Transonic Flow past Finite Wedges

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1. Introduction. In an earlier paper [16] the authors described how hodograph methods could be applied to determine the steady, two-dimensional flow of a compressible fluid with subsonic upstream velocity past a finite wedge. By generalizing the known solution for incompressible flow past such a wedge an expression was obtained for the stream function Ψ which satisfied exactly the equations of non-viscous compressible flow. This solution represented uniform flow at infinity at a given subsonic velocity in the direction of the axis of symmetry (conditions always being considered in a plane containing a normal section of the wedge); the streamline $\Psi = 0$ was identified as the axis of symmetry up to the tip of the wedge, after which it was continued as the sloping side of the wedge. A singularity of the transformation to the hodograph variables was invariably found to occur for a point on the wedge side, the point being either the sonic point or, under certain circumstances, a point of subsonic velocity. When the singularity occurred at the sonic point it was identified as the shoulder of the wedge and it was suggested that the flow around the corner might be obtained by means of characteristics, a local Prandtl-Meyer expansion being inserted at the shoulder. The generalized hodograph solution was only to be used up to (say) the sonic line. In this way it was intended to avoid the physically unreal limit line which always occurs in the analytical solution in this case, and to allow the characteristics to indicate the occurrence of shock-waves. It introduces questions, however, of over-specification of boundary conditions and also of the upstream effect of the subsonic flow behind the shock-wave which closes the supersonic region. The object of Part I of this paper is to discuss these questions in more detail and also to examine the physical picture when the singularity on the wedge side occurs at a subsonic velocity.

Further investigation and development of the solution obtained in [16] are opportune in view of the work which has recently appeared in American journals on flow past thin wedges at high subsonic Mach numbers. This includes papers by COLE [6], GUDERLEY & YOSHIHARA [9], IMAI [11], TRILLING [21], TRILLING & WALKER [22] and WEINSTEIN [24]. These writers all use the Tricomi approximation, *i.e.* an approximation of the full equations of motion valid in the neighbourhood of sonic speed. It follows that only thin wedges in near-sonic streams can be treated by this means. Even under these conditions a somewhat severe approximation is necessary to represent the stagnation point which always exists at the tip of the wedge in a subsonic stream. In §§5, 6 below an exact expression is found for the drag on the wedge as given by the generalized hodograph solution; it is obtained in the form of a contour integral together with a residue term. It is shown how this expression reduces to substantially the same form as is obtained by the methods using Tricomi's equation for thin wedges at sonic or high subsonic free stream velocities. Numerically, the latter methods give a drag which is of the order of 5–10% higher than is obtained in this paper. A discussion is given of the effect on the drag coefficient of the choice of counteracting function (see §2 below) made when generalizing the complex potential of the incompressible flow according to the established hodograph method.

In addition the drag of a blunt wedge is computed directly from the contour integral for a number of subsonic free stream Mach numbers. The results obtained substantiate a working hypothesis due to MACCOLL [15]; he argued that since the pressure at the stagnation point bears a fixed ratio to the critical pressure p^* (*i.e.*, pressure at sonic speed) and since the pressure is actually critical at the wedge shoulder when the speed is sonic at that point, the drag on a blunt body may be computed with sufficient accuracy by assuming that p/p^* is a universal function of r/R where p is the pressure at a point on the wedge at distance r from the apex and R is the length of the sloping edge of the wedge. The values of the drag coefficient obtained in this paper are in excellent agreement with Maccoll's values obtained by iterative computations.

In an appendix the analytical connection between the forms of the stream function given by the exact and the approximate equations of motion is illustrated.

PART I. *General remarks on transonic flow past finite wedges*

2. Résumé and discussion of the Mackie-Pack solution†

2.1. As already stated above, the method used in the earlier paper by the authors was the so-called "hodograph method", first exploited by CHAPLYGIN [2] and in recent years developed independently and at considerable length by LIGHTHILL [13, 14] and CHERRY [3, 4]. The equation of two-dimensional motion of a non-viscous gas, flowing adiabatically, is linear when expressed in terms of the hodograph variables q and θ , q being the resultant velocity at any point and θ the inclination of the stream at the point to some given line (here the axis of symmetry of the normal wedge-section); thus, the principle of superposition of solutions is available. Solutions obtained in the hodograph plane must, of course, be subsequently interpreted in the physical plane.

In the following paragraphs a résumé is given of the principal ideas of the hodograph theory and, in particular, of the results for wedge flow obtained by the authors [16] in their earlier work on this subject.

2.2. Suppose that the complex potential for the flow of incompressible fluid past a certain still obstacle is W_i . The hodograph method derives from W_i a pseudo-"complex potential" W of which the imaginary part is a genuine stream function in compressible flow. The shape of the obstacle for which this stream function is valid, while of the same general character as that of the original

† Many different notations have been used in the studies of transonic flow. Throughout this paper the suffix 0 refers to conditions at infinity upstream, the suffix 1 to conditions at the stagnation point and the asterisk is used for values corresponding to sonic velocity.

incompressible flow, is in fact different, the difference increasing with increasing Mach number. If W_i may be expanded in the form

$$W_i = \sum d_n (q/q_0)^n e^{-ni\theta}$$

when q is less than the undisturbed uniform velocity q_0 at infinity upstream, the d_n 's being coefficients independent of q and θ , then W is given by

$$W = \sum d_n f_n(\tau_0) \psi_n(\tau) e^{-ni\theta};$$

here $\tau = q^2/q_m^2$, q_m being the maximum possible speed in the gas, τ_0 is the value of τ in the undisturbed flow at infinity upstream and $\psi_n(\tau)$ is the function

$$\psi_n = \tau^{\frac{1}{2}} F(a_n, b_n; n+1; \tau)$$

with

$$a_n + b_n = n - \beta, \quad a_n b_n = \frac{-\beta n(n+1)}{2} \quad \text{and} \quad \beta = \frac{1}{(\gamma - 1)}.$$

$F(a_n, b_n; n+1; \tau)$ is a hypergeometric function defined in the usual way and γ is the ratio of specific heats of the gas. (In this notation, with $\gamma = 1.4$, the speed of sound corresponds to $\tau = \frac{1}{6}$.)

The series for W converges only for $\tau < \tau_0$ and the stream function for those parts of the velocity field where $\tau > \tau_0$ has to be found by analytic continuation. The function $f_n(\tau_0)$, which may be termed a *countering function*, has so to be chosen that $W \rightarrow W_i$ as $\tau, \tau_0, \beta \rightarrow 0$ simultaneously, the condition for incompressibility, and so that the series for W behaves near $\tau = \tau_0$ in the same way as W_i behaves near $q = q_0$. An additional condition which will be imposed here is that the function shall have no poles for $\Re(n) \geq 0$ when considered as a function of the suffix n . The simplest functions satisfying these conditions are:

$$(i) \quad f_n(\tau_0) = e^{-ns_0},$$

where s_0 is the value at $\tau = \tau_0$ of the function

$$s(\tau) = \sigma + (\tau^*)^{\frac{1}{2}} \tanh^{-1} \{(\tau^* - \tau)/(1 - \tau)\}^{\frac{1}{2}} - \tanh^{-1} \{(1 - \tau/\tau^*)/(1 - \tau)\}^{\frac{1}{2}}$$

in which τ^* denotes the value of τ corresponding to sonic speed (denoted by τ_s in [16])[†] and

$$\sigma = s(\tau^*) = -(1/\tau^*)^{\frac{1}{2}} \tanh^{-1} (\tau^*)^{\frac{1}{2}} - \frac{1}{2} \log_e (\frac{1}{2}\beta).$$

The manner in which this function arises is shown in the papers of Lighthill [13] and in full detail in a summarizing series of lectures by one of the authors [18].

$$(ii) \quad f_n(\tau_0) = 1/\psi_n(\tau_0).$$

[†] See footnote on the foregoing page.

This form was used by CHAPLYGIN. It is not always the best function to use since its singularities, when considered as a function of a continuous complex variable n lead to undesirably clumsy analytical continuations of W . Nevertheless, it is very important in the application of the hodograph method to the study of any flow in which a free boundary has to be preserved.

It is an unsatisfactory feature of the hodograph method in gas dynamics at the present time that the precise effect of variations in the choice of $f_n(\tau_0)$ is not known; however, both of the functions given here will be considered in the investigations of wedge drag in §6.

The appropriate form of W for $\tau < \tau_0$ obtained by generalizing the complex potential for incompressible flow past a finite wedge of semi-angle π/ξ is

$$W = -c^2 \sum_{n=1}^{\infty} f_n(\tau_0) \psi_n(\tau) e^{-in\xi\theta}, \quad (2.2.1)$$

where c^2 is a scale factor. The stream function Ψ is seen to vanish identically when $\theta = 0$ and when $\theta = \pi/\xi$. The stream function for those parts of the velocity-field where τ exceeds τ_0 may be obtained by means of an intermediate integral. The actual form of W for an unspecified $f_n(\tau_0)$ satisfying the conditions laid down above is

$$W = \frac{c^2}{2i\xi} \int_C \frac{\psi_\nu(\tau) f_\nu(\tau_0) \exp(-\nu i\theta + \pi i\nu/\xi)}{\sin(\pi\nu/\xi)} d\nu. \quad (2.2.2)$$

The integral is taken along a path C stretching from $\nu = -i\infty$ to $\nu = +i\infty$ such that the simple poles of the integrand at $\nu = n\xi$ ($n = 1, 2, 3, \dots$) lie to the right of the path and all other poles to the left. The integral converges for $\tau \leq \tau^*$ and $0 < \theta < 2\pi/\xi$ and will be the chief subject for study in considering the wedge drag. The integral yields the series (2.2.1) for W when $\tau < \tau_0$; for $\tau > \tau_0$ it gives a stream function which again vanishes identically for $\theta = \pi/\xi$ but which is non-zero for general values of τ when $\theta = 0$.

It is seen that, at least as far as the singularity already mentioned in the introduction, the wedge profile is preserved in the generalization by the hodograph method.

A relation of some importance in the subsequent work is now recalled. If ρ is the density of a gas and ρ_1 its value at a stagnation point, and if M denotes the local Mach number, then on the sloping edge of the wedge the variation of the velocity with distance r from the apex is given by

$$\frac{dr}{d\tau} = \rho_1(M^2 - 1)\Psi_\theta/2\rho q_m \tau^{\frac{3}{2}}. \quad (2.2.3)$$

The origin of Cartesian coordinates will be taken at the apex of the wedge; the x -axis will be taken along the axis of symmetry with its positive direction pointing downstream. The Jacobian $J = \partial(x, y)/\partial(q, \theta)$ of the transformation to the hodograph plane is

$$J = (\rho_1^2/\rho^2 \tau^{\frac{3}{2}} q_m^{\frac{3}{2}}) \{(M^2 - 1)\Psi_\theta^2 - 4\tau^2 \Psi_\tau^2\}. \quad (2.2.4)$$

2.3. The hodograph method which has just been outlined possesses two main disadvantages. The first, which has already been mentioned, is that the shape of the body in the compressible flow obtained by the generalization is different from the shape of the body in the original incompressible flow. In the present case, however, the wedge profile is preserved in general up to the shoulder of the wedge. The second disadvantage is the latent danger of singularities occurring in the transformations. These singularities may either deprive the solution of a physically acceptable interpretation or affect the flow pattern in a manner which is not easily foreseen. The occurrence of limit lines has been too well discussed in the literature to need detailed explanation here, but the manner in which it affects the present solution is considered in §4. The presence of a less well known type of isolated singularity which arises under certain conditions in the wedge problem is now investigated.

3. Interpretation of the solution when a subsonic singularity occurs*

3.1. At subsonic velocities the Jacobian J can only vanish at isolated points where Ψ_r and Ψ_θ vanish simultaneously. On the wedge side, since the streamline is straight, $\Psi_r \equiv 0$; J therefore vanishes on the wedge-side when $M = 1$ or when $\Psi_\theta = 0$. From (2.2.3), $dr/d\tau$ is seen to vanish along with J . The case $M = 1$ is that of the limit line discussed in the next section. We may quickly dismiss the possibility of Ψ_θ vanishing on the axis of symmetry, which is another straight streamline; this follows immediately from the series form of $[\Psi_\theta]_{\theta=0}$ derived from (2.2.1) and the result (obtained in [13]) that $\psi_n(\tau) > 0$ for real positive values of n when $\tau > 0$. The stagnation point at the tip of the wedge is a singularity ($J = -\infty$) of the same nature as that which occurs in the corresponding incompressible flow.

With regard to the behaviour of Ψ_θ on the wedge side the following results can be established.* For any given (subsonic) free stream velocity, Ψ_θ can be made to vanish on the wedge side for some subsonic value of τ provided that the angle of the wedge is sufficiently small. Alternatively, if the angle of the wedge is given, Ψ_θ will vanish on the wedge side for some subsonic value of τ if the free stream velocity is sufficiently low. In neither case, however, does Ψ_θ vanish before the gas, accelerating up the wedge from the stagnation point at the tip, has attained its free stream velocity. Finally, values of τ_0 and ξ do exist such that $\Psi_\theta \neq 0$ on the wedge side for $0 < \tau < \tau^*$. These results are to be expected on physical grounds. When the wedge angle or the free stream velocity is too small, no supersonic region is possible and the flow, unable to turn a convex corner [1], presumably breaks away from the shoulder giving a free streamline.

Since $dr/d\tau$ changes sign when Ψ_θ changes sign and since r is a single-valued

* A more detailed account of the singularity considered in this section and of other aspects of the singularities occurring at the shoulder of the wedge is contained in the doctoral thesis presented by one of the authors to the University of St. Andrews—"Hodograph methods applied to flow past finite wedges", by A. G. MACKIE.

function of τ it follows that the streamline cannot continue straight after this singularity. It apparently turns back on itself. Cusped streamlines are known to occur when limit lines are present [19], but here the singularity is isolated. Streamlines arbitrarily close to the wedge side encounter no singularity. The only possible explanation is that the second set of values of r , corresponding to values of τ greater than that at which the singularity occurs, which apparently lies over the first, belongs in fact to another branch of the solution and that the streamline corresponding to the branch actually considered is continued in a non-singular way but is no longer straight. The actual verification of this by analytical methods is impracticable in view of the extremely complicated nature of the solution for values of $\tau > \tau_0$. However, considerable support is given to this idea by the consideration of a model solution which exhibits the same main features as far as the occurrence of a subsonic singularity is concerned.

3.2. The particular solution of the equations of motion in the hodograph variables chosen to give a type of flow analogous to that occurring in the wedge problem is

$$\Psi = -\Psi_{-\frac{27}{2}}(\tau) \sin \frac{27}{2}\theta + 10^6 \psi_{-\frac{9}{2}}(\tau) \sin \frac{9}{2}\theta. \quad (3.2.1)$$

The position coordinates can be found directly and the whole field of flow computed. The particular choice of (3.2.1) was made because the functions involved were available in tables computed by HUCKEL [10]. [These tables contain an error, pointed out to the authors by Professor T. M. CHERRY, in the values of the derivative of $Y_n(\tau) = F(a_n, b_n; n+1; \tau)$. The function tabulated as $dY_n/d\tau$ is actually $(dY_n/d\tau)/(\mp 5n/4)$.] When $\theta = 0$, $\Psi = 0$ and $y = 0$. The distance x measured along $\theta = 0$ and the velocity are related by the equation corresponding to (2.2.3), namely

$$\left[\frac{dx}{d\tau} \right]_{\theta=0} = \rho_1(M^2 - 1)[\Psi_\theta]_{\theta=0}/2q_m \rho \tau^{\frac{3}{2}}. \quad (3.2.2)$$

Now

$$[\Psi_\theta]_{\theta=0} = -\frac{27}{2} \psi_{-\frac{27}{2}}(\tau) + 10^6 \times \frac{9}{2} \psi_{-\frac{9}{2}}(\tau). \quad (3.2.3)$$

For small values of τ this expression is negative. It follows that initially x increases from $-\infty$ as τ increases from zero. However, at a certain value of τ ($= \tau'$) (3.2.3) changes sign; τ' lies between .07 and .08. Thereafter as τ increases x decreases and the streamline apparently turns back on itself.

As has already been remarked this cannot be interpreted as a breakdown of the flow such as occurs with a limit line, for $\Psi_\tau \neq 0$ off the straight streamline and the singularity is isolated. The explanation is that $\tau = \tau'$, $\theta = 0$ is a branch-point singularity and the set of values of x corresponding to $\tau > \tau'$, $\theta = 0$ although coinciding with the set corresponding to $\tau < \tau'$, $\theta = 0$ belongs to a different branch of the solution. Moreover it must be possible to continue $\Psi = 0$

in a physically acceptable manner. Obviously it cannot be continued with θ still equal to zero.

Investigation of (3.2.1) shows that the streamline $\Psi = 0$ can be continued for non-zero values of θ and values of $\tau < \tau'$. The shape of the streamline is shown in Figure 1. τ and θ are continuous along it. It is straight up to B where $\tau = \tau'$ and then the flow decelerates along the curved portion BC which becomes asymptotically straight as $\tau \rightarrow 0$, having the direction $\theta = -2\pi/27$. It can be shown that $\tau < \tau'$ in the whole region above $\Psi = 0$.

3.3. Although it is not of application to the wedge problem it is of some interest to examine briefly the solution below $\Psi = 0$. In Figure 1 $A'BC$ is now the upper boundary of the flow. A' is the same point as A in the physical plane but is associated with the flow below $\Psi = 0$. B is the singular point on the streamline and C is a typical point on the curved part of $\Psi = 0$. Thus the lower solution represents flow decelerating round a concave bend (from A' towards C). The two flows cannot be joined together along $\theta = 0$ but are continuous over

FIGURE 1

the curved part of $\Psi = 0$. On $\Psi = -0$, τ and $-x$ increase together, but at $\tau = \tau^*$ a limit line occurs. The flow cannot therefore be continued back along the streamline BA' past the sonic point.

In the hodograph plane the situation is somewhat clearer. The singularity at B is a saddle point. Figure 2 shows the streamline $\Psi = 0$ in the hodograph plane, the points A, A', B, C corresponding to the points lettered in the same way in Figure 1. O is the origin of coordinates corresponding to $\tau = 0$. OY corresponds to the line $\theta = 0$ and OX to the line $\theta = -2\pi/27$, the θ -scale having been considerably magnified. $OABCO$ corresponds to $\Psi = +0$, $YA'BCO$ to $\Psi = -0$. A, A' are, of course, distinct points in the hodograph plane.

This affords an interesting example of the singularity listed in Theorem 4 of the paper by CRAGGS [7] on the singularities of the hodograph transformation (note that his u, v correspond to x, y of the above, while his x, y correspond to velocity components here). In particular it illustrates how a 360° neighbourhood of the physical plane at B maps into a 180° neighbourhood of the hodograph plane. Mention of a singularity of this type has recently been made by CHERRY [5].

3.4. It is now possible to interpret these results in terms of the wedge problem. In this and in the model solution the flow accelerates along a straight wall until it encounters at a subsonic velocity a point where $\Psi_\theta = \Psi_\tau = 0$. In the simple problem the critical streamline has been shown to continue through the singularity as a curved line which becomes asymptotically straight at infinity. It may therefore be conjectured with some confidence that the streamline $\Psi = 0$ in the wedge problem continues as a curved line which tends to become straight ultimately. Examination of the series and integral forms of the position co-ordinates shows that at $\tau = \tau_0$, $\theta = 0$, x becomes infinite. This then gives the ultimate behaviour of the streamline.

If the correctness of this interpretation is assumed, then the solution obtained by the hodograph method when no local supersonic region occurs may be summarized as follows: The solution represents flow past a profile which has a straight sloping edge up to a certain point after which it gently begins to slope back to the

FIGURE 2

direction of the free stream velocity (or axis of symmetry). When the singularity occurs near $\tau = \tau_0$ on the wedge side conditions will approximate the free stream (constant pressure) break-away from the wedge shoulder.

4. Discussion of the solution with singularity at sonic speed

The analysis of §2 shows that a singularity must always occur on the straight sloping face of the wedge. If the speed at infinity is great enough, J vanishes first at the point where the speed of sound is reached. There is then a limit line in the physical plane, its locus corresponding to the image in this plane of the curve in the hodograph plane given by

$$(M^2 - 1)\Psi_\theta^2 - 4\tau^2\Psi_\tau^2 = 0.$$

The stream function with which we are concerned in this paper gives bounded values of $|z| = |x + iy|$ for subsonic τ ($\neq \tau_0$) and for $\tau = \tau^*$. The solution therefore includes a supersonic region embedded in a subsonic region for the

conditions at infinity demand $\tau \sim \tau_0$, $\theta \sim 0$ as in the analogous incompressible flow. It is true that large values of $|z|$ are obtained corresponding to supersonic values of τ near $\tau = 1$. The reason for this is that τ and θ are many-valued functions of x and y and there will be values of x and y in the subsonic region which are also associated with supersonic values of τ . As far as this multiple mapping is concerned, the situation is believed to be similar to that which arises in the special exact solution discovered by RINGLEB [19]. What is of concern here is that branch of the solution which gives subsonic flow at infinity. The sonic singularity on the wedge side is now identified with the shoulder of the wedge, but the streamline $\Psi = 0$ is not continued as the straight upper face of the wedge since, for $\tau > \tau_0$, Ψ does not vanish identically when $\theta = 0$.

There must however be a limiting streamline beyond which (in the sense of away from the upper face of the wedge) the flow is non-singular, because a limit line is necessarily confined to the supersonic region of the physical plane. It has been known for some years [20] that the occurrence of a limit line renders useless the accompanying analytical solution downstream of this line and, further, that no solution of the equations of motion can be patched to the given solution across the limit line to give a continuation of the flow. As a result, in order to obtain a possible physical picture of the flow downstream from the shoulder of the wedge the introduction of at least one shock-wave is essential. (This also follows as a result of some theorems by NIKOLSKY & TAGANOFF [17].)

With this in mind the authors suggested in their earlier paper that a point-localized Prandtl-Meyer expansion should be inserted at the singular point and should be used in conjunction with the solution given by the hodograph method up to the sonic line; the flow field in the neighbourhood of the corner might then be built up by the method of characteristics. By means of this the location of a shock-wave could be determined and it is believed that this method would give a fair picture of conditions near the corner. It would not, however, give a solution compatible with the subsonic field downstream (mathematically, the problem is now over-prescribed).

Under certain circumstances the insertion of a point-localized Prandtl-Meyer expansion can lead to an exact solution in a problem in which the boundary conditions have been altered. For example, suppose a nozzle possesses a region of shock-free supersonic flow downstream from the sonic line and that this region extends across the whole width of the nozzle. If the nozzle is sawn off at a section within this region there is an overpressure where with respect to the outer medium, a point-localized Meyer expansion inserted at each of the edges and beginning at the forward Mach line in each case may be used, provided that these Mach lines do not meet the sonic line, to continue the flow by means of the method of characteristics. The solution so obtained is exact. In the wedge problem, however, the flow is not strictly supersonic downstream, the supersonic flow being bounded laterally by a subsonic field. It is further restricted by the presence of the solid boundary. Every streamline passing through the shock-wave has its entropy changed and it cannot be expected that the flow

downstream of the shock-wave for these streamlines can be made both to fit exactly the non-singular subsonic solution containing the streamlines which do not meet the shock-wave and, at the same time, to have the face of the wedge as the streamline $\Psi = 0$. This lack of adjustment may be considered as a 'disturbance' which will be propagated upstream through the simply-connected subsonic field and hence will modify it. This in turn will lead to an adjustment of the supersonic field, and thus may be looked upon as the beginning of an iterative process by which, if the mathematical analysis were feasible, the exact solution for the flow past a wedge, including the shock-wave system, might be found. What is important for the present paper is an assessment of the validity of the approximate solution obtained by the hodograph method for the computation of the drag of a wedge.

As the speed at infinity increases it becomes increasingly difficult for disturbances to get back against the stream. When the speed at infinity is actually sonic, the sonic line stretches from the shoulder to infinity and presents an impassable barrier to upstream influence by downstream disturbances, so long as attention is restricted, as here, to non-viscous fluids. (In viscous flow the boundary layer provides a channel by means of which disturbances may out-flank the sonic barrier.) For drag calculations, even when $M_0 < 1$, it is reasonable to suppose that the effects of modifications downstream will be at their least significant; for drag is determined by the pressure on the sloping face of the wedge, and this face, being, so to speak, 'protected' by the supersonic region, can only be affected by disturbances which have passed round the outside of the supersonic region. There is another way of looking at the problem of drag, mentioned by TRILLING & WALKER [22], which lends support to this theory. The drag on the wedge may be represented by the force on a stream-tube a long way away from the wedge. If the flow at large distances is adequately represented there should therefore be but a small error in the drag obtained from the approximate solution.

It is interesting evidence in favour of these conjectures that computations of drag made by TRILLING & WALKER differ by little from computations by COLE [6]. Both used Tricomi's approximation to the differential equation of motion, but COLE represented the sonic line by a straight line through the shoulder perpendicular to the axis of the wedge while TRILLING & WALKER made no restriction on the sonic line but ensured that the Prandtl-Meyer expansion at the corner was correctly included in their solution.

The MACKIE-PACK solution, while encountering the difficulties already outlined in the region downstream from the corner, has the advantage of being derived from the full equation of motion, and certain other advantages which will be discussed in the sections on drag which follow.

It is also of importance for the discussion of this section that for a blunt body (180° wedge) the method of this paper gives a drag which is in very close agreement with that obtained by MACCOLL, using an iterative method, for flow behind a detached shock-wave with the sonic line determined by the assumption that

streamlines cut it at right angles (!)—and that calculations made on the blunt body confirm MACCOLL's hypothesis about the insensitiveness of the profiles of p/p^* for flow over wedges (see §7 below).

PART II. *Drag of a wedge in a sonic or subsonic stream*

5. Exact analytic expression for the drag

In this section an exact expression is found for the drag on the wedge in a sonic or subsonic stream. The only restriction is that the free stream velocity is assumed to be high enough for sonic velocity to be attained at the shoulder (see §3).

Let F_D denote the total force perpendicular to one of the sloping faces. Then $F_D = \int_0^R p \, dr$ where R is the length of the side of the wedge, *i.e.*

$$F_D = \int_0^{\tau^*} p \frac{dr}{d\tau} d\tau.$$

Now $p = p_1(1 - \tau)^{\beta+1}$ by Bernoulli's Theorem and $dr/d\tau$ is given by (2.2.3). Hence

$$\begin{aligned} F_D &= \int_0^{\tau^*} p_1(1 - \tau)^{\beta+1} (2\rho_1/\rho q_m \tau^{\frac{3}{2}}) (M^2 - 1) [\Psi_\theta]_{\theta=\pi/\xi} d\tau \\ &= \left(\frac{p_1}{2q_m \tau^*} \right) \int_0^{\tau^*} \{(\tau - \tau^*)/\tau^{\frac{3}{2}}\} [\Psi_\theta]_{\theta=\pi/\xi} d\tau. \end{aligned} \quad (5.1)$$

The expression for Ψ is given by the contour integral (2.2.1). The contour C is along the imaginary axis from $-i\infty$ to $+i\infty$, indented so as to pass to the right of $\nu = 0$. The integral converges uniformly with respect to θ in $0 < \epsilon \leq \theta \leq (2\pi/\xi) - \epsilon$. Hence, by differentiation under the integral sign,

$$[\Psi_\theta]_{\theta=\pi/\xi} = \Im \left[\frac{-c^2}{2\xi} \int_C \frac{\nu f_\nu(\tau_0) \psi_\nu(\tau)}{\sin(\pi\nu/\xi)} d\nu \right]. \quad (5.2)$$

When this expression is substituted in (5.1) it is found necessary for the convergence of ensuing integrals at $\tau = 0$ (see below) that $\Re \nu > 1$. The contour C is therefore replaced by C' where C' is the straight line running from $-i\infty + \delta$ to $+i\infty + \delta$ and $1 < \delta < 2$. It is easily verified that the contributions at the ends of the range vanish for the particular choice of $f_\nu(\tau_0)$ which is made and hence that the value of (5.2) is unchanged by this deformation of the contour of integration. It also follows from the asymptotic forms of $\psi_n(\tau)$ given by LIGHT-HILL [13] that (5.2) is uniformly convergent in the range $0 \leq \tau \leq \tau^*$. It is therefore permissible to invert the order of integration in (5.1). Accordingly

$$F_D = -\Im \frac{p_1 c^2}{4q_m \xi} \int_{C'} \frac{\nu f_\nu(\tau_0) I_1}{\sin(\pi \nu / \xi)} d\nu, \quad (5.3)$$

where $I_1 = \int_0^{\tau^*} \{(\tau - \tau^*)/\tau^* \tau^{\frac{1}{2}}\} \psi_\nu(\tau) d\tau$.

The differential equation satisfied by $\psi_\nu(\tau)$ may be written as

$$2\tau\tau^*(1 - \tau)^{\beta+1} \frac{d}{d\tau} \{2\tau\psi'_\nu(\tau)(1 - \tau)^{-\beta}\} + \nu^2(\tau - \tau^*)\psi_\nu(\tau^*) = 0.$$

Hence (with dashes denoting differentiation with respect to τ)

$$\begin{aligned} I_1 &= -2\nu^{-2} \int_0^{\tau^*} \tau^{-\frac{1}{2}}(1 - \tau)^{\beta+1} \frac{d}{d\tau} \{2\tau\psi'_\nu(\tau)(1 - \tau)^{-\beta}\} d\tau \\ &= -[4\tau^{\frac{1}{2}}(1 - \tau)\psi'_\nu(\tau)/\nu^2]_0^{\tau^*} - 2\nu^{-2} \int_0^{\tau^*} (1/\tau^* \tau^{\frac{1}{2}})(\tau + \tau^*)\psi'_\nu(\tau) d\tau \\ &= -\nu^{-2} [4\tau^{\frac{1}{2}}(1 - \tau)\psi'_\nu(\tau) + (2/\tau^* \tau^{\frac{1}{2}})(\tau + \tau^*)\psi_\nu(\tau)]_0^{\tau^*} + I_1 \nu^{-2}, \\ \therefore I_1 &= -4M_\nu(\tau^*)/(\nu^2 - 1), \end{aligned}$$

where

$$M_\nu(\tau^*) = (\tau^*)^{-\frac{1}{2}} \{ \tau^*(1 - \tau^*)\psi'_\nu(\tau^*) + \psi_\nu(\tau^*) \}. \quad (5.4)$$

Then from (5.3)

$$\begin{aligned} F_D &= \Im \frac{p_1 c^2}{q_m \xi} \int_{C'} \frac{\nu f_\nu(\tau_0) M_\nu(\tau^*)}{(\nu^2 - 1) \sin(\pi \nu / \xi)} d\nu \\ &= \Im \frac{p_1 c^2}{q_m \xi} \int_C \frac{\nu f_\nu(\tau_0) M_\nu(\tau^*)}{(\nu^2 - 1) \sin(\pi \nu / \xi)} d\nu + \frac{\pi p_1 c^2}{q_m \xi} \frac{f_1(\tau_0) M_1(\tau^*)}{\sin(\pi / \xi)}. \end{aligned}$$

Since the integrand in this expression is now regular at $\nu = 0$ it is no longer necessary to indent the contour and the integration may be performed along the imaginary axis. It is also apparent from symmetry that the integral term is purely imaginary. Hence, finally,

$$F_D = \frac{p_1 c^2}{q_m \xi i} \int_{-i\infty}^{i\infty} \frac{\nu f_\nu(\tau_0) M_\nu(\tau^*)}{(\nu^2 - 1) \sin(\pi \nu / \xi)} d\nu + \frac{\pi p_1 c^2}{q_m \xi} \frac{f_1(\tau_0) M_1(\tau^*)}{\sin(\pi / \xi)}. \quad (5.5)$$

It is now necessary to connect the scale factor c^2 with the length R of the wedge. This is achieved through the relation

$$\begin{aligned} R &= \int_0^{\tau^*} \frac{dr}{d\tau} d\tau \\ &= \frac{1}{2q_m \tau^*} \int_0^{\tau^*} \frac{(\tau - \tau^*)[\Psi_\theta]_{\theta=\pi/\xi}}{(1 - \tau)^{\beta+1} \tau^{\frac{1}{2}}} d\tau, \end{aligned}$$

using (2.2.3) as before. Thus

$$R = -\Im \frac{c^2}{4q_m \xi} \int_{c'} \frac{\nu f_\nu(\tau_0) I_2}{\sin(\pi \nu / \xi)} d\nu,$$

where

$$I_2 = \int_0^{\tau^*} \frac{(\tau - \tau^*) \psi_\nu(\tau)}{\tau^* \tau^{\frac{3}{2}} (1 - \tau)^{\beta+1}} d\tau.$$

I_2 is evaluated in exactly the same way as I_1 and is given by

$$I_2 = \frac{-4\{\tau^*(1 - \tau^*)\psi_\nu'(\tau^*) + \frac{1}{2}(1 - \tau^*)\psi_\nu(\tau^*)\}}{(\nu^2 - 1)(\tau^*)^{\frac{3}{2}}(1 - \tau^*)^{\beta+1}}.$$

Hence

$$R = \Im \frac{c^2}{(1 - \tau^*)^{\beta+1} q_m \xi} \int_{c'} \frac{\nu f_\nu(\tau_0) N_\nu(\tau^*)}{(\nu^2 - 1) \sin(\pi \nu / \xi)} d\nu,$$

where

$$N_\nu(\tau^*) = (\tau^*)^{-\frac{1}{2}} \{\tau^*(1 - \tau^*)\psi_\nu'(\tau^*) + \frac{1}{2}(1 - \tau^*)\psi_\nu(\tau^*)\}. \quad (5.6)$$

The contour may be deformed as before and, on using $p^* = p_1(1 - \tau^*)^{\beta+1}$, a formula similar to (5.5) is obtained:

$$Rp^* = \frac{p_1 c^2}{q_m \xi i} \int_{-i\infty}^{i\infty} \frac{\nu f_\nu(\tau_0) N_\nu(\tau^*)}{(\nu^2 - 1) \sin(\pi \nu / \xi)} d\nu + \frac{\pi p_1 c^2}{q_m \xi} \frac{f_1(\tau_0) N_1(\tau^*)}{\sin(\pi / \xi)}. \quad (5.7)$$

The total drag per unit length of the wedge, measured from the atmospheric pressure (*i.e.*, pressure at infinity) and in the direction of the axis is

$$(F_D - Rp_0) \frac{\sin(\pi / \xi)}{R} = p^* \left\{ \frac{F_D}{Rp^*} - \left(\frac{1 - \tau_0}{1 - \tau^*} \right)^{\beta+1} \right\} \sin(\pi / \xi).$$

It is worth noting that because I_1 and I_2 can be integrated explicitly, the total pressure up to any point with given velocity on the wedge can be found. This is applicable in particular to finding the drag on a wedge in a dead air theory, provided the velocity of the stream is known along the free streamline from the shoulder.

6. Drag of a thin wedge in a near-sonic stream

When π/ξ , the semi-angle of the wedge, is small, the drag coefficient C_D is usually defined by

$$\begin{aligned} C_D &= \frac{\pi}{R\xi} \int_0^R \frac{(p - p_0)}{\frac{1}{2}\rho_0 q_0^2} dr \\ &= \frac{2}{\gamma M_0^2} \cdot \frac{\pi}{\xi} \cdot \left\{ \frac{p^*}{p_0} \frac{F_D}{Rp^*} - 1 \right\}. \end{aligned} \quad (6.1)$$

In the work which follows, the function $f_\nu(\tau_0)$ will be taken as $1/\psi_\nu(\tau_0)$. This appears to be the most satisfactory form when, as here, the continuation for values of $\tau > \tau_0$ is not sought explicitly. To the degree of approximation required in this section the alternative forms for $f_\nu(\tau_0)$ possessing the same asymptotic behaviour as $1/\psi_\nu(\tau_0)$ give the same result. These asymptotic forms are proportional to $e^{-\nu\sigma_0}$ for $\tau_0 < \tau^*$ and $\nu^{-\frac{1}{2}} e^{-\nu\sigma}$ for $\tau_0 = \tau^*$.

When $M_0 = 1$, (6.1) gives for the drag coefficient $C_D = C_D^*$ where

$$\begin{aligned} C_D^* &= \frac{2\pi}{\gamma\xi} \left\{ \frac{F_D - Rp^*}{Rp^*} \right\} \\ &= \frac{2\pi}{\gamma\xi} \frac{\Im \int_{C'} \frac{\nu \{M_\nu(\tau^*) - N_\nu(\tau^*)\}}{(\nu^2 - 1)\psi_\nu(\tau^*) \sin(\pi\nu/\xi)} d\nu}{\Im \int_{C'} \frac{\nu N_\nu(\tau^*)}{(\nu^2 - 1)\psi_\nu(\tau^*) \sin(\pi\nu/\xi)} d\nu}. \end{aligned} \quad (6.2)$$

It is clear that in the integrals in (6.2) the contour C' can be deformed into a contour C'' without traversing any singularities of the integrands, such that $|\nu| > \frac{1}{2}\xi$ everywhere on C'' . Then, since ξ is large, $\nu^2 - 1$ can be replaced by ν^2 in the denominators and $\psi_\nu(\tau^*)$, $\psi'_\nu(\tau^*)$ can be replaced by their asymptotic expansions for large values of $|\nu|$ in order to obtain the first order term of (6.2). C'' may then be replaced by C' .

These forms are obtained from the formulae given by CHERRY [4] together with the leading terms in the asymptotic formulae for $J_\nu(\nu)$ and $J'_\nu(\nu)$ for large $|\nu|$. These Bessel functions occur in CHERRY'S formulae and the required asymptotic forms are given by WATSON [23]. CHERRY'S formulae are actually for functions $\chi_\nu(\tau)$, $\chi'_\nu(\tau)$ which are linearly related to $\psi_\nu(\tau)$, $\psi'_\nu(\tau)$. The results required are

$$\psi_\nu(\tau^*) = N(1 - \tau^*)^\beta e^{\nu\sigma} \{A_1 \nu^{\frac{1}{2}} + O(\nu^{-\frac{1}{2}})\} \quad (6.3)$$

$$2\tau^* \psi'_\nu(\tau^*) = N(1 - \tau^*)^\beta e^{\nu\sigma} \{B_1 Q_0 \nu^{\frac{1}{2}} + O(\nu^{\frac{1}{2}})\}. \quad (6.4)$$

N is a known constant which need not be determined explicitly. Q_0 is a constant occurring in CHERRY'S formulae (q_0^* in his notation) and has the value $\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}$.

A_1 and B_1 are the leading coefficients in the asymptotic expansions of $J_\nu(\nu)$ and $J'_\nu(\nu)$ respectively. The values of these constants are $A_1 = \Gamma(\frac{1}{3})\pi^{-1}2^{-\frac{1}{3}}3^{-\frac{1}{6}}$ and $B_1 = \Gamma(\frac{2}{3})\pi^{-1}2^{-\frac{1}{3}}3^{-\frac{1}{6}}$.

Then, when the substitutions indicated above are made in (6.2) it is found, after some simplification, that

$$C_D^* = \left(\frac{2A_1\pi}{B_1 Q_0 \xi} \right) \Im \int_{C'} \frac{d\nu}{\nu \sin(\pi\nu/\xi)} / \Im \int_{C'} \frac{d\nu}{\nu^{\frac{1}{2}} \sin(\pi\nu/\xi)}. \quad (6.5)$$

Now let

$$\frac{\xi^{\alpha-1}}{2i} \int_{C'} \frac{\nu^{-\alpha} d\nu}{\sin(\pi\nu/\xi)} = n(\alpha)$$

with the ν -plane assumed cut from 0 to $-\infty$ and $\arg \nu = 0$ on the positive real ν -axis. Then $n(\alpha)$ is an integral function. Moreover it can be shown that the integral vanishes when taken round the right hand infinite semi-circle based on C' as diameter, provided $\Re(\alpha) > 1$. $n(\alpha)$ can then be evaluated by summing the series generated by those poles of $\operatorname{cosec}(\pi\nu/\xi)$ for which $\Re(\nu) > 0$. Hence for $\Re(\alpha) > 1$,

$$n(\alpha) = \sum_{r=1}^{\infty} (-1)^{r+1} r^{-\alpha} = (1 - 2^{1-\alpha})\zeta(\alpha),$$

where $\zeta(\alpha)$ is the Riemann Zeta Function.

From the simple properties of the Zeta Function, it follows that $(1 - 2^{1-\alpha})\zeta(\alpha)$ is also an integral function. Thus for all α ,

$$n(\alpha) = (1 - 2^{1-\alpha})\zeta(\alpha).$$

The particular values of $n(\alpha)$ required are obtained from the tables of Zeta Functions in JAHNKE & EMDE [12]. An exception is the value of $n(1)$ which is equal to $\log_e 2$.

Evaluation of (6.5) then gives

$$C_D^* = \frac{2A_1 \pi}{B_1 Q_0 \xi} \cdot \frac{1}{\xi^{\frac{2}{3}}} \cdot \frac{n(1)}{n(\frac{1}{3})}.$$

On further simplification

$$C_D^* = \lambda(\pi/\xi)^{\frac{2}{3}} \left\{ \frac{n(1)}{n(\frac{1}{3})} \right\}, \quad (6.6)$$

where

$$\lambda = 2\pi^{\frac{1}{3}} / \{(\gamma + 1)^{\frac{1}{3}} 3^{\frac{1}{3}} \Gamma^2(\frac{2}{3}) \sin \frac{1}{3}\pi\}.$$

For $\gamma = 1.4$ this gives a value of $C_D^* = 1.55 (\gamma + 1)^{-\frac{1}{3}} (\pi/\xi)^{\frac{2}{3}}$ as compared with the value $1.67 (\gamma + 1)^{-\frac{1}{3}} (\pi/\xi)^{\frac{2}{3}}$ obtained by COLE [6] and $1.75 (\gamma + 1)^{-\frac{1}{3}} (\pi/\xi)^{\frac{2}{3}}$ obtained by GUDERLEY & YOSHIHARA [9].

There is a very interesting connection between the form in which C_D^* is obtained in (6.6) and the form which COLE obtained. COLE's expression is

$$C_D^* = \lambda(\pi/\xi)^{\frac{2}{3}} \left\{ \frac{n(\frac{2}{3})}{n(0)} \right\},$$

where $n(0)$ is equal to $\frac{1}{2}$. It follows that COLE's form is exactly the same as would be obtained by a limiting process from the MACKIE-PACK solution if the function $f_\nu(\tau^*)$ is taken as $\nu^{\frac{1}{3}}/\psi_\nu(\tau^*)$ instead of $1/\psi_\nu(\tau^*)$, *i.e.*, if for $\tau < \tau^*$, Ψ is given by

$$\Psi = c^2 \sum_{n=1}^{\infty} n^{\frac{1}{3}} \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau^*)} \sin n\xi\theta. \quad (6.7)$$

Investigation of this solution shows that it does satisfy the appropriate uniform conditions at infinity upstream and that $\Psi = 0$ when $\theta = 0$ or $\theta = \pi/\xi$. But it is less easy to see to what the flow corresponds for values of $\tau > \tau^*$. A certain arbitrariness exists in the choice of function $f_\nu(\tau_0)$ in the general case, as has already been mentioned in §2. The important condition to satisfy is that the series for Ψ (and hence for the position coordinates) has a radius of convergence $\tau = \tau_0$. This identifies τ_0 as the (generalized) velocity at infinity. The position coordinates corresponding to the solution for Ψ can be written down as Fourier series (see [16]). Infinite values are obtained at $\tau = \tau_0$ but only for $\theta = 0$. This ensures that the flow at infinity upstream is parallel to the axis of the wedge. The actual form of $x(\tau)$ along the axis of symmetry is

$$x(\tau) = -E(\tau) \sum_{n=1}^{\infty} \frac{n\xi}{n^2\xi^2 - 1} f_{n\xi}(\tau_0) \{\psi_{n\xi}(\tau) + 2\tau\psi'_{n\xi}(\tau)\}$$

where

$$E(\tau) = c^2/\tau^{\frac{1}{2}}(1 - \tau)^{\beta}.$$

It follows from (6.3), (6.4) that this will become negative infinite as $\tau \rightarrow \tau_0$ if $f_{n\xi}(\tau_0)$ is of the form $n^\alpha/\psi_{n\xi}(\tau_0)$, where $\alpha \geq -\frac{2}{3}$. This is satisfied by (6.7) for the case $\tau_0 = \tau^*$. However there do appear to be grounds for preferring to take $f_\nu(\tau_0)$ of the form $1/\psi_\nu(\tau_0)$ or its asymptotic form for large $|\nu|$. For, far away from the wedge where the flow is almost uniform and compressibility effects are negligible, the flow obtained in this case is known to have exactly the same form as that of incompressible flow past the required shape of body. If another form of $f_\nu(\tau_0)$ is used, the flow field at a distance is similar to the field far away from some other type of body. The same remarks apply in the neighbourhood of the stagnation point at the apex of the wedge. It should be noted that a latent arbitrariness also exists in COLE'S approach from the transonic approximation, since he has selected that particular form for which a Fourier-Bessel inverse exists. Thus his choice is governed by mathematical rather than physical requirements.

Before this subject is left, further numerical comparisons can be made with the results obtained by approximating to the equation of motion. A second approximation to the expression for C_D for near-sonic velocities of the free stream is given by

$$C_D = [C_D]_{M_0=1} + (M_0^2 - 1)[dC_D/d(M_0^2)]_{M_0=1} + \dots \quad (6.8)$$

The second term is obtained by differentiating (6.1) with respect to M_0^2 and putting $M_0 = 1$. The factor multiplying $(M_0^2 - 1)$ in (6.8) turns out to be $\frac{2\pi}{(\gamma + 1)\xi} \cdot \frac{n(-\frac{1}{3})n(1)}{\{n(\frac{1}{3})\}^2}$. Thus

$$C_D = \frac{1.55}{(\gamma + 1)^{\frac{1}{2}}} \left(\frac{\pi}{\xi}\right)^{\frac{3}{2}} + \frac{1.78}{\gamma + 1} (M_0^2 - 1) \frac{\pi}{\xi} + \dots$$

The corresponding expression obtained by COLE is

$$C_D = \frac{1.67}{(\gamma + 1)^{\frac{1}{2}}} \left(\frac{\pi}{\xi} \right)^{\frac{3}{2}} + \frac{2}{\gamma + 1} (M_0^2 - 1) \frac{\pi}{\xi} + \dots$$

Finally comparison can be made with the work of IMAI [11] who calculated the drag on a wedge in a sonic stream, using the transonic approximation but assuming that the flow broke away from the shoulder leaving a dead air region behind the wedge. The exact solution of this problem for incompressible flow is

$$W_i = - \sum_{n=1}^{\infty} n(q/q_0)^{n\frac{1}{2}} e^{-in\xi\theta},$$

where q_0 is the velocity at infinity. This is now generalized as before. It is essential to take $f_\nu(\tau_0) = 1/\psi(\tau_0)$ in this case to preserve the free streamline. C_D is found in exactly the same manner as before by approximating for large ξ and the result is

$$C_D = \frac{1.78}{(\gamma + 1)^{\frac{1}{2}}} \left(\frac{\pi}{\xi} \right)^{\frac{3}{2}}.$$

The corresponding result obtained by IMAI is

$$C_D = \frac{1.94}{(\gamma + 1)^{\frac{1}{2}}} \left(\frac{\pi}{\xi} \right)^{\frac{3}{2}}.$$

These results show that there is an appreciable increase in the drag coefficient when the flow breaks away from the shoulder.

It is interesting to note that the actual form of C_D as obtained by approximating for large ξ from the MACKIE-PACK solution of the full equation of motion is the same as that obtained from the transonic approximation but that the numerical coefficients are in every case slightly smaller. However, the problem of break-away flow past a wedge is one which is exactly solved by the hodograph method used in this paper; one may therefore more confidently suppose that the smaller coefficient obtained for the wedge drag in the earlier problem is significant.

7. Drag of a blunt wedge

While the equations (5.5) and (5.7) together give an exact expression for the drag of a wedge according to the MACKIE-PACK solution, the actual evaluation of the integrals in particular cases is not easy. The integrals can be written as infinite series but, since $\tau_0 < \tau^*$ in general, these series are generated by the residues at the poles occurring in the left-hand part of the complex ν -plane and involve complicated and untabulated functions. It seems that the best approach is a direct numerical integration up the imaginary axis. To illustrate this, the pressure on a blunt wedge was calculated for a range of subsonic free stream Mach numbers. This form of wedge was chosen partly for the purpose of com-

parison with the work of MACCOLL [15] and partly because the convergence of the integrals is most rapid for a blunt wedge. Before this work is described however, a brief summary is given of those of MACCOLL's results which are relevant here, as his paper is not generally available.

MACCOLL considers the case of flow past a blunt two-dimensional wedge with varying free stream Mach number but with constant stagnation pressure in the gas. His work is restricted to those cases where sonic velocity occurs at the shoulder. The fundamental working hypothesis is as follows. Since the pressure at the tip is fixed as the stagnation pressure and the pressure at the shoulder is fixed as the pressure corresponding to sonic velocity, the total absolute pressure on the wedge face will not vary greatly for varying Mach number and may be taken as constant.

Consequently, if the drag coefficient C_D is defined as before (with $\sin \pi/\xi$ replacing π/ξ , which was used as an approximation when thin wedges were considered), then

$$\begin{aligned} C_D &= \frac{2}{\gamma M_0^2} \left\{ \frac{F_D - Rp_0}{Rp_0} \right\} \sin \frac{\pi}{\xi} \\ &= \frac{2}{\gamma M_0^2} \left\{ K \frac{p_1}{p_0} - 1 \right\} \sin \frac{\pi}{\xi} \end{aligned}$$

where K is a constant on the hypothesis and equal to F_D/Rp_1 . Using a relaxation process, MACCOLL gave the value of K as 0.928. By means of the hodograph approach an independent set of values can be found for F/Rp_1 , the accuracy of the assumption of constant K to some extent checked and an independent numerical value obtained.

To do this, F_D and Rp^* have first to be evaluated numerically with $\xi = 2$ for a range of values of M_0 . The forms in (5.5) and (5.7) were used, $f_r(\tau_0)$ being taken as $e^{-\nu\tau_0}$ for simplicity, and the residue terms calculated immediately from the tables of FERGUSON and LIGHTHILL [8]. The integral term in (5.5) was first rewritten as

$$\frac{-p_1 c^2}{q_m} \int_0^\infty \frac{[\cos y s_0 \Re\{M_{iy}(\tau^*)\} + \sin y s_0 \Im\{M_{iy}(\tau^*)\}] y dy}{(1 + y^2) \sinh \frac{1}{2} \pi y}.$$

Because of the damping term $\sinh \frac{1}{2} \pi y$ in the denominator, the integral was evaluated only up to $y = 4$ to give the required accuracy. $M_{iy}(\tau^*)$ was calculated directly from the series expansions of $\psi_{iy}(\tau)$, $\psi'_{iy}(\tau)$ for $\tau = \frac{1}{8}$, the value of τ^* corresponding to $\gamma = 1.4$, and for $y = 0, 1, 2, 3, 4$. From the values of the integrand at these five points a double application of Simpson's rule was used to evaluate the integral numerically.

An exactly similar process was carried out with (5.7). The two expressions were evaluated for $\tau_0 = .08, .10, .12, .14$ and $.16$. By means of the relation $K = F_D/Rp_1 = F_D(1 - \tau^*)^{\delta+1}/Rp^*$, a table of the results (Table 1) was constructed.

Examination of Table 1 shows that there is very good confirmation of MACCOLL'S results both with regard to the supposition that K is constant and to its actual value; for K varies by less than 1 % over the tabulated range of Mach numbers, and never differs from MACCOLL'S value (0.928) by as much as $1\frac{1}{2}$ %. Because of the errors of the numerical approximation, the last decimal place in the table of values of K may not be accurate. However, the monotonic trend is significant.

TABLE 1

τ_0	M_0	K
.08	.659	0.941
.10	.745	0.938
.12	.826	0.936
.14	.902	0.934
.16	.976	0.933

For it is to be expected that as the free stream Mach number increases, the acceleration up the wedge side will be more rapid and higher velocities will appear slightly earlier on the wedge side although the velocities at the tip and shoulder are unchanged. This tends to reduce the value of the total absolute drag and hence of K as M_0 increases. This tendency is apparent in Table 1.

The further application of these methods would be considerably facilitated by the construction of tables of $\psi_\nu(\tau^*)$, $\psi'_\nu(\tau^*)$ for purely imaginary values of ν .

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8. Appendix

The similarity between the form in which the drag on the wedge was obtained in §6 and the corresponding form obtained by COLE [6] indicates that there must be an essential connection between the MACKIE-PACK solution and the various forms of the stream function obtained by different authors from the transonic approximation. It is the purpose of the present appendix to establish this connection.

This may be done by transforming the basic equation for $\psi_n(\tau)$ and considering the behaviour of the solution in the neighbourhood of $\tau = \tau^*$. By means of the substitutions $\zeta = \frac{n}{3} \left(\frac{2\beta + 1}{2\beta} \right)^{\frac{1}{3}} \left(1 - \frac{\tau}{\tau^*} \right)^{\frac{1}{3}}$ and $\chi = \left(\frac{2}{3\zeta} \right)^{\frac{1}{3}} \psi$, the equation for $\psi_n(\tau)$ can be reduced in this neighbourhood to

$$\frac{d^2 \chi}{d\zeta^2} + \frac{1}{\zeta} \frac{d\chi}{d\zeta} - \chi \left(1 + \frac{1}{9\zeta^2} \right) = 0. \quad (8.1)$$

It is now necessary to look for a solution χ of (8.1) such that $(3\zeta/2)^{\frac{1}{3}} \chi(\zeta)$ behaves like $\psi_n(\tau)$ near $\tau = \tau^*$. A necessary condition is that $(3\zeta/2)^{\frac{1}{3}} \chi(\zeta)$ should be finite and non-zero at $\zeta = 0$, for $\tau = \tau^*$ does not correspond everywhere to $\Psi = 0$. If $\chi(\zeta)$ is taken as $I_{-\frac{1}{3}}(\zeta)$ this condition is satisfied. Thus $\psi_n(\tau)$ may be replaced by $g(n)(3\zeta/2)^{\frac{1}{3}} I_{-\frac{1}{3}}(\zeta)$ near $\tau = \tau^*$ ($\zeta = 0$), where $g(n)$ must be chosen so as to give the correct asymptotic forms for large $|n|$. It is noted that it is sufficient to ensure this correspondence for large values of $|n|$ only since the contour C in (2.2.2) can be deformed for thin wedges so that $|\nu|$ is always large. It can be shown that if $g(n)$ is taken as $An^{\frac{1}{3}} e^{n\sigma}$, where A is a certain real constant, then the correct correspondence is obtained for sonic and subsonic velocities.

If the counteracting function is now taken as $1/\psi_\nu(\tau_0)$ and $\psi_\nu(\tau)$ is replaced by the function indicated above, (2.2.2) gives

$$W = \frac{c^2}{2i\xi} \int_C \frac{I_{-\frac{1}{3}}(\zeta) \zeta^{\frac{1}{3}} e^{-\nu i\theta + \pi i\nu/\xi}}{I_{-\frac{1}{3}}(\zeta_0) \zeta_0^{\frac{1}{3}} \sin(\pi\nu/\xi)} d\nu, \quad (8.2)$$

where ζ_0 is the value of ζ corresponding to the slightly subsonic free stream velocity.

Now let $\zeta = \frac{2}{3}\nu\eta^{\frac{1}{3}}$ so that η is independent of ν . Then (8.2) reduces to

$$W = \frac{c^2 \eta^{\frac{1}{3}}}{2i\xi\eta_0^{\frac{1}{3}}} \int_C \frac{I_{-\frac{1}{3}}(\frac{2}{3}\eta^{\frac{1}{3}}\nu) e^{-\nu i\theta + \pi i\nu/\xi}}{I_{-\frac{1}{3}}(\frac{2}{3}\eta_0^{\frac{1}{3}}\nu) \sin(\pi\nu/\xi)} d\nu.$$

This simplifies to a purely imaginary expression and gives

$$\Psi = \eta^{\frac{1}{3}} \int_0^\infty \frac{J_{-\frac{1}{3}}(\frac{2}{3}\eta^{\frac{1}{3}}t) \sinh t(\pi/\xi - \theta)}{J_{-\frac{1}{3}}(\frac{2}{3}\eta_0^{\frac{1}{3}}t) \sinh(\pi t/\xi)} dt,$$

apart from a constant multiplier.

This bears a strong resemblance to the form obtained by WEINSTEIN [24] for

the stream function. The difference in the constant multiplier is significant only in fixing the scale. Apart from this the differences may be removed entirely by an appropriate alternative choice of the counteracting function in the above. By another suitable choice of counteracting function COLE's solution can be recovered.

The following remarks are made in conclusion. The literature now provides a range of solutions, all satisfying essentially the same boundary conditions but different in that they employ different counteracting functions. None of these solutions describes adequately conditions at the shoulder and the most satisfactory would be that which best exhibits upstream the influence of the downstream region behind the shock at the shoulder. The present authors have selected a solution based on the known incompressible flow past the given profile. While this appears to be a satisfactory basis for a solution from the point of view of assessing the drag, the authors feel that the chief merit of their solution lies in the fact that it satisfies the exact equation of motion; it thus does not require the approximation to the stagnation point necessary in the other solutions and hence is applicable to wedge shapes of any angle for a wide range of free stream velocities.