*The Elementary Basic Principles of the Unified Theory of Relativity. B_2^**

VÁCLAV HLAVATÝ

Graduate Institute for Mathematics and Mechanics, Indiana University, Bloomington, Indiana

CONTENTS

	Page
1. The problem to be solved.	248
I. First Class	
2. Prerequisites.	251
3. Auxilliary lemmas.	252
4. Non-holonomic components of S , U and Y	257
5. Continuation.	260
II. Second Class	
6. Prerequisites.	263
7. The tensors S , U and Y	265
III. Third Class	
8. Auxilliary theorems.	267
9. Non-holonomic frame.	270
10. Auxilliary results.	272
11. The tensors S , U and I	273
Bibliography.	277

* Prepared under joint contract with the Office of Naval Research and the Army Office of Ordnance Research.

1. The problem to be solved. 1) The unified field theory as sketched by EINSTEIN [1a] is based on a real general tensor $g_{\lambda\mu}$ *. Let us denote by $h_{\lambda\mu} = g_{(\lambda\mu)}$ its symmetric part and by $k_{\lambda\mu} = g_{[\lambda\mu]}$ its skew-symmetric part. The determinants of $g_{\lambda\mu}$, $h_{\lambda\mu}$, $k_{\lambda\mu}$ will be denoted by g , h , k respectively. Throughout this paper we assume

$$(1.1a) \quad g < 0.$$

Moreover we assume that $h_{\lambda\mu}$ is of signature $---+$, so that

$$(1.1b) \quad h < 0.$$

The tensor $h_{\lambda\mu}$ will play the role of the metric tensor. Its Christoffel symbols are $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ and ∇_μ is the symbol of the covariant derivative with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$. Later on we shall need the scalars

$$\begin{aligned} g &= g(1) \stackrel{\text{def}}{=} g/h^\dagger, & k &\stackrel{\text{def}}{=} k/h, \\ 2K &\stackrel{\text{def}}{=} \frac{1}{2} k_{\alpha\beta} k^{\alpha\beta}, \\ D &\stackrel{\text{def}}{=} K^2 - k. \end{aligned}$$

There are only three different algebraic classes for the tensor $g_{\lambda\mu}$, namely [4]

$$\left. \begin{array}{l} \text{first} \\ \text{second} \\ \text{third} \end{array} \right\} \quad \text{class for} \quad \begin{array}{l} Dk \neq 0, \\ D \neq 0 \quad k = 0, \\ D = k = 0. \end{array}$$

Every object built up from $g_{\lambda\mu}$ of the p^{th} class will be termed an object of the p^{th} class.

2) The connection $\Gamma_{\lambda\mu}^\nu$ of the unified field as has been defined by EINSTEIN is a solution of

$$(1.2a) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha}.$$

It is known [9] that a necessary and sufficient condition for the existence and uniqueness of such a solution is

* All Greek indices run from I to IV. Unless otherwise stated the Latin indices run as follows: i, j, k, l, m from 1 to 4, a, b, c, d from 1 to 2, e, f, g, h from 3 to 4. The indices for which the usual Einstein summation rule is not valid will be denoted by x, y, z, t . They run from 1 to 4 unless stated otherwise.

† The reason for the use of the symbol $g(1)$ will appear later.

$$(1.2b) \quad \begin{aligned} g &\neq 0 && \text{whenever} && k \neq 0, \\ g(g-2) &\neq 0 && \text{whenever} && k = 0. \end{aligned}$$

Throughout this paper we assume the validity of (1.2b). The solution of (1.2a) is of the form [5]

$$(1.3a) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}^{\nu} + U_{\lambda\mu}^{\nu},$$

where

$$(1.3b) \quad S_{\lambda\mu}^{\nu} \stackrel{\text{def}}{=} \Gamma_{[\lambda\mu]}^{\nu}; \quad U_{\lambda\mu}^{\nu} \stackrel{\text{def}}{=} 2h^{\nu\alpha} S_{\alpha(\lambda}^{\beta} k_{\mu)\beta}.$$

The tensor $S_{\lambda\mu}^{\nu}$ is the unique solution of the system of 24 equations

$$(1.3c) \quad 2S_{\alpha\beta\gamma} X_{\omega\mu\nu}^{\alpha\beta\gamma} = K_{\omega\mu\nu},$$

where

$$(1.4) \quad \begin{aligned} K_{\omega\mu\nu} &\stackrel{\text{def}}{=} \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu} + \nabla_{\nu} k_{\omega\mu} \\ &= 3\nabla_{[\omega} k_{\nu\mu]} + 2\nabla_{\nu} k_{\omega\mu}, \end{aligned}$$

and

$$(1.5) \quad X_{\omega\mu\nu}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} \delta_{[\omega\mu]}^{[\alpha\beta]\gamma} - 2\delta_{\nu}^{[\alpha} k_{[\omega}^{\beta]} k_{\mu]}^{\gamma]} - 2\delta_{[\mu}^{[\alpha} k_{\omega]}^{\beta]} k_{\nu}^{\gamma]*}.$$

It is not difficult to exhibit a construction of the unique tensor $Y_{\xi\eta\zeta}^{\omega\mu\nu}$ [5] such that

$$(1.6) \quad X_{\omega\mu\nu}^{\alpha\beta\gamma} Y_{\xi\eta\zeta}^{\omega\mu\nu} = \delta_{[\xi\eta]}^{[\alpha\beta]\gamma}{}_{\zeta};$$

then (1.3c) yields the solution

$$(1.3d) \quad S_{\alpha\beta\gamma} = \frac{1}{2} K_{\omega\mu\nu} Y_{\alpha\beta\gamma}^{\omega\mu\nu}.$$

Although the construction of $Y_{\xi\eta\zeta}^{\omega\mu\nu}$ has been explicitly described in [5], its actual form in terms of $g_{\lambda\mu}$ has not been given.

3) The main goal of this paper is to find an explicit form of S , U and Y in terms of $g_{\lambda\mu}$. It turns out that the use of a convenient non-holonomic frame yields the components of S , U , Y and Γ in a very simple form as a function of $g_{\lambda\mu}$.†

The results of this paper open a new way to the solution of several problems

* We define

$$\delta_{\beta_1 \dots \beta_r}^{\alpha_1 \dots \alpha_r} = \delta_{\beta_1}^{\alpha_1} \dots \delta_{\beta_r}^{\alpha_r},$$

where δ_{β}^{α} is the usual Kronecker delta.

† A non-tensorial form of the solution of (1.2a) has been given in [1b].

Let us mention here at least the problem of the conformal invariants of the transformation

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \quad \sigma \neq 0 \quad (\text{cfr. [10]}).$$

The paper is divided into three chapters devoted in turn to the three algebraic classes of the tensor $g_{\lambda\mu}$.

Note. For further information about the mathematics of the unified field $g_{\lambda\mu}$ as well as its physical applications *cfr.* [1]–[17].

I. FIRST CLASS

2. Prerequisites. The condition (1.1a) enables us to find the unique tensor $*g^{\lambda\nu}$ satisfying the condition

$$(2.1) \quad g_{\lambda\mu} *g^{\lambda\nu} = g_{\mu\lambda} *g^{\nu\lambda} = \delta_{\mu}^{\nu}$$

while [4]

$$(2.2) \quad G_{\lambda}^{\nu} \stackrel{\text{def}}{=} g_{\mu\lambda} *g^{\nu\mu} \neq \delta_{\lambda}^{\nu}.$$

The eigenvectors e^{ν} and the eigenvalues ρ of G_{λ}^{ν} satisfy the relation

$$(2.3a) \quad \rho e^{\nu} = G_{\lambda}^{\nu} e^{\lambda}$$

equivalent to

$$(2.3b) \quad (\rho g_{\nu\omega} - g_{\omega\nu}) e^{\nu} = 0.$$

Hence we must have

$$(2.4a) \quad \text{Det } |\rho g_{\nu\omega} - g_{\omega\nu}| = 0.$$

Comparing (1.1a, b) and (2.4a) we see that $\rho \neq 0, -1$. Moreover in this chapter we confine ourselves to the first class, so that $k \neq 0$ and hence $\rho \neq 1$. On the other hand

$$(2.5) \quad \begin{aligned} \rho g_{\nu\omega} - g_{\omega\nu} &= (\rho - 1)h_{\nu\omega} - (\rho + 1)k_{\omega\nu} \\ &= -(\lambda h_{\nu\omega} + k_{\omega\nu})(\rho + 1), \end{aligned}$$

where

$$-\lambda = \frac{\rho - 1}{\rho + 1}$$

so that (2.4a) is equivalent to

$$(2.4b) \quad \begin{aligned} g(\lambda) &\stackrel{\text{def}}{=} \frac{1}{\mathfrak{h}} \text{Det } (\lambda h_{\nu\omega} + k_{\omega\nu}) \\ &= \lambda^4 + 2\lambda^2 K + k = 0. \end{aligned}$$

Hence the eigenvectors e of G_{λ}^{ν} are also the eigenvectors of k_{λ}^{ν} :

$$(2.5a) \quad \lambda e^{\nu} = k_{\lambda}^{\nu} e^{\lambda}.$$

Let us denote the roots (2.4b) by $\lambda_1, \dots, \lambda_4$; thus

$$(2.6) \quad \begin{aligned} \lambda_1 &= i\sqrt{\sqrt{D} + K} = -\lambda_2, \\ \lambda_3 &= \sqrt{\sqrt{D} - K} = -\lambda_4, \end{aligned}$$

The roots λ_3, λ_4 are always real, while the other two are complex conjugate.* The corresponding eigenvectors ${}_1e'', \dots, {}_4e''$ satisfy the equations

$$(2.5b) \quad \lambda_x {}_xe'' = k_{\lambda} {}_xe''$$

$$(2.7a) \quad \begin{aligned} h_{xx} &\stackrel{\text{def}}{=} h_{\lambda\mu} {}_xe'' {}_xe'' = 0, \\ h_{xy} &\stackrel{\text{def}}{=} h_{\lambda\mu} {}_ye'' {}_xe'' = 0, \quad \begin{cases} y = 1, 2, \\ z = 3, 4. \end{cases} \end{aligned}$$

Moreover the factors of proportionality of these vectors may be chosen in such a way that

$$(2.7b) \quad \begin{aligned} h_{yz} &\stackrel{\text{def}}{=} h_{xy} {}_xe'' {}_ye'' = 1, \\ xy &= 12 \quad \text{or} \quad xy = 34. \end{aligned}$$

Hence the eigenvectors ${}_xe''$ are defined up to the transformation

$$(2.8) \quad {}'_xe'' = \sigma_x {}_xe'', \quad \sigma_1 \sigma_2 = \sigma_3 \sigma_4 = 1.$$

The directions of ${}_1e''$ and ${}_2e''$ are complex conjugate. Hence we may assume without loss of generality that ${}_1e''$ and ${}_2e''$ are complex conjugate and ${}_3e'', {}_4e''$ are real. This property will be preserved by (2.8) provided that

$$(2.9) \quad \begin{aligned} \sigma_1 &= e^{i\varphi}, & \sigma_2 &= e^{-i\varphi}, \\ \sigma_3 &= e^{\psi}, & \sigma_4 &= e^{-\psi}. \end{aligned}$$

Later on we shall see the geometrical significance of (2.9).

3. Auxiliary lemmas. Let us first consider the scalars

$$(3.1) \quad \lambda_{ijk} \stackrel{\text{def}}{=} 1 + \lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i$$

with the understanding that *at least two of the three indices ijk are different*.

Lemma (3.1). *The scalars λ_{ijk} have the following properties:*

- (a) *They are symmetric in their indices.*
- (b) *There are two scalars S_{pq} such that for any $i = 1, \dots, 4$*

$$(3.2) \quad S_{pq} \stackrel{\text{def}}{=} S_{(pq)} = 1 + \lambda_p \lambda_q = \lambda_{ipq}, \quad pq = 12 \quad \text{or} \quad pq = 34.$$

* One has to keep in mind that $k < 0$; consequently $\sqrt{D} > |K|$.

(c) *A necessary and sufficient condition for*

$$(3.3) \quad \lambda_{ijk} \neq 0^*$$

is that

$$(3.4) \quad g \neq 0.$$

Proof. The statement (a) is obvious; statement (b) follows from (2.6) by an easy inspection. In order to prove (c) let us consider the product

$$S_{12} S_{34} = 1 + \lambda_1 \lambda_2 + \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_3 \lambda_4.$$

From (2.4b) (or from (2.6)) we have

$$2K = \lambda_1 \lambda_2 + \lambda_3 \lambda_4$$

$$k = \lambda_1 \lambda_2 \lambda_3 \lambda_4,$$

so that

$$(3.5) \quad S_{12} S_{34} = 1 + 2K + k = g(1) = g.$$

This equation together with statement (b) proves statement (c) for three different indices i, j, k , and for $i, j, k = 1, 2$ or $i, j, k = 3, 4$. To complete the proof we have to consider the remaining scalars λ_{pxx} , where $p = 1, 2, x = 3, 4$ or $p = 3, 4, x = 1, 2$. We have

$$(3.6) \quad \lambda_{pxx} = 1 + 2\lambda_p \lambda_x + \lambda_x^2.$$

This is a complex number. Its imaginary part is

$$(3.7) \quad 2\lambda_p \lambda_x = 2\epsilon i \sqrt{-k} = 2\epsilon \sqrt{k} \neq 0 \quad (\epsilon = \pm 1),$$

and this completes the proof.

We shall need this lemma in connection with non-holonomic coordinates (components). The eigenvectors ${}_x e^\nu$ being linearly independent (which follows from (2.7)), we may look upon them as intermediate components U_x^ν of the unit tensor. Its remaining intermediate components U_ν^x result from either one of the following two systems:

$$(3.8) \quad \text{a) } U_x^\nu U_\nu^\mu = \delta_x^\mu, \quad \text{b) } U_j^\nu U_\lambda^j = \delta_\lambda^\nu.$$

By non-holonomic components of a vector $v^\nu(w_\lambda)$ we understand the scalars

$$v^i = U_\lambda^i v^\lambda, \quad w_i = U_i^\lambda w_\lambda.$$

The non-holonomic components of tensors are defined in a similar way. Thus, for

* This inequality has to hold for any set of indices i, j, k which are not all equal.

instance, the h_{ij} defined by (2.7) are non-holonomic components of $h_{\lambda\mu}$. The products

$$h^{ij} = h^{\lambda\mu} U_{\lambda}^i U_{\mu}^j$$

are non-holonomic components of $h^{\lambda\nu}$. We have in particular

$$(3.9) \quad ((h_{ij})) = ((h^{ij})) = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$U_{\lambda}^i = {}_j e^{\mu} h_{\lambda\mu} h^{ij}.$$

Lemma (3.2). Let the $\left\{ \begin{smallmatrix} \text{contra-} \\ \text{co-} \end{smallmatrix} \right\}$ variant index j appear $\left\{ \begin{smallmatrix} n_j \\ N_j \end{smallmatrix} \right\}$ times in a non-holonomic component $T^{:::}$ of a tensor T and put

$$N_1 - N_2 - (n_1 - n_2) = n_{12},$$

$$N_3 - N_4 - (n_3 - n_4) = n_{34};$$

then the non-holonomic component $T^{:::}$ of a known tensor T is known up to the factor $F \equiv e^{n_{12}\varphi + n_{34}\psi}$, i.e.

$${}'T^{:::} = FT^{:::}.$$

Proof. The equation (2.8) yields

$${}'U_x^{\nu} = \sigma_x U_x^{\nu}, \quad {}'U_{\lambda}^x = \sigma_x^{-1} U_{\lambda}^x$$

with σ 's defined by (2.9). The proof follows easily from these facts.

Note. Lemma (3.2) could be interpreted in the following way. Let v_i be non-holonomic components of a vector v_{λ} and put

$$v_1 = \alpha + i\beta, \quad v_2 = \bar{v}_1 = \alpha - i\beta, \quad v_3 = \gamma + \delta, \quad v_4 = \gamma - \delta;$$

then α, β (γ, δ) are known up to a rotation (Lorentz rotation), the angle of rotation being $\varphi(\psi)$.

Lemma (3.3). The non-holonomic components of k_{λ}^{ν} are

$$(3.11) \quad k_x^{\nu} = \lambda_x \delta_x^{\nu}.$$

Proof. Equation (2.5b) yields

$$(3.12) \quad \lambda_x U_x^{\nu} = k_{\lambda}^{\nu} U_x^{\lambda},$$

so that by virtue of (3.8)

$$k_{\lambda}^{\nu} U_x^{\lambda} U_{\nu}^y = k_x^y = \lambda_x U_x^{\nu} U_{\nu}^y = \lambda_x \delta_x^y.$$

Lemma (3.4a). *Let x, y, z be three different indices. The non-holonomic components X_{xyz}^{ijk*} of X which are not identically zero are the components X_{xyz}^{ijk} , where ijk is a permutation of xyz . The remaining components X_{xyz}^{ijk*} are equal to zero. In particular†*

	$X_{...}^{xyz}$	$X_{...}^{yxz}$	$X_{...}^{zxy}$
(3.13a)	$X_{xyz}^{\dots}$	$\frac{1}{2}(\lambda_{xyz} - \lambda_x \lambda_y)$	$-\frac{1}{2}\lambda_x \lambda_y$
	$X_{yxz}^{\dots}$	$-\frac{1}{2}\lambda_y \lambda_z$	$\frac{1}{2}(\lambda_{xyz} - \lambda_y \lambda_z)$
	$X_{zxy}^{\dots}$	$-\frac{1}{2}\lambda_z \lambda_x$	$-\frac{1}{2}\lambda_z \lambda_x$

Proof. First instead of (1.5) we may write

$$(3.14) \quad \begin{aligned} 2X_{\omega\mu\nu}^{\alpha\beta\gamma} &= \delta_{\omega\mu\nu}^{\alpha\beta\gamma} - \delta_{\mu\omega\nu}^{\alpha\beta\gamma} \\ &- [\delta_{\nu}^{\alpha}(k_{\omega}^{\beta} k_{\mu}^{\gamma} - k_{\mu}^{\beta} k_{\omega}^{\gamma}) - \delta_{\nu}^{\beta}(k_{\omega}^{\alpha} k_{\mu}^{\gamma} - k_{\mu}^{\alpha} k_{\omega}^{\gamma})] \\ &- k_{\nu}^{\gamma}(\delta_{\mu}^{\alpha} k_{\omega}^{\beta} - \delta_{\omega}^{\alpha} k_{\mu}^{\beta} - \delta_{\mu}^{\beta} k_{\omega}^{\alpha} + \delta_{\omega}^{\beta} k_{\mu}^{\alpha}). \end{aligned}$$

Now using the permutation of indices

$$\begin{pmatrix} xyz\alpha\beta\gamma \\ \omega\mu\nu xyz \end{pmatrix}$$

as well as equation (3.11), we obtain

$$\begin{aligned} 2X_{xyz}^{xyz} &= \delta_{xyz}^{xyz} - \lambda_x \delta_z^z (-\delta_x^x \lambda_y \delta_y^y - \delta_y^y \lambda_x \delta_x^x), \\ &= 1 + \lambda_x \lambda_y + \lambda_z \lambda_x = \lambda_{xyz} - \lambda_x \lambda_y. \end{aligned}$$

The remaining components in the row $X_{xyz}^{\dots}$ of (3.13a) are obtained in the same way. The remaining two rows may be obtained either by the same method, or by suitably interchanging the indices in the first row. Comparing (3.11) and (3.14), we see that X_{xyz}^{ijk} is identically equal to zero whenever ijk is not a permutation of xyz . Hence the table exhibits the only components $X_{xyz}^{ijk\dagger}$ which are not identically equal to zero.

* Of course we have

$$X_{xyz}^{ijk} = -X_{yxz}^{ijk} = X_{yxz}^{jik} = -X_{xzy}^{ijk}.$$

† The Table (3.13) is to be understood in the following way: The component X_{pqr}^{ijk} is at the intersection of the row $X_{pqr}^{\dots}$ with the column $X_{...}^{ijk}$.

‡ Cfr. the preceding footnote.

Lemma (3.4b). *Let $y = z \neq x$. The non-holonomic components X_{xy}^{ijk} which are not identically equal to zero are*

$$(3.13b) \quad X_{xyy}^{xyy} = \frac{1}{2} \lambda_{xyy}.$$

The remaining components X_{xyy}^{ijk} are equal to zero.*

Proof. Using the permutation

$$\begin{pmatrix} xy y \alpha \beta \gamma \\ \omega \mu \nu x y y \end{pmatrix}$$

as well as the equation (3.11), one obtains from (3.14)

$$\begin{aligned} 2X_{xyy}^{xyy} &= \delta_{xyy}^{xyy} + \delta_y^y \lambda_x \delta_x^x \lambda_y \delta_y^y - \lambda_y \delta_y^y (-\delta_x^x \lambda_y \delta_y^y - \delta_y^y \lambda_x \delta_x^x), \\ &= 1 + 2\lambda_x \lambda_y + \lambda_y^2 = \lambda_{xyy}. \end{aligned}$$

On the other hand the equations (3.11) and (1.5) show that X_{xyy}^{ijk} is identically equal to zero whenever ijk is not a permutation of the indices x, y, y .

Note. Table (3.13a) and equation (3.13b) yield all components

$$X_{ijk}^{pqr} = -X_{jik}^{pqr} = X_{jik}^{qpr} = -X_{ijk}^{qpr} \quad (p, q, r = 1, \dots, 4)$$

not identically equal to zero.

The last Lemma† of this section deals with the construction of the frame U_i^r, U_r^i by means of the eigenvalues $\lambda_1, \dots, \lambda_4$, given by (2.6). Let us introduce the scalar

$$(3.15) \quad \Lambda_x(\lambda) \stackrel{\text{def}}{=} (\lambda^2 + \lambda_x^2 + 2K) (\lambda + \lambda_x)$$

and the tensor

$$(3.16) \quad E_\mu^r(\lambda) \stackrel{\text{def}}{=} \sum_{x=1}^{x=4} \Lambda_x(\lambda) U_x^r U_\lambda^x.$$

Lemma (3.5). *The tensor $E_\mu^r(\lambda)$ has the following properties. a) It may be expressed in terms of k_λ^r as follows:*

$$(3.17) \quad E_\mu^r(\lambda) = (\lambda \delta_\mu^\beta + k_\mu^\beta) [k_\alpha^r k_\beta^\alpha + (\lambda^2 + 2K) \delta_\beta^r].$$

b) *If a^λ, b_μ are two arbitrary vectors, then*

$$(3.18a) \quad a^\mu E_\mu^r(\lambda_x) = \alpha_x U_x^r,$$

$$(3.18b) \quad b_\nu E_\mu^r(\lambda_x) = \beta_x U_\mu^x,$$

* Cf. the second footnote on the preceding page.

† This Lemma inserted long after the completion of the paper has been inspired by the results quoted on p. 30 of the book J. A. SCHOUTEN: Ricci Calculus (1954).

where

$$(3.19a) \quad \alpha_x \stackrel{\text{def}}{=} a^x \Lambda_x(\lambda_x),$$

$$(3.19b) \quad \beta_x \stackrel{\text{def}}{=} b_x \Lambda_x(\lambda_x).$$

Proof. An inspection based on

$$k_\lambda{}^\nu = \sum_{x=1}^{x=4} \lambda_x U_x^\nu U_\lambda^x$$

shows the equivalence of (3.16) and (3.17). Moreover we have from (3.15)

$$\Lambda_x(\lambda_x) \neq 0, \quad \Lambda_x(\lambda_y) = 0, \quad y \neq x,$$

and these relations together with (3.16) yield (3.18) and (3.19) at once.

Remark. This Lemma shows that there is always a suitable vector a^ν such that the equations (2.7) are satisfied by

$$U_x^\nu \equiv {}_x e^\nu \stackrel{\text{def}}{=} a^\alpha (\delta_\alpha^\beta \lambda_x + k_\alpha^\beta) [k_\gamma{}^\nu k_\beta{}^\gamma + \delta_\beta^\nu (\lambda_x^2 + 2K)].$$

A similar remark holds for U_λ^x which, on the other hand, may be obtained also from

$$U_\lambda^i = {}_j e^\mu h_{\lambda\mu} h^{ij}.$$

4. Non-holonomic components of S , U and Y .

Theorem (4.1a). *Let xyz be three different indices. Then the non-holonomic components S_{xyz} of $S_{\omega\mu\nu}$ are*

$$(4.1) \quad S_{xyz} = \frac{1}{2\lambda_{xyz}} [K_{xyz} + 3K_{[xyz]} \lambda_x \lambda_y],^*$$

so that

$$(4.2) \quad S_{[xyz]} = \tfrac{1}{2} K_{[xyz]}.$$

Proof. Using (1.3c) and Lemma (3.4a) we obtain

$$\begin{aligned} S_{ijk} X_{xyz}^{ijk} &\equiv S_{xyz}(\lambda_{xyz} - \lambda_x \lambda_y) - S_{yzz} \lambda_x \lambda_y - S_{zxy} \lambda_x \lambda_y = \tfrac{1}{2} K_{xyz}, \\ (4.3) \quad S_{ijk} X_{yzx}^{ijk} &\equiv -S_{xyz} \lambda_y \lambda_z + S_{yzz}(\lambda_{xyz} - \lambda_y \lambda_z) - S_{zxy} \lambda_y \lambda_z = \tfrac{1}{2} K_{yzx}, \\ S_{ijk} X_{zxy}^{ijk} &\equiv -S_{xyz} \lambda_z \lambda_x - S_{yzz} \lambda_z \lambda_x + S_{zxy}(\lambda_{xyz} - \lambda_z \lambda_x) = \tfrac{1}{2} K_{zxy}. \end{aligned}$$

* Footnote added in proof: A similar but more complicated formula in terms of matrix calculus has been given in a paper by S. N. Bose, "The affine connection in Einstein's new unitary field theory," *Annals Math.* Vol. **59**, No. 1, January, 1954, pp. 171-176.

The determinant of this system is

$$\Delta_{xyz} \stackrel{\text{def}}{=} \begin{vmatrix} \lambda_{xyz} - \lambda_x \lambda_y & -\lambda_x \lambda_y & -\lambda_x \lambda_y \\ -\lambda_y \lambda_z & \lambda_{xyz} - \lambda_y \lambda_z & -\lambda_y \lambda_z \\ -\lambda_z \lambda_x & -\lambda_z \lambda_x & \lambda_{xyz} - \lambda_z \lambda_x \end{vmatrix}.$$

Let us apply the following elementary operations to Δ_{xyz} : 1) add the last two rows to the first one; 2) subtract the second column from the third one; 3) add the second row to the third one. In this way we easily obtain

$$(4.4) \quad \Delta_{xyz} = (\lambda_{xyz})^2 \neq 0,$$

so that (4.3) yields

$$(\lambda_{xyz})^2 S_{xyz} = \frac{1}{2} \begin{vmatrix} K_{xyz} & -\lambda_x \lambda_z & -\lambda_x \lambda_y \\ K_{yzx} & \lambda_{xyz} - \lambda_y \lambda_z & -\lambda_y \lambda_z \\ K_{zxy} & -\lambda_z \lambda_x & \lambda_{xyz} - \lambda_z \lambda_x \end{vmatrix}.$$

Let us apply the following elementary operations to the determinant on the right hand side: 1) add the last two rows to the first one; 2) add the third row to the second one; 3) subtract the second column from the last one. In this way we easily obtain

$$(4.5) \quad (\lambda_{xyz})^2 S_{xyz} = \frac{1}{2} [3K_{[xyz]}(1 + \lambda_x \lambda_y) - 2K_{z[xy]}] \lambda_{xyz}.$$

On the other hand

$$3K_{[xyz]} - 2K_{z[xy]} = K_{xyz} + K_{yzx} + K_{zxy} - K_{zxy} + K_{zyx} = K_{xyz},$$

so that (4.5) leads easily to (4.1). Using this equation, we have

$$\begin{aligned} 3S_{[xyz]} &= S_{xyz} + S_{yzx} + S_{zxy} \\ &= \frac{1}{2\lambda_{xyz}} \{K_{xyz} + K_{yzx} + K_{zxy} + 3K_{[xyz]} \lambda_x \lambda_y + 3K_{[yzx]} \lambda_y \lambda_z + 3K_{[zxy]} \lambda_z \lambda_x\} \\ &= \frac{1}{2\lambda_{xyz}} 3K_{[xyz]} (1 + \lambda_x \lambda_y + \lambda_y \lambda_z + \lambda_z \lambda_x) = \frac{3}{2} K_{[xyz]}. \end{aligned}$$

Theorem (4.1b). *The formula (4.1) holds for any arbitrary set of indices $x, y, z = 1, \dots, 4$ for which $x \neq y$.*

Proof. Put $z = y$ in (4.1). Then this equation reduces to

$$(4.6) \quad S_{xyy} = \frac{K_{xyy}}{2\lambda_{xyy}}.$$

On the other hand using equation (1.3c) as well as Lemma (3.4b) we obtain

$$S_{ijk} X_{xyy}^{ijk} = \lambda_{xyy} S_{xyy} = \frac{1}{2} K_{xyy},$$

and this equation is equivalent to (4.6).

Theorem (4.2). *The non-holonomic components of the tensor Y involved in (1.3d) are*

$$(4.7) \quad Y_{xyz}^{ijk} = \frac{1}{\lambda_{xyz}} (\delta_{[xy]z}^{[ij]k} + 3\delta_z^{[k} k_{[x}{}^i k_{y]}{}^{j]})$$

Proof. By virtue of (3.11) equation (4.1) may be written in the form

$$\begin{aligned} 2S_{xyz} \lambda_{xyz} &= K_{xyz} + 3K_{[xyz]} k_x^x k_y^y \\ &= K_{ijk} \{ \delta_{[xy]z}^{ijk} + 3\delta_z^{[k} k_{[x}{}^i k_{y]}{}^{j]} \}, \end{aligned}$$

so that

$$S_{xyz} = \frac{1}{2} K_{ijk} \frac{1}{\lambda_{xyz}} (\delta_{[xy]z}^{ijk} + 3\delta_z^{[k} k_{[x}{}^i k_{y]}{}^{j]}).$$

Comparing this equation with (1.3d) one obtains (4.7).

Note. The presence of the factor $1/\lambda_{xyz}$ in (4.1) and (4.7) is responsible for the fact that it was “next to impossible” [17]₁ to obtain the (non-holonomic) components of S and Y (in terms of $g_{\lambda\mu}$) *by means of a pure tensorial method*. However, it is possible to write (4.1) in tensorial form.

Theorem (4.3). *Let*

$$(4.8) \quad \begin{aligned} L_{xyz} &\stackrel{\text{def}}{=} \frac{K_{xyz}}{\lambda_{xyz}} \quad \text{for } x \neq y, \\ L_{xxz} &\stackrel{\text{def}}{=} 0 \end{aligned}$$

be non-holonomic components of a tensor. Then

$$(4.9) \quad S_{ijk} = \frac{1}{2} (L_{ijk} + 3L_{[lmk]} k_i^l k_j^m).$$

The proof is obvious.

There is also the possibility of writing both equations (4.1) and (4.7) in tensorial form.

Theorem (4.4). *Let the non-holonomic components of a tensor $M_{\omega\mu\nu}^{\alpha\beta\gamma}$ be defined in the following way:*

$$(4.10a) \quad M_{xyz}^{ijk} \stackrel{\text{def}}{=} \frac{1}{\lambda_{xyz}} \delta_{xyz}^{ijk} \quad \text{for } x \neq y,$$

$$(4.10b) \quad M_{xxz}^{ijk} \stackrel{\text{def}}{=} 0.$$

Then

$$(4.11) \quad S_{pqr} = \frac{1}{2} M_{pqr}^{ijk} (K_{ijk} + 3K_{[lm]l} k_i^l k_j^m), \quad *$$

$$(4.12) \quad Y_{ijk}^{pqr} = M_{ijk}^{efg} (\delta_{[ef]g}^{[pq]r} + 3\delta_g^{[r} k_{[e}^p k_{f]}^q]).$$

The proof is obvious.

Theorem (4.4a). *The non-holonomic components of the tensor U defined by (1.3b) are*

$$(4.13a) \quad U_{xy} = \frac{1}{2\lambda_{xyz}} (\lambda_y K_{zxy} + \lambda_x K_{zyx}).$$

Proof. First of all we have by virtue of (3.11)

$$\begin{aligned} U_{xy} &= S_{zxi} k_y^i + S_{zyi} k_x^i \\ &= S_{zxy} \lambda_y + S_{zyx} \lambda_x, \end{aligned}$$

so that according to (4.1)

$$U_{xy} = \frac{1}{2\lambda_{xyz}} \{ \lambda_y K_{zxy} + 3K_{[xyz]} \lambda_z \lambda_x \lambda_y + \lambda_x K_{zyx} + 3K_{[xyz]} \lambda_z \lambda_y \lambda_x \},$$

and this equation reduces to (4.13a).

Theorem (4.4b). *The components U_{xy} may also be written*

$$(4.13b) \quad U_{xy} = -k_{(y}^i L_{x)zi}.$$

Proof. First we have

$$\frac{1}{2} (\lambda_y K_{zxy} + \lambda_x K_{zyx}) = -\frac{1}{2} (k_y^i K_{xzi} + k_x^i K_{yzi}) = -k_{(y}^i K_{x)zi}.$$

Now using the definition of the tensor L , we obtain (4.13b) at once.

Note. The components U_{xy} could be also expressed in terms of the tensor M defined by (4.10).

Theorem (4.5). *The non-holonomic components of S , U dealt with in this section are known up to the factors F involved in Lemma (3.2).*

The proof follows at once from Lemma (3.2).

5. Continuation. In this section we shall deal with the non-holonomic components of the tensors S , Y and U and also with the holonomic as well as the non-holonomic components of the connection, defined by (1.3).

Theorem (5.1). *Put*

$$\begin{aligned} M_{\omega\mu\nu}^{\alpha\beta\gamma} &= M_{ijk}^{pqr} U_{\omega}^i U_{\mu}^j U_{\nu}^k U_p^{\alpha} U_q^{\beta} U_r^{\gamma} \\ (5.1) \quad &= \sum_{xyz} \frac{x \mp y}{\lambda_{xyz}} U_{\omega}^x U_{\mu}^y U_{\nu}^z U_x^{\alpha} U_y^{\beta} U_z^{\gamma} \quad (p, q, r = 1, \dots, 4) \end{aligned}$$

* All Latin indices used here run from 1 to 4.

and

$$(5.2) \quad \begin{aligned} L_{\omega\mu\nu} &= L_{ijk} U_{\omega}^i U_{\mu}^j U_{\nu}^k \\ &= \sum_{\substack{x+y \\ xyz}} \frac{K_{xyz}}{\lambda_{xyz}} U_{\omega}^x U_{\mu}^y U_{\nu}^z; \end{aligned}$$

then

$$(5.3) \quad Y_{\omega\mu\nu}^{\alpha\beta\gamma} = M_{\omega\mu\nu}^{\xi\eta\zeta} (\delta_{[\xi\eta]}^{\alpha\beta} \gamma_{\zeta}^{\gamma} + 3\delta_{\zeta}^{[\gamma} k_{[\xi}^{\alpha} k_{\eta]}^{\beta]}),$$

$$(5.4a) \quad S_{\omega\mu\nu} = \frac{1}{2} M_{\omega\mu\nu}^{\xi\eta\zeta} (K_{\xi\eta\zeta} + 3K_{[\zeta\alpha\beta]} k_{\xi}^{[\alpha} k_{\eta}^{\beta]}),$$

$$(5.4b) \quad S_{\omega\mu\nu} = \frac{1}{2} (L_{\omega\mu\nu} + 3L_{[\alpha\beta\nu]} k_{\omega}^{[\alpha} k_{\mu}^{\beta]}),$$

$$(5.5) \quad U_{\nu\omega\mu} = -k_{(\mu}^{\alpha} L_{\nu)\alpha}.$$

The proof follows at once from (4.11), (4.12), (4.9) and (4.13).

Note. Equations (5.3)–(5.5) give the non-holonomic components of the tensors Y , S , U in a surveyable tensorial form. On the other hand equations (5.1) and (5.2) show clearly that it is impossible to obtain (5.3)–(5.5) by a straightforward tensorial method.

In order to deal with the connection of the unified field we need first a (well known) lemma. In it for obvious reasons we put

$${}_j e^{\nu} \equiv U_j^{\nu}, \quad {}^i e_{\lambda} \equiv U_{\lambda}^i.$$

Lemma (5.1). *The non-holonomic components $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ of $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ are*

$$(5.6) \quad \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} = -{}_j e^{\nu} {}_k e^{\mu} \nabla_{\mu} {}^i e_{\nu}.$$

Proof. The transformation rule

$$\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} = U_j^{\lambda} U_k^{\mu} U_{\nu}^i \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + U_{\nu}^i U_k^{\mu} \partial_{\mu} U_j^{\nu}$$

may be rewritten as

$$\begin{aligned} \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} &= {}^i e_{\nu} \left[\partial_{\mu} {}_j e^{\nu} + \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} {}_j e^{\lambda} \right] {}_k e^{\mu}, \\ &= {}^i e_{\nu} {}_k e^{\mu} \nabla_{\mu} {}_j e^{\nu}, \\ &= -{}_j e^{\nu} {}_k e^{\mu} \nabla_{\mu} {}^i e_{\nu}. \end{aligned}$$

Theorem (5.2a). *The non-holonomic components Γ_{jk}^i of $\Gamma_{\lambda\mu}^{\nu}$ are **

* In (5.7a) we put $z^* = \begin{cases} 3 - z & \text{for } z = 1, 2, \\ 7 - z & \text{for } z = 3, 4. \end{cases}$

$$(5.7a) \quad \Gamma_{xy}^z = -{}_x e^\nu{}_y e^\mu \nabla_\mu {}^z e_\nu + \frac{1}{2\lambda_{xyz^*}} \{K_{xyz^*} + 3K_{[xyz^*]} \lambda_x \lambda_y + \lambda_y K_{z^*xy} + \lambda_x K_{z^*yx}\},$$

and this equation is equivalent to

$$(5.7b) \quad \Gamma_{jk}^i = -{}_j e^\nu{}_k e^\mu \nabla_\mu {}^i e_\nu + \frac{1}{2}(L_{jk}{}^i + 3L_{[pqr]} k_j{}^p k_k{}^q h^{ri}) - k_{(j}{}^m L_{k)}{}^i{}_m.$$

Proof. The equation (1.3a) may be written in non-holonomic form

$$(5.8) \quad \Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + S_{jk}{}^i + U^i{}_{jk};$$

moreover according to (3.9) we have

$$(5.9) \quad S_{jk}{}^m h^{mi} = S_{jk}{}^{i*}.$$

These equations together with (4.1) and (4.13a) lead to (5.7a); equations (5.8), (4.9) and (4.13b) lead to (5.7b).

Theorem (5.2b). *The holonomic components $\Gamma_{\lambda\mu}^\nu$ may be written*

$$(5.7c) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2}(L_{\lambda\mu}{}^\nu + 3L_{[\alpha\beta\gamma]} k_\lambda{}^\alpha k_\mu{}^\beta h^{\gamma\nu}) - k_{(\lambda}{}^\alpha L_{\mu)}{}^\nu{}_\alpha.$$

The proof follows at once from (5.7b).

II. SECOND CLASS

6. Prerequisites. The tensor $g_{\lambda\mu}$ of the second class is characterized by

$$(6.1) \quad k = 0 \quad D = K^2 \neq 0.$$

Hence the equations (2.4b) and (2.6) reduce to

$$(6.2) \quad g(\lambda) = \lambda^2(\lambda^2 + 2K) = 0$$

and

$$\begin{aligned} \lambda_1 &= -\lambda_2 = i\sqrt{\sqrt{K^2} + K}, \\ \lambda_3 &= -\lambda_4 = \sqrt{\sqrt{K^2} - K}. \end{aligned}$$

If $K > 0$ we have

$$(6.3a) \quad \lambda_1 = -\lambda_2 = \sqrt{-2K}, \quad \lambda_3 = \lambda_4 = 0,$$

while if $K < 0$ we have

$$(6.3b) \quad \lambda_1 = \lambda_2 = 0 \quad \lambda_3 = -\lambda_4 = \sqrt{-2K}.$$

The significance of (6.3) may be shown best by means of a geometrical description.* Let T_4 be the linear tangent space to space-time at a point P , and let I_3 be the projective ideal space of T_4 . Then $h_{\lambda\mu}$ defines a non-degenerate non-ruled quadric H in I_3 and $k_{\lambda\mu}$ defines a linear complex K in I_3 . This complex is a special one, for $k = 0$. Hence $k_{\lambda\mu}$ defines a straight line L in I_3 . Denote by $*L$ its conjugate polar with respect to H . The lines L and $*L$ are skew, for we are dealing with the second class. Every point e^λ incident with L satisfies the equation

$$(6.4a) \quad e^\lambda k_\lambda{}^\nu = 0.$$

In particular, when $K > 0$ there are two real points ${}_3e''$, ${}_4e'' \neq {}_3e''$ satisfying (6.4a) and

$$(6.4b) \quad e^\lambda e^\mu h_{\lambda\mu} = 0,$$

while $*L$ intersects H in two complex conjugate points ${}_1e''$, ${}_2e''$ which belong to the roots λ_1 , λ_2 and satisfy (6.4b). When however $K < 0$ then the intersection points ${}_1e''$, ${}_2e'' \neq {}_1e''$ of L and H which satisfy (6.4a, b) (and belong to the roots

* Cfr. [7].

λ_1, λ_2) are complex conjugate, while the intersection points ${}_3e''$ and ${}_4e''$ of $*L$ and H are real. They belong to the roots λ_3, λ_4 .

It is appropriate for our purpose to deal with both cases simultaneously. To this effect, from now on we denote by ${}_1e'', {}_2e''$ the eigenvectors belonging to

$$(6.5a) \quad \lambda_1 = -\lambda_2 = \sqrt{-2K} \neq 0,$$

while we reserve the symbols ${}_3e'', {}_4e''$ for the eigenvectors satisfying (6.4), *i.e.* belonging to the eigenvalues

$$(6.5b) \quad \lambda_3 = -\lambda_4 = 0.$$

Moreover we choose the factors of proportionality of the e 's in such a way that again we have

$$(6.6a) \quad \begin{array}{llll} \sigma_1 = e^{i\varphi}, & \sigma_2 = e^{-i\varphi}, & \sigma_3 = e^\psi, & \sigma_4 = e^{-\psi} \quad \text{for } K > 0 \\ \sigma_1 = e^\psi, & \sigma_2 = e^{-\psi}, & \sigma_3 = e^{i\varphi}, & \sigma_4 = e^{-i\varphi} \quad \text{for } K < 0. \end{array}$$

Hence if we introduce again the numbers n_{12} and n_{34} as defined in Lemma (3.2) we may state the following

Lemma (6.1). *The component T''' as mentioned in Lemma (3.2) is known up to the factor of proportionality*

$$'F = \begin{cases} e^{n_{12}\varphi + n_{34}\psi} & K > 0 \\ e^{n_{12}\psi + n_{34}\varphi} & K < 0 \end{cases} \quad \text{for}$$

i.e.

$$'T''' = FT'''$$

The proof is obvious.

The next Lemma will concern the scalars λ_{ijk} defined by (3.1) for at least two different indices.

Lemma (6.2). *The scalars λ_{ijk} , symmetric in their indices, have the following values*

$$(6.7a) \quad \lambda_{abi} = g \quad (a \neq b),$$

$$(6.7b) \quad \lambda_{xxa} = g \quad (a \neq x),$$

$$(6.7c) \quad \lambda_{xxe} = 2 - g,$$

$$(6.7d) \quad \lambda_{efi} = 1$$

$$(i, j, k = 1, \dots, 4, \quad a, b, x = 1, 2, \quad e, f = 3, 4),$$

so that a necessary and sufficient condition for

$$(6.7e) \quad \lambda_{ijk} \neq 0$$

is

$$(6.8) \quad g(g - 2) \neq 0.$$

Proof. First of all we have from (6.1) and (2.4b)

$$g \equiv g(1) = 1 + 2K,$$

so that according to (6.5a)

$$1 + \lambda_1 \lambda_2 = 1 + 2K = g$$

$$1 + \lambda_1^2 = 1 + \lambda_2^2 = 1 - 2K = (2 - g).$$

Combining these results with (6.5b) we obtain easily (6.7a-d). Then the statement about (6.7e) is obvious.

In subsequent considerations we shall again identify ${}_3e^\nu$ with the intermediate components U_i^ν of the unit tensor. Moreover we denote by U_λ^i its remaining intermediate components. Hence we may use the same definition of non-holonomic components as previously.

Lemma (6.3). *The non-holonomic components of k_λ^ν are given by the matrix*

$$(6.9) \quad ((k_i^j)) = \begin{pmatrix} \sqrt{-2K} & 0 & 0 & 0 \\ 0 & -\sqrt{-2K} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The proof follows at once from (3.11) (which is obviously valid also for the second class) and (6.5).

Lemma (6.4). *Put*

$$(6.10) \quad E_\mu^\nu(\lambda) \stackrel{\text{def}}{=} \lambda \delta_\mu^\nu + k_\mu^\nu$$

If a^ν, b_λ are two arbitrary vectors perpendicular to ${}_3e^\nu, {}_4e^\nu$, then for $x = 1, 2$

$$(6.11) \quad a^\mu E_\mu^\nu(\lambda_x) = 2a^x \lambda_x U_x^\nu, \quad b_\nu E_\mu^\nu(\lambda_x) = 2b_x \lambda_x U_\mu^x.$$

7. Tensors S, U and Y .

Theorem (7.1). *Theorems (4.1)–(4.4) as well as (5.1)–(5.2) hold for the tensor $g_{\lambda\mu}$ of second class.*

Proof. In proving these theorems we did not use the fact that all eigenvalues are different from zero. Hence these theorems must hold also for the case characterized by (6.5).

Theorem (7.2). *The non-holonomic components of S and U dealt with in this section are known up to the factor F involved in Lemma (6.1).*

The proof is obvious.

Note. It might be advantageous for special purposes to use Lemma (6.2) in connection with Theorem (7.1). Thus for instance we obtain

$$\begin{aligned}
 2S_{abi} &= \frac{1}{g} (K_{abi} + 6K K_{[abi]}), \\
 2S_{xfx} &= \frac{1}{2-g} K_{xfx}, \\
 2S_{afb} &= \frac{1}{g} K_{afb}, \quad a \neq b, \\
 2S_{afe} &= K_{afe}, \\
 2S_{efi} &= K_{efi} \\
 (a, b, c, x &= 1, 2, \quad e, f = 3, 4, \quad i = 1, \dots, 4),
 \end{aligned}
 \tag{7.1}$$

and

$$\begin{aligned}
 Y_{abm}^{ijk} &= \frac{1}{g} (\delta_{[ab]m}^{ij\ k} + 6\delta_m^{[k} \delta_{[a}^i \delta_{b]}^j] K), \\
 Y_{afg}^{ijk} &= \delta_{[af]g}^{ij\ kk}, \quad Y_{[af]b}^{ij\ k} = \frac{1}{g} \delta_{[af]b}^{ij\ k}, \quad a \neq b, \\
 Y_{xfx}^{ijk} &= \frac{1}{2-g} \delta_{[xf]x}^{ij\ k}, \\
 Y_{efm}^{ijk} &= \delta_{[ef]m}^{ij\ k} \\
 (i, j, k, m &= 1, \dots, 4, \quad a, b, x = 1, 2, \quad e, f = 3, 4),
 \end{aligned}
 \tag{7.2}$$

and so on.

III. THIRD CLASS

8. Auxiliary theorems. The tensor $g_{\lambda\mu}$ of the third class is characterized by

$$(8.1a) \quad k = 0,$$

$$(8.1b) \quad D = K^2 = 0,$$

so that (2.4b) reduces to

$$(8.2a) \quad g(\lambda) \equiv \lambda^4 = 0,$$

and in particular

$$(8.2b) \quad g(1) = 1.$$

In this section we shall derive some geometrical consequences of (8.1) in I_3 . These results will serve as a basis for subsequent considerations.

According to (8.1) $k_{\lambda\mu}$ defines a real straight line L tangent to the quadric H (which is defined by $h_{\lambda\mu}$).^{*} Its harmonic polar $*L$ (with respect to H) meets L at its tangential point. Let us denote this point by ${}_4e'$ (and its tangential plane by ${}_4e_\lambda = {}_4e^\mu h_{\lambda\mu}$) and choose an arbitrary point ${}_3e' \neq {}_4e'$ on L . Then we have obviously

$$(8.3a) \quad {}_3e^\lambda k_\lambda{}^\nu = 0$$

$$(8.3b) \quad {}_4e^\lambda {}_4e^\mu h_{\lambda\mu} = 0$$

$$(8.3c) \quad {}_3e^\lambda {}_3e^\mu h_{\lambda\mu} \neq 0.$$

Theorem (8.1). *Let v' be an arbitrary point in I_3 not incident with L . Then $v^\lambda k_\lambda{}^\nu$ is a point incident with $*L$.*

Proof. Let $a_\mu \neq {}_4e_\mu$ be an arbitrary plane incident with L . Choosing properly the factor of proportionality of this plane, we may write

$$(8.4) \quad k_{\lambda\mu} = {}_4e_\lambda a_\mu - {}_4e_\mu a_\lambda,$$

so that

$$(8.5) \quad v^\lambda k_{\lambda\mu} = a_\mu (v^\lambda {}_4e_\lambda) - {}_4e_\mu (v^\lambda a_\lambda).$$

Hence the plane $v^\lambda k_{\lambda\mu}$ is incident with L , and consequently its pole $v^\lambda k_\lambda{}^\nu$ (with respect to H) is incident with $*L$.

^{*} Cfr. [4].

Theorem (8.2). *Let $1e''$, $2e''$ be two arbitrary points such that the matrix $((e''))$ is of rank 4. Then the matrix $\kappa \stackrel{\text{def}}{=} ((e^\lambda k_\lambda''))$ is of rank 2.*

Proof. The highest possible rank of the matrix κ is obviously 2 (cfr. (8.3a)). Hence it is sufficient to prove that it cannot be 0 or 1.

The only points v^λ for which $v^\lambda k_\lambda'' = 0$ are the points incident with L . The points $3e''$ and $4e'' \neq 3e''$ are already incident with L , so that neither one of the points ae'' could be chosen on L . Hence κ can not have the rank 0.

Assume now that the rank of κ is 1. Then there is a point X^λ such that

$$(8.6) \quad ae^\lambda k_\lambda'' = \sigma_a X''$$

where $\sigma_a \neq 0$. According to Theorem (8.1) the point X'' is incident with $*L$. On the other hand we have obviously from (8.6)

$$X^\lambda ae^\mu h_{\lambda\mu} = 0.$$

Hence the point X'' is incident also with the conjugate polar $*M$ to the line M defined by $1e''$ and $2e''$, and consequently $*M$ and $*L$ meet (in X''). Therefore we see that the polar plane X_λ (of X'') which is incident with L (for X'' is incident with $*L$) is also incident with M . Hence L and M define a plane incident with $1e''$, $\dots$, $4e''$, and consequently the matrix $((e''))$ cannot have the rank 4 as required.

Definition (8.1). *Let ae'' be two real points such that*

- 1) *the matrix $((e''))$ is of rank 4,*
- 2) *the points $ae^\lambda k_\lambda''$ are two of the four points e'' .*

Then we say that the points e'' constitute a projective point frame in I_3 , closed with respect to $k_{\lambda\mu}$.*

Theorem (8.3). *A necessary and sufficient condition for e'' to be a closed projective point frame is that one of the points $1e''$, $2e''$, say $2e'' \neq 4e''$, is incident with $*L$, while the remaining point $1e''$ is incident with the polar plane $2e_\lambda = h_{\lambda\mu} 2e^\mu$ of $2e''$ but not incident with L .*

Proof. Assume that $2e'' \neq 4e''$ is on $*L$. Then its polar plane $2e_\lambda = 2e^\mu h_{\mu\lambda}$ is incident with L , so that choosing conveniently the factor of proportionality of $4e''$ we have instead of (8.4)

$$(8.7) \quad k_{\lambda\mu} = 4e_\lambda 2e_\mu - 4e_\mu 2e_\lambda.$$

Therefore

$$(8.8a) \quad 2e^\lambda k_{\lambda\mu} = (4e_\lambda 2e^\lambda) 2e_\mu - (2e_\lambda 2e^\lambda) 4e_\mu.$$

Remembering that $2e^\lambda$ is incident with $*L$ but not with its own polar plane, we have

$$4e_\lambda 2e^\lambda = 0, \quad 2e_\lambda 2e^\lambda \neq 0.$$

* We use the word projective in order to show that in I_3 we disregard the factors of proportionality.

We may assume without loss of generality that the factor of proportionality of ${}_2e^\lambda$ has been chosen in such a way that

$${}_2e^\lambda {}_2e^\lambda = -1^*.$$

Consequently (8.8a) reduces to

$$(8.9) \quad {}_2e^\lambda k_\lambda{}^\nu = {}_4e^\nu.$$

Assume now that ${}_1e^\nu$ is incident with ${}_2e_\lambda$ but not with ${}_4e_\lambda$ † so that ${}_1e^\lambda {}_4e_\lambda \neq 0$, ${}_1e^\lambda {}_2e_\lambda = 0$. Hence if we choose properly the factor of proportionality of ${}_1e^\lambda$ we obtain from (8.7)

$$(8.10) \quad {}_1e^\lambda k_\lambda{}^\nu = {}_2e^\nu.$$

Assume now that (8.3) and (8.10) hold.‡ Then the latter equation together with Theorem (8.1) shows that ${}_2e^\nu$ is incident with *L , and consequently ${}_2e_\lambda$ is incident with L . Moreover we have from (8.10)

$${}_1e^\lambda {}_1e^\mu k_{\lambda\mu} = {}_1e^\lambda {}_2e_\lambda = 0,$$

so that ${}_1e^\nu \neq {}_4e^\nu$, ${}_3e^\nu$ is incident with the plane ${}_2e_\lambda$ through L . Hence we may assume $k_{\lambda\mu}$ in the form (8.7), and this equation leads obviously to (8.9).

The construction of a closed frame requires some data which are arbitrary and cannot be deduced from the given data (*i.e.* from $g_{\lambda\mu}$). Such data will be called *external data*.

The following theorems will be proved simultaneously.

Theorem (8.4a). *There is no projective closed point frame whose construction requires only one external parameter.*

Theorem (8.4b). *There are projective closed frames with two null points whose construction requires two external parameters.*

Theorem (8.4c). *Any projective closed point frame as mentioned in Theorem (8.4b) is constituted by points ${}_3e^\nu$, ${}_4e^\nu$ and ${}_2e^\nu$, ${}_1e^\nu$, where ${}_2e^\nu$ is the point mentioned in Theorem (8.3) while ${}_1e^\nu$ is the intersection point of H with the harmonic polar *N of the line N defined by ${}_2e^\nu$ and ${}_3e^\nu$.*

Proof. The points ${}_2e^\nu$ on *L and ${}_3e^\nu$ on L are arbitrary. Hence there is a two parametric set of lines N (not incident with ${}_4e^\nu$ and) incident with ${}_4e_\lambda$, each yielding not only the required points $NL = {}_3e^\nu$, $N {}^*L = {}_2e^\nu$ but also a unique harmonic polar *N (incident with ${}_4e^\nu$ but not with ${}_4e_\lambda$) which is the intersection of ${}_2e_\lambda$ and ${}_3e_\lambda$. Hence by virtue of Theorem (8.3) any point $\neq {}_4e^\nu$ on *N may be thought of as ${}_1e^\nu$, and this is true also for the intersection point of *N and H . (The construction of N in ${}_4e_\lambda$ requires *two external parameters*, and both *N and its intersection point ($\neq {}_4e^\nu$) with H are uniquely determined by N and H .)

* The signature of $h_{\lambda\mu}$ is $---+$ and ${}_2e^\nu$ is outside of H .

† The plane ${}_4e_\lambda$ is already incident with ${}_4e^\nu$, ${}_3e^\nu$, ${}_2e^\nu$.

‡ By virtue of Theorem (8.1) the points ${}_ae^\lambda k_\lambda{}^\nu$ are incident with *L .

9. Non-holonomic frame.

Definition (9.1). A set of vectors ${}_i e^\nu$ ($i = 1, \dots, 4$) in T_4 whose ideal points e^ν in I_3 constitute a projective closed point frame as mentioned in Theorem (8.4c) will be termed a closed vector frame in T_4 provided that ${}_2 e^\nu$ and ${}_3 e^\nu$ are unit vectors.

In the next theorem we shall use the scalars

$$(9.1) \quad h_{ij} \stackrel{\text{def}}{=} h_{\lambda\mu} {}_i e^\lambda {}_j e^\mu.$$

Theorem (9.1). A closed vector frame satisfies the conditions

$$(9.2) \quad ((h_{ij})) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(9.3) \quad {}_1 e^\lambda k_\lambda{}^\nu = {}_2 e^\nu, \quad {}_2 e^\lambda k_\lambda{}^\nu = {}_4 e^\nu, \quad {}_3 e^\lambda k_\lambda{}^\nu = 0.$$

Proof. The equations (9.3) follow at once from (8.3), (8.9) and (8.10). On the other hand the vectors ${}_2 e^\nu$ and ${}_3 e^\nu$ being unit vectors we have

$$(9.4a) \quad h_{22} = h_{33} = -1^*.$$

Therefore we obtain from (9.3)

$$(9.4b) \quad {}_1 e^\lambda {}_2 e^\mu k_{\lambda\mu} = -1 = -{}_2 e^\lambda {}_1 e^\mu k_{\lambda\mu} = -h_{11}.$$

The remaining h_{ij} in (9.2) follow easily from Theorem (8.4c).

Theorem (9.2). Let ${}_i e^\nu, {}'_i e^\nu$ be two arbitrary closed vector frames. Then

$$(9.5a) \quad {}'_j e^\nu = A_j^i {}_i e^\nu$$

with

$$(9.5b) \quad \begin{array}{ccccc} & A_1^1 & A_1^2 & A_1^3 & A_1^4 \\ A_i^1 & \epsilon & B & C & \frac{1}{2}\epsilon(B^2 + C^2) \\ A_i^2 & 0 & \epsilon & 0 & B \\ A_i^3 & 0 & 0 & \epsilon' & \epsilon\epsilon'C \\ A_i^4 & 0 & 0 & 0 & \epsilon \end{array},$$

where $\epsilon, \epsilon' = \pm 1$ and B, C are two arbitrary (external) scalars.

* Cfr. the first footnote on the preceding page.

Proof. Both frames being closed, we have obviously

$$\begin{aligned}
 {}'_1e^\nu &= A {}_1e^\nu + B {}_2e^\nu + C {}_3e^\nu + D {}_4e^\nu, \\
 {}'_2e^\nu &= E {}_2e^\nu + F {}_4e^\nu, \\
 {}'_3e^\nu &= G {}_3e^\nu + H {}_4e^\nu, \\
 {}'_4e^\nu &= I {}_4e^\nu,
 \end{aligned}
 \tag{9.6}$$

where the scalars $A, \dots, I$ are to be determined by the requirement that (9.2) and (9.3) hold also for $'_je^\nu$. The requirement about (9.4a) leads at once to

$$E = \epsilon, \quad G = \epsilon', \quad \epsilon, \epsilon' = \pm 1, \tag{9.7a}$$

and consequently the requirement concerning (9.3) yields

$$A = \epsilon, \quad I = \epsilon, \quad B = F. \tag{9.7b}$$

On the other hand $'h_{12} = 0^*$ is satisfied because of (9.7b) while $'h_{13} = 0$ leads to

$$\epsilon\epsilon' C = H. \tag{9.7c}$$

The requirement concerning (9.4b) is satisfied by virtue of (9.7b) while $'h_{11} = 0$ gives

$$2D = \epsilon(B^2 + C^2). \tag{9.7d}$$

Moreover we have $'h_{23} = 'h_{24} = 'h_{f4} = 0$.

Note. Theorem (9.2) shows clearly two external parameters B, C defining a closed vector frame in terms of an arbitrary closed vector frame (cf. also Theorem (8.4b)).

As in previous chapters we again introduce the non-holonomic frame by means of the definition

$$U_i^\nu \stackrel{\text{def}}{=} {}_ie^\nu, \tag{9.8a}$$

which leads to the remaining components U_λ^i :

$$U_\lambda^j U_i^\lambda = \delta_i^j, \quad U_\lambda^i U_i^\nu = \delta_\lambda^\nu. \tag{9.8b}$$

The intermediate components U_i^ν, U_λ^i are defined up to the transformations

$$\begin{aligned}
 {}'U_i^\nu &= A_i^j U_j^\nu, \\
 {}'U_\lambda^i &= B_i^j U_\lambda^j,
 \end{aligned}
 \tag{9.9a}$$

where the A 's are defined by (9.5b) and the B 's are obtained from

$$B_j^i A_k^j = \delta_k^i. \tag{9.9b}$$

* We put $'h_{ij} = h_{\lambda\mu} {}'_ie^\lambda {}'_je^\mu$.

Defining the non-holonomic components $T_j^{i\cdots}$ of a tensor $T_\lambda^{r\cdots}$ in the usual way by means of the U 's we may prove easily the following

Theorem (9.3). *The non-holonomic components $T_j^{i\cdots}$ of a known tensor are known up to transformations*

$$(9.10) \quad {}'T_j^{i\cdots} = A_j^k B_m^i \dots T_k^{m\cdots},$$

where the A 's are defined by (9.5b) while the B_j^i are defined by (9.9b).

The proof is obvious.

Note. It is interesting to compare Lemma (3.2), Lemma (6.1) and Theorem (9.3). In all these three cases the non-holonomic components of known tensors are known up to transformations which require two external parameters. There the similarity stops: The difference may be described best in terms of the ideal space I_3 . The group of transformations involved in Lemmas (3.2) and (6.1) leaves the coordinate tetrahedron invariant* while in the case of the third class only a point (*i.e.* ${}_4e^r$), its polar plane (${}_4e_\lambda = h_{\lambda\mu} {}_4e^\mu$) and two lines incident with ${}_4e^r$ and ${}_4e_\lambda$ are left invariant.

It is also interesting to consider the non-holonomic components g_{ij} of $g_{\lambda\mu}$.

Theorem (9.4). *The non-holonomic components g_{ij} of $g_{\lambda\mu}$ are given by*

$$(9.11) \quad ((g_{ij})) = \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}.$$

Proof. According to (9.3) we have

$$(9.12) \quad k_1^2 = k_2^4 = 1 \quad \text{the remaining} \quad k_i^j = 0.$$

Combining these equations with

$$k_i^m h_{mj} = k_{ij}$$

one obtains by virtue of (9.2) the matrix (9.11).

10. Auxiliary results. Let us prove first a theorem about the Christoffel symbols $\{_{jk}^i\}$ as defined by (5.6), where of course the ${}_4e^r$ and ${}_4e_\lambda = U_\lambda^i$ constitute the frame involved in the previous section.

Theorem (10.1). *We have in every closed frame*

$$(10.1) \quad \left\{ \begin{matrix} 1 \\ 4i \end{matrix} \right\} = 0 \quad \left\{ \begin{matrix} x \\ xi \end{matrix} \right\} = 0 \quad \left\{ \begin{matrix} 4 \\ 1i \end{matrix} \right\} = 0, \quad x = 2, 3.$$

* Strictly speaking we have two groups in the case of the second class, one for $K > 0$, one for $K < 0$.

Proof. Let us start with

$$(10.2) \quad 0 = \nabla_i h_{yy} = \partial_i h_{yy} - 2 \left\{ \begin{matrix} j \\ yi \end{matrix} \right\} h_{jy}.$$

Remembering that (9.2) holds in every closed frame we obtain (10.1) by substituting into (10.2) $y = 4, 3, 2, 1$ respectively.

In the next theorem we shall deal with some non-holonomic components of the tensor K defined by (1.4).

Theorem (10.2). *We have in every closed frame*

$$(10.3a) \quad K_{4p4} = 0, \quad p = 2, 3,$$

$$(10.3b) \quad K_{422} = 0,$$

$$(10.3c) \quad K_{f[24]} = 0,$$

$$(10.3d) \quad K_{124} + K_{412} = 0.$$

Proof. First we obtain from (1.4) (10.1) and (9.11)

$$K_{4p4} = 2\nabla_4 k_{4p} = -2 \left[\left\{ \begin{matrix} 1 \\ 44 \end{matrix} \right\} k_{1p} + \left\{ \begin{matrix} j \\ p4 \end{matrix} \right\} k_{4j} \right] = 0.$$

In a similar manner

$$(10.4a) \quad K_{422} = 2\nabla_2 k_{42} = -2 \left\{ \begin{matrix} 1 \\ 42 \end{matrix} \right\} k_{12} = 0,$$

$$(10.4b) \quad K_{f24} - K_{f42} = \nabla_f k_{42} + \nabla_2 k_{f4} + \nabla_4 k_{f2} - \nabla_f k_{24} - \nabla_4 k_{f2} - \nabla_2 k_{f4},$$

$$(10.5a) \quad = 2\nabla_f k_{42} = -2 \left\{ \begin{matrix} 1 \\ 4f \end{matrix} \right\} k_{12} = 0,$$

and

$$(10.5b) \quad K_{124} + K_{412} = \nabla_1 k_{42} + \nabla_2 k_{14} + \nabla_4 k_{12} + \nabla_4 k_{21} + \nabla_1 k_{42} + \nabla_2 k_{41} \\ = 2\nabla_1 k_{42} = -2 \left\{ \begin{matrix} 1 \\ 41 \end{matrix} \right\} k_{12} = 0.$$

The invariance of (10.3b, c, d) follows at once from (10.4), (10.5) and Theorem (10.1). On the other hand the Table (9.5b) shows clearly that K_{4p4} is invariant but for the sign.

11. The tensors S , U and X . In subsequent considerations we shall use with advantage the starred indices defined by the following rule: If

$$a = \begin{matrix} 1 \\ 2 \end{matrix} \text{ then } a^* = \begin{matrix} 2 \\ 4 \end{matrix}.$$

Theorem (11.1). *The non-holonomic components of k_λ^ν are*

$$(11.1) \quad k_f^i = 0 \quad k_a^i = \delta_{a*}^i.$$

The proof follows at once from (9.12).

Theorem (11.2). *The non-holonomic components of the tensor X involved in (1.3c) are*

$$(11.2a) \quad X_{fmg}^{ijk} = \delta_{[fm]g}^{ij\ k},$$

$$(11.2b) \quad X_{fga}^{ijk} = \delta_{[fg]a}^{ij\ k},$$

$$(11.2c) \quad X_{fab}^{ijk} = \delta_{[fa]b}^{ij\ k} + \delta_{[fa*]b*}^{ij\ k},$$

$$(11.2d) \quad X_{abm}^{ijk} = \delta_{[ab]m}^{ij\ k} - 2\delta_{m[a*b*]}^{[ij]k} + \delta_m^c \delta_{c*}^k (\delta_{[ab*]}^{[ij]} + \delta_{[a*b]}^{[ij]}).$$

Proof. We see from (11.1) that the second row on the right hand side of (3.14) is equal to zero whenever at least one of the indices $\omega\mu$ is replaced by $f = 3, 4$. In the remaining cases we obtain for this row

$$- [2\delta_m^i k_{[a}^j k_{b]}^k - 2\delta_m^j k_{[a}^i k_{b]}^k] = -4\delta_{m[a*b*]}^{[ij]k}.$$

On the other hand the third row on the right hand side of (3.14) is equal to zero whenever ν is replaced by $e = 3, 4$ or μ are replaced by $f, g = 3, 4$. In the remaining cases we obtain for this row

$$- k_b^k (2\delta_f^{[j} k_a^{i]} = 2\delta_{[fa*]b*}^{[ij]k}.$$

and

$$- 2k_m^k (k_a^{[j} \delta_b^{i]} - \delta_a^{[i} k_b^{j]}) = 2\delta_m^c (\delta_{ab*}^{[ij]} + \delta_{a*b}^{[ij]}) \delta_{c*}^k.$$

The proof follows easily from these facts and from (3.14).

Theorem (11.3). *The non-holonomic components $S_{ijk} = -S_{jik}$ of the tensor S are*

$$(11.3a) \quad 2 S_{fmg} = K_{fmg},$$

$$(11.3b) \quad 2 S_{fga} = K_{fga},$$

$$(11.3c) \quad 2 S_{fab} = K_{fab} - K_{fa*b*},$$

$$(11.3d) \quad 2 S_{abf} = K_{abf} + 2 K_{f[a*b*]},$$

$$(11.3e)_1 \quad 2 S_{121} = K_{121} + K_{124} + 2 K_{412},$$

$$(11.3e)_2 \quad 2 S_{122} = K_{122} + K_{422} + K_{414}.$$

Proof. Consider the equations derived from (1.3c), namely

$$(11.4a) \quad 2 S_{ijk} X_{fmg}^{ijk} = K_{fmg},$$

$$(11.4b) \quad 2 S_{ijk} X_{fga}^{ijk} = K_{fga},$$

$$(11.4c) \quad 2 S_{ijk} X_{fab}^{ijk} = K_{fab},$$

$$(11.4d) \quad 2 S_{ijk} X_{abf}^{ijk} = K_{abf},$$

$$(11.4e) \quad 2 S_{ijk} X_{abc}^{ijk} = K_{abc}.$$

The equations (11.3a, b) follow at once from (11.2a, b) and (11.4a, b). On the other hand the equations (11.2c) and (11.4c) lead to

$$2 S_{fab} + 2 S_{fa^*b^*} = K_{fab}.$$

Using (10.3a) and (11.3a, b), we may solve this system in the form (11.3c). In a similar manner we obtain from (11.4d, e), (11.2d) and (11.3a, b, c) the equations (11.3d, e_1 , e_2).

The main theorem of this section is

Theorem (11.4). *The system (11.3) is equivalent to*

$$(11.5) \quad 2 S_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^{\alpha} K_{\omega]\alpha\beta} k_{\nu}{}^{\beta}.$$

Proof. First we obtain from (10.3b) and (11.3e)₂

$$(11.6a) \quad 2 S_{122} = K_{122} - K_{144} = K_{122} - K_{12^*2^*}.$$

Moreover we have by virtue of (10.3c) and (11.3d)

$$(11.6b) \quad 2 S_{abf} = K_{abf}.$$

Finally the equation (10.3d) leads to

$$K_{124} + 2 K_{412} = K_{412} = -K_{142}$$

so that (11.3e)₁ reduces to

$$(11.6c) \quad 2 S_{121} = K_{121} - K_{142} = K_{121} - K_{12^*1^*}.$$

The equations (11.3a, b, c) and (11.6a, b, c) may be condensed into one system

$$(11.7) \quad 2 S_{ijk} = K_{ijk} - 2 k_{[j}{}^l K_{i]lm} k_k{}^m.$$

This equation is clearly in tensorial form and therefore equivalent to (11.5).

Corollary (11.1). *We have**

$$(11.8) \quad 2 S_{[\omega\mu\nu]} = K_{[\omega\mu\nu]} (= -\partial_{[\omega} k_{\mu\nu]}).$$

The proof based on (10.3b) and (11.7) is obvious.

* Cf. [5] equation (7.25). (There is an obvious misprint in this equation.)

Theorem (11.2). *The tensor Y involved in (1.3d) has the holonomic components*

$$(11.9) \quad Y_{\omega\mu\nu}^{\gamma\alpha\beta} = \delta_{[\omega\mu]\nu}^{\gamma\alpha\beta} - 2 \delta_{[\omega}^{\gamma} k_{\mu]}^{\alpha} k_{\nu}^{\beta}.$$

The proof follows at once from (1.3d) and from (11.5) written in the form

$$S_{\omega\mu\nu} = \frac{1}{2} K_{\gamma\alpha\beta} (\delta_{[\omega\mu]\nu}^{\gamma\alpha\beta} - 2 k_{\nu}^{\beta} \delta_{[\omega}^{\gamma} k_{\mu]}^{\alpha}).$$

Theorem (11.3). *The tensor U defined by (1.3b) has the holonomic components*

$$(11.10) \quad U_{\nu\omega\mu} = K_{\nu(\omega}{}^{\alpha} k_{\mu)\alpha} + k_{\gamma}{}^{\beta} [k_{(\mu}{}^{\gamma} K_{\omega)\alpha\beta} k_{\nu}{}^{\alpha} - K_{\nu\alpha\beta} k_{(\omega}{}^{\alpha} k_{\mu)}{}^{\gamma}].$$

Proof. This equation follows at once from (11.5) and (1.3b).

Theorem (11.4). *The connection $\Gamma_{\lambda\mu}^{\nu}$ defined by (1.3c) is*

$$(11.11) \quad \Gamma_{\omega\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \omega\mu \end{matrix} \right\} + \frac{1}{2} (K_{\omega\mu}{}^{\nu} - 2 k_{[\mu}{}^{\alpha} K_{\omega]\alpha\beta} k^{\nu\beta}) \\ + h^{\nu\xi} \{ K_{\xi(\omega}{}^{\alpha} k_{\mu)\alpha} + k_{\gamma}{}^{\beta} [k_{(\mu}{}^{\gamma} K_{\omega)\alpha\beta} k_{\xi}{}^{\alpha} - K_{\xi\alpha\beta} k_{(\omega}{}^{\alpha} k_{\mu)}{}^{\gamma}] \}.$$

The proof follows at once from (1.3a), (11.5) and (11.10).

BIBLIOGRAPHY

- [1a] EINSTEIN, ALBERT, *The Meaning of Relativity*, 3rd ed., Princeton, 1950, p. 162; 4th ed., Princeton, 1953.
- [1b] EINSTEIN, A. & KAUFMAN, B., "Algebraic Properties of the Field in the Relativistic Theory of the Asymmetric Field," *Annals of Mathematics*, Vol. **59**, No. 2, March, 1954, pp. 230-244.
- [2] HLAVATÝ, VÁCLAV, "The Elementary Basic Principles of the Unified Theory of Relativity," *Proceedings of the National Academy of Sciences*, Vol. **38**, No. 3, March, 1952, pp. 343-347.
- [3] —, "The Einstein Connection of the Unified Theory of Relativity," *Proceedings of the National Academy of Sciences*, Vol. **38**, No. 5, May, 1952, pp. 415-419.
- [4] —, "The Elementary Basic Principles of the Unified Theory of Relativity. A," *Journal of Rational Mechanics and Analysis*, Vol. **1**, No. 4, October, 1952, pp. 539-562.
- [5] —, "The Elementary Basic Principles of the Unified Theory of Relativity. B," *Journal of Rational Mechanics and Analysis*, Vol. **2**, No. 1, January, 1953, pp. 1-52.
- [6] —, "The Schrödinger Final Affine Field Laws," *Proceedings of the National Academy of Sciences*, Vol. **38**, No. 12, December, 1952, pp. 1052-1058.
- [7] —, "The Spinor Connection in the Unified Einstein Theory of Relativity," *Proceedings of the National Academy of Sciences*, Vol. **39**, No. 6, June, 1953, pp. 501-506.
- [8] —, "Connections Between Einstein's Two Unified Theories of Relativity," *Proceedings of the National Academy of Sciences*, Vol. **39**, No. 6, June, 1953, pp. 507-510.
- [9] HLAVATÝ, VÁCLAV & SÁENZ, A. W., "Uniqueness Theorems in the Unified Theory of Relativity," *Journal of Rational Mechanics and Analysis*, Vol. **2**, No. 3, July, 1953, pp. 523-536.
- [10] HLAVATÝ, VÁCLAV, "The Elementary Basic Principles of the Unified Theory of Relativity. C₁ Introduction," *Journal of Rational Mechanics and Analysis*, Vol. **3**, No. 1, January, 1954, pp. 103-146.
- [11] —, "The Elementary Basic Principles of the Unified Theory of Relativity. C₂: Applications I," *Journal of Rational Mechanics and Analysis*, Vol. **3**, No. 2, March, 1954, pp. 147-179.
- [12] —, "Maxwell's Field in the Einstein Unified Field Theory," *Nieuw Archief voor Wiskunde*, Series III, 1954, pp. 103-114.
- [13] —, "Report on the Recent Einstein Unified Field Theory," *Seminario Matematico Università di Padova*, Vol. **23**, 1954, pp. 316-332.
- [14] —, "The Elementary Basic Principles of the Unified Theory of Relativity. C₃: Applications II," *Journal of Rational Mechanics and Analysis*, Vol. **3**, No. 6, November, 1954, pp. 645-689.
- [15] —, "The Law of Inertia in the Unified Field Theory," *Seminario Matematico Università di Torino*, Vol. **13**, 1953-1954, pp. 1-14.
- [16] LICHNEROWICZ, ANDRÉ, "Compatibilité des équations de la théorie unitaire du champ d'Einstein," *Journal of Rational Mechanics and Analysis*, Vol. **3**, No. 5, September, 1954, pp. 487-521.
- [17] SCHRÖDINGER, ERWIN, "The final affine field laws I-III," *Proc. R. Irish Acad.* Vol. **51**, 1947, pp. 163-171; Vol. **51**, 1948, pp. 205-216; Vol. **52**, 1948, pp. 1-9.