*Uniqueness of Axially Symmetric Subsonic Flow past a Finite Body**

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We shall be concerned here with the steady irrotational flow of an ideal compressible fluid past a fixed body, and in particular with its unique dependence on the Mach number at infinity. For the case of *plane subsonic* flow there is an essentially complete solution of this problem due to BERS [1], who has shown that with the circulation prescribed, either directly or by means of the Kutta-Joukowski condition, there is at most one subsonic flow past the body with given Mach number at infinity. In his proof, which by its nature seems limited to plane flows, BERS makes essential use of the hodograph transformation and of the properties of pseudo-analytic functions and quasi-conformal mappings. However, when the flow is *symmetric* about a line parallel to the uniform stream, these analytical tools are unnecessary and the uniqueness theorem can be proved by elementary means based on the maximum principle for elliptic differential equations (*cf.* [2]). In this paper we shall extend the same method to prove uniqueness of axially symmetric flows. The possibility of such an extension is of interest in itself in view of the fact that these flows cannot be treated by methods based on the hodograph transformation.

Specifically, we consider axially symmetric subsonic flows past a certain class

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of walls, including as a special case flows past a finite body placed in a uniform stream. It will be proved that such flows depend uniquely on the Mach number at infinity and on the maximum local Mach number. Except for the channel flows discussed in [2], this seems to be the first treatment of the uniqueness problem for non-plane flows. In the course of the proof we establish that the stream function ψ in the flow past a finite body has the expected limit behavior at infinity: $\psi - \frac{1}{2}Q_0 y^2 \rightarrow 0$, Q_0 being the free stream mass flow. This limit theorem is all that is needed to extend to axially symmetric flows the principal results on plane flows obtained in [2] and [4], and hence, as far as the class of problems treated by comparison methods is concerned, the status of the theory is now essentially the same for both plane and axially symmetric subsonic flows.

The method of proof parallels closely that for plane flows in [2], and, although there are important differences in detail, there are no unusual difficulties arising from the singular behavior of the equation of axially symmetric flow. As in the plane problem, the maximum principle is the only essential tool.

1. A theorem of Phragmén-Lindelöf type. The theorem we prove in this section generalizes to singular elliptic partial differential equations a known result for harmonic functions. The special case $p = 0$ was considered by GILBARG [2] in proving the uniqueness of symmetric plane flows.

Theorem 1. *Let p be a non-negative constant and let the linear partial differential equation*

$$(1) \quad L(u) = Au_{xx} + 2Bu_{xy} + C\left(u_{yy} - p\frac{u_y}{y}\right) + Du_x = 0$$

be uniformly elliptic¹ in an infinite region R lying in the half plane $y > 0$. We suppose the coefficients A , B , C approach finite limits at infinity, and that $yD \rightarrow 0$ at infinity. Let $u = u(x, y)$ be a twice continuously differentiable solution of (1) in R such that $u = o(r^{p+1})$ as $r = \sqrt{x^2 + y^2} \rightarrow \infty$. Then if u is continuous on the boundary of R and has a limit u_0 when infinity is approached along the boundary, it has the same limit as infinity is approached from the interior of R .

Proof. By means of an appropriate linear coordinate transformation we may assume that

$$A \rightarrow 1, \quad B \rightarrow 0, \quad C \rightarrow 1 \quad \text{as } (x, y) \rightarrow \infty,$$

and also it can be assumed that $u_0 = 0$. Let $\epsilon > 0$ be fixed during the following discussion. Now choose r_0 so large that $u \geq -\epsilon$ on the portion $\tilde{R}$ of the boundary lying outside the circle $r = r_0$, and also so large that the quantities $A - 1$, B , $C - 1$, yD differ from zero by a suitably small quantity (to be specified later) in the part $\tilde{R}$ of the region R outside $r = r_0$. We seek to determine a function $F(x, y)$ in $\tilde{R}$ with the properties

¹ That is, there exists a positive constant k such that at every point of R , $AC - B^2 \geq k$. It is assumed that A , B , C , D are continuous in R .

$$(2) \quad L(F) \leq 0, \quad F > 0, \quad \text{and} \quad F \rightarrow 0 \quad \text{as } (x, y) \rightarrow \infty.$$

From this will follow the theorem.

We set $v = y^{p+1}/r^{p+2}$. The following results are then easily verified:

$$(3) \quad v_{xx} + v_{yy} - p \frac{v_y}{y} = 0, \quad \frac{v(v_{xx}^2 + v_{xy}^2)^{\frac{1}{2}}}{v_x^2 + v_y^2} \leq p + 2, \quad \frac{v |v_x|}{v_x^2 + v_y^2} \leq y.$$

Let us take F in the form $F = f(v)$, whence

$$(4) \quad L(F) = (Av_x^2 + 2Bv_x v_y + Cv_y^2)f'' + L(v)f'.$$

For $r \geq r_0$ we have

$$(5) \quad Av_x^2 + 2Bv_x v_y + Cv_y^2 \geq (1 - \delta_1)(v_x^2 + v_y^2)$$

and

$$(6) \quad L(v) = (A - C)v_{xx} + 2Bv_{xy} + Dv_x \leq \delta_2[(v_{xx}^2 + v_{xy}^2)^{\frac{1}{2}} + y^{-1} |v_x|];$$

here δ_1 and δ_2 are non-negative constants which can be arbitrarily small if r_0 is large enough; in particular, r_0 can be chosen so that

$$0 \leq \sigma \equiv \frac{(p+3)\delta_2}{1-\delta_1} < 1.$$

Now choose $f(v)$ as a solution of the ordinary differential equation

$$\frac{f''}{f'} = -\frac{\sigma}{v} \quad \text{with} \quad f' > 0.$$

In view of (3)–(6), the first condition of (2) is automatically satisfied. A solution obeying the second and third conditions of (2) is

$$f(v) = v^{1-\sigma} = \left(\frac{y^{p+1}}{r^{p+2}} \right)^{1-\sigma},$$

and $F(x, y)$ is thereby determined.

We now show that $\lambda > 0$ can be chosen so that

$$(7) \quad u(x, y) + \lambda F(x, y) \geq -2\epsilon \quad \text{on the boundary of } \tilde{R}.$$

From the choice of r_0 relation (7) is obviously true on $\tilde{R}$ whatever the value of λ , and therefore it remains only to choose λ so that (7) holds on the arcs of $r = r_0$ which form the rest of the boundary of $\tilde{R}$. This can certainly be done if these arcs have a minimum ordinate $y_0 > 0$, for then F is bounded from zero on them. On the other hand, if the arcs extend to $y = 0$ we can use the continuity of u and the fact that the endpoints of the arcs lie on $\tilde{R}$ to infer the truth of (7) at points with ordinate less than some value $y_0 > 0$; λ can then be chosen

so that (7) holds also on the remainder of the arcs ($y \geq y_0$). With λ thus determined we assert that

$$(8) \quad u(x, y) + \lambda F(x, y) \geq -2\epsilon \quad \text{in } \tilde{R}.$$

For set $U = u + \lambda F + 2\epsilon$; from (1), (2), and (7) we have $L(U) \leq 0$ in $\tilde{R}$, $U \geq 0$ on the boundary of $\tilde{R}$, and since $u = o(r^{p+1})$ we have also

$$(9) \quad \lim_{r \rightarrow \infty} \frac{U}{r^{p+1}} = 0.$$

The conditions of Theorem 3 of [3] are satisfied by the function U , hence $U \geq 0$ in $\tilde{R}$, and (8) is proved.²

Since $F \rightarrow 0$ as $(x, y) \rightarrow \infty$ we have from (8)

$$(10) \quad u(\infty) \geq -2\epsilon,$$

in an obvious notation. Applying the same argument to the function $-u$ we obtain also

$$(11) \quad u(\infty) \leq 2\epsilon.$$

But ϵ was arbitrary to begin with, so we conclude from (10) and (11) that $u(\infty) = 0$, and the theorem is proved.

In the next section Theorem 1 will be applied to a function which is $o(1 + y^{p+1})$ as $r \rightarrow \infty$; (this condition clearly implies that the function is also $o(r^{p+1})$ as $r \rightarrow \infty$). Accordingly, it would be sufficient for the purposes of this paper to have only a weaker form of Theorem 1 in which the hypothesis $u = o(r^{p+1})$ is replaced by $u = o(1 + y^{p+1})$. To prove this weaker form of Theorem 1 it is no longer necessary to use the results of [3] in the above proof. Indeed, (9) may in this case be replaced by

$$(9') \quad \lim_{r \rightarrow \infty} \frac{U}{1 + y^{p+1}} = 0,$$

and the result $U \geq 0$, obtained before from Theorem 3 of [3], may now be derived by the following elementary argument:

Choose a small value $\epsilon > 0$; then if r is sufficiently large we have $U + \epsilon(1 + y^{p+1}) \geq 0$. Since $L(1 + y^{p+1}) = 0$, we have $L[U + \epsilon(1 + y^{p+1})] \leq 0$ in $\tilde{R}$, and evidently $U + \epsilon(1 + y^{p+1}) \geq 0$ on the boundary of $\tilde{R}$. From these facts and the maximum principle it follows at once that $U + \epsilon(1 + y^{p+1}) \geq 0$ throughout $\tilde{R}$. Since ϵ is arbitrary we conclude, as desired, that $U \geq 0$ throughout $\tilde{R}$.

² Actually, from [3] we need only Lemma 5, which is used in the proof of Theorem 3. The term Du_x in $L(u)$ does not appear in [3], but the proof of Lemma 5 is easily altered to account for this additional term.

It should be mentioned that a similar simplification can be introduced in the proof of the uniqueness of plane symmetric flows in [2].

2. Uniqueness of axially symmetric subsonic flows past a wall.³ Let x denote the axial and y the radial coordinate ($y > 0$) in some meridian half plane of a subsonic axially symmetric flow. A flow of this type is completely specified by its stream function $\psi(x, y)$ which satisfies the quasi-linear partial differential equation

$$(12) \quad (d^2 - V^2)\psi_{xx} + 2UV\psi_{xy} + (d^2 - U^2)\psi_{yy} - d^2 \frac{\psi_y}{y} = 0,$$

where $U = y^{-1}\psi_x$, $V = y^{-1}\psi_y$, and $d = \rho c$ is a known function of $Q \equiv (U^2 + V^2)^{\frac{1}{2}}$. The density ρ and the sound speed c are positive, monotone decreasing differentiable functions of Q , determined from the equation of state and Bernoulli's law. The axial and radial velocity components, denoted by u and v , respectively, and the flow speed q are given in terms of ψ by the formulas

$$\rho u = V, \rho v = -U, \text{ and } \rho q = Q.$$

The subsonic character of the flow implies the relation $d > Q$, which shows that equation (12) is of elliptic type. We assume in the following that the free stream velocity is subsonic, but do not exclude the possibility that sonic speed is achieved on the boundary. Hence (12) is uniformly elliptic in the exterior of any sufficiently large circle. Finally, it is useful to note that (12) may be written in the alternate form

$$(13) \quad (d^2 - V^2)\psi_{xx} + 2UV\left(\psi_{xy} - \frac{\psi_x}{2y}\right) + (d^2 - U^2)\left(\psi_{yy} - \frac{\psi_y}{y}\right) = 0.$$

Theorem 2. *Let $\psi(x, y)$ be the stream function of an axially symmetric subsonic flow past a wall with equal finite ordinates at $x = \pm \infty$, and let $U = \psi_x/y \rightarrow 0$, $V = \psi_y/y \rightarrow Q_0$ at infinity. Then the function $\psi(x, y) - \frac{1}{2}Q_0 y^2$ has a finite limit as (x, y) approaches infinity from within the flow region.*

Proof. For convenience let us take $\psi = 0$ on the wall, which is assumed to have the ordinate $y = y_0$ at infinity. Consider the linear elliptic equation

$$(14) \quad A\psi_{xx} + 2B\left(\psi_{xy} - \frac{\psi_x}{2y}\right) + C\left(\psi_{yy} - \frac{\psi_y}{y}\right) = 0$$

whose coefficients are the functions of x, y obtained by inserting ψ in the coefficients of (13). This equation is obviously of the same form as (1), with $p = 1$ and $D = -B/y$, and the coefficients satisfy the conditions of Theorem 1. Of course ψ is a solution of (14) in the flow region R and so is y^2 .

In order to apply Theorem 1 we shall establish that the function

³ In reality such a flow occupies the exterior of a tube of variable circular cross section. A finite body is the special case of a wall whose ordinate is zero for large r .

$\psi - \frac{1}{2}Q_0(y^2 - y_0^2) = o(1 + y^2)$ as $r \rightarrow \infty$.⁴ This fact is seen easily if we integrate the relation

$$(15) \quad \psi_\theta - Q_0 r^2 \sin \theta \cos \theta = \epsilon r^2 \sin \theta, \quad \epsilon \rightarrow 0 \text{ as } r \rightarrow \infty,$$

along the circle of radius r from any interior point (r, θ) to the boundary; (15) is a consequence of the identity

$$\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = -U \sin \theta + V \cos \theta$$

together with the hypothesis $U \rightarrow 0$, $V \rightarrow Q_0$. If the flow is one past a finite body, we obtain at once

$$\begin{aligned} |\psi - \tfrac{1}{2}Q_0 y^2| &\leq \epsilon r^2(1 - |\cos \theta|) \leq \epsilon y^2, \quad \epsilon \rightarrow 0 \text{ as } r \rightarrow \infty, \\ &= o(y^2) \quad \text{as } r \rightarrow \infty, \end{aligned}$$

while for the flow past a wall the details are slightly altered, and we get

$$|\psi - \tfrac{1}{2}Q_0(y^2 - y_0^2)| = o(1 + y^2) \quad \text{as } r \rightarrow \infty.$$

The conditions of Theorem 1 are now satisfied by the solution $\psi - \frac{1}{2}Q_0(y^2 - y_0^2)$ of (14) and we have therefore the required result that $\psi - \frac{1}{2}Q_0 y^2$ has the limit $-\frac{1}{2}Q_0 y_0^2$ at infinity.

Now let ψ and $\bar{\psi}$ be two stream functions satisfying the conditions of Theorem 2, and let both ψ and $\bar{\psi}$ be normalized to zero on the wall. Then by Theorem 2,

$$\psi - \tfrac{1}{2}Q_0(y^2 - y_0^2) \rightarrow 0, \quad \bar{\psi} - \tfrac{1}{2}Q_0(y^2 - y_0^2) \rightarrow 0, \quad \text{as } (x, y) \rightarrow \infty$$

and $\bar{\psi} - \psi \rightarrow 0$ at infinity. But since $\psi - \bar{\psi} = 0$ on the wall the maximum principle (which applies to the difference of two solutions of (12), cf. [2]) gives the result $\bar{\psi} = \psi$, and therefore the flows are identical. Observing that the Mach number $m = q/c = Q/d$ is a monotone increasing function of Q , we therefore have

Theorem 3. *Let Γ be a wall with equal finite ordinates at $x = \pm \infty$. Then there can be at most one axially symmetric subsonic flow past Γ with prescribed Mach number at infinity. In particular, an axially symmetric subsonic flow past a finite body is uniquely determined by the Mach number at infinity.*

The proof of the following theorem can be carried over word for word from the proof of Theorem 7 of [2].

Theorem 4. *The flows considered in Theorem 3 are uniquely determined by the maximum value of the local Mach number, and the axially symmetric subsonic flow*

⁴The boundedness of $\psi - \frac{1}{2}Q_0 y^2$ has already been proved (although not explicitly stated) on p. 250 of [2]. This fact, in conjunction with Theorem 1 of the present paper, suffices for proof of Theorem 2. However, for the sake of completeness, and also in order to allow application of the remarks at the close of Section 1, we proceed as above.

past a finite body is uniquely determined by the maximum Mach number on the body.

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