Gravitational Waves and One-dimensional Einsteinian Gas-Dynamics

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1. Introduction. It has long been known [1] that Einstein's equations of general relativity possess certain approximate solutions representing so-called gravitational waves. By this is meant that the gravitational potentials corresponding to a zero distribution of matter are solutions of the classical wave-equation, the velocity of propagation being equal to that of light. It has been argued by BERGMANN [2] that waves of this kind are emitted whenever mass points undergo accelerations but the example he gives—referring to plane waves—contains no indication that the gravitational effect of mass points is taken into account. Exact solutions of Einstein's equations of wave-form are also known: EINSTEIN & ROSEN [3] find that there are no plane waves but that there are cylindrical waves. This curious result is interpreted in terms of the energy carried by the waves but this interpretation is suspect because the concept of energy is artificial in general relativity. To introduce it, non-tensor quantities have to be used. In an important paper A. H. TAUB [4] has examined the exact solutions of plane wave type and has shown that they are spurious because by a coordinate transformation the waves can be removed. His method depends on assuming *a priori* that the energy-tensor is zero except perhaps in singular regions. It may be however that there are distributions of matter that tend to zero with the lapse of time and that significant gravitational waves may then result. Such cases are not necessarily covered by TAUB's method and this is one question that is examined in this paper. It will be said that a gravitational wave solution of Einstein's equations in their exact form is significant if it can be shown to lead to a zero energy-tensor and, at the same time, to a curved space-time. When the space-time attained is flat, and a coordinate-transformation can thus reduce it to the MINKOWSKI space-time of special relativity, the gravitational waves will be regarded as physically spurious.

At the time when TAUB carried out his investigation the method of reduction

of Einstein's equations to those of classical gas-dynamics was not known [5]. Using this method it is now possible to show that gravitational waves of a kind are possible in classical theory also, though their velocity is a function of the time and is not constant. The final problem that will be solved in this paper is that of the general relativity analogues of the one-dimensional classical gas-motions known as "sound waves of finite amplitude," since there is a natural connection between such motions and gravitational waves.

2. Einstein's equations for plane symmetry. When the ten Einsteinian gravitational potentials are denoted by $g_{\mu\nu}$, ($\mu, \nu = 1, 2, 3, 4$) the metric of space-time has the general form

$$(2.1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 1, 2, 3, 4)$$

and Einstein's equations are, for a zero value of the cosmical constant,

$$(2.2) \quad -8\pi G T^\mu_\nu = R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R.$$

Here G is the constant of gravitation; R^μ_ν and R are, respectively, the mixed contracted Riemann-Christoffel tensor and the scalar curvature for the metric (2.1) and δ^μ_ν is the Kronecker delta. If the material which is represented by the space-time (2.1) is a perfect fluid of density ρ , pressure p and having the velocity 4-vector u^μ , ($\mu = 1, 2, 3, 4$), then the energy-tensor is

$$(2.3) \quad T^\mu_\nu = g_{\nu\alpha} T^{\mu\alpha} = g_{\nu\alpha} \left\{ \left(\rho + \frac{p}{c^2} \right) u^\mu u^\alpha - g^{\mu\alpha} \frac{p}{c^2} \right\}.$$

The velocity 4-vector is a unit vector and therefore satisfies the identity

$$(2.4) \quad 1 = g_{\mu\nu} u^\mu u^\nu.$$

A plane symmetrical distribution of perfect fluid is one in which ρ , p and u^μ are functions of a linear coordinate x with dimensions of length and a time coordinate t . Hence at the instant t , a component of the energy-tensor has the same value at all points in a plane $x = \text{constant}$. The metric of the corresponding space-time is known from the work of A. H. TAUB [4] to be

$$(2.5) \quad ds^2 = e^{2f} dt^2 - \frac{1}{c^2} (e^{2f} dx^2 + e^{2h} dy^2 + e^{2h} dz^2),$$

where f and h are functions of x and t . The calculation of Einstein's equations for such a case may be performed by the use of DINGLE's formulae [6], the index 4 being assigned to t , while 1, 2, and 3 refer to x , y , and z , respectively. The gravitational potentials in DINGLE's notation are

$$(2.6) \quad D = Ac^2 = e^{2f}, \quad B = C = \frac{e^{2h}}{c^2},$$

and, since ρ and p are supposed to be expressed in grams per cubic centimeter and dynes per square centimeter, his constant 8π must be replaced by $8\pi G$ where G is the constant of gravitation in c.g.s. units. Inspection of the formulae for T^{12} , T^{13} , T^{23} , T^{24} and T^{34} shows, in virtue of equation (2.6), that all these components are identically zero. This implies that $u^2 = 0$ and $u^3 = 0$, and the identity satisfied by the velocity 4-vector, therefore, becomes

$$(2.7) \quad 1 = D(u^4)^2 - A(u^1)^2.$$

It is also convenient to introduce the following symbols:

$$(2.8) \quad L = e^f u^4, \quad U = e^f u^1, \quad \sigma = \rho + \frac{p}{c^2}, \quad \kappa = \frac{8\pi G}{c^2},$$

and to denote partial derivatives of f and h with respect to x and t by subscripts. The surviving Einstein equations given below correspond to the energy-tensor components T^{11} , T^{22} (or T^{33}), T^{44} and T^{14} ($= T^{41}$), respectively:

$$(2.9) \quad \kappa(\sigma U^2 + p) = e^{-2f} \{-2h_{tt} + c^2(h_x^2 + 2h_x f_x) - (3h_t^2 - 2h_t f_t)\},$$

$$(2.10) \quad \kappa p = e^{-2f} \{-h_{tt} - f_{tt} + c^2(h_{xx} + f_{xx}) + c^2 h_x^2 - h_t^2\},$$

$$(2.11) \quad -\kappa \left(\sigma L^2 - \frac{p}{c^2} \right) = e^{-2f} \left\{ 2h_{xx} + 3h_x^2 - 2h_x f_x - \frac{1}{c^2} (h_t^2 + 2h_t f_t) \right\},$$

$$(2.12) \quad -\kappa \sigma U L = 2e^{-2f} \{-h_{xt} - h_x h_t + h_x f_t + h_t f_x\},$$

and equation (2.7) is

$$(2.13) \quad 1 = L^2 - \frac{U^2}{c^2}.$$

An algebraic solution of these equations gives

$$(2.14) \quad \kappa c^2 \rho = e^{-2f} \{c^2(f_{xx} - h_{xx}) + h_{tt} - f_{tt} - 3c^2 h_x^2 + 3h_t^2\},$$

$$(2.15) \quad \kappa p = e^{-2f} \{c^2(f_{xx} + h_{xx}) - h_{tt} - f_{tt} + c^2 h_x^2 - h_t^2\},$$

$$(2.16) \quad \kappa \sigma = \frac{2e^{-2f}}{c^2} \{c^2 f_{xx} - f_{tt} - c^2 h_x^2 + h_t^2\},$$

$$(2.17) \quad \kappa \sigma U^2 = e^{-2f} \{-c^2(f_{xx} + h_{xx}) - h_{tt} + f_{tt} - 2h_t^2 + 2h_t f_t + 2c^2 h_x f_x\},$$

$$(2.18) \quad \kappa \sigma L^2 = \frac{e^{-2f}}{c^2} \{c^2(f_{xx} - h_{xx}) - f_{tt} - h_{tt} - 2c^2(h_x^2 - h_x f_x) + 2f_t h_t\},$$

$$(2.19) \quad \kappa \sigma U L = 2e^{-2f} \{h_{xt} + h_x h_t - h_x f_t - h_t f_x\},$$

of which, of course, (2.16) is a consequence of (2.14) and (2.15).

If $\sigma = 0$, then *each* of the expressions in curly brackets on the right-hand sides

of equations (2.17), (2.18) and (2.19) are separately zero. If $\sigma \neq 0$ it is possible to eliminate σ , U and L from the three equations and obtain the *consistency relation* which must be satisfied by f and h if algebraic inconsistency is to be avoided, *viz*:—

$$\begin{aligned}
 (2.20) \quad & 4\{h_{xt} + h_x h_t - f_t h_x - f_x h_t\}^2 \\
 &= \frac{1}{c^2} \{h_{tt} - f_{tt} + c^2(h_{xx} + f_{xx} - 2f_x h_x) + 2(h_t^2 - f_t h_t)\} \\
 &\quad \times \{h_{tt} + f_{tt} + c^2(h_{xx} - f_{xx} + 2h_x^2 - 2f_x h_x) - 2f_t h_t\}.
 \end{aligned}$$

Of course, even if $\sigma = 0$, this equation is still true because each of the three expressions in curly brackets found in it vanishes independently.

For reasons to be explained in Section 3, attention will be concentrated on those space-times for which

$$(2.21) \quad h = - \int \psi(X) dX, \quad f = - \int \chi(X) dX,$$

where

$$(2.22) \quad X = x - \int v(t) dt,$$

and ψ , χ , v are arbitrary functions of their arguments. Constants may be added to the right-hand sides of the equations (2.21), if required, in order to make e^{2h} and e^{2f} physically dimensionless. If a prime denote a total derivative of a function with respect to its argument, then

$$\begin{aligned}
 (2.23) \quad & h_x = -\psi, & f_x &= -\chi, \\
 & h_{xx} = -\psi', & f_{xx} &= -\chi', \\
 & h_{xt} = v\psi', & f_t &= v\chi, \\
 & h_t = v\psi, & f_{tt} &= v'\chi - v^2\chi'. \\
 & h_{tt} &= v'\psi - v^2\psi',
 \end{aligned}$$

It is also convenient to define the functions Ψ and Φ by

$$\begin{aligned}
 (2.24) \quad & \Psi = \psi' - \chi' - 2\psi^2 + 2\chi\psi, \\
 & \Phi = \psi' + \chi' + 2\chi\psi,
 \end{aligned}$$

whence

$$\begin{aligned}
 (2.25) \quad & \Psi + \Phi = 2(\psi' - \psi^2 + 2\chi\psi), \\
 & \Psi - \Phi = -2(\chi' + \psi^2).
 \end{aligned}$$

Hence (2.14) to (2.19) now become

$$(2.26) \quad \kappa c^2 \rho = e^{-2f} \{v'(\psi - \chi) + (c^2 - v^2)(\psi' - \chi' - 3\psi^2)\},$$

$$(2.27) \quad \kappa p = e^{-2f} \{-v'(\psi + \chi) + (c^2 - v^2)(-\psi' - \chi' + \psi^2)\},$$

$$(2.28) \quad \kappa \sigma = \frac{e^{-2f}}{c^2} \{-2v' \chi + (c^2 - v^2)(\Psi - \Phi)\},$$

$$(2.29) \quad \kappa \sigma U^2 = e^{-2f} \{v'(\chi - \psi) + c^2 \Phi + v^2 \Psi\},$$

$$(2.30) \quad \kappa \sigma L^2 = \frac{e^{-2f}}{c^2} \{-v'(\chi + \psi) + v^2 \Phi + c^2 \Psi\},$$

$$(2.31) \quad \kappa \sigma UL = e^{-2f} v(\Phi + \Psi).$$

The consistency relation (2.20) becomes, after some calculation,

$$(2.32) \quad \begin{aligned} & -(c^2 - v^2)^2 \Phi \Psi \\ & = v'[v'(\psi^2 - \chi^2) - v^2\{(\chi + \psi)\Psi + (\psi - \chi)\Phi\} + c^2\{(\chi - \psi)\Psi - (\chi + \psi)\Phi\}]. \end{aligned}$$

Hence the five equations (2.9) to (2.13) are reducible when f , h are given by (2.21) and (2.22), to the following five independent equations: (a) the equations (2.26) and (2.27) for ρ and p , respectively; (b) the consistency relation (2.32) and the condition (2.13), which together can replace equations (2.30) and (2.31); (c) the equation (2.29) which, provided that $\sigma \neq 0$, may be written

$$(2.33) \quad U^2 = c^2 \frac{v'(\chi - \psi) + c^2 \Phi + v^2 \Psi}{-2v' \chi + (c^2 - v^2)(\Psi - \Phi)}.$$

Finally the coordinate-velocity of the perfect fluid may be calculated. This velocity is, by (2.8) and (2.13),

$$u = \frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = \frac{u^1}{u^4} = \frac{U}{L} = \frac{U}{(1 + U^2/c^2)^{\frac{1}{2}}},$$

and therefore by (2.33)

$$(2.34) \quad u^2 = c^2 \frac{v'(\chi - \psi) + c^2 \Phi + v^2 \Psi}{-v'(\chi + \psi) + v^2 \Phi + c^2 \Psi},$$

which is valid so long as $\sigma \neq 0$.

It is important to detect those combinations of functions f and h in (2.5) which correspond to the Minkowski space-time of special relativity. When this happens, it is in theory possible to find a transformation from (t, x, y, z) to the coordinates (T, X, Y, Z) of an inertial system that will turn (2.5) into

$$(2.35) \quad ds^2 = dT^2 - \frac{1}{c^2} (dX^2 + dY^2 + dZ^2).$$

A simpler way of detecting space-times reducible to (2.35) is to use the Riemann-Christoffel (curvature) tensor $R_{\alpha\beta\gamma\delta}$, for, if all components of this tensor are identically zero, but not otherwise, then (2.5) is convertible into (2.35). The formulae for $R_{\alpha\beta\gamma\delta}$ can be evaluated from the work of EISENHART [7];—assigning the index 4 to t and the indices 1, 2, 3 to x, y, z , respectively, the independent non-vanishing components of the tensor are ($\gamma = 2$ or 3):—

$$(2.36) \quad \begin{aligned} R_{1\gamma\gamma 4} &= -\frac{1}{c^2} e^{2h} (h_{xt} + h_x h_t - h_x f_t - h_t f_x) \\ &= -\frac{e^{2h}}{c^2} v(\psi' - \psi^2 + 2\chi\psi), \end{aligned}$$

$$(2.37) \quad R_{4114} = -\frac{e^{2f}}{c^2} (f_{tt} - c^2 f_{xx}) = -\frac{e^{2f}}{c^2} \{v' \chi + (c^2 - v^2)\chi'\},$$

$$(2.38) \quad \begin{aligned} R_{4\gamma\gamma 4} &= -\frac{e^{2h}}{c^2} (h_{tt} + h_t^2 - h_t f_t - c^2 h_x f_x) \\ &= -\frac{e^{2h}}{c^2} \{v' \psi + v^2(\psi^2 - \psi' - \chi\psi) - c^2 \chi\psi\}, \end{aligned}$$

$$(2.39) \quad \begin{aligned} R_{1\gamma\gamma 1} &= -\frac{e^{2h}}{c^4} \{c^2(h_{xx} + h_x^2 - h_x f_x) - h_t f_t\} \\ &= -\frac{e^{2h}}{c^4} \{c^2(\psi^2 - \psi' - \chi\psi) - v^2 \chi\psi\}. \end{aligned}$$

3. Newtonian gravitational waves. The author has proved elsewhere [8] that a Newtonian approximation to equations (2.14) to (2.19) is possible in which the functions h and f degenerate to a single function φ . If x now stands for Euclidean distance and t , for absolute Newtonian time, and if ρ_n , p_n and u_n stand for the Newtonian density, pressure and fluid velocity, respectively, then the approximation in question is

$$(3.1) \quad \begin{cases} \rho_n = -\varphi_{xx}, & p_n = -\varphi_{tt} + \frac{\varphi_{xt}^2}{\varphi_{xx}} - 2\pi G \varphi_x^2, \\ u_n = -\frac{\varphi_{xt}}{\varphi_{xx}}. \end{cases}$$

It is easily verified that these expressions for ρ_n , p_n , u_n satisfy the equations of continuity and of motion

$$(3.2) \quad \begin{cases} \frac{\partial \rho_n}{\partial t} + \frac{\partial \rho_n u_n}{\partial x} = 0, \\ \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} = -\frac{1}{\rho_n} \frac{\partial p_n}{\partial x} + 4\pi G \frac{\partial \varphi}{\partial x}, \end{cases}$$

for all functions φ . If $V = 4\pi G\varphi$, the first formula in (3.1) is $-4\pi G\rho_n = V_{xx}$, which is Poisson's equation and shows that V is the gravitational potential of the distribution of fluid of density ρ_n . Then equations (3.2) indicate that the motion of the fluid is controlled by two forces: the pressure-gradient and the gravitational self-attraction of the fluid mass.

The equations (3.1) were obtained on the assumption that ρ_n , and therefore φ_{xx} , were not zero. It has however been checked that the equations remain valid as $\rho_n \rightarrow 0$, provided that the limits, as $\varphi_{xx} \rightarrow 0$, of $\varphi_{xt}/\varphi_{xx}$ and of $\varphi_{xt}^2/\varphi_{xx}$ are both finite. It will now be shown that this happens, for example, when φ is the function appropriate to "sound waves of finite amplitude," viz:—

$$(3.3) \quad \begin{aligned} \varphi &= - \int \psi(X) dX - \int \left[\int P(t) dt \right] dt, \\ X &= x - \int v(t) dt, \end{aligned}$$

where ψ , P and v are arbitrary functions of their arguments. For we now have

$$\begin{aligned} \varphi_x &= -\psi, & \varphi_t &= v\psi - \int P(t) dt, \\ \varphi_{xx} &= -\psi', & \varphi_{tt} &= v'\psi - v^2\psi' - P, \\ \varphi_{xt} &= v\psi', \end{aligned}$$

and therefore equations (3.1) become

$$\begin{aligned} \rho_n &= \psi', & p_n &= -v'\psi + v^2\psi' + P - v^2\psi' \left(\frac{\psi'}{\psi'} \right) - 2\pi G\psi^2, \\ u_n &= v \left(\frac{\psi'}{\psi'} \right). \end{aligned}$$

Hence as $\rho_n \rightarrow 0$ ($\psi' \rightarrow 0$) the factors (ψ'/ψ') tend to unity and, in the limit of zero density, we have

$$0 = \psi', \quad p_n = P - v'\psi - 2\pi G\psi^2, \quad u_n = v.$$

Thus ψ must be a constant, say, ψ_0 ; also if the density is zero the pressure is presumably zero also and therefore the functions P and v are related by

$$P(t) = \psi_0 v'(t) + 2\pi G\psi_0^2.$$

Hence the function φ is now

$$(3.4) \quad \begin{aligned} \varphi &= -\psi_0 X - \psi_0 \int v(t) dt - \pi G\psi_0^2 t^2 \\ &= -\psi_0 \{x + \pi G\psi_0 t^2\}. \end{aligned}$$

The conclusion is this: if φ is given by (3.4) then the corresponding distribution of matter has zero density and pressure, and therefore the material cannot be regarded as present in any physical sense. Yet its velocity survives, since $u_n = v$, and so does its gravitational potential, neither of which necessarily vanishes with p_n and ρ_n . Moreover this surviving potential is propagated as an accelerated wave, whose velocity is clearly, by (3.4), equal to $-2\pi G\psi_0 t$.

It is not, of course, claimed that this solution of the hydrodynamical equations, with gravitational self-attraction taken into account, has any physical validity. But if "disembodied" gravitational potentials, propagated by wave-motions, are regarded as significant in the "gravitational wave" solutions of the exact Einstein equations, it is not easy to see why corresponding solutions of these equations in their degenerate Newtonian forms, should be ignored.

Returning to the general case in which φ is given by (3.3) but $\rho_n \neq 0$, it is to be noticed that a zero value of $P(t)$ is possible. Indeed the inclusion of $P(t)$ is made in (3.3) in order that, for certain selections of the function ψ , boundary conditions may be satisfied. An example will be found in [8, §iv, i] where $P(t)$ must be a constant different from zero; but in [8, §iv, ii] another illustration is given wherein P could be zero. The selection of the forms (2.21) for h and f therefore corresponds to the choice of Newtonian fluid motions in which $P(t) \equiv 0$, a mathematical simplification which will not unduly restrict the argument. Assuming then that $P(t) = 0$, it is proved in [8, §iv, iv] that *adiabatic* Newtonian motions for which $\varphi = -\int \psi(X)dX$ are such that

$$\begin{aligned} \rho_n &= \psi', & p_n &= -\alpha\psi - 2\pi G\psi^2, \\ (3.5) \quad v &= u_n = \alpha t + \beta, \\ X &= x - \frac{1}{2}\alpha t^2 - \beta t, \end{aligned}$$

where α, β are positive or negative constants. In these equations x, t are Euclidean distance and Newtonian absolute time, respectively. Hence again we obtain $p_n = 0, \rho_n = 0$ if $\psi = \psi_0 = \text{constant}$ and $\alpha = -2\pi G\psi_0$. The surviving gas-velocity and gravitational potential are now $u_n = \beta - 2\pi G\psi_0 t$ and $\varphi = -\psi_0(x + \pi G\psi_0 t^2 - \beta t)$, respectively, results of the same class as those given by (3.4).

4. Ultimately empty space-times. In seeking for gravitational wave solutions of the exact Einstein equations (2.26) to (2.31) two methods may be adopted. The customary one is to impose *a priori* the conditions $p = 0, \rho = 0$ and then to find solutions in which h and f are functions of $X = x - ct$. A different method will be adopted here in that it will be assumed, firstly, that X may have the more general form (2.22), and, secondly, that p and ρ must tend to zero as t tends to infinity but need not vanish for all t . Space-times in which this second assumption holds good will be called "ultimately empty" space-times. Thirdly, it will also be assumed that, in this ultimate state, X degenerates to $x - \beta t$, where the constant β need not however equal c numerically.

If, for large t , the pressure and density are to be zero, σ must then also vanish. But since X is to degenerate to $x - \beta t$, it follows from (2.22) that v must tend to a constant value different from zero. Hence by (2.31), if σ ultimately vanishes, it must be because

$$(4.1) \quad \Phi + \Psi = 0,$$

which, by (2.25) is also

$$(4.2) \quad \psi' - \psi^2 + 2\chi\psi = 0.$$

But this is an equation in X , a variable independent of t , and therefore (4.2) is always valid, whether t is large or small. Thus ultimately empty space-times are such that the product σU is always zero, because by (2.13), L cannot be less than unity. Now, by (2.29) and (4.1) the permanent vanishing of σU entails the restriction that, for all t ,

$$(4.3) \quad v'(\chi - \psi) - (c^2 - v^2)\Psi = 0.$$

Thus, in ultimately empty space-times, v , χ and ψ satisfy the two equations (4.2) and (4.3) and such space-times may be classified into the following four types:

(i) In this class $v = \text{constant} = \beta \neq \pm c$, and therefore $X = x - \beta t$. Moreover, by (4.3), $\Psi = 0$ also, which entails $\Phi = 0$. Thus by the second equation of (2.25) and the equation (4.2), χ and ψ satisfy

$$(4.4) \quad \chi' = -\psi^2, \quad \psi' = \psi^2 - 2\chi\psi.$$

There is obviously a solution in which $\chi = 0$, $\psi = 0$, which leads immediately to the Minkowski space-time (2.35) with $(t, x, y, z) \equiv (T, X, Y, Z)$. The other solutions are obtained by eliminating χ from (4.4) and, writing $\zeta = \psi/\psi'$, there comes

$$(4.5) \quad \frac{d\zeta}{d\psi} = -2\psi\zeta^3 - \zeta^2,$$

which is Abel's differential equation of the first kind [9]. If further $\omega = 2\psi\zeta = 2\psi^2/\psi'$, the equation becomes

$$\frac{d\omega}{d\psi} = -\frac{\omega(\omega+2)(\omega-1)}{2\psi},$$

and a first integral is

$$(4.6) \quad \omega^{-3}(\omega+2)(\omega-1)^2 = A\psi^{-3},$$

where A is a constant of integration. A further integration does not appear to be possible in terms of elementary functions. However, using (4.4) and the definition of ω , (2.26) and (2.27) give

$$\kappa c^2 \rho = 2(c^2 - \beta^2) e^{-2f} \psi^2 \left(\frac{1 - \omega}{\omega} \right),$$

$$\kappa p = -2(c^2 - \beta^2) e^{-2f} \psi^2 \left(\frac{1 - \omega}{\omega} \right).$$

Since the density and pressure are thus functions of X alone, they cannot tend to zero as t tends to infinity, for any arbitrary x . Moreover ρ and p always have opposite signs unless $\omega = 1$ (*i.e.* $A = 0$), when both are permanently zero. In this case

$$\chi = -\frac{1}{4} \frac{1}{x_0 - X}, \quad \psi = \frac{1}{2} \frac{1}{x_0 - X},$$

where x_0 is the constant of integration. By a suitable addition of constants in (2.21), the metric of the corresponding space-time may be written

$$(4.7) \quad \begin{cases} ds^2 = \left(1 - \frac{X}{x_0}\right)^{-1} \left(dt^2 - \frac{dx^2}{c^2}\right) - \frac{1}{c^2} \left(1 - \frac{X}{x_0}\right) (dy^2 + dz^2) \\ X = x - \beta t, \end{cases}$$

which is not flat because, by (2.21) and (2.37) with $v = \beta$,

$$R_{4114} = \frac{c^2 - \beta^2}{4c^2 x_0^2} \left(1 - \frac{X}{x_0}\right)^{-5/2},$$

and this is not zero. Thus (4.7) apparently gives a genuine case of gravitational waves, propagated in empty space with speed β in the direction of the x -axis. But this is illusory for two reasons: firstly, because the gravitational potentials have a singularity at $x = x_0 + \beta t$ and are imaginary if $x > x_0 + \beta t$. At the singularity, the energy-tensor is undefined and therefore it cannot be said that $p = 0$, $\rho = 0$ when $x \geq x_0 + \beta t$. Thus the space-time (4.7) represents the region outside some unknown distribution of matter, just as the Schwarzschild space-time represents the region outside a sphere. Secondly, and more importantly, the "wave" appearance of the gravitational potentials in (4.7) is due to the use of a special coordinate-system. If indeed t and x are transformed to τ and ξ , where

$$t = \frac{\tau + \beta \xi / c^2}{(1 - \beta^2 / c^2)^{1/2}}, \quad x = \frac{\xi + \beta \tau}{(1 - \beta^2 / c^2)^{1/2}},$$

the metric becomes

$$(4.8) \quad \begin{cases} ds^2 = (1 + k\xi)^{-1} \left(d\tau^2 - \frac{d\xi^2}{c^2}\right) - \frac{(1 + k\xi)}{c^2} (dy^2 + dz^2), \\ k = \frac{-\left(1 - \frac{\beta^2}{c^2}\right)^{1/2}}{x_0}, \end{cases}$$

a statical space-time discovered by A. H. TAUB.* Hence there are no solutions of type (i) which tend to empty gravitational wave space-times as the time increases. Instead we arrive at a solution in which p and ρ are always zero and gravitational waves can be eliminated by a coordinate transformation.

(ii) Here $v = \text{constant} = \pm c$ and it will be sufficient to consider $v = +c$, which implies $X = x - ct$. The equation (4.3) is now an identity and χ, ψ need satisfy (4.2) only. The pressure and density are again permanently zero but for a different reason from that found in case (i). The vanishing is now due to the zero values of the factors v' and $(c^2 - v^2)$ that occur in (2.26) and (2.27). Equation (4.2) is soluble in two ways, viz:—

$$(4.9) \quad (a) \quad \psi = 0, \quad \chi \text{ arbitrary};$$

$$(4.10) \quad (b) \quad \psi \neq 0, \quad \chi = \frac{\psi^2 - \psi'}{2\psi}.$$

Inspection of the formulae (2.36) to (2.39) reveals that the curvature tensor is zero in case (a), whatever χ may be and therefore (4.9) defines the Minkowski space-time, though the coordinates are not those of an inertial system. Indeed we have

$$ds^2 = e^{-2\int \chi dx} \left(dt^2 - \frac{dx^2}{c^2} \right) - \frac{1}{c^2} (dy^2 + dz^2),$$

$$X = x - ct,$$

as a possible form for the metric of the Minkowski space-time in which the “gravitational potentials” are propagated as waves with speed c . But this is merely due to using a non-inertial coordinate system.

In case (b) we have, by (2.21) with a constant, $\frac{1}{2} \log x_0$, introduced into the definition of f ,

$$e^{2h} = e^{-2\int \psi dx}, \quad e^{2f} = x_0 \psi e^{-\int \psi dx},$$

and since $X = x - ct$, the space-time

$$(4.11) \quad ds^2 = x_0 \psi e^{-\int \psi dx} \left(dt^2 - \frac{dx^2}{c^2} \right) - \frac{e^{-2\int \psi dx}}{c^2} (dy^2 + dz^2),$$

appears to be one in which gravitational waves occur and are propagated with speed c . But this result is again illusory for the coordinates t, x may be transformed to τ, ξ where

$$-\xi + c\tau = x_0 e^{-\int \psi dx}, \quad \xi + c\tau = x + ct,$$

* Apart from the notation, (4.11) is identical with equation (8.21) in [4].

and (4.11) becomes

$$(4.12) \quad ds^2 = d\tau^2 - \frac{d\xi^2}{c^2} - \frac{(\xi - c\tau)^2}{x_0^2 c^2} (dy^2 + dz^2).$$

It is then easy to verify from (2.36) to (2.39) that the curvature tensor is again identically zero and that (4.12) is therefore still another form of the metric of the Minkowski space-time expressed in non-inertial coordinates. Hence case (ii) only gives various forms of the Minkowski space-time, the pressure and density are always identically zero, and the appearance of gravitational waves is due to the use of non-inertial coordinates.

(iii) In this case v is not a constant and $\chi = \psi$, an equation that implies, by (2.24), $\Psi = 0$. Hence the only relation imposed on ψ (or χ) is (4.2), viz:—

$$(4.13) \quad \psi' + \psi^2 = 0,$$

whose solution is

$$(4.14) \quad \psi = (X - x_0)^{-1},$$

where x_0 is the constant of integration. The metric of space-time is therefore

$$(4.15) \quad \begin{cases} ds^2 = x_0^2(x_0 - X)^{-2} \left\{ dt^2 - \frac{1}{c^2} (dx^2 + dy^2 + dz^2) \right\} \\ X = x - \int v(t) dt, \end{cases}$$

and it is not flat, for by (2.37) for example,

$$R_{4114} = -\frac{1}{c^2} \frac{x_0^2}{(X - x_0)^4} \{v'(X - x_0) - (c^2 - v^2)\},$$

which is not zero. Equations (2.26), (2.27) and (2.34) become

$$(4.16) \quad \begin{aligned} 8\pi G\rho &= 3(v^2 - c^2)/x_0^2, \\ 8\pi Gp/c^2 &= \{2v'(x_0 - X) - 3(v^2 - c^2)\}/x_0^2, \\ u &= 0. \end{aligned}$$

Thus the material is at rest in the coordinate-system and $\rho \geq 0$, $p \geq 0$ so long as

$$(4.17) \quad v^2 \geq c^2, \quad x \leq x_0 - \frac{3(v^2 - c^2)}{2v'} + \int v(t) dt.$$

This type of solution can therefore apparently tend to “gravitational waves” with $X = x - ct$ provided that, as t increases, (a) $v(t)$ decreases to the value c ; (b) $\int v(t) dt$ tends to ct ; and (c) v' tends to zero. However, in this ultimate condition, (4.15) has degenerated to the Minkowski space-time since all the

components of the curvature tensor are then zero. The physical situation in the earlier stages of the motion is also very artificial: the density, though positive, depends on the time alone and not on position. The pressure depends on both x and t and vanishes when $x = x_0 - \frac{3(v^2 - c^2)}{2v'} + \int v(t) dt$, being negative on one side, and positive on the other side, of this plane. An example of a form for v which satisfies (4.17) is

$$v = c(1 + e^{-t/t_0}),$$

where t_0 is a constant. Here

$$8\pi G\rho = 3c^2 e^{-t/t_0}(2 + e^{-t/t_0})/x_0^2,$$

$$8\pi G\frac{p}{c^2} = 6c^2 e^{-t/t_0} \left\{ \frac{1}{3} \frac{x - x_0}{ct_0} - \frac{1}{3} \frac{t}{t_0} - \frac{1}{6} e^{-t/t_0} - 1 \right\} / x_0^2,$$

and the plane on which $p = 0$ is, at time t ,

$$x = x_0 + (3t_0 + t)c + \frac{1}{2}ct_0 e^{-t/t_0},$$

the pressure being negative for values of x smaller than this.

(iv) The last case is one in which v is not a constant and $\chi \neq \psi$. Hence (4.3) is

$$\frac{v'}{c^2 - v^2} = \frac{\Psi}{\chi - \psi},$$

an equation, one of whose members is a function of t , the other, of X . Their equality therefore implies that

$$(4.18) \quad \frac{v'}{c^2 - v^2} = \frac{\alpha}{c^2} = \frac{\Psi}{\chi - \psi},$$

where α is a non-zero constant. Hence the function v is

$$(4.19) \quad v = c \tanh \left(\frac{\alpha t + \beta}{c} \right),$$

where β/c is the constant of integration, and therefore

$$(4.20) \quad X = x - \int v(t) dt = x - \frac{c^2}{\alpha} \log \cosh \left(\frac{\alpha t + \beta}{c} \right).$$

The functions χ, ψ satisfy (4.2) and the second of equations (4.18), i.e.

$$(4.21) \quad \begin{cases} \psi' - \psi^2 + 2\chi\psi = 0, \\ \psi' - 2\psi^2 - \chi' + 2\chi\psi = \frac{\alpha}{c^2}(\chi - \psi). \end{cases}$$

A solution is

$$(4.22) \quad \psi = 0, \quad \chi = e^{-(\alpha/c^2)x},$$

but, since $c^2\chi' + \alpha\chi = 0$, it is easy to show, using (4.19) also, that the components of the curvature tensor given by (2.36) to (2.39) are all identically zero, which means, of course, that the pressure and density are so also. By a suitable choice of additive constants in (2.21) still another form of the Minkowski space-time in non-inertial coordinates is

$$(4.23) \quad \begin{cases} ds^2 = \exp(e^{-(\alpha/c^2)x}) \left(dt^2 - \frac{dx^2}{c^2} \right) - \frac{1}{c^2} (dy^2 + dz^2), \\ X = x - \frac{c^2}{\alpha} \log \cosh \left(\frac{\alpha t + \beta}{c} \right). \end{cases}$$

Here the "gravitational waves" are propagated with the variable velocity $v = c \tanh [(\alpha t + \beta)/c]$, a situation analogous to the Newtonian cases discussed in Section 3.

A second solution of (4.21), in which $\psi \neq 0$, is obtained by writing the equations in the equivalent forms

$$(4.24) \quad \chi = \frac{\psi^2 - \psi'}{2\psi}, \quad \chi' = -\psi^2 + \frac{\alpha}{2c^2} \left(\psi + \frac{\psi'}{\psi} \right),$$

and then eliminating χ . If $\xi = \psi/\psi'$, the resulting equation is

$$(4.25) \quad \frac{d\xi}{d\psi} = - \left(2\psi - \frac{\alpha}{c^2} \right) \xi^3 - \left(1 - \frac{\alpha}{c^2} \frac{1}{\psi} \right) \xi^2,$$

which is again Abel's differential equation of the first kind [9]. Owing to the nature of the coefficients of ξ^3 and ξ^2 , it appears to have no simple solutions. By (2.21) and (4.24) the metric is

$$(4.26) \quad \begin{aligned} ds^2 &= x_0 \psi e^{-\int \psi dx} \left(dt^2 - \frac{dx^2}{c^2} \right) - \frac{e^{-2\int \psi dx}}{c^2} (dy^2 + dz^2), \\ X &= x - \frac{c^2}{\alpha} \log \cosh \left(\frac{\alpha t + \beta}{c} \right), \end{aligned}$$

and the space-time is curved because, by (2.37) for example,

$$R_{4114} = -x_0 \psi^2 \left(\frac{\alpha}{c^2} - \psi \right) e^{\int \psi dx} \operatorname{sech}^2 \left(\frac{\alpha t + \beta}{c} \right),$$

which is not zero for finite values of t . It is also easily verified that $\chi = 0$, which leads to $\psi = (x_0 - X)^{-1}$, cannot be a solution of both the equations (4.24). Thus division by χ is legitimate and equations (2.26), (2.27) and (2.34) become, with the help of (4.19) and (4.24),

$$(4.27) \quad \begin{cases} 8\pi G\rho = \frac{c^2}{x_0} e^{\int \psi dx} \left(\frac{\psi'}{\psi} - 2\psi \right) \operatorname{sech}^2 \left(\frac{\alpha t + \beta}{c} \right), \\ 8\pi G \frac{p}{c^2} = \frac{c^2}{x_0} e^{\int \psi dx} \left\{ -\frac{2\alpha}{c^2} - \left(\frac{\psi'}{\psi} - 2\psi \right) \right\} \operatorname{sech}^2 \left(\frac{\alpha t + \beta}{c} \right), \\ u = 0. \end{cases}$$

The last equation shows that the material is at rest in the coordinate system and, clearly, as t becomes large, both ρ and p tend to zero. The metric (4.26) also tends to the form (4.11) for large t , because, in the former, $X \rightarrow x - ct + (c^2/\alpha) \log 2$, which apart from a trivial additive constant, is the expression for X in the latter. Thus the space-time (4.26) eventually ends as the Minkowski space-time with $\rho = 0$ and $p = 0$. In the absence of a solution of (4.25), the signs of ρ and p for small t are unknown, but it can be said that both are positive only if

$$0 < \frac{1}{x_0} \left(\frac{\psi'}{\psi} - 2\psi \right) < \frac{2}{c^2} \left(-\frac{\alpha}{x_0} \right).$$

The general conclusion of this section is that ultimately empty space-times can occur in cases (iii) and (iv) though, for small t , the distributions of matter have somewhat curious physical properties. The eventual empty state however means that the space-time has degenerated to the Minkowski space-time of special relativity expressed in terms of non-inertial coordinates. It is the use of these coordinates that gives rise to the illusion of gravitational waves, waves that may be propagated with either constant or variable velocities. In case (ii), where the velocity of propagation is c , only the Minkowski space-time is possible and no ultimately empty space-times exist. Case (i), wherein the velocity of propagation is a constant not equal to c , does include TAUB's solution (4.7) which is permanently empty and also curved. But even here the wave appearance of the potentials can be removed by a coordinate transformation. It therefore does not appear that genuine gravitational wave solutions of Einstein's equations exist when the potentials have the form (2.21); such waves are illusions due to the use of non-inertial coordinates in the Minkowski space-time.

5. Einsteinian gas-dynamics. The ultimately empty space-times considered in the preceding section are characterised by the forms $X = x - \beta t$ and $X = x - (c^2/\alpha) \log \cosh [(\alpha t + \beta)/c]$, of which the latter becomes $X = x - \frac{1}{2}(\alpha t + \beta)^2$ when $1/c^2 \rightarrow 0$. Reference to (3.5) indicates that these are precisely the forms for X occurring in adiabatic Newtonian one-dimensional gas motions of the type denoted "sound waves of finite amplitude." Therefore it is to be anticipated that solutions of the equations (2.26) to (2.31), with the condition $\Phi + \Psi = 0$ removed, will give the analogues in general relativity of the aforesaid Newtonian motions.

If $\Phi + \Psi \neq 0$, then neither σ nor U in (2.31) can vanish and the consistency relation (2.32) comes into play. We deal first with the case when $X = x - (c^2/\alpha) \log \cosh [(\alpha t + \beta)/c]$ which means that $v = c \tanh [(\alpha t + \beta)/c]$. In equation (2.32), the functions of X and of t must be separable and the equation must reduce to the following two equations:—

$$(5.1) \quad \Psi(\chi + \psi) + \Phi(\psi - \chi) = -\Psi(\psi - \chi) - \Phi(\psi + \chi),$$

$$(5.2) \quad -\Psi\Phi = \frac{\alpha}{c^2} \left[\frac{\alpha}{c^2} (\psi^2 - \chi^2) - \{\Psi(\psi - \chi) + \Phi(\psi + \chi)\} \right].$$

But (5.1) is $(\Psi + \Phi)\psi = 0$ and since, by hypothesis, $\Psi + \Phi \neq 0$ it follows that $\psi = 0$ and therefore by (2.24) $\Psi = -\Phi = -\chi'$. Thus $\psi = 0$ implies $\Psi + \Phi = 0$ and therefore it may be concluded that, if $v = c \tanh [(\alpha t + \beta)/c]$, only the ultimately empty space-times of Section 4 (iii) and (iv) are possible. These then are the analogues in general relativity of the Newtonian gas-flows (3.5). Incidentally, the vanishing of u in (4.16) and (4.27) does not mean that the corresponding Newtonian velocity u_n is zero. The coordinates employed in the relativistic formulae may well be co-moving as is indicated by the fact that the corresponding metrics, (4.15) and (4.26) are not statical. A similar situation arises in Newtonian, as compared with relativistic, cosmology [10]; in the one case, the radial velocity is non-zero, in the other, it vanishes because of the employment of a co-moving coordinate system.

Turning now to the alternative $v = \beta \neq c$, the consistency relation (2.32) becomes $\Psi\Phi = 0$ and therefore *either*

$$(5.3) \quad \Psi = 0, \quad \Phi \neq 0,$$

or

$$(5.4) \quad \Phi = 0, \quad \Psi \neq 0,$$

and these two cases will be considered separately.

(i) $\Psi = 0$ means by (2.24) that

$$\chi' - 2\chi\psi = \psi' - 2\psi^2$$

whence

$$(5.5) \quad \chi = \psi + Ae^{2\int\psi dX},$$

where A is the constant of integration. This equation may also be written, by (2.21),

$$f' = h' - Ae^{-2h}.$$

Thus the relation between f and h , apart from additive constants, is

$$(5.6) \quad f = h - A \int e^{-2h} dX, \quad e^{2h} = e^{-2\int\psi dX}, \quad X = x - \beta t.$$

Since however $\Phi \neq 0$, and by (2.25)

$$(5.7) \quad \Phi = 2(\chi' + \psi^2)$$

when $\Psi = 0$, the only restriction on the function ψ is that

$$(5.8) \quad \psi' + 2A\psi e^{2\int \psi dx} + \psi^2 \neq 0.$$

Dealing next with the equations (2.26) to (2.28), we obtain

$$(5.9) \quad 8\pi G\rho = (\beta^2 - c^2)e^{-2f}(2A\psi e^{2\int \psi dx} + 3\psi^2),$$

$$(5.10) \quad 8\pi G \frac{p}{c^2} = (\beta^2 - c^2)e^{-2f}(2\psi' - \psi^2 + 2Ae^{2\int \psi dx}),$$

$$\kappa\sigma = \frac{\beta^2 - c^2}{c^2} e^{-2f} \Phi \neq 0.$$

But the last equation shows that (2.29) and (2.30) become, respectively,

$$U^2 = \frac{c^4}{\beta^2 - c^2}, \quad L^2 = \frac{\beta^2}{\beta^2 - c^2},$$

and therefore U and L are real only if

$$(5.11) \quad \beta^2 > c^2.$$

Finally, the coordinate-velocity of the gas is by (2.34)

$$(5.12) \quad u = c^2/\beta,$$

which, in virtue of (5.11), means that $u < c$. The fact that ρ and p given by (5.9), (5.10), (5.5) and (5.6) involve one undetermined function $\psi(X)$ is to be expected. For Einstein's equations in the one-dimensional case are a generalization of the equations of continuity and of motion (3.2) of Newtonian theory. The latter, even if φ is regarded as a known function, are but two equations for the three unknowns ρ_n , p_n , u_n and are therefore indeterminate to the extent of one arbitrary function of the coordinates. The generalization performed by EINSTEIN has therefore not destroyed this indeterminacy.

In (5.6), f and h are functions of $X = x - \beta t$, the gravitational potentials thus depending on both x and t . But by means of the transformation*

$$(5.13) \quad c\tau = (x - \beta t)/\gamma, \quad \xi = \left(ct - \frac{\beta}{c}x\right)/\gamma, \quad \gamma = \left(\frac{\beta^2}{c^2} - 1\right)^{\frac{1}{2}}$$

* I am indebted to Dr. A. H. TAUB for pointing out to me that such transformations should exist.

we obtain

$$X = c\gamma\tau$$

and therefore f and h become functions of τ whilst the metric of space-time transforms into

$$(5.14) \quad ds^2 = e^{2f(c\gamma\tau)} \left(d\tau^2 - \frac{d\xi^2}{c^2} \right) - \frac{e^{2h(c\gamma\tau)}}{c^2} (dy^2 + dz^2).$$

The arrangement of plus and minus signs in this formula indicates that τ and ξ are, respectively, a time and a space coordinate just as much as are t and x . Since ρ and p are scalars, formulae (5.9) and (5.10) can still be used on replacing X by $c\gamma\tau$. Thus, as in the cosmological models of the universe [10], the pressure and density are functions of a time-coordinate alone and (5.14) might therefore be described as the metric of a universe possessing plane, instead of the usual spherical, symmetry. The coordinate-velocity u^1 transforms to $\bar{u}^1$ under (5.13), by the usual rule for contravariant vectors, where

$$\begin{aligned} \bar{u}^1 &= \frac{\partial \xi}{\partial x} u^1 + \frac{\partial \xi}{\partial t} u^4 = e^{-f} \left(-\frac{\beta}{c\gamma} U + \frac{c}{\gamma} L \right) \\ &= e^{-2f} \left(-\frac{\beta c}{\gamma} + \frac{\beta c}{\gamma} \right) (\beta^2 - c^2)^{-1} \\ &= 0. \end{aligned}$$

Hence the coordinate system (τ, ξ, y, z) is co-moving with the material.

(ii) The second possibility is $\Phi = 0$, $\Psi \neq 0$ and the detailed analysis follows closely that of case (i). The results only will be quoted, in the order in which they were obtained in (i). We now have, if $Q = \exp(-2\int \psi dX)$ and B is a constant of integration,

$$\chi = BQ - Q \int (\psi'/Q) dX,$$

$$\Psi = -2(\chi' + \psi^2) \neq 0,$$

$$(5.15) \quad 8\pi G\rho = (c^2 - \beta^2)e^{-2f} \left\{ 2\psi' - 3\psi^2 + 2B\psi Q - 2\psi Q \int (\psi'/Q) dX \right\},$$

$$(5.16) \quad 8\pi G \frac{p}{c^2} = (c^2 - \beta^2)e^{-2f} \left\{ \psi^2 + 2B\psi Q - 2\psi Q \int (\psi'/Q) dX \right\},$$

$$\kappa\sigma = \frac{c^2 - \beta^2}{c^2} e^{-2f} \Psi \neq 0, \quad U^2 = \frac{c^2 \beta^2}{c^2 - \beta^2}, \quad L^2 = \frac{c^2}{c^2 - \beta^2},$$

$$\beta^2 < c^2, \quad u = \beta(<c).$$

The coordinate-transformation is now

$$(5.17) \quad \tau = \left(t - \frac{\beta}{c^2} x\right) / \gamma, \quad \xi = (x - \beta t) / \gamma, \quad \gamma = \left(1 - \frac{\beta^2}{c^2}\right)^{\frac{1}{2}},$$

and the metric of space-time becomes

$$(5.18) \quad ds^2 = e^{2f(\gamma\xi)} \left(d\tau^2 - \frac{d\xi^2}{c^2}\right) - \frac{e^{2h(\gamma\xi)}}{c^2} (dy^2 + dz^2),$$

and, since ξ is clearly a spatial coordinate, the space-time is statical. Thus the density and pressure become, by (5.15) and (5.16) with $X = \gamma\xi$, functions of the spatial coordinate ξ alone. The transform of u^1 under (5.17) is

$$\begin{aligned} \bar{u}^1 &= \frac{\partial \xi}{\partial x} u^1 + \frac{\partial \xi}{\partial t} u^4 = e^{-f} \left(\frac{U}{\gamma} - \frac{\beta}{\gamma} L \right) \\ &= e^{-f} \left(\frac{c\beta}{\gamma} - \frac{c\beta}{\gamma} \right) (c^2 - \beta^2)^{-\frac{1}{2}} \\ &= 0, \end{aligned}$$

and therefore the material is at rest in the coordinate system (τ, ξ, y, z) .

The conclusion of this section is that, when $v = \beta \neq c$, then $\beta^2 > c^2$ corresponds to purely time-dependent distributions of matter in motion analogous to those occurring in cosmology. But if $\beta^2 < c^2$ the distributions are statical and there is no motion. The Minkowski space-time corresponds, as was shown in Section 4 (ii), to $\beta^2 = c^2$ and this permanently empty space-time thus separates the purely time-dependent from the purely space-dependent solutions of Einstein's equations found in the present section. If the Minkowski space-time may be regarded as the seat of "gravitational waves," such waves act as separators between the two kinds of solution.

Summary. Exact solutions of Einstein's equations in general relativity are obtained under the assumption of plane symmetry. They correspond to sound waves of finite amplitude in classical gas-dynamics. Limiting cases, when the energy-tensor tends to zero with the time, are examined in detail in order to elucidate the problem of gravitational waves. Such waves turn out to be spurious, in general, in the sense that they can be removed by a coordinate transformation leading to a flat space-time. In one case, waves of variable velocity are found and have analogues in Newtonian theory. The flat space-time of special relativity separates gas-flows that are purely time-dependent from flows that are purely space-dependent.

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