ_2 determine uniquely the elements of $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ except insofar as they may be altered by certain orthogonal transformations applied simultaneously to both of the matrices. Since any scalar invariant $\mathcal{J}$ formed from the elements of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ can be expressed as a single-valued function of the elements of $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ and is unaltered by an orthogonal transformation applied simultaneously to both of the matrices, it must be expressible as a single-valued function of the ten scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}, \mathcal{Q}_{1122}$ and Δ_2 .

35. Functionally independent scalar invariants of any number of symmetric 3×3 matrices. Let us consider the $6R - 3$ scalar invariants consisting of the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ defined in terms

of the elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ according to equations (33.1), together with the $6R - 12$ scalar invariants $\mathfrak{Q}_r, \mathfrak{Q}_{1r}, \mathfrak{Q}_{11r}, \mathfrak{Q}_{2r}, \mathfrak{Q}_{12r}, \mathfrak{Q}_{112r}$ ($r = 3, 4, \dots, R$) formed from the elements of the R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ according to the definitions

$$\begin{aligned} \mathfrak{Q}_r &= \text{tr } \mathbf{A}_r, & \mathfrak{Q}_{1r} &= \text{tr } \mathbf{A}_1 \mathbf{A}_r, & \mathfrak{Q}_{11r} &= \text{tr } \mathbf{A}_1^2 \mathbf{A}_r, \\ \mathfrak{Q}_{2r} &= \text{tr } \mathbf{A}_2 \mathbf{A}_r, & \mathfrak{Q}_{12r} &= \text{tr } \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_r, & \mathfrak{Q}_{112r} &= \text{tr } \mathbf{A}_1^2 \mathbf{A}_2 \mathbf{A}_r. \end{aligned} \quad (35.1)$$

Applying an orthogonal transformation, which transforms $\mathbf{A}_1$ into the diagonal matrix $\bar{\mathbf{A}}_1 = (\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)})$ say, to all R matrices, let us suppose that $\mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_R$ become $\bar{\mathbf{A}}_2, \bar{\mathbf{A}}_3, \dots, \bar{\mathbf{A}}_R$, where $\bar{\mathbf{A}}_r = \|\bar{a}_{ij}^{(r)}\|$, with $\bar{a}_{ij}^{(r)} = \bar{a}_{ji}^{(r)}$, ($r = 2, 3, \dots, R$). The scalar invariants defined by (33.1) and (35.1) may now be expressed in terms of the elements of the matrices $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_R$ by equations (33.2) and

$$\begin{aligned} \mathfrak{Q}_r &= \bar{a}_{ii}^{(r)}, \quad \mathfrak{Q}_{1r} = \bar{a}_{11}^{(1)} \bar{a}_{11}^{(r)} + \bar{a}_{22}^{(1)} \bar{a}_{22}^{(r)} + \bar{a}_{33}^{(1)} \bar{a}_{33}^{(r)}, \\ \mathfrak{Q}_{11r} &= (\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(r)} + (\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(r)} + (\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(r)}, \\ \mathfrak{Q}_{2r} &= \bar{a}_{ij}^{(2)} \bar{a}_{ji}^{(r)}, \quad \mathfrak{Q}_{12r} = \bar{a}_{11}^{(1)} \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(r)} + \bar{a}_{22}^{(1)} \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(r)} + \bar{a}_{33}^{(1)} \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(r)} \end{aligned} \quad (35.2)$$

and

$$\mathfrak{Q}_{112r} = (\bar{a}_{11}^{(1)})^2 \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(r)} + (\bar{a}_{22}^{(1)})^2 \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(r)} + (\bar{a}_{33}^{(1)})^2 \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(r)}.$$

It can readily be shown that provided the eigen-values of $\mathbf{A}_1$ are all unequal and none of the off-diagonal components of $\mathbf{A}_2$ are zero, the $6R - 3$ scalar invariants defined by equations (33.2) and (35.2) are functionally independent. We form the Jacobian matrix $\mathbf{J}$ of the $6R - 3$ scalar invariants considered as functions of the $6R - 3$ elements $\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}, \bar{a}_{33}^{(1)}, \bar{a}_{11}^{(r)}, \bar{a}_{22}^{(r)}, \bar{a}_{33}^{(r)}, \bar{a}_{23}^{(r)}, \bar{a}_{31}^{(r)}$ and $\bar{a}_{12}^{(r)}$ ($r = 2, 3, \dots, R$) in accordance with equations (33.2) and (35.2). We have

$$\mathbf{J} = \left\| \begin{array}{ccccc} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} & \mathbf{B}_{14} & \\ 0 & \mathbf{B}_{22} & \mathbf{B}_{23} & \mathbf{B}_{24} & \\ 0 & 0 & \mathbf{B}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{B}_{44} & 0 \\ & & & & \\ 0 & 0 & 0 & 0 & \mathbf{B}_{RR} \end{array} \right\|, \quad (35.3)$$

where

$$\mathbf{B}_{11} = \left\| \begin{array}{ccc} 1, & 2\bar{a}_{11}^{(1)}, & 3(\bar{a}_{11}^{(1)})^2 \\ 1, & 2\bar{a}_{22}^{(1)}, & 3(\bar{a}_{22}^{(1)})^2 \\ 1, & 2\bar{a}_{33}^{(1)}, & 3(\bar{a}_{33}^{(1)})^2 \end{array} \right\|,$$

$$\mathbf{B}_{1r} = \left\| \begin{array}{cccccc} 0, & \bar{a}_{11}^{(r)}, & 2\bar{a}_{11}^{(1)} \bar{a}_{11}^{(r)}, & 0, & \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(r)}, & 2\bar{a}_{11}^{(1)} \bar{a}_{1j}^{(2)} \bar{a}_{j1}^{(r)} \\ 0, & \bar{a}_{22}^{(r)}, & 2\bar{a}_{22}^{(1)} \bar{a}_{22}^{(r)}, & 0, & \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(r)}, & 2\bar{a}_{22}^{(1)} \bar{a}_{2j}^{(2)} \bar{a}_{j2}^{(r)} \\ 0, & \bar{a}_{33}^{(r)}, & 2\bar{a}_{33}^{(1)} \bar{a}_{33}^{(r)}, & 0, & \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(r)}, & 2\bar{a}_{33}^{(1)} \bar{a}_{3j}^{(2)} \bar{a}_{j3}^{(r)} \end{array} \right\|$$

$$(r = 2, 3, \dots, R),$$

$$\mathbf{B}_{2r} = \left\| \begin{array}{cccccc} 0, & 0, & 0, & \bar{a}_{11}^{(r)}, & \bar{a}_{11}^{(1)} \bar{a}_{11}^{(r)}, & (\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(r)} \\ 0, & 0, & 0, & \bar{a}_{22}^{(r)}, & \bar{a}_{22}^{(1)} \bar{a}_{22}^{(r)}, & (\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(r)} \\ 0, & 0, & 0, & \bar{a}_{33}^{(r)}, & \bar{a}_{33}^{(1)} \bar{a}_{33}^{(r)}, & (\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(r)} \\ 0, & 0, & 0, & 2\bar{a}_{23}^{(r)}, & \bar{a}_{23}^{(r)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{23}^{(r)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\ 0, & 0, & 0, & 2\bar{a}_{31}^{(r)}, & \bar{a}_{31}^{(r)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{31}^{(r)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\ 0, & 0, & 0, & 2\bar{a}_{12}^{(r)}, & \bar{a}_{12}^{(r)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & \bar{a}_{12}^{(r)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2] \end{array} \right\|$$

$$(r = 3, 4, \dots, R),$$

$$\mathbf{B}_{rr} = \left\| \begin{array}{cccccc} 1, & \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2, & \bar{a}_{11}^{(2)}, & \bar{a}_{11}^{(1)} \bar{a}_{11}^{(2)}, & (\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(2)} \\ 1, & \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2, & \bar{a}_{22}^{(2)}, & \bar{a}_{22}^{(1)} \bar{a}_{22}^{(2)}, & (\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(2)} \\ 1, & \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2, & \bar{a}_{33}^{(2)}, & \bar{a}_{33}^{(1)} \bar{a}_{33}^{(2)}, & (\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(2)} \\ 0, & 0, & 0, & 2\bar{a}_{23}^{(2)}, & \bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & \bar{a}_{23}^{(2)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\ 0, & 0, & 0, & 2\bar{a}_{31}^{(2)}, & \bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & \bar{a}_{31}^{(2)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\ 0, & 0, & 0, & 2\bar{a}_{12}^{(2)}, & \bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & \bar{a}_{12}^{(2)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2] \end{array} \right\|$$

$$(r = 3, 4, \dots, R),$$

and

$$\mathbf{B}_{22} = \left\| \begin{array}{cccccc} 1, & \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2, & 2\bar{a}_{11}^{(2)}, & 2\bar{a}_{11}^{(1)} \bar{a}_{11}^{(2)}, & 2(\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(2)} \\ 1, & \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2, & 2\bar{a}_{22}^{(2)}, & 2\bar{a}_{22}^{(1)} \bar{a}_{22}^{(2)}, & 2(\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(2)} \\ 1, & \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2, & 2\bar{a}_{33}^{(2)}, & 2\bar{a}_{33}^{(1)} \bar{a}_{33}^{(2)}, & 2(\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(2)} \\ 0, & 0, & 0, & 4\bar{a}_{23}^{(2)}, & 2\bar{a}_{23}^{(2)}(\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}), & 2\bar{a}_{23}^{(2)}[(\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2] \\ 0, & 0, & 0, & 4\bar{a}_{31}^{(2)}, & 2\bar{a}_{31}^{(2)}(\bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}), & 2\bar{a}_{31}^{(2)}[(\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2] \\ 0, & 0, & 0, & 4\bar{a}_{12}^{(2)}, & 2\bar{a}_{12}^{(2)}(\bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}), & 2\bar{a}_{12}^{(2)}[(\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2] \end{array} \right\|.$$

$$(35.4)$$

A necessary and sufficient condition for functional independence of the $6R - 3$ scalar invariants defined by equations (33.2) and (35.2) is that $\det \mathbf{J} \neq 0$. Now, from (35.3) and (35.4), we have

$$\begin{aligned}
\det \mathbf{J} &= \det \mathbf{B}_{11} \prod_{r=2}^R \det \mathbf{B}_{rr} \\
&= 24(2^R) \begin{vmatrix} 1, & \bar{a}_{11}^{(1)}, & (\bar{a}_{11}^{(1)})^2 \\ 1, & \bar{a}_{22}^{(1)}, & (\bar{a}_{22}^{(1)})^2 \\ 1, & \bar{a}_{33}^{(1)}, & (\bar{a}_{33}^{(1)})^2 \end{vmatrix}^R \begin{vmatrix} 1, & \bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)}, & (\bar{a}_{22}^{(1)})^2 + (\bar{a}_{33}^{(1)})^2 \\ 1, & \bar{a}_{33}^{(1)} + \bar{a}_{11}^{(1)}, & (\bar{a}_{33}^{(1)})^2 + (\bar{a}_{11}^{(1)})^2 \\ 1, & \bar{a}_{11}^{(1)} + \bar{a}_{22}^{(1)}, & (\bar{a}_{11}^{(1)})^2 + (\bar{a}_{22}^{(1)})^2 \end{vmatrix}^{R-1} \\
&\quad \times (\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)})^{R-1} \\
&= 24(2^R) [(\bar{a}_{22}^{(1)} - \bar{a}_{33}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})]^{2R-1} (\bar{a}_{23}^{(2)} \bar{a}_{31}^{(2)} \bar{a}_{12}^{(2)})^R.
\end{aligned} \tag{35.5}$$

Thus, the $6R - 3$ scalar invariants defined by equations (33.1) and (35.1) will be functionally independent if and only if the eigen-values of $\bar{\mathbf{A}}_1$ are all unequal and none of the off-diagonal components of $\bar{\mathbf{A}}_2$ are zero. If these conditions are satisfied by the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ then it can be seen that an arbitrary scalar invariant formed from the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ is functionally related to the set of scalar invariants defined by equations (33.1) and (35.1).

It has already been seen in §33 that if the nine scalar invariants $\mathcal{Q}_1, \mathcal{Q}_{11}, \mathcal{Q}_{111}, \mathcal{Q}_2, \mathcal{Q}_{12}, \mathcal{Q}_{112}, \mathcal{Q}_{22}, \mathcal{Q}_{122}$ and $\mathcal{Q}_{1122}$ are specified, then the elements of $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are not, in general, uniquely determined. However, if in addition to these scalar invariants, the scalar invariant Δ_2 , defined by (33.15), is specified, they are uniquely determined apart from the effects of certain orthogonal transformations applied simultaneously to $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$. These are of the types described by (33.13) and (33.14). If, furthermore, the $6R - 12$ scalar invariants described by equations (35.1) or (35.2) are specified, then the elements of $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_R$ are uniquely determined, apart from the effects of orthogonal transformations of the types (33.13) and (33.14) applied simultaneously to $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_R$. From the expressions (35.2) for $\mathcal{Q}_r, \mathcal{Q}_{1r}$ and $\mathcal{Q}_{11r}$, we have

$$\bar{a}_{11}^{(r)} = - \frac{\mathcal{Q}_{11r} - (\bar{a}_{22}^{(1)} + \bar{a}_{33}^{(1)})\mathcal{Q}_{1r} + \bar{a}_{22}^{(1)} \bar{a}_{33}^{(1)} \mathcal{Q}_r}{(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})}. \tag{35.6}$$

$\bar{a}_{22}^{(r)}$ and $\bar{a}_{33}^{(r)}$ are obtained from (35.6) by cyclic permutation of $\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$. From the expressions (35.2) for $\mathcal{Q}_{2r}, \mathcal{Q}_{12r}$ and $\mathcal{Q}_{112r}$, we obtain

$$\bar{a}_{23}^{(2)} \bar{a}_{33}^{(r)} = - \frac{[\beta \mathcal{Q}_1 - \gamma - \frac{1}{2} \alpha (\mathcal{Q}_1^2 - \mathcal{Q}_{11})] - \beta \bar{a}_{11}^{(1)} + \alpha (\bar{a}_{11}^{(1)})^2}{(\bar{a}_{11}^{(1)} - \bar{a}_{22}^{(1)})(\bar{a}_{33}^{(1)} - \bar{a}_{11}^{(1)})}, \tag{35.7}$$

where

$$\begin{aligned}
\alpha &= \frac{1}{2} [\mathcal{Q}_{2r} - \bar{a}_{11}^{(2)} \bar{a}_{11}^{(r)} - \bar{a}_{22}^{(2)} \bar{a}_{22}^{(r)} - \bar{a}_{33}^{(2)} \bar{a}_{33}^{(r)}], \\
\beta &= \mathcal{Q}_{12r} - \bar{a}_{11}^{(1)} \bar{a}_{11}^{(2)} \bar{a}_{11}^{(r)} - \bar{a}_{22}^{(1)} \bar{a}_{22}^{(2)} \bar{a}_{22}^{(r)} - \bar{a}_{33}^{(1)} \bar{a}_{33}^{(2)} \bar{a}_{33}^{(r)} \\
&\text{and}
\end{aligned} \tag{35.8}$$

$$\gamma = \mathcal{Q}_{112r} - (\bar{a}_{11}^{(1)})^2 \bar{a}_{11}^{(2)} \bar{a}_{11}^{(r)} - (\bar{a}_{22}^{(1)})^2 \bar{a}_{22}^{(2)} \bar{a}_{22}^{(r)} - (\bar{a}_{33}^{(1)})^2 \bar{a}_{33}^{(2)} \bar{a}_{33}^{(r)}.$$

Thus, if the elements of $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are uniquely determined, then the elements of $\bar{\mathbf{A}}_3, \bar{\mathbf{A}}_4, \dots, \bar{\mathbf{A}}_R$ are uniquely determined provided $\bar{a}_{11}^{(1)}, \bar{a}_{22}^{(1)}$ and $\bar{a}_{33}^{(1)}$ are all unequal and none of the elements $\bar{a}_{23}^{(2)}, \bar{a}_{31}^{(2)}, \bar{a}_{12}^{(2)}$ are zero. It can readily be seen that if $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are transformed by the orthogonal transformations (33.13) or (33.14), then the new matrices $\bar{\mathbf{A}}_3, \bar{\mathbf{A}}_4, \dots, \bar{\mathbf{A}}_R$ given by equations (35.6) and (35.7) are also obtained by the application of the same transformation to the original matrices.

It is thus seen that provided the eigen-values of $\mathbf{A}_1$ are all unequal and none of the off-diagonal components of $\mathbf{A}_2$ are zero, then Δ_2 and the $6R - 3$ scalar invariants defined by equations (33.1) and (35.1) determine the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ uniquely apart from the effects of an orthogonal transformation applied simultaneously to all R matrices. Therefore, provided the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the specified conditions, any scalar invariant formed from the elements of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ must be expressible as a single-valued function of Δ_2 and the $6R - 3$ scalar invariants defined by equations (33.1) and (35.1). This set of invariants is not functionally independent, but the set obtained from this by deleting Δ_2 is.

If one of the off-diagonal components of $\bar{\mathbf{A}}_2$ is zero while the remaining two are non-zero and the eigen-values of $\mathbf{A}_1$ are all unequal, then the set of $5R - 2$ scalar invariants formed by omitting the scalar invariants $\mathcal{Q}_{112r}$ ($r = 2, 3, \dots, R$) from the set of $6R - 3$ scalar invariants defined by equations (33.1) and (35.1) are functionally independent. However, an arbitrary scalar invariant formed from the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ is not necessarily functionally related to this set of $5R - 2$ scalar invariants. Again, if any two of the off-diagonal components of $\bar{\mathbf{A}}_2$ are zero, the remaining off-diagonal component of $\bar{\mathbf{A}}_2$ being non-zero, and the eigen-values of $\mathbf{A}_1$ are all unequal, then the set of $4R - 1$ scalar invariants formed by omitting the scalar invariants $\mathcal{Q}_{112r}$ and $\mathcal{Q}_{12r}$ ($r = 2, 3, \dots, R$) from the set of $6R - 3$ scalar invariants defined by equations (33.1) and (35.1) are functionally independent, but an arbitrary scalar invariant formed from the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ is not necessarily functionally related to these. Further, if all three of the off-diagonal components of $\bar{\mathbf{A}}_2$ are zero, but the eigen-values of $\mathbf{A}_1$ are all unequal, the set of $3R$ scalar invariants formed by omitting $\mathcal{Q}_{2r}, \mathcal{Q}_{12r}$ and $\mathcal{Q}_{112r}$ ($r = 2, 3, \dots, R$) is functionally independent, but an arbitrary scalar invariant formed from the elements of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$ is not necessarily functionally related to these.

CHAPTER V. REPRESENTATIONS FOR THE STRESS MATRIX IN TERMS OF KINEMATIC MATRICES

36. Introduction. It has been seen in §15 that if the components of stress t_{ij} at a point of a body of isotropic material undergoing deformation are assumed to depend only on the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration at that point and at the instant of time considered, then the stress must be expressible as an isotropic tensor function of $n + 1$ tensors $C_{kl}, A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}$. These are defined in terms of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration gradients referred to a fixed rectangular Cartesian co-ordinate system x_i by

$$C_{kl} = \frac{\partial x_k}{\partial X_j} \frac{\partial x_l}{\partial X_j},$$

and

$$A_{kl}^{(r)} = 2d_{kl}^{(r)} + \sum_{s=1}^{r-1} \binom{r}{s} \frac{\partial v_i^{(r-s)}}{\partial x_k} \frac{\partial v_i^{(s)}}{\partial x_l} \quad (r = 1, 2, \dots, n), \quad (36.1)$$

where

$$d_{kl}^{(r)} = \frac{1}{2} \left(\frac{\partial v_k^{(r)}}{\partial x_l} + \frac{\partial v_l^{(r)}}{\partial x_k} \right). \quad (36.2)$$

Here $v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(n)}$ denote the velocity, acceleration, second acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration respectively of a material particle in the axial system x_i at the instant of time considered. Also, x_i denotes the position of a material particle at time t , which was at X_i before the deformation of the body commenced. The displacement of this material particle at time t is thus $x_i - X_i$. We shall now employ the notation

$$\mathbf{T} = \| t_{ij} \|, \quad \mathbf{C} = \| C_{kl} \| \quad \text{and} \quad \mathbf{A}_r = \| A_{kl}^{(r)} \| \quad (r = 1, 2, \dots, n). \quad (36.3)$$

We shall call $\mathbf{T}$ the stress matrix, $\mathbf{C}$ the strain matrix, $\mathbf{A}_1$ the velocity-strain matrix, $\mathbf{A}_2$ the acceleration-strain matrix and $\mathbf{A}_r$ ($r = 3, 4, \dots, n$) the r^{th} acceleration strain matrix.

If the components of stress t_{ij} at a point of a body of isotropic material undergoing deformation are assumed to depend only on the gradients of velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration at that point and at the instant of time considered, then it is seen in §15 that the stress must be expressible as an isotropic tensor function of the n tensors $A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}$ defined by (36.1).

From the results obtained in chapters III and IV, it can be shown that a linear relation can be established between the stress matrix and certain simple isotropic matrix functions of the kinematic matrices, the coefficients in this relation being expressible as scalar invariants of the kinematic matrices. In all the linear relations discussed here, the coefficient of $\mathbf{T}$ is zero if certain relations between the kinematic matrices obtain. It will, however, be shown that it is possible to obtain a linear relation between the stress matrix and a set of isotropic matrix functions of the kinematic matrices which provides a representation for $\mathbf{T}$ except under conditions which can exist in the volume element under consideration only instantaneously.

37. Relations between the stress matrix and the kinematic matrices. We shall first assume that the components of the stress t_{ij} depend only on the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration. Then, it is seen from §15 that the stress tensor t_{ij} is an isotropic tensor function of the $n + 1$ tensors C_{kl} , $A_{kl}^{(1)}$, $A_{kl}^{(2)}$, $\dots$, $A_{kl}^{(n)}$ defined by equations (36.1). From the definition of an isotropic matrix function of a number of matrices, given in §21, it is seen that the stress matrix $\mathbf{T}$ is an isotropic matrix function of the $n + 1$ kinematic matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$.

We see immediately from §31(ii) that a relation of the form

$$\begin{aligned} \varphi \mathbf{T} + \varphi_1 \mathbf{C} + \varphi_2 \mathbf{C}^2 + \varphi_3 \mathbf{A}_1 + \varphi_4 \mathbf{A}_1^2 + \varphi_5 (\mathbf{C} \mathbf{A}_1 + \mathbf{A}_1 \mathbf{C}) \\ + \varphi_6 (\mathbf{C}^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{C}^2) + \varphi_7 \mathbf{I} = 0 \end{aligned} \quad (37.1)$$

is valid, in which φ , φ_1 , φ_2 , $\dots$, φ_7 are scalar invariants of $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ and $\varphi = 0$ if and only if $\mathbf{C}$ has a pair of equal eigen-values or $\mathbf{C}$ and $\mathbf{A}_1$ have a principal direction in common.

We can classify the relations between $\mathbf{C}$ and $\mathbf{A}_1$ for which $\varphi = 0$ in equation (37.1) in the following manner:

- (a) no two eigen-values of $\mathbf{C}$ are equal and
 - (i) $\mathbf{C}$ and $\mathbf{A}_1$ have one and only one principal direction in common, or
 - (ii) $\mathbf{C}$ and $\mathbf{A}_1$ have three principal directions in common;
- (b) $\mathbf{C}$ has a single pair of equal eigen-values but no two eigen-values of $\mathbf{A}_1$ are equal and
 - (i) $\mathbf{C}$ and $\mathbf{A}_1$ have no principal direction in common, or
 - (ii) $\mathbf{C}$ and $\mathbf{A}_1$ have a principal direction in common which is perpendicular to the plane defined by the pair of equal eigen-values of $\mathbf{C}$, or
 - (iii) $\mathbf{C}$ and $\mathbf{A}_1$ have a single principal direction in common which lies in the plane defined by the pair of equal eigen-values of $\mathbf{C}$;
- (c) $\mathbf{C}$ has a single pair of equal eigen-values and $\mathbf{A}_1$ has two equal eigen-values;
- (d) all three eigen-values of $\mathbf{C}$ are equal.

Now, let us consider in each case that the relations persist throughout a finite interval of time.

In case *a(i)* it is seen from §18(i) that $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have a principal

direction in common. From §30, it follows that $\mathbf{T}$, $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have a principal direction in common. Hence, from §31(v) we see that a relation of the form (37.1) with $\varphi \neq 0$ and $\varphi_4 = \varphi_5 = \varphi_6 = 0$ is valid.

In case $a(ii)$, it is seen from §18(i) that $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have all their principal directions in common. From §30, it follows that $\mathbf{T}$, $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have all their principal directions in common. Hence, from §31(vii) we see that a relation of the form (37.1) with $\varphi \neq 0$ and $\varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$ is valid.

In case c , it is seen from §18(ii) that each of the matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ has a pair of equal eigen-values and that the principal planes defined by these pairs coincide. It follows from §30, that $\mathbf{T}$ also has a pair of equal eigen-values and that the principal plane defined by these coincides with the principal plane defined by the pairs of equal eigen-values of $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$. From §31(viii) we see that a relation of the form (37.1) with $\varphi \neq 0$ and $\varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$ is valid.

In case d , it is seen from §18(ii) that each of the matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ has all three of its eigen-values equal. It follows from §30 that the three eigen-values of $\mathbf{T}$ are equal and from §31(ix) that a relation of the form (37.1) with $\varphi \neq 0$ and $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$ is valid.

In case $b(ii)$ it is seen from §18(iv) that each of the matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ has a pair of equal eigen-values and that the principal planes defined by these pairs of equal eigen-values coincide. This implies that the relation $b(ii)$ between the matrices cannot persist throughout a finite interval of time. It follows from §30 that $\mathbf{T}$ also has a pair of equal eigen-values and that the principal planes defined by these coincides with the principal planes defined by the equal eigen-values of $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$. From §31(viii) we see that a relation of the form (37.1) with $\varphi \neq 0$ and $\varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$ is valid.

From the foregoing results it is seen that a relation of the form (37.1) with $\varphi \neq 0$ is valid, except perhaps instantaneously, unless the condition $b(i)$ or $b(iii)$ exists throughout a finite time interval.

We can readily obtain a relation between the stress matrix $\mathbf{T}$ and the kinematic matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$, the validity of which, as a representation for $\mathbf{T}$, is not limited by the condition $b(i)$. From §31(iii), we see that a relation of the form

$$\begin{aligned} \varphi \mathbf{T} + \varphi_1 \mathbf{C} + \varphi_2 \mathbf{C}^2 + \varphi_3 \mathbf{A}_1 + \varphi_4 \mathbf{A}_1^2 + \varphi_5 (\mathbf{A}_1 \mathbf{C} + \mathbf{C} \mathbf{A}_1) \\ + \varphi_6 (\mathbf{C}^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{C}^2) + \varphi_7 (\mathbf{A}_1^2 \mathbf{C} + \mathbf{C} \mathbf{A}_1^2) + \varphi_8 \mathbf{I} = 0, \end{aligned} \quad (37.2)$$

with $\varphi \neq 0$ is valid provided that both $\mathbf{C}$ and $\mathbf{A}_1$ do not have a pair of equal eigen-values and $\mathbf{C}$ and $\mathbf{A}_1$ do not have a principal direction in common. Thus, $\varphi \neq 0$ if the relation $b(i)$ exists between the matrices $\mathbf{C}$ and $\mathbf{A}_1$. Since the relation (37.1) is a particular case of (37.2) we readily see that a relation of the form (37.2) with $\varphi \neq 0$ is valid, except perhaps instantaneously, unless throughout a finite interval of time the relation $b(iii)$ exists between the matrices $\mathbf{C}$ and $\mathbf{A}_1$.

We shall now obtain a relation between the stress matrix $\mathbf{T}$ and the kinematic

matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2, \dots, \mathbf{A}_n$ which is always valid, except instantaneously for certain relations between the kinematic matrices. We see from §31(iv) that a relation of the form

$$\begin{aligned} \varphi \mathbf{T} + \varphi_1 \mathbf{C} + \varphi_2 \mathbf{C}^2 + \varphi_3 \mathbf{A}_1 + \varphi_4 \mathbf{A}_1^2 + \varphi_5 (\mathbf{A}_1 \mathbf{C} + \mathbf{C} \mathbf{A}_1) + \varphi_6 (\mathbf{C}^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{C}^2) \\ + \varphi_7 (\mathbf{A}_1^2 \mathbf{C} + \mathbf{C} \mathbf{A}_1^2) + \varphi_8 \mathbf{A}_2 + \varphi_9 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \varphi_{10} \mathbf{I} = 0, \end{aligned} \quad (37.3)$$

with $\varphi \neq 0$, is valid unless the condition

$$\begin{vmatrix} C_{11}, & A_{11}^{(1)}, & A_{1j}^{(1)} A_{j1}^{(1)}, & A_{11}^{(2)}, & 2A_{1j}^{(1)} A_{j1}^{(2)}, & 1 \\ C_{22}, & A_{22}^{(1)}, & A_{2j}^{(1)} A_{j2}^{(1)}, & A_{22}^{(2)}, & 2A_{2j}^{(1)} A_{j2}^{(2)}, & 1 \\ C_{33}, & A_{33}^{(1)}, & A_{3j}^{(1)} A_{j3}^{(1)}, & A_{33}^{(2)}, & 2A_{3j}^{(1)} A_{j3}^{(2)}, & 1 \\ C_{23}, & A_{23}^{(1)}, & A_{2j}^{(1)} A_{j3}^{(1)}, & A_{23}^{(2)}, & A_{2j}^{(1)} A_{j3}^{(2)} + A_{2j}^{(2)} A_{j3}^{(1)}, & 0 \\ C_{31}, & A_{31}^{(1)}, & A_{3j}^{(1)} A_{j1}^{(1)}, & A_{31}^{(2)}, & A_{3j}^{(1)} A_{j1}^{(2)} + A_{3j}^{(2)} A_{j1}^{(1)}, & 0 \\ C_{12}, & A_{12}^{(1)}, & A_{1j}^{(1)} A_{j2}^{(1)}, & A_{12}^{(2)}, & A_{1j}^{(1)} A_{j2}^{(2)} + A_{1j}^{(2)} A_{j2}^{(1)}, & 0 \end{vmatrix} = 0 \quad (37.4)$$

is satisfied and either each of $\mathbf{C}$ and $\mathbf{A}_1$ has a pair of equal eigen-values or $\mathbf{C}$ and $\mathbf{A}_1$ have a principal direction in common.

We see immediately that a relation of the form (37.3), with $\varphi \neq 0$, is valid unless the condition $b(iii)$ is satisfied by $\mathbf{C}$ and $\mathbf{A}_1$, since equation (37.2) is a special case of the relation (37.3). By an appropriate orthogonal transformation, we can transform $\mathbf{C}$ into the diagonal matrix $\bar{\mathbf{C}} = (\bar{C}_{11}, \bar{C}_{22}, \bar{C}_{33})$ say. Let us suppose that the same orthogonal transformation transforms $\mathbf{A}_1$ and $\mathbf{A}_2$ into the symmetric matrices $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ respectively, where $\bar{\mathbf{A}}_1 = \|\bar{A}_{ij}^{(1)}\|$ and $\bar{\mathbf{A}}_2 = \|\bar{A}_{ij}^{(2)}\|$. Then, if the condition $b(iii)$ is satisfied by the $\mathbf{C}$ and $\mathbf{A}_1$, we may take $\bar{C}_{11} = \bar{C}_{22}$ and $\bar{A}_{12}^{(1)} = \bar{A}_{13}^{(1)} = 0$. Writing the condition (37.4) in terms of the elements of $\bar{\mathbf{C}}$, $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$, instead of the elements of $\mathbf{C}$, $\mathbf{A}_1$ and $\mathbf{A}_2$, it becomes

$$\begin{vmatrix} \bar{C}_{11}, & \bar{A}_{11}^{(1)}, & (\bar{A}_{11}^{(1)})^2, & \bar{A}_{11}^{(2)}, & 2\bar{A}_{1j}^{(1)} \bar{A}_{j1}^{(2)}, & 1 \\ \bar{C}_{11}, & \bar{A}_{22}^{(1)}, & (\bar{A}_{22}^{(1)})^2 + (\bar{A}_{23}^{(1)})^2, & \bar{A}_{22}^{(2)}, & 2\bar{A}_{2j}^{(1)} \bar{A}_{j2}^{(2)}, & 1 \\ \bar{C}_{33}, & \bar{A}_{33}^{(1)}, & (\bar{A}_{33}^{(1)})^2 + (\bar{A}_{23}^{(1)})^2, & \bar{A}_{33}^{(2)}, & 2\bar{A}_{3j}^{(1)} \bar{A}_{j3}^{(2)}, & 1 \\ 0, & \bar{A}_{23}^{(1)}, & \bar{A}_{23}^{(1)}(\bar{A}_{22}^{(1)} + \bar{A}_{33}^{(1)}), & \bar{A}_{23}^{(2)}, & \bar{A}_{2j}^{(1)} \bar{A}_{j3}^{(2)} + \bar{A}_{2j}^{(2)} \bar{A}_{j3}^{(1)}, & 0 \\ 0, & 0, & 0, & \bar{A}_{31}^{(2)}, & \bar{A}_{3j}^{(1)} \bar{A}_{j1}^{(2)} + \bar{A}_{3j}^{(2)} \bar{A}_{j1}^{(1)}, & 0 \\ 0, & 0, & 0, & \bar{A}_{12}^{(2)}, & \bar{A}_{1j}^{(1)} \bar{A}_{j2}^{(2)} + \bar{A}_{1j}^{(2)} \bar{A}_{j2}^{(1)}, & 0 \end{vmatrix} = 0, \quad (37.5)$$

i.e. either

$$\begin{vmatrix} \bar{C}_{11}, & \bar{A}_{11}^{(1)}, & (\bar{A}_{11}^{(1)})^2, & 1 \\ \bar{C}_{11}, & \bar{A}_{22}^{(1)}, & (\bar{A}_{22}^{(1)})^2 + (\bar{A}_{23}^{(1)})^2, & 1 \\ \bar{C}_{33}, & \bar{A}_{33}^{(1)}, & (\bar{A}_{33}^{(1)})^2 + (\bar{A}_{23}^{(1)})^2, & 1 \\ 0, & \bar{A}_{23}^{(1)}, & \bar{A}_{23}^{(1)}(\bar{A}_{22}^{(1)} + \bar{A}_{33}^{(1)}), & 0 \end{vmatrix} = 0, \quad (37.6)$$

or

$$\begin{vmatrix} \bar{A}_{31}^{(2)}, & \bar{A}_{3j}^{(1)} \bar{A}_{j1}^{(2)} + \bar{A}_{3j}^{(2)} \bar{A}_{j1}^{(1)} \\ \bar{A}_{12}^{(2)}, & \bar{A}_{1j}^{(1)} \bar{A}_{j2}^{(2)} + \bar{A}_{1j}^{(2)} \bar{A}_{j2}^{(1)} \end{vmatrix} = 0. \quad (37.7)$$

Now, it was seen in §18(iii) that if the condition $b(iii)$ persists over a finite time interval, $\bar{A}_{11}^{(1)} = \bar{A}_{22}^{(1)}$ and $\bar{A}_{12}^{(1)} = \bar{A}_{13}^{(1)} = \bar{A}_{12}^{(2)} = 0$. With these relations, the condition expressed by equation (37.6) becomes

$$(\bar{A}_{23}^{(1)})^3 (\bar{C}_{11} - \bar{C}_{33}) = 0,$$

i.e.

$$\bar{A}_{23}^{(1)} = 0 \quad \text{or} \quad \bar{C}_{11} = \bar{C}_{33}. \quad (37.8)$$

Similarly, the condition expressed by equation (37.7) becomes

$$(\bar{A}_{13}^{(2)})^2 \bar{A}_{32}^{(1)} = 0,$$

i.e.

$$\bar{A}_{13}^{(2)} = 0 \quad \text{or} \quad \bar{A}_{32}^{(1)} = 0. \quad (37.9)$$

Now, if $\bar{A}_{32}^{(1)} = 0$, $\mathbf{C}$ and $\mathbf{A}_1$ have all their principal directions in common, in violation of condition $b(iii)$, in which case we have already seen that the relation (37.3) provides a representation for $\mathbf{T}$.

If, on the other hand, $\bar{A}_{13}^{(2)} = 0$, $\mathbf{C}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ have a principal direction in common. Provided that the conditions which lead to this conclusion persist throughout a finite interval of time, we see from the result of §18(iii) that $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have a principal direction in common. From §30, we see that this implies that $\mathbf{T}$, $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\dots$, $\mathbf{A}_n$ have a principal direction in common.

We have already seen that in this case we can obtain a representation for $\mathbf{T}$ of the form (37.3).

If $\bar{C}_{11} = \bar{C}_{33}$, then all three eigen-values of $\mathbf{C}$ are equal in violation of condition $b(iii)$, so that in this case again we can obtain a representation for $\mathbf{T}$ of the form (37.3).

We thus conclude that if the components of stress t_{ij} depend only on the gradients of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration, then there exists a relation of the form (37.3) with $\varphi \neq 0$, except perhaps at isolated instants of

time. The coefficients $\varphi, \varphi_1, \dots, \varphi_{10}$ will, of course, be scalar invariants of the $n + 1$ matrices $\mathbf{C}, \mathbf{A}_1, \dots, \mathbf{A}_n$.

In a similar manner it can be shown that if the components of stress depend only on the gradients of the velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration, and if $n \geq 3$, then there exists a relation of the form

$$\begin{aligned} \varphi \mathbf{T} + \varphi_1 \mathbf{A}_1 + \varphi_2 \mathbf{A}_1^2 + \varphi_3 \mathbf{A}_2 + \varphi_4 \mathbf{A}_2^2 + \varphi_5 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ + \varphi_6 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \varphi_7 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) + \varphi_8 \mathbf{A}_3 \\ + \varphi_9 (\mathbf{A}_2 \mathbf{A}_3 + \mathbf{A}_3 \mathbf{A}_2) + \varphi_{10} \mathbf{I} = 0, \end{aligned} \quad (37.10)$$

where $\varphi \neq 0$, except perhaps at isolated instants of time, and $\varphi, \varphi_1, \dots, \varphi_{10}$ are scalar invariants of the matrices $\mathbf{A}_1, \dots, \mathbf{A}_n$.

It is, of course, apparent that any relation of the form (37.3) implies that the stress components are expressible as single-valued functions of the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration and any relation of the form (37.10) implies that the stress components are expressible as single-valued functions of the gradients of velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration.

If the components of the stress t_{ij} depend on the gradients of the displacement and velocity only, then it follows that the stress matrix $\mathbf{T}$ must be an isotropic matrix function of the kinematic matrices $\mathbf{C}$ and $\mathbf{A}_1$ only. It is seen immediately, from the result obtained in Appendix I, that a representation of $\mathbf{T}$ of the form (37.2) may always be obtained. The scalar coefficients in such a representation may, of course, be expressed as scalar invariants of $\mathbf{C}$ and $\mathbf{A}_1$ only. In a similar manner, it can be seen that if the stress components t_{ij} depend on the gradients of the velocity and acceleration only, then a representation of $\mathbf{T}$ of the form (37.10), with $\varphi_8 = \varphi_9 = 0$, may always be obtained. The scalar coefficients in this representation can, of course, be expressed as scalar invariants of $\mathbf{A}_1$ and $\mathbf{A}_2$ only.

38. The coefficients in the representations for the stress matrix. It was seen in §37 that if we assume for an isotropic material that the stress components in a rectangular Cartesian co-ordinate system are single-valued functions of the gradients of displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration, then the stress matrix $\mathbf{T}$ may be expressed by equation (37.3) as a linear combination of certain simple isotropic matrix functions of the kinematic matrices $\mathbf{C}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ defined by equations (36.1), (36.2) and (36.3). The coefficients $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{10}$ in equation (37.3) are scalar invariants of the matrices $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2, \dots, \mathbf{A}_n$ defined by equations (36.1), (36.2) and (36.3). It has been shown in §37 that $\varphi \neq 0$, except perhaps instantaneously.

Now, it has been seen from the results of §35 that, provided certain conditions are satisfied by the matrices $\mathbf{C}$ and $\mathbf{A}_1$, any scalar invariant of the matrices $\mathbf{C}, \mathbf{A}_1, \dots, \mathbf{A}_n$ may be expressed as a single-valued function of the scalar in-

variants $\text{tr } \mathbf{C}$, $\text{tr } \mathbf{C}^2$, $\text{tr } \mathbf{C}^3$, $\text{tr } \mathbf{A}_r$, $\text{tr } \mathbf{C} \mathbf{A}_r$, $\text{tr } \mathbf{C}^2 \mathbf{A}_r$, $\text{tr } \mathbf{A}_1 \mathbf{A}_r$, $\text{tr } \mathbf{C} \mathbf{A}_1 \mathbf{A}_r$, $\text{tr } \mathbf{C}^2 \mathbf{A}_1 \mathbf{A}_r$ ($r = 1, 2, \dots, n$) and $\det \mathbf{A}_1$. The conditions under which it has been proven that any scalar invariant of $\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2, \dots, \mathbf{A}_n$ may be so expressed are:

- (i) no two eigen-values of $\mathbf{C}$ are equal and
- (ii) if $\mathbf{C}$ and $\mathbf{A}_1$ are referred to the principal directions of $\mathbf{C}$, then none of the off-diagonal components of $\mathbf{A}_1$ is zero.

It has been seen in §18 that if certain relations exist between elements of the matrices $\mathbf{C}$ and $\mathbf{A}_1$ throughout a finite interval of time, then certain restrictions are thereby imposed on the matrices $\mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$. It would no doubt be possible, by taking advantage of this fact, to obtain a set of scalar invariants of $\mathbf{C}, \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ in terms of which an arbitrary scalar invariant of the matrices $\mathbf{C}, \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ could be expressed as a single-valued function, except under conditions existing only instantaneously. We shall not, however, pursue this course further in the present paper.

If the stress components in a rectangular Cartesian co-ordinate system are single-valued functions of the gradients of velocity, acceleration, $\dots$, $(n-1)^{\text{th}}$ acceleration, then the stress matrix $\mathbf{T}$ may be expressed by equation (37.10). The coefficients $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{10}$ in this equation are scalar invariants of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$. $\varphi \neq 0$, except perhaps instantaneously. In this case, it follows from the results of §35 that, provided certain conditions are satisfied by the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, any scalar invariant of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ may be expressed as a single-valued function of the scalar invariants $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$, $\text{tr } \mathbf{A}_1^3$, $\text{tr } \mathbf{A}_1 \mathbf{A}_r$, $\text{tr } \mathbf{A}_1^2 \mathbf{A}_r$, $\text{tr } \mathbf{A}_2 \mathbf{A}_r$, $\text{tr } \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_r$, $\text{tr } \mathbf{A}_1^2 \mathbf{A}_2 \mathbf{A}_r$ ($r = 1, 2, \dots, n$) and $\det \mathbf{A}_2$. The conditions under which this result has been proven are identical with (i) and (ii), if $\mathbf{C}$ and $\mathbf{A}_1$ are replaced by $\mathbf{A}_1$ and $\mathbf{A}_2$.

39. Discussion of some special cases.

- (i) *Stress dependent on the displacement gradients only.*

If the stress components t_{ij} in a rectangular Cartesian coordinate system are dependent only on the displacement gradients, then it has been seen in §13 that the stress must be expressible as an isotropic tensor function of the tensor C_{ij} defined in (36.1). It follows, of course, that the stress matrix $\mathbf{T}$ ($= \| t_{ij} \|$) must be an isotropic matrix function of $\mathbf{C}$ ($= \| C_{ij} \|$) only and consequently the principal directions of $\mathbf{T}$ and $\mathbf{C}$ coincide. A linear relation can then be established between $\mathbf{T}$ and any three linearly independent isotropic matrix functions of $\mathbf{C}$. We have

$$\varphi \mathbf{T} + \varphi_1 \mathbf{C} + \varphi_2 \mathbf{C}^2 + \varphi_3 \mathbf{I} = 0, \quad (39.1)$$

where $\varphi, \varphi_1, \varphi_2$ and φ_3 are scalar invariants of $\mathbf{C}$ alone. Since any scalar invariant of $\mathbf{C}$ must be expressible as a single-valued function of the three scalar invariants $\text{tr } \mathbf{C}$, $\text{tr } \mathbf{C}^2$ and $\text{tr } \mathbf{C}^3$, we see that $\varphi, \varphi_1, \varphi_2$ and φ_3 must be single-valued functions of $\text{tr } \mathbf{C}$, $\text{tr } \mathbf{C}^2$ and $\text{tr } \mathbf{C}^3$. It can be readily shown that as a result of the fact that $\mathbf{T}$ is an isotropic matrix function of $\mathbf{C}$ only, any possible degeneracy in $\mathbf{C}$ (such

as the equality of a pair, or all three, of its eigen-values) is reflected in a corresponding degeneracy of $\mathbf{T}$. As a consequence of this it can be shown that a relation of the form (39.1), with $\varphi \neq 0$, must always be valid.

RICHTER (1948) has given a derivation of equations equivalent to (39.1) which bears some resemblance to, but is less complete than that given here.

(ii) *Stress dependent on velocity gradients only.*

It has been seen in §14 that if the stress components t_{ij} in a rectangular Cartesian coordinate system depend only on the velocity gradients, then the stress tensor must be an isotropic tensor function of the velocity-strain tensor $A_{ij}^{(1)}$ defined in (36.1). It follows, of course, that the stress matrix $\mathbf{T}$ must be an isotropic matrix function of $\mathbf{A}_1$ ($= \| A_{ij}^{(1)} \|$). By reasoning analogous to that employed in (i) we see that a relation of the form

$$\varphi \mathbf{T} + \varphi_1 \mathbf{A}_1 + \varphi_2 \mathbf{A}_1^2 + \varphi_3 \mathbf{I} = 0, \quad (39.2)$$

with $\varphi \neq 0$ exists, in which φ , φ_1 , φ_2 and φ_3 are single-valued functions of the three scalar invariants $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$ and $\text{tr } \mathbf{A}_1^3$.

Relations of the form (39.1) and (39.2) have been obtained previously on the basis of slightly different assumptions from those employed here. REINER (1945, 1948) has *assumed* that the stress matrix is expressible as a polynomial in a single kinematic matrix. If the kinematic matrix is taken as $\mathbf{A}_1$ (REINER 1945) then, by the use of the Hamilton-Cayley theorem, a relation of the form (39.2) is obtained in which $\varphi = 1$ and φ_1 , φ_2 and φ_3 are expressible as polynomials in $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$ and $\text{tr } \mathbf{A}_1^3$.

RIVLIN (1948) has shown that, if it is assumed that the principal directions of the stress and velocity-strain matrices coincide and the eigen-values of the stress matrix are expressible as polynomials in the eigen-values of the velocity strain matrix, then the stress matrix may be expressed in the form (39.2), with $\varphi = 1$, while φ_1 , φ_2 and φ_3 are polynomials in $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$ and $\text{tr } \mathbf{A}_1^3$.

It can be shown (see Appendix II) that if we assume the stress components t_{ij} in any rectangular Cartesian co-ordinate system are expressible as polynomials in the velocity-gradients only with constant coefficients then REINER's point of departure follows, if we take the scalar coefficients in his initial equation as polynomials in the basic set of invariants.

In an analogous manner, it can be shown that if we assume that the stress components t_{ij} in a rectangular Cartesian coordinate system are expressible as polynomials in the displacement gradients only with constant coefficients, then the stress matrix $\mathbf{T}$ is expressible as a polynomial in the matrix $\mathbf{C}$ with coefficients which are expressible as polynomials in the scalar invariants $\text{tr } \mathbf{C}$, $\text{tr } \mathbf{C}^2$ and $\text{tr } \mathbf{C}^3$.

40. Appendix I. Isotropic matrix functions of two symmetric matrices. It has been shown in §27 that if the symmetric 3×3 matrix $\mathbf{T}$ is an isotropic matrix function of R symmetric 3×3 matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_R$, then we can obtain a representation of $\mathbf{T}$ of the form

$$\begin{aligned} \alpha \mathbf{T} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2 + \alpha_4 \mathbf{A}_2^2 + \alpha_5 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ + \alpha_6 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \alpha_7 (\mathbf{A}_2^2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2^2) + \alpha_8 \mathbf{I} = 0 \end{aligned} \quad (40.1)$$

except perhaps when $\mathbf{A}_1$ and $\mathbf{A}_2$ have one or more principal directions in common.

It will be shown in this section that if $\mathbf{T}$ is an isotropic matrix function of only two symmetric 3×3 matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, then we can always obtain a representation of $\mathbf{T}$ of the form (40.1).

It is shown in §30 that, in this case, every common principal direction of $\mathbf{A}_1$ and $\mathbf{A}_2$ is a common principal direction of $\mathbf{T}$. Then, from §29(i), it follows that a representation of $\mathbf{T}$ of the form (40.1) can be obtained unless each of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ has a pair of equal eigen-values or $\mathbf{A}_1$ and $\mathbf{A}_2$ have all their principal directions in common.

If $\mathbf{A}_1$ and $\mathbf{A}_2$ have all their principal directions in common, then, from §30, each of the principal directions is a principal direction of $\mathbf{T}$. It then follows from §29(ii) that a representation of $\mathbf{T}$ of the form (40.1) can be obtained in this case.

We now have to consider the case when each of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ has two and only two equal eigen-values and the principal directions determined by these pairs of equal eigen-values do not coincide. In this case we can choose as a rectangular Cartesian reference system x_i one in which the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ take the forms

$$\mathbf{A}_1 = \begin{vmatrix} a_{11}^{(1)} & 0 & 0 \\ 0 & a_{11}^{(1)} & 0 \\ 0 & 0 & a_{33}^{(1)} \end{vmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{vmatrix} a_{11}^{(2)} & 0 & 0 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{23}^{(2)} & a_{33}^{(2)} \end{vmatrix}. \quad (40.2)$$

The x_1 -axis is chosen in the direction of the line of intersection of the principal planes corresponding to the pairs of equal eigen-values of the two matrices and the x_3 -axis is normal to the principal plane corresponding to the pair of equal eigen-values of $\mathbf{A}_1$. Since $\mathbf{A}_1$ has only one pair of equal eigen-values,

$$a_{11}^{(1)} \neq a_{33}^{(1)}.$$

Since $\mathbf{A}_2$ has a single pair of equal eigen-values and the x_1 -axis is parallel to the principal plane determined by these, two of the eigen-values of $\mathbf{A}_2$ must be equal to $a_{11}^{(2)}$ and the third must be different from $a_{11}^{(2)}$. This requires that

$$(a_{22}^{(2)} - a_{11}^{(2)})(a_{33}^{(2)} - a_{22}^{(2)}) - (a_{23}^{(2)})^2 = 0$$

and (40.3)

$$(a_{22}^{(2)} - a_{33}^{(2)})^2 \neq (a_{23}^{(2)})^2.$$

From §30, it is seen that the x_1 -axis is a principal direction of $\mathbf{T}$, so that in the coordinate system x_i , $\mathbf{T}$ may be expressed in the form

$$\mathbf{T} = \begin{bmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{bmatrix}. \quad (40.4)$$

From (40.2) it is apparent that

$$\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1 = \begin{bmatrix} 2a_{11}^{(1)} a_{11}^{(2)} & 0 & 0 \\ 0 & 2a_{11}^{(1)} a_{22}^{(2)} & a_{23}^{(2)}(a_{11}^{(1)} + a_{33}^{(1)}) \\ 0 & a_{23}^{(2)}(a_{11}^{(1)} + a_{33}^{(1)}) & 2a_{33}^{(1)} a_{33}^{(2)} \end{bmatrix}. \quad (40.5)$$

From (40.2), (40.5) and (40.4), it is seen that we can obtain a representation of the matrix $\mathbf{T}$ in terms of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1$ and $\mathbf{I}$, *i.e.* a representation of the form (40.1), provided that these are linearly independent. The condition for linear independence of these four matrices is

$$a_{23}^{(2)}(a_{22}^{(2)} - a_{11}^{(2)})(a_{11}^{(1)} - a_{33}^{(1)})^2 \neq 0.$$

Since we have assumed $a_{11}^{(1)} \neq a_{33}^{(1)}$ this condition is equivalent to

$$a_{23}^{(2)}(a_{22}^{(2)} - a_{11}^{(2)}) \neq 0. \quad (40.6)$$

If $a_{22}^{(2)} = a_{11}^{(2)}$, then, from (40.3), $a_{23}^{(2)} = 0$. This implies that $\mathbf{A}_1$ and $\mathbf{A}_2$ have three common principal directions. It then follows from §30 that the principal directions of $\mathbf{A}_1$ and $\mathbf{A}_2$ are also principal directions of $\mathbf{T}$. We have already seen that in this case a representation of $\mathbf{T}$ of the form (40.1) can be obtained.

It is thus seen that if $\mathbf{T}$ is an isotropic matrix function of only two symmetric 3×3 matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, then we can always obtain a representation of $\mathbf{T}$ of the form (40.1).

We have not discussed the continuity properties of the scalar coefficients in the representations which have been mentioned. It is trivial to establish some of these. For example, if the scalar coefficients in (40.1) are defined as in §27, then the quantities of α_i/α ($i = 1, \dots, 8$) are finite rational functions of the elements of $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{A}_2$, hence analytic functions of these variables, for all values of these matrices such that $\mathbf{A}_1$ and $\mathbf{A}_2$ do not have a common principal direction. Similar statements can be made concerning the other representations discussed in this paper. Unfortunately, our analysis does not provide answers to all such questions. For example, it is not clear whether, in (40.1), one can define α_i and α in such a way that α_i/α is an analytic function of the elements of $\mathbf{T}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ for all values of these arguments.

41. Appendix II. Expression of the stress tensor as a polynomial in the kinematic tensors. We assume that the stress components t_{ij} in any rectangular Cartesian co-ordinate system x_i are expressible as polynomials in the gradients of the displacement, velocity, acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration. It is seen from §15 that each of the stress components must be expressible as a poly-

nomial in the components of the tensors C_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $\dots$, $A_{ij}^{(n)}$, defined by equations (15.9) and (15.10). It is seen, from §15, that the polynomial expressions for t_{ij} have the following property:

$$\begin{aligned} t_{pq}(C_{kl}, A_{kl}^{(1)}, A_{kl}^{(2)}, \dots, A_{kl}^{(n)}) \\ = t_{ij}(C_{mn} a_{km} a_{ln}, A_{mn}^{(1)} a_{km} a_{ln}, A_{mn}^{(2)} a_{km} a_{ln}, \\ \dots, A_{mn}^{(n)} a_{km} a_{ln}) a_{ip} a_{jq}, \end{aligned} \quad (41.1)$$

where a_{ij} satisfy the relations $a_{ij} a_{ik} = \delta_{jk}$.

Since t_{pq} is a polynomial in the variables C_{kl} , $A_{kl}^{(1)}$, $A_{kl}^{(2)}$, $\dots$, $A_{kl}^{(n)}$, we have as a typical term

$$\begin{aligned} \alpha_{pqk_1 l_1 k_2 l_2 \dots k_r l_r m_1 n_1 m_2 n_2 \dots m_s n_s \dots p_1 q_1 p_2 q_2 \dots p_t q_t} C_{k_1 l_1} C_{k_2 l_2} \dots C_{k_r l_r} \\ \cdot A_{m_1 n_1}^{(1)} A_{m_2 n_2}^{(1)} \dots A_{m_s n_s}^{(1)} \dots A_{p_1 q_1}^{(n)} A_{p_2 q_2}^{(n)} \dots A_{p_t q_t}^{(n)}, \end{aligned} \quad (41.2)$$

in which it is assumed that the constant coefficient

$$\alpha_{pqk_1 l_1 k_2 l_2 \dots k_r l_r m_1 n_1 m_2 n_2 \dots m_s n_s \dots p_1 q_1 p_2 q_2 \dots p_t q_t}$$

is unchanged by the following permutations:

- (i) permutation of p and q ;
- (ii) permutation of k_i and l_i , of m_i and n_i , $\dots$ of p_i and q_i ;
- (iii) permutation of $k_i l_i$ and $k_k l_k$, of $m_i n_i$ and $m_k n_k$, $\dots$ of $p_i q_i$ and $p_k q_k$.

As a result of the relation (41.1) we must have

$$\begin{aligned} \alpha_{pqk_1 l_1 k_2 l_2 \dots k_r l_r m_1 n_1 m_2 n_2 \dots m_s n_s \dots p_1 q_1 p_2 q_2 \dots p_t q_t} C_{k_1 l_1} C_{k_2 l_2} \dots C_{k_r l_r} \\ \cdot A_{m_1 n_1}^{(1)} A_{m_2 n_2}^{(1)} \dots A_{m_s n_s}^{(1)} \dots A_{p_1 q_1}^{(n)} A_{p_2 q_2}^{(n)} \dots A_{p_t q_t}^{(n)} \\ = a_{ip} a_{jq} \alpha_{ijk_1 l_1 k_2 l_2 \dots k_r l_r m_1 n_1 m_2 n_2 \dots m_s n_s \dots p_1 q_1 p_2 q_2 \dots p_t q_t} \\ C_{a_1 b_1} C_{a_2 b_2} \dots C_{a_r b_r} a_{k_1 a_1} a_{l_1 b_1} a_{k_2 a_2} a_{l_2 b_2} \dots a_{k_r a_r} a_{l_r b_r} \\ A_{c_1 d_1}^{(1)} A_{c_2 d_2}^{(1)} \dots A_{c_s d_s}^{(1)} a_{m_1 c_1} a_{n_1 d_1} a_{m_2 c_2} a_{n_2 d_2} \dots a_{m_s c_s} a_{n_s d_s} \\ \dots \\ \dots \\ A_{e_1 f_1}^{(n)} A_{e_2 f_2}^{(n)} \dots A_{e_t f_t}^{(n)} a_{p_1 e_1} a_{q_1 f_1} a_{p_2 e_2} a_{q_2 f_2} \dots a_{p_t e_t} a_{q_t f_t} \\ = a_{ip} a_{jq} \alpha_{ija_1 b_1 a_2 b_2 \dots a_r b_r c_1 d_1 c_2 d_2 \dots c_s d_s \dots e_1 f_1 e_2 f_2 \dots e_t f_t} \\ C_{k_1 l_1} C_{k_2 l_2} \dots C_{k_r l_r} a_{a_1 k_1} a_{b_1 l_1} a_{a_2 k_2} a_{b_2 l_2} \dots a_{a_r k_r} a_{b_r l_r} \\ A_{m_1 n_1}^{(1)} A_{m_2 n_2}^{(1)} \dots A_{m_s n_s}^{(1)} a_{c_1 m_1} a_{d_1 n_1} a_{c_2 m_2} a_{d_2 n_2} \dots a_{c_s m_s} a_{d_s n_s} \\ \dots \\ \dots \\ A_{p_1 q_1}^{(n)} A_{p_2 q_2}^{(n)} \dots A_{p_t q_t}^{(n)} a_{e_1 p_1} a_{f_1 q_1} a_{e_2 p_2} a_{f_2 q_2} \dots a_{e_t p_t} a_{f_t q_t}. \end{aligned} \quad (41.3)$$

We see immediately that

$$\begin{aligned}
 \alpha_{pqk_1 l_1 k_2 l_2 \cdots k_r l_r m_1 n_1 m_2 n_2 \cdots m_s n_s \cdots p_1 q_1 p_2 q_2 \cdots p_t q_t} \\
 = a_{ip} a_{jq} \alpha_{ija_1 b_1 a_2 b_2 \cdots a_r b_r c_1 d_1 c_2 d_2 \cdots c_s d_s \cdots e_1 f_1 e_2 f_2 \cdots e_t f_t} \\
 a_{a_1 k_1} a_{b_1 l_1} a_{a_2 k_2} a_{b_2 l_2} \cdots a_{a_r k_r} a_{b_r l_r} \\
 a_{c_1 m_1} a_{d_1 n_1} a_{c_2 m_2} a_{d_2 n_2} \cdots a_{c_s m_s} a_{d_s n_s} \\
 \cdots \\
 a_{e_1 p_1} a_{f_1 q_1} a_{e_2 p_2} a_{f_2 q_2} \cdots a_{e_t p_t} a_{f_t q_t}.
 \end{aligned} \tag{41.4}$$

Equation (41.4) implies that

$$\alpha_{pqk_1 l_1 k_2 l_2 \cdots k_r l_r m_1 n_1 m_2 n_2 \cdots m_s n_s \cdots p_1 q_1 p_2 q_2 \cdots p_t q_t} \tag{41.5}$$

is an isotropic tensor of even rank $2(1 + r + s + \cdots + t)$. Now the most general isotropic tensor of rank $2(1 + r + s + \cdots + t)$ can be expressed as a linear combination with scalar coefficients of the outer products of $1 + r + s + \cdots + t$ Kronecker δ 's. These outer products are

$$\delta_{pq} \delta_{k_1 l_1} \delta_{k_2 l_2} \cdots \delta_{k_r l_r} \delta_{m_1 n_1} \delta_{m_2 n_2} \cdots \delta_{m_s n_s} \cdots \delta_{p_1 q_1} \delta_{p_2 q_2} \cdots \delta_{p_t q_t} \tag{41.6}$$

and the tensors obtained by all possible permutations of the indices in (41.6). A number of proofs of this result have already been given, but none of these appear to be entirely satisfactory. A further proof by R. TOUPIN & J. L. ERICKSEN will be published shortly.

It is readily seen that if (41.5) is expressed as a linear combination of these outer products in the expression (41.2), we obtain the element in the p^{th} row and q^{th} column of a matrix formed from the matrices $\mathbf{C}$ ($= \| C_{ij} \|$), $\mathbf{A}_r$ ($= \| A_{ij}^{(r)} \|$), ($r = 1, 2, \cdots, n$), and the unit matrix by (i) matrix multiplication and addition and (ii) multiplication by scalar invariants of the matrices $\mathbf{C}$, $\mathbf{A}_1, \cdots, \mathbf{A}_n$ which are expressible as polynomials in the elements of these matrices. We shall describe such an expression as a matrix polynomial in $\mathbf{C}$, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$.

Thus, if the stress components t_{ij} in the rectangular Cartesian co-ordinate system x_i are expressible as polynomials in the gradients of displacement, velocity, acceleration, $\cdots$, $(n - 1)^{\text{th}}$ acceleration, the stress matrix $\mathbf{T}$ may be expressed as a matrix polynomial in the matrices $\mathbf{C}$, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$, with coefficients which scalar invariants of these matrices expressible as polynomials in their elements.

In a similar way, it can be shown that if the stress components t_{ij} are expressible as polynomials in the gradients of velocity, acceleration, $\cdots$, $(n - 1)^{\text{th}}$ acceleration, the stress matrix $\mathbf{T}$ may be expressed as a matrix polynomial in the

matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$, with coefficients which are scalar invariants of these matrices expressible as polynomials in their elements.

If the stress components t_{ij} are expressible as polynomials in the velocity gradients only, it is seen that the stress matrix $\mathbf{T}$ may be expressed as a matrix polynomial in $\mathbf{A}_1$, with coefficients which are scalar invariants of $\mathbf{A}_1$ expressible as polynomials in the elements of $\mathbf{A}_1$.

Again, if the stress components t_{ij} are expressible as polynomials in the displacement gradients only, it is seen that the stress matrix $\mathbf{T}$ may be expressed as a matrix polynomial in $\mathbf{C}$, with coefficients which are scalar invariants of $\mathbf{C}$ expressible as polynomials in the elements of $\mathbf{C}$.

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