Hypo-elasticity

C. TRUESDELL

Graduate Institute for Mathematics and Mechanics, Indiana University, Bloomington, Indiana*

CONTENTS

1. Definition of the hypo-elastic bodies.....	83
2. Isotropic hypo-elastic bodies.....	85
3. Relation to the theory of elasticity.....	87
4. Some special hypo-elastic bodies.....	88
5. Some properties of $\tilde{\mathbf{s}}$	89
6. Homogeneous stress.....	91
7. Hydrostatic stress.....	92
8. Simple shear of compressible bodies.....	98
9. Simple shear of incompressible bodies.....	119
10. Simple extension of compressible bodies.....	120
11. The kinematics and dynamics of torsion.....	125
12. Remarks on the torsion of compressible hypo-elastic bodies of grade one	126
13. Torsion of an incompressible hypo-elastic body of grade zero.....	128
14. Concluding remarks.....	131

1. Definition of the hypo-elastic bodies. In the paper called *The mechanical foundations of elasticity and fluid dynamics*¹ I drew attention toward the concept

* This paper was written for the Applied Mathematics Branch, Naval Research Laboratory, Washington, D. C. The numerical calculations and the figures drawn from them were prepared by Messrs. HARRISON HANCOCK and B. G. ZIMMERMAN of the Computer Section; the final drawings were made by the Graphic Arts Division.

¹ See Chapter IV-D as revised, pp. 603-609 of Vol. 2 of this *Journal* (1953). The present memoir is to be taken as correcting in some respects and especially as simplifying and broadening the ideas of the passage quoted. The ground thought has been explained more fully in a note called *The simplest rate theory of pure elasticity*, presented at the Second Ordnance Symposium, Chicago, April, 1954. This preliminary note, which concerns only what are here called *bodies of grade zero* (§4), is to appear in the *Communications on Pure and Applied Mathematics N. Y. U.*, Vol. 8, No. 1, January, 1955.

of elasticity embodied in the principle

$$(1.1) \quad \text{rate of stress} = f(\text{rate of deformation}).$$

Here I begin a thorough search of the theory which rests upon it.

As I have shown, for a dynamically sound theory “rate of stress” shall mean the tensor $\dot{\mathbf{s}}^{ij}$ defined by

$$(1.2) \quad \dot{\mathbf{s}}^{ij} \equiv \dot{s}^{ij} - s^{ik} v^j_{,k} - s^{kj} v^i_{,k} + s^{ij} v^k_{,k},$$

where s^{ij} is the spatial stress tensor referred for convenience to twice the shear modulus as the unit of stress, the dot denotes the material derivative, the comma denotes the covariant derivative, and v^i is the velocity. The “rate of deformation”, as usual, is

$$(1.3) \quad d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

Finally, it makes for economy to replace (1.1) by a still more general concept of elasticity, according to which the response to deformation may be moderated by the magnitude of the reigning stress field. Thus we propose to study the ideal material defined by the constitutive equation

$$(1.4) \quad \dot{\mathbf{s}} = f(\mathbf{d}, \mathbf{s}).$$

(As is usual in continuum mechanics, quantities distinguishing any preferred directions with which we may wish to endow the body are understood to enter (1.4) without being written explicitly. Also, if it is desired to include dependence on the temperature θ as well, the ratio θ/θ_0 , where θ_0 is a reference temperature, may be inserted.)

In the way I introduced into continuum mechanics some years ago, we now render the functional form of $\mathbf{f}$ in (1.4) more explicit by applying dimensional analysis. The stress tensor $\mathbf{s}$, being related to the usual stress tensor $\mathbf{t}$ by

$$(1.5) \quad \mathbf{s} \equiv \frac{\mathbf{t}}{2\mu},$$

is dimensionless. Hence not only $\mathbf{d}$ but also $\mathbf{s}$ is of the dimension $\mathbf{T}^{-1}$, where $\mathbf{T}$ is a unit of time.

At this point we add another constitutive restriction: just as in the classical theory, *no modulus of our ideal material shall carry a dimension independent of the dimension of stress*. The dimensional moduli may be tensors rather than scalars, as must indeed be the case for materials endowed with no symmetries; our restriction asserts that the physical components of these tensors must carry dimension $[\text{stress}]^\alpha$, for some α . In particular, the absence of a modulus of dimension $\mathbf{T}$ makes what are usually called “relaxation effects” impossible in our present theory. From (1.5) and the above assumption we conclude that the function $\mathbf{f}$ in (1.4) must be of dimension $\mathbf{T}^{-1}$ and that for its specification only

dimensionless parameters in addition to its explicit arguments $\mathbf{d}$ and $\mathbf{s}$ may occur. Finally, we add the restriction that $\mathbf{f}$ must be an *analytic* function of the components d_j^i and s_j^i for all values of these components and in all co-ordinate systems. For dimensional invariance it is then necessary that (1.4) shall reduce to a relation which is homogeneous of degree 1 in the d_j^i :

$$(1.6) \quad \mathfrak{s}_j^i = A_{j\ l}^{i\ k} d_k^l,$$

where the tensor $A_{j\ l}^{i\ k}$ is a dimensionless function of the components s_m^n .

That our easy inferences may not pass as merely facile, I pause to assert with vehemence that the definitive form (1.6) has fallen to us only by our stated assumptions: for a material possessing a time constant, or for a relation (1.4) which is not at least differentiable with respect to d_j^i near $\mathbf{d} = 0$, nothing so simple as (1.6) can be expected even for sufficiently small values of $\mathbf{d}$. But for the ideal material we define here the constitutive equation (1.6) is in no sense an approximation: for all types of motion, it expresses our concept and is consistent with the principles of mechanics. We shall name the class of ideal materials defined by (1.6) *hypo-elastic bodies*.

Now I shall be asked what physical materials are to be regarded as hypo-elastic. I answer, if we are to discard this theory on physical grounds, let us at least wait until we have first learned what it predicts. To this I add only what must be obvious from the derivation: a hypo-elastic body is a material which may soften or harden in strain but in general has neither preferred state nor preferred stress; in its response to deformation it is, analytically speaking, entirely smooth. I repeat, moreover, that the definition of the hypo-elastic bodies differs from most employed in plasticity theory in that it is dynamically sound. It is at least possible to ask how a hypo-elastic body behaves under large strains, large rotations, or quick stress changes, with some chance of getting an answer which is a theorem rather than an opinion. In this same spirit will we pursue its theory, seeking only exact solutions.

2. Isotropic hypo-elastic bodies. An ideal material is said to be *isotropic*¹ if its constitutive equations are expressed by isotropic functions. One way of

¹ At the present writing the invariance principles of general continuum mechanics furnish a field of vigorous and earnest research. In particular, concepts of isotropy broader than that in the *Mechanical foundations* have been introduced by RIVLIN & ERICKSEN and by NOLL, who have proposed also additional restrictions upon all constitutive equations. According to these new restrictions, a relation of type (1.4) must always reduce to (2.1), or, in other words, the definition of hypo-elastic bodies given here is appropriate *only* to the isotropic case, unless we add to the arguments of (1.4) quantities specifying the local rotation from a preferred initial state. More generally, it appears from these new restrictions that anisotropy is a concept appropriate only to materials which have a preferred state. See W. NOLL, *On the continuity of the solid and fluid states*, preceding in this issue, and R. S. RIVLIN & J. L. ERICKSEN, *Stress-deformation relations for isotropic materials*, to appear in the next issue.

delimiting the special class of functions $\mathbf{f}(\mathbf{d}, \mathbf{s})$ which are isotropic is to say that no quantities specifying preferred directions shall enter the list of arguments of $\mathbf{f}$. That is, the function symbol is to be taken almost literally: the components of $\mathbf{f}$ are to be functions *only* of the components of $\mathbf{d}$ and $\mathbf{s}$ and unspecified scalars, and the relation is to be tensorially invariant.

The concept of an isotropic function, while introduced by CAUCHY, is neglected in algebra, and only recently has there been an attempt to construct the general theory. Using an unduly restricted definition of an isotropic function, some years ago² I concluded that an analytic isotropic relation $\tilde{\mathbf{s}} = \mathbf{f}(\mathbf{d}, \mathbf{s})$ between symmetric 3×3 matrices $\tilde{\mathbf{s}}, \mathbf{d}, \mathbf{s}$ must necessarily be expressible in the form

$$\begin{aligned} \tilde{\mathbf{s}} = & \mathfrak{G}_0 \mathbf{1} + \mathfrak{G}_1 \mathbf{d} + \mathfrak{G}_2 \mathbf{d}^2 \\ (2.1) \quad & + \mathfrak{G}_3 \mathbf{s} + \mathfrak{G}_4 \mathbf{s}^2 + \frac{1}{2} \mathfrak{G}_5 (\mathbf{d}\mathbf{s} + \mathbf{s}\mathbf{d}) \\ & + \frac{1}{2} \mathfrak{G}_6 (\mathbf{d}\mathbf{s}^2 + \mathbf{s}^2 \mathbf{d}) + \frac{1}{2} \mathfrak{G}_7 (\mathbf{s}\mathbf{d}^2 + \mathbf{d}^2 \mathbf{s}) \end{aligned}$$

where the $\mathfrak{G}_r$ are analytic scalar functions of $\mathbf{d}$ and $\mathbf{s}$. Since then RIVLIN & ERICKSEN³ have given an adequate definition and have proved that (2.1) is fully general.⁴ For an isotropic hypo-elastic body (1.6) and (2.1) must be consistent. Hence, since the $\mathfrak{G}_r$ are power series in the components of $\mathbf{d}$, we must have $\mathfrak{G}_2 = \mathfrak{G}_7 = 0$; $\mathfrak{G}_1, \mathfrak{G}_5$, and $\mathfrak{G}_6$ must be independent of $\mathbf{d}$ and so functions of $\mathbf{s}$ only; while $\mathfrak{G}_0, \mathfrak{G}_3$, and $\mathfrak{G}_4$ must be of degree 1 in the components d_{ij} . Write

$$(2.2) \quad \delta \equiv d^k_k, \quad M \equiv s^i_j d^j_i, \quad N \equiv s^i_j s^j_k s^k_i.$$

Then, since the $\mathfrak{G}_r$ must also be power series in $\mathbf{s}$, our result is

$$\begin{aligned} \tilde{\mathbf{s}} = & \delta_{g0} \mathbf{1} + g_1 \mathbf{d} + \delta_{g2} \mathbf{s} + M_{g3} \mathbf{1} \\ (2.3) \quad & + \frac{1}{2} g_4 (\mathbf{d}\mathbf{s} + \mathbf{s}\mathbf{d}) + \delta_{g5} \mathbf{s}^2 + M_{g6} \mathbf{s} \\ & + N_{g7} \mathbf{1} + \frac{1}{2} g_8 (\mathbf{d}\mathbf{s}^2 + \mathbf{s}^2 \mathbf{d}) + M_{g9} \mathbf{s}^2 + N_{g10} \mathbf{s} + N_{g11} \mathbf{s}^2, \end{aligned}$$

where the g_r are dimensionless analytic functions of the three principal invariants of $\mathbf{s}$:

$$\begin{aligned} (2.4) \quad g_r = & g_r(I, II, III), \\ I \equiv & s^k_k, \quad II \equiv \frac{1}{2} \delta^{ij}_k s^k_i s^l_j, \quad III \equiv \frac{1}{6} \delta^{ijk}_{lmn} s^l_i s^m_j s^n_k. \end{aligned}$$

² See §26 of "A new definition of a fluid. II. The Maxwellian fluid," *J. math. pures appl.* (9) **30**, 111–158 (1951).

³ *Op. cit. ante.*

⁴ It does not follow from the work of RIVLIN & ERICKSEN that our assumed analyticity of (1.7) implies that the $\mathfrak{G}_r$ also are necessarily analytic. However, Dr. ERICKSEN has shown me an independent argument which proves that in the case considered here, *viz.*, $\mathfrak{G}_2 = \mathfrak{G}_7 = 0$, the remaining $\mathfrak{G}_r$ are indeed analytic. Thus (2.3) with analytic g_r is fully general for the present theory. The case when (1.6) is not analytic near $\mathbf{d} = 0, \mathbf{s} = 0$ is far from devoid of interest but will not be considered in this paper.

The equations (2.3) are the constitutive equations of the isotropic hypo-elastic body.

In an isochoric motion, $\delta = 0$. An incompressible body is one susceptible only of isochoric motions. We might require (2.3) to hold for incompressible bodies also, but in so doing we should lose the possibility of its being consistent with a variational principle or of including conservative deformations as a special case. It seems better, asserting analogy to other models of incompressible bodies in continuum mechanics, to replace (2.3) in the incompressible case by

$$(2.5) \quad \begin{aligned} \tilde{\mathbf{s}}' = & g_{11} \mathbf{d} + \frac{1}{2} g_{12} (\mathbf{d}\mathbf{s} + \mathbf{s}\mathbf{d}) + M_{g6} \mathbf{s} + \frac{1}{2} g_{88} (\mathbf{d}\mathbf{s}^2 + \mathbf{s}^2 \mathbf{d}) \\ & + M_{g9} \mathbf{s}^2 + N_{g10} \mathbf{s} + N_{g11} \mathbf{s}^2, \end{aligned}$$

where $\mathbf{s}' \equiv \mathbf{s} + p\mathbf{1}$, p being an arbitrary hydrostatic pressure.

The full set of differential equations of hypo-elasticity is obtained by adding to (2.3) or (2.5) the definition (1.3) and the usual equations of mass and motion:

$$(2.6) \quad \dot{\rho} + \rho v^k_{,k} = 0,$$

$$(2.7) \quad 2\mu s^{ij}_{,j} = \rho(f^i - \dot{v}^i),$$

where ρ is the density and where f^i , the field of extrinsic force, is to be regarded as given. Thus if we eliminate $\mathbf{d}$ by (1.3), we obtain a system of ten non-linear equations for ten unknowns. For compressible bodies the unknowns are ρ , s^{ij} , and v^i ; for incompressible bodies they are p , s^{ij} , and v^i .

It would seem natural to employ in hypo-elasticity the same boundary conditions, referring to stress, displacement, or both, as in elasticity. A displacement condition is complicated because it requires integration of the velocity field $\mathbf{v}$, however, and in many cases it will be equally natural to prescribe $\mathbf{v}$ instead. The main analytical difference from elasticity theory comes in the appearance of $\partial \mathbf{s} / \partial t$ in addition to $\mathbf{s}$ in the differential equations. Thus, beyond boundary conditions such as are usual in elasticity, we may expect to be able to satisfy the condition of *arbitrary initial stress*: $\mathbf{s} = \mathbf{S}$, say, when $t = 0$, where $\mathbf{S}$ is any solution of (2.7) consistent with whatever boundary conditions are employed.

In this paper we make no attempt to prove a general theorem of existence and uniqueness. A special theorem is obtained in §6.

3. Relation to the theory of elasticity. In a displacement $\mathbf{u}$ which is sufficiently small, sufficiently nearly irrotational, and sufficiently nearly homogeneous, we may employ the approximations

$$(3.1) \quad \mathbf{s} dt \approx \mathbf{s}(t + dt) - \mathbf{s}(t), \quad \mathbf{u} \approx \mathbf{v} dt,$$

thus in the case $\mathbf{s}(t) = 0$ recovering the classical infinitesimal theory of elasticity. *That is, the equations of the theory of hypo-elasticity reduce to those of the classical linear theory of elasticity under the assumptions usual in formulating that theory.*

There is no obvious connection between the classical theory of finite strain

and the theory of hypo-elasticity. Indeed, it was originally my intention to find a new concept of elastic behavior, mutually exclusive with the theory of finite strain except in the linearized case. However, Dr. WALTER NOLL¹ has announced a proof that *every isotropic elastic body is also an isotropic hypo-elastic body*. Thus hypo-elasticity is a generalization of the classical theory of finite strain.

Beyond this greater generality, there are two important differences. First, for the elastic case the coefficient functions in (2.3) will have special forms. The hypo-elastic theory for these special cases is not equivalent to elasticity, however, since in elasticity only a special class of initial stresses, those obtainable by elastic strain from the natural state, are admissible, while in hypo-elasticity any initial stress satisfying the equations of motion may be employed. Thus even for bodies whose response to strain from an unstressed state is elastic, the theory of hypo-elasticity offers the possibility of a more liberal treatment of initially stressed configurations.²

Second, if the strain energy in the elastic case has an especially simple form, no particularly simple form of the resulting hypo-elastic coefficients in (2.3) is to be expected. More important, the especially simple hypo-elastic bodies to be defined in the next section will generally fail to fall within the elastic special case. Thus the theory of hypo-elasticity offers some simple models for elastic behavior in large deformation which are not included in the classical theory of finite strain.

4. Some special hypo-elastic bodies. To include the classical linear theory as a special case in first approximation, we take (*cf.* (1.5))

$$(4.1) \quad g_1(0, 0, 0) = 1, \quad g_0(0, 0, 0) = \frac{\lambda}{2\mu} = \frac{\nu}{1 - 2\nu},$$

where λ and μ are the usual Lamé constants and ν is the usual Poisson modulus.

We shall say a hypo-elastic body is of *grade zero* if the right hand side of (2.3) or (2.5) is independent of $\mathbf{s}$:

$$(4.2) \quad g_0 \equiv \frac{\lambda}{2\mu}, \quad g_1 \equiv 1, \quad g_2 \equiv g_3 \equiv \cdots g_{11} \equiv 0, \quad \text{for all } \mathbf{s};$$

of *grade one* if

¹ In a colloquium at Indiana University in May, 1954. After hearing this result, I realized that equation (55.4) in *The mechanical foundations* gives a strong hint of it in the case when there is a strain energy. The proof is given in §15b of NOLL's paper cited in §2 above.

² It now seems to me that the theory of the hypo-elastic body of grade zero, with the inertial terms in the equations of motion neglected, is what CAUCHY intended by his much misunderstood theory of initial stress (see §55 of *The mechanical foundations*).

$$\begin{aligned}
(4.3) \quad g_0 &\equiv \frac{\lambda}{2\mu} + \alpha_0 I, & g_1 &\equiv 1 + \alpha_1 I, \\
g_2 &\equiv \alpha_2, & g_3 &\equiv \alpha_3, & g_4 &\equiv \alpha_4, \\
g_5 &\equiv g_6 \equiv \cdots g_{11} \equiv 0, & & \text{for all } \mathbf{s},
\end{aligned}$$

where the α_r are dimensionless constants; of *grade two* if

$$\begin{aligned}
(4.4) \quad g_0 &\equiv \frac{\lambda}{2\mu} + \alpha_0 I + \beta_0 I^2 \beta'_0 II, \\
g_1 &\equiv 1 + \alpha_1 I + \beta_1 I^2 + \beta'_1 II, \\
g_2 &\equiv \alpha_2 + \beta_2 I, & g_3 &\equiv \alpha_3 + \beta_3 I, & g_4 &\equiv \alpha_4 + \beta_4 I, \\
g_5 &\equiv \beta_5, & g_6 &\equiv \beta_6, & g_7 &\equiv \beta_7, & g_8 &\equiv \beta_8, \\
g_9 &\equiv g_{10} \equiv g_{11} \equiv 0, & & \text{for all } \mathbf{s},
\end{aligned}$$

where the β_r are dimensionless constants; *etc.* If we think of our theory as extending the classical linear one, the grades we have defined would suggest a perturbation scheme with respect to a slightly stressed state. However, no such view is necessary. Each special choice of the functions g_r in (2.3) or (2.5) defines a special hypo-elastic body, consistent with the principles of mechanics for deformations and stresses of any magnitude whatever. According to this view (4.2), (4.3), and (4.4) merely represent interesting simple kinds of hypo-elastic body.

5. Some properties of $\tilde{\mathbf{s}}$. It simplifies solutions in special cases to notice that the affine connection need not be known in order to calculate $\tilde{\mathbf{s}}$: in all co-ordinate systems we have

$$(5.1) \quad \tilde{\mathbf{s}}^{ij} = \partial_t s^{ij} + v^k \partial_k s^{ij} - s^{ik} \partial_k v^j - s^{kj} \partial_k v^i + (s^{ij}/\sqrt{g}) \partial_k (\sqrt{g} v^k),$$

where ∂_t denotes $\partial/\partial t$ with x^k held constant, ∂_k denotes $\partial/\partial x^k$, and $g \equiv \det g_{ij}$, the g_{ij} being the covariant metric components.

Now we shall solve the equation

$$(5.2) \quad \tilde{\mathbf{s}} = 0$$

for a given velocity field $\mathbf{v}$. Let the motion be $x^i = x^i(X^1, X^2, X^3, t)$, where the X^α are any three independent functions of the initial co-ordinates. (In other words, let the functions x^i be any three independent integrals of the system $dx^i = v^i dt$.) Then $v^i = \partial x^i / \partial t$. Let $\mathbf{c}^{-1}$ be the reciprocal of Cauchy's deformation tensor calculated from the configuration X^α :

$$(5.3) \quad (c^{-1})^{ij} = G^{\alpha\beta} x^i_{,\alpha} x^j_{,\beta},$$

where the $G^{\alpha\beta}$ are the contravariant components of any second order tensor field depending only on $\mathbf{X}$. Then it is easy to show that¹

$$(5.4) \quad \overline{(c^{-1})^{ij}} = (c^{-1})^{ik} v^j_{,k} + (c^{-1})^{kj} v^i_{,k}.$$

Hence

$$(5.5) \quad \overline{\rho(c^{-1})^{ij}} = \rho[(c^{-1})^{ik} v^j_{,k} - (c^{-1})^{kj} v^i_{,k} - (c^{-1})^{ij} v^k_{,k}].$$

Comparison with (1.2) shows that $\rho(c^{-1})^{ij}$ is a solution of (5.2). Since the X^α are arbitrary independent functions of the initial coordinates, the quantities $x^i_{,a}$ at time $t = 0$ are not necessarily Kronecker deltas but may take on arbitrary values. Hence the quantities $\rho(c^{-1})^{ij}$ take on arbitrary initial values, and thus furnish a solution of the initial value problem for (5.2). Since (5.2) is linear in $\mathbf{s}$, this solution is unique. Hence

$$(5.6) \quad s^{ij} = \rho(c^{-1})^{ij}$$

is the general solution of (5.2).

For bodies of grade zero we can separate the boundary value problem from the initial value problem. For, as Dr. ERICKSEN has remarked, if $\mathbf{s}$ satisfies the constitutive equation (1.6) in the case when $A^{i,k}_{j,l}$ is independent of $\mathbf{s}$, then so does $\mathbf{s} + {}_0\mathbf{s}$, where ${}_0\mathbf{s}$ is any solution of (5.2). But, as we have just seen, (5.6) provides a solution of (5.2) satisfying arbitrary initial data. Let $\mathbf{s}$ be a function which satisfies

1. The constitutive equations;
2. The equations

$$(5.7) \quad 2\mu s^{ij}_{,j} = \rho(f^i + {}_0s^{ij}_{,j} - \dot{v}^i);$$

3. The boundary conditions

$$(5.8) \quad s^{ij} n_j = b^i - {}_0s^{ij} n_j,$$

where n_j is the unit normal and b^i is the prescribed surface traction; and

4. The initial condition $s^{ij} = 0$.

Then $\mathbf{s} + {}_0\mathbf{s}$ satisfies the original combined problem.

From (5.6) it is evident that a solution of $\tilde{\mathbf{s}} = 0$ which satisfies the equations of motion (2.7) at $t = 0$ will generally fail to do so subsequently. This fact is reflected in the foregoing analysis by the appearance of ${}_0s^{ij}_{,j}$ as an effective extrinsic force in (5.7). If, however, we have a particular motion and a particular initial condition such that (5.6) yields ${}_0s^{ij}_{,j} = 0$, then we can formulate a superposition principle: if $\mathbf{s}$ satisfies all the differential equations of the theory, so does $\mathbf{s} + {}_0\mathbf{s}$.

¹ Use $\mathbf{c}\mathbf{c}^{-1} = \mathbf{1}$ and the special case $\mathbf{r} = 0$ of eq. (22.3) in *The mechanical foundations*. Note that for the present argument $G^{\alpha\beta}$ need not be a metric tensor of space.

In the equation (2.5), appropriate to incompressible bodies, appears the quantity $\mathbf{s}' \equiv \mathbf{s} + p\mathbf{1}$. Our solution (5.6) of the equation $\tilde{\mathbf{s}} = 0$ shows that any deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ such that $(c^{-1})_{i,j}^{j,i} = p_{,i}$ for some p , yields a solution $\mathbf{s}' = \mathbf{c}^{-1} - p\mathbf{1}$ of the equation $\tilde{\mathbf{s}}' = 0$. Among such deformations are all those which can be produced by suitable surface loads in any incompressible elastic body and all those similarly possible in RIVLIN'S "neo-Hookean" material.¹ If one of these is a solution for an incompressible hypo-elastic body of grade zero, and if $\mathbf{s}$ is any possible stress in such a body, then $\mathbf{s} + \alpha(\mathbf{c}^{-1} - p\mathbf{1})$ for arbitrary α is also a solution for that body.

6. Homogeneous stress. We recall² that a linear velocity field

$$(6.1) \quad \mathbf{v} = \mathbf{a}\mathbf{x} + \mathbf{b},$$

where $\mathbf{a}$ is a square matrix and the co-ordinate $\mathbf{x}$ is rectangular Cartesian, corresponds to a motion without acceleration if and only if $\mathbf{a}$ and $\mathbf{b}$ have the forms

$$(6.2) \quad \mathbf{a} = \frac{\mathbf{A}}{\mathbf{1} + \mathbf{A}t}, \quad \mathbf{b} = \frac{\mathbf{B}}{\mathbf{1} + \mathbf{A}t}.$$

The constant matrix $\mathbf{A}$ and vector $\mathbf{B}$ are the initial values of the velocity gradient and the velocity at the origin. Among the linear velocity fields (6.1) it is only these which can correspond to a system of stresses and forces under which equilibrium is possible.

In particular, consider a state of homogeneous stress $\mathbf{s}(t)$ subject to no extrinsic force. For a velocity field satisfying (6.1) and (6.2), the dynamical equations (2.7) are satisfied, and the constitutive equations (1.6) reduce to an ordinary differential system:

$$(6.3) \quad \dot{\mathbf{s}} = \mathbf{g}\left(\mathbf{s}, \frac{\mathbf{A}}{\mathbf{1} + \mathbf{A}t}\right),$$

where $\mathbf{g}$ is analytic in its arguments. The second argument exists and is continuously differentiable in the interval $t_- < t < t_+$, where $-1/t_+(-1/t_-)$ is the least negative (greatest positive) proper number of $\mathbf{A}$, or, if there is no such, $t_+ = +\infty$ ($t_- = -\infty$). The second argument is analytic for $|t| < t_0$, where $1/t_0$ is that among the absolute values of the proper numbers of $\mathbf{A}$ which is not surpassed by any other. From the general theory of ordinary differential equations follows then the

Existence and uniqueness theorem for homogeneous stress. *Consider the class of linear accelerationless motions given by (6.1) and (6.2), $\mathbf{A}$ being the initial velocity gradient and $\mathbf{B}$ the initial velocity of the origin. Then, whatever is $\mathbf{B}$, to each $\mathbf{A}$ and to arbitrary homogeneous initial stress $\mathbf{S}$ there corresponds for any given*

¹ See §54 of *The mechanical foundations*.

² These results are proved in the note cited in §1.

hypo-elastic body, a unique homogeneous stress system $\mathbf{s}(t)$ in $t_- < t < t_+$. For $|t| < t_0$ this stress system $\mathbf{s}(t)$ is an analytic function of t .

In proving the foregoing theorem, we have considered only the compressible case. For incompressible bodies we may assign $p(t)$ as an arbitrary continuously differentiable or analytic function, and the argument in the two cases goes through in just the same way. The presence of this arbitrary pressure enables us to satisfy boundary conditions more easily in the incompressible case.

Most of the remainder of this paper will consider special examples of the class of solutions just obtained. It is interesting and important to notice the occurrence of movable singularities. The numbers t_- , t_+ , t_0 are determined by $\mathbf{A}$, the matrix of *initial velocity gradients*. Thus a homogeneous stress distribution exists for a certain time for any linear motion; how long it continues to exist, while not influenced by the initial stress, depends on how the motion is started.

7. Hydrostatic stress. We begin with the simplest possible case: $d^i_j = (\delta/3)\delta^i_j$, $s^i_j = -\varpi\delta^i_j$, where δ and ϖ are functions of time only. Such a solution is not dynamically possible unless the acceleration field vanishes. Hence the density ρ must satisfy

$$(7.1) \quad \rho = \frac{\rho_0}{1 + k_0 t},$$

where k_0 is an arbitrary constant. Since $\delta = -d \log \rho / dt$, the constitutive equation (1.17) yields

$$(7.2) \quad \begin{aligned} \log \frac{\rho}{\rho_0} &= \int_{\varpi_0}^{\varpi} \frac{d\xi}{f(\xi)}, \\ f(\varpi) &\equiv g_0 + \frac{1}{3}g_1 + \varpi(\frac{1}{3} - g_2 - g_3 - \frac{1}{3}g_4) \\ &\quad + \varpi^2(g_5 + g_6 + g_7 + \frac{1}{3}g_8) - \varpi^3(g_9 + g_{10}) + \varpi^4 g_{11}, \end{aligned}$$

where the arguments of the g_r are -3ϖ , $3\varpi^2$, $-\varpi^3$. Thus if the pressure is applied in such a way that (7.1) holds, or, equivalently, the volume V shall satisfy $V/V_0 = (1 + k_0 t)^3$, there is then the definite relation (7.2) connecting pressure $p = 2\mu\varpi$ with density ρ , determinate to within a pair of arbitrary initial values ϖ_0 , ρ_0 .

The amount of pressure required to produce a given condensation will generally differ from the amount of tension required for a rarefaction of equal magnitude. Moreover, to satisfy the physical requirement $d\rho/d\varpi \geq 0$ we must have

$$(7.3) \quad f(\varpi) \geq 0,$$

a restriction upon the class of admissible coefficient functions g_r , upon the range of admissible pressures, or upon both.

When $\varpi > 0$ ($\varpi < 0$) we shall say the body *hardens* or *softens* according as

the curve of ρ against ϖ is curved downward or upward (upward or downward). Thus, for example, only if the curvature changes sign at $\varpi = 0$ can the body harden both in pressure and in tension. Since

$$(7.4) \quad \frac{1}{\rho} \frac{d^2 \varpi}{d\rho^2} = \frac{1 - f'}{f^2},$$

the hardening or softening of a body can be determined from the sign of $1 - f'$, except possibly at points where $\rho = 0$ or $d\rho/d\varpi = \infty$.

For bodies of grade zero, one, or two we have

$$(7.5) \quad \begin{aligned} f(\varpi) &= \kappa + \alpha_p \varpi + \beta_p \varpi^2, \\ \kappa &\equiv \frac{\lambda + \frac{2}{3}\mu}{2\mu} = \frac{k}{2\mu} > 0, \\ \alpha_p &\equiv \frac{1}{3} - 3\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \frac{1}{3}\alpha_4, \\ \beta_p &\equiv 9\beta_0 + 3\beta'_0 + 3\beta_1 + \beta'_1 + 3\beta_2 + 3\beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \frac{1}{3}\beta_8. \end{aligned}$$

For a body of grade zero, $\alpha_p = \frac{1}{3}$ and $\beta_p = 0$; grade one, $\alpha_p \neq \frac{1}{3}$ and $\beta_p = 0$; grade two, $\beta_p \neq 0$. In all cases the sign of $1 - \alpha_p - 2\beta_p \varpi$ determines hardening or softening, except possibly at a root of $(7.5)_1$ or where $\rho = 0$.

For a body of grade zero $(7.2)_2$ becomes

$$(7.6) \quad \frac{\rho}{\rho_0} = \left(\frac{\kappa + \frac{1}{3}\varpi}{\kappa + \frac{1}{3}\varpi_0} \right)^3.$$

This body softens in compression but hardens in tension. As $\varpi \rightarrow \infty$, so does ρ , but at the tension $\varpi = -3\kappa$ the density vanishes. In an application to a given physical material, we must expect rupture or yield before such a tension is obtained. For a material whose Poisson modulus is $\frac{1}{3}$, this tension is 4, a very large value indeed.

For a body of grade one we have hardening (softening) or softening (hardening) in compression (tension) if $\alpha_p > 1$ ($\alpha_p < 1$); (7.6) is to be replaced by

$$(7.7) \quad \frac{\rho}{\rho_0} = \begin{cases} \left(\frac{\kappa + \alpha_p \varpi}{\kappa + \alpha_p \varpi_0} \right)^{1/\alpha_p}, & \alpha_p \neq 0, \\ \exp \frac{\varpi - \varpi_0}{\kappa}, & \alpha_p = 0. \end{cases}$$

In compression (tension) there is hardening or softening (softening or hardening) according as $\alpha_p > 1$ or $\alpha_p < 1$; if $\alpha_p = 1$, (7.7) reduces to the classical linear formula. If $\alpha_p > 0$ there is a tension $\varpi = -\kappa/\alpha_p$ at which $\rho = 0$. Negative values of the α_r tend to reduce the magnitude of this tension from the value $\varpi = -3\kappa$ which is appropriate to the body of grade zero. If $\alpha_p < 0$ there is a pressure

$\varpi = \kappa/(-\alpha_p)$ at which $\rho = \infty$, suggesting that such a body will collapse under pressure but not under tension. Thus physical application for the case $\alpha_p > 0$, especially when $\alpha_p \geq 1$, is more likely than for $\alpha_p < 0$. Figure 7.1 shows our results concerning bodies of grade zero and grade one for various values of α_p in the case $\varpi_0 = 0$, $\kappa = \frac{4}{3}$.

FIGURE 7.1

Hydrostatic pressure in the hypo-elastic bodies of grades 0 and 1. The numbers on the curves are the appropriate values of α_p .

For the bodies of grade two, put

$$(7.8) \quad \bar{\beta} \equiv +\sqrt{|\alpha_p^2 - 4\kappa\beta_p|}.$$

Then, formally,

$$(7.9) \quad \frac{\rho}{\rho_0} = \begin{cases} \left[\frac{2\beta_p \varpi + \alpha_p - \bar{\beta}}{2\beta_p \varpi + \alpha_p + \bar{\beta}} \cdot \frac{2\beta_p \varpi_0 + \alpha_p + \bar{\beta}}{2\beta_p \varpi_0 + \alpha_p - \bar{\beta}} \right]^{1/\bar{\beta}}, \\ \exp \left[\frac{4\kappa(\varpi - \varpi_0)}{(2k + \alpha_p \varpi_0)(2k + \alpha_p \varpi)} \right], \\ \exp \left[\frac{2}{\bar{\beta}} \left(\operatorname{Arctan} \frac{2\beta_p \varpi + \alpha_p}{\bar{\beta}} - \operatorname{Arctan} \frac{2\beta_p \varpi_0 + \alpha_p}{\bar{\beta}} \right) \right], \end{cases}$$

according as $4\kappa\beta_p \leq \alpha_p^2$. A large variety of possible behavior is included in this result. The fourteen essentially different cases are delimited and shown in Figure 7.2 for typical values of α_p and β_p when $\varpi_0 = 0$ and $\kappa = \frac{4}{3}$. In all graphs the light line is the result from the classical linear theory. Dashed lines are asymptotes; circles locate points of inflection. At one extreme lie the very soft bodies of types 6–10, which must give way easily either in pressure or in tension; at the other, the extremely hard bodies of types 12–14, which approach a limit density as $\varpi \rightarrow \infty$ and which yield in tension only after very great stress is applied. Between these extremes lie bodies which yield both in pressure and in tension, with greater or lesser ease, or in one but not in the other. In some, such as types 1 and 10, the body first hardens in compression and then softens after sufficient pressure has been applied. In these results we see for the first time pressure-density relations which are qualitatively like those of many actual materials yet emerge as rigorous solutions in a dynamically possible theory of continuum mechanics.

Thus far in our examples we have supposed $\varpi_0 = 0$. Since the internal pressure in a test specimen is unknown, and since in any case our assumed homogeneity of the pressure field will in a test situation be at best an idealization, it is possible that ϖ_0 might be left in some cases a parameter to be adjusted to fit the experimental data. If, however, a definite value for ϖ_0 is known in advance, we can determine the effect of initial pressure or tension in a simple way. From (7.2) we have

$$(7.10) \quad \log \frac{\rho}{\rho_0} = \int_0^{\varpi - \varpi_0} \frac{d\eta}{g(\eta)}, \quad g(\eta) \equiv f(\eta + \varpi_0);$$

i.e., the density increment corresponding to a pressure increment $\varpi - \varpi_0$ from initial pressure ϖ_0 in the body defined by $f(\varpi)$ is the same as that corresponding to an equal pressure increment from an unstressed state in the body defined by $g(\varpi) \equiv f(\varpi + \varpi_0)$. For example, in the bodies of grade zero, one, and two we have

$$(7.11) \quad \begin{aligned} g(\varpi) &= \kappa^* + \alpha^* \varpi + \beta^* \varpi^2, \\ \kappa^* &\equiv \kappa + \alpha_p \varpi_0 + \beta_p \varpi_0^2, & \alpha^* &\equiv \alpha_p + 2\beta_p \varpi_0, & \beta^* &\equiv \beta_p. \end{aligned}$$

For a body of grade zero we get the same result as in the classical theory of elasticity: initial hydrostatic pressure ϖ_0 has the effect of raising the dimensionless

FIGURE 7.2, Types 1-6
Hydrostatic pressure in typical hypo-elastic bodies of grade 2

FIGURE 7.2, Types 7-12
Hydrostatic pressure in typical hypo-elastic bodies of grade 2

FIGURE 7.2, Types 13-14
Hydrostatic pressure in typical hypo-elastic bodies of grade 2

compressibility by the amount $\frac{1}{3}\alpha_0$. For a body of grade two in which $\beta_p > 0$ any sufficiently large initial pressure or tension increases the effective compressibility, while for grades zero and one initial pressure and tension must necessarily have opposite effect on the response of the body. Further results concerning the effect of initial pressure can be read off by substituting (7.11)₂–(7.11)₄ into the conditions defining the fourteen types illustrated in Figure 7.2. Figure 7.3 gives some examples. For bodies of grades zero and one initial pressure affects only the quantity, not the quality of the response, while for a body of grade two sufficiently large initial pressure converts the response of the body into a form typical of another of the fourteen basic types.

8. Simple shear of compressible bodies. Another case evidently compatible with homogeneous stress is simple shear, where

$$(8.1) \quad v_x = 2ky, \quad v_y = v_z = 0, \quad k = \text{const.}$$

If we put

$$(8.2) \quad \tau \equiv kt, \quad \theta \equiv \text{Arctan } \tau,$$

then θ is the angle through which a plane initially normal to the x -axis has been rotated at time t , and τ is a convenient dimensionless measure of the shear. Guided by the analogous problem in finite elasticity theory, we seek as a solution a stress field such that $s_{xz} = s_{yz} = 0$. The constitutive equations (2.3) then

FIGURE 7.3

The effect of initial pressure on the response of certain hypo-elastic bodies of grades 0, 1, and 2 to hydrostatic pressure. The numbers on the curves are the appropriate values of ω_0 .

yield the following system of four ordinary differential equations:

$$\begin{aligned}
 \frac{d}{d\tau} \begin{vmatrix} s_{xx} & s_{xy} & 0 \\ \cdot & s_{yy} & 0 \\ \cdot & \cdot & s_{zz} \end{vmatrix} &= \begin{vmatrix} 2s_{xy} & -s_{xy} & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{vmatrix} + g_1 \begin{vmatrix} 0 & 1 & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{vmatrix} \\
 &+ 2s_{xy}[g_3 + (s_{xx} + s_{yy})g_7] \begin{vmatrix} 1 & 0 & 0 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 1 \end{vmatrix} \\
 &+ g_4 \begin{vmatrix} s_{xy} & \frac{1}{2}(s_{xx} + s_{yy}) & 0 \\ \cdot & s_{xy} & 0 \\ \cdot & \cdot & 0 \end{vmatrix} \\
 (8.3) \quad &+ 2s_{xy}[g_6 + (s_{xx} + s_{yy})g_{10}] \begin{vmatrix} s_{xx} & s_{yy} & 0 \\ \cdot & s_{yy} & 0 \\ \cdot & \cdot & s_{zz} \end{vmatrix} \\
 &+ g_8 \begin{vmatrix} s_{xy}(s_{xx} + s_{yy}) & \frac{1}{2}(s_{xx}^2 + s_{yy}^2) + s_{xy}^2 & 0 \\ \cdot & s_{xy}(s_{xx} + s_{yy}) & 0 \\ \cdot & \cdot & 0 \end{vmatrix} \\
 &+ 2s_{xy}[g_9 + (s_{xx} + s_{yy})g_{11}] \begin{vmatrix} s_{xx}^2 + s_{xy}^2 & s_{xy}(s_{xx} + s_{yy}) & 0 \\ \cdot & s_{yy}^2 + s_{xy}^2 & 0 \\ \cdot & \cdot & s_{zz}^2 \end{vmatrix},
 \end{aligned}$$

where the arguments of the g_i are $s_{xx} + s_{yy} + s_{zz}$, $s_{xx}s_{yy} + s_{yy}s_{zz} + s_{zz}s_{xx} - s_{xy}^2$, $(s_{xx}s_{yy} - s_{xy}^2)s_{zz}$. A number of results follow by inspection:

1. As in finite elasticity theory, shearing stress alone is always insufficient to maintain simple shearing.

2. The coefficients g_0 , g_2 , and g_5 have no influence on the response of the body to simple shear.

3. $s_{zz} = 0$ is a possible solution if and only if $g_3 \equiv g_7 \equiv 0$.

4. $s_{yy} = 0$ is a possible solution if and only if $g_3 \equiv g_4 \equiv g_7 \equiv g_8 \equiv g_9 \equiv g_{10} \equiv g_{11} \equiv 0$.

A body of grade zero satisfies both the conditions 3 and 4. For it we get the extremely interesting result

$$(8.4) \quad s_{xy} - S_{xy} = \tau, \quad s_{xx} - S_{xx} = \tau^2 + 2\tau S_{xy},$$

where the capital letters denote initial stresses. For an initially unstressed specimen, the classical law of proportionality between shear stress and shear strain continues to hold precisely with τ as the measure of shear, but in addition a normal tension equal to the square of the shear stress must be supplied on the $x = \text{const.}$ planes. For an initially rectangular block with faces parallel to the co-ordinate planes, the stresses which must act upon the faces inclined at angle $\frac{1}{2}\pi - \theta$ to the shearing planes are:

$$\begin{aligned}
 n &\equiv \text{Normal tension} = \frac{s_{xx} - 2\tau s_{xy} + \tau^2 s_{yy}}{1 + \tau^2}, \\
 &= -\frac{\tau^2}{1 + \tau^2} = -\sin^2 \theta; \\
 s &\equiv \text{Shearing stress} = \frac{(s_{xx} - s_{yy})\tau + s_{xy}(1 - \tau^2)}{1 + \tau^2}, \\
 &= \frac{\tau}{1 + \tau^2} = \frac{1}{2}\sin 2\theta.
 \end{aligned}
 \tag{8.5}$$

Thus as the shearing proceeds the shear stress on these slanted faces builds up to the value $\frac{1}{2}$ when $\theta = \frac{1}{4}\pi$ but falls off thereafter to 0, while the normal stress is a pressure which increases steadily to 1. A similar phenomenon in the finite strain theory of elasticity is called "the Poynting effect."

For a body of grade one the general solution of (8.3) is

$$\begin{aligned}
 s_{xy} &= \frac{A}{\sqrt{|\alpha_s|}} \frac{\sinh}{\sin} \sqrt{|\alpha_s|} \tau + S_{xy} \frac{\cosh}{\cos} \sqrt{|\alpha_s|} \tau, \\
 \frac{1}{2}\alpha_s &\equiv \alpha_1 + \alpha_3 + \alpha_4 + 3\alpha_1 \alpha_3 + \alpha_1 \alpha_4 + \alpha_3 \alpha_4 + \frac{1}{2}\alpha_4^2, \\
 A &\equiv 1 + (\alpha_1 + \frac{1}{2}\alpha_4)S_{xx} + (1 + \alpha_1 + \frac{1}{2}\alpha_4)S_{yy} + \alpha_1 S_{zz}, \\
 \frac{s_{zz} - S_{zz}}{2\alpha_3} &= \frac{s_{yy} - S_{yy}}{2\alpha_3 + \alpha_4} = \frac{s_{xx} - S_{xx}}{2 + 2\alpha_3 + \alpha_4} = \frac{s_{xx} - s_{yy} - (S_{xx} - S_{yy})}{2}, \\
 &= \int_0^\tau s_{xy}(\xi) d\xi = \frac{2A}{|\alpha_s|} \frac{\sinh^2}{\sin^2} \frac{1}{2}\sqrt{|\alpha_s|} \tau + \frac{S_{xy}}{\sqrt{|\alpha_s|}} \frac{\sinh}{\sin} \sqrt{|\alpha_s|} \tau,
 \end{aligned}
 \tag{8.6}$$

where the upper alternatives refer to the case $\alpha_s > 0$, the lower to $\alpha_s < 0$; if $\alpha_s = 0$ these same formulae still hold provided we replace (8.6)₁ and (8.6)₈ by

$$\begin{aligned}
 s_{xy} - S_{xy} &= A\tau, \\
 \int_0^\tau s_{xy}(\xi) d\xi &= \frac{1}{2}A\tau^2 + S_{xy} \tau,
 \end{aligned}
 \tag{8.7}$$

respectively.

Henceforth we consider only the case of an initially unstressed block. Then $A = 1$, and we have

$$(8.8) \quad s_{xy} = \begin{cases} \frac{1}{\sqrt{|\alpha_s|}} \frac{\sinh \sqrt{|\alpha_s|} \tau}{\sin \sqrt{|\alpha_s|} \tau}, & \alpha_s \neq 0, \\ \tau, & \alpha_s = 0, \end{cases}$$

for the shear stress on the shear planes. Thus if $\alpha_s > 0$ the body stiffens indefinitely in shear, the shear stress being ultimately exponential. If, on the contrary, $\alpha_s < 0$, the body softens until the shear stress increases to its maximum value $1/\sqrt{|\alpha_s|}$. While according to (8.6) the shear stress will decrease for greater angles of shear, we take this fact as indication that the body will collapse or yield at or before this strain. Thus we may call α_s the *shear stiffness* of the body of grade one and refer to the body itself as *soft*, *neutral*, or *hard* in shear according as $\alpha_s \leq 0$. For a soft body we may call the maximum shear stress, viz. $1/\sqrt{|\alpha_s|}$, its *shear strength* s_s , and the corresponding angle

$$(8.9) \quad \theta_s = \text{Arctan} \frac{\pi}{2 \sqrt{|\alpha_s|}}$$

its *ultimate shear angle*. Henceforth without further mention we limit all discussion of soft bodies to the case when the ultimate shear angle is not exceeded.

Figure 8.1 shows the hyperboloid $\alpha_s = 0$ in the space whose co-ordinates are $\alpha_1, \alpha_3, \alpha_3 + \frac{1}{2}\alpha_4$. Points inside the hyperboloid represent soft bodies; outside, hard bodies.

Next we consider the normal forces acting upon the shear planes and upon the planes $y = \text{const.}$, these latter being of all the planes normal to the shear planes the only ones which are preserved in the deformation. The normal forces s_{yy} and s_{zz} are furnished by (8.6)₄ and (8.6)₅. From the fact that the last expression in (8.6) is positive for $\tau \neq 0$ (recall that $S_{xy} = 0$ and $A = 1$) follow dynamic interpretations for the coefficients α_3 and $\alpha_3 + \frac{1}{2}\alpha_4$.

First, the sign of s_{zz} is the same as the sign of α_3 . If we fail to supply the necessary tension s_{zz} to a specimen in shear, we may expect that it will draw together or spread out in a direction perpendicular to the shear planes according as $\alpha_3 > 0$ or $\alpha_3 < 0$. Thus we may call α_3 the *shear elusiveness*. For a given body, s_{zz} has the same sign for all values of the shear angle θ and is a monotone function of $|\theta|$.

Second, the sign of s_{yy} is the same as the sign of $\alpha_3 + \frac{1}{2}\alpha_4$. If we fail to supply the necessary tension s_{yy} to the planes being sheared, we may expect that these planes will draw together or spread apart from one another according as $\alpha_3 + \frac{1}{2}\alpha_4 > 0$ or $\alpha_3 + \frac{1}{2}\alpha_4 < 0$. Thus we may call $\alpha_3 + \frac{1}{2}\alpha_4$ the *shear tenseness*. For a given body, s_{yy} has the same sign for all values of the shear angle θ and is a monotone function of $|\theta|$.

FIGURE 8.1

The hyperboloid $\alpha_s = 0$. Points inside the hyperboloid represent soft bodies; outside, hard bodies.

From (8.6) we have as $\tau \rightarrow 0$

$$(8.10) \quad \begin{aligned} \frac{s_{zz}}{\alpha_3} &= \frac{s_{yy}}{\alpha_3 + \frac{1}{2}\alpha_4} = \tau^2 + O(\tau^4), \\ &= s_{xy}^2 + O(\tau^4): \end{aligned}$$

For small shears, the normal forces are proportional to the square¹ of the shearing stress and their magnitudes are independent of α_s , being determined entirely by α_3 and $\alpha_3 + \frac{1}{2}\alpha_4$, respectively. For large shears we must distinguish between hard, neutral, and soft bodies. If $\alpha_s < 0$, $|s_{zz}|$ and $|s_{yy}|$ increase steadily to finite upper limits, obtained at the ultimate shear angle, when

$$(8.11) \quad \frac{s_{zz}}{\alpha_3} = \frac{s_{yy}}{\alpha_3 + \frac{1}{2}\alpha_4} = \frac{2}{|\alpha_s|} = 2s_s^2:$$

for a soft body, the ratios of normal force to the square of the shearing stress increase from α_3 and $\alpha_3 + \frac{1}{2}\alpha_4$ for small shears to twice those values at the ultimate shear angle. If $\alpha_s = 0$, however, we have

$$(8.12) \quad \frac{s_{zz}}{\alpha_3} = \frac{s_{yy}}{\alpha_3 + \frac{1}{2}\alpha_4} = \tau^2 = s_{xy}^2,$$

so that for a neutral body the ratios between normal forces and the square of the shear stress remain precisely α_3 and $\alpha_3 + \frac{1}{2}\alpha_4$, for all shears. If $\alpha_s > 0$, we have as $\tau \rightarrow \infty$

$$(8.13) \quad \begin{aligned} \frac{s_{zz}}{\alpha_3} &= \frac{s_{yy}}{\alpha_3 + \frac{1}{2}\alpha_4} \sim \frac{1}{\alpha_s} \exp(\sqrt{\alpha_s} \tau), \\ &\sim \frac{2}{\sqrt{\alpha_s}} s_{xy}: \end{aligned}$$

For hard bodies, the initial proportionality between normal stress and the square of the shear stress, independent of α_s , is converted gradually with increasing shear into proportionality to the shear stress itself, in the ratio $2/\sqrt{\alpha_s}$. Thus, in particular, the harder is the body the less normal stress relative to shear stress, other things being equal, will be required for large shears.

We turn now to the normal and tangential forces n and s acting upon the slanted faces of the block. These forces are easily calculated from (8.5)₂ and (8.5)₅. To begin with, from (8.5)₂ and (8.6)₇ we observe that for a given angle of shear, s is a functional of s_{xy} . Hence s , like s_{xy} , is a function of α_s and τ only: the shear stiffness α_s determines the shearing stress on all faces of a sheared block, independently of the shear elusiveness and the shear tenseness, which determine

¹ Recall that the components s_{ij} are dimensionless. To interpret these results in terms of the usual stress t_{ij} , multiply (8.10) by $4\mu^2$, obtaining $2\mu t_{zz}/t_{xy}^2 = \alpha_3 + O(\tau^4)$, etc.

the normal forces on the shear planes and on the planes $z = \text{const.}$ When $\alpha_s \neq 0$ we get

$$\begin{aligned}
 s &= \frac{\frac{4\tau}{|\alpha_s|} \frac{\sinh^2}{\sin^2} \frac{1}{2} \sqrt{|\alpha_s|} \tau + \frac{1 - \tau^2}{\sqrt{|\alpha_s|}} \frac{\sinh}{\sin} \sqrt{|\alpha_s|} \tau}{1 + \tau}, \\
 (8.14) \quad &= \frac{\frac{2\tau}{\alpha_s} [\sqrt{1 + \alpha_s s_{xy}^2} - 1] + (1 - \tau^2) s_{xy}}{1 + \tau^2}, \\
 &= \left[1 - \tau^2 + \frac{2\tau}{\sqrt{|\alpha_s|}} \frac{\tanh \frac{1}{2} \sqrt{|\alpha_s|} \tau}{\tan \frac{1}{2} \sqrt{|\alpha_s|} \tau} \right] \frac{s_{xy}}{1 + \tau^2}, \\
 &= \frac{s_{xy}}{1 + \tau^2} + O(\tau^4) \text{ as } \tau \rightarrow 0;
 \end{aligned}$$

while if $\alpha_s = 0$ we get

$$(8.15) \quad s = \frac{1}{2} \sin 2\theta = \frac{s_{xy}}{1 + s_{xy}^2},$$

just as for a body of grade zero. For small shear, of course, $s \approx s_{xy}$; however, (8.14)₄ shows that $|s| < |s_{xy}|$ for small shears, and we may ask if s can decrease to zero and change sign. For neutral bodies, (8.15) shows that in fact s decreases steadily to zero. When $\alpha_s \neq 0$, (8.14)₃ gives as necessary and sufficient for $s = 0$ when $\tau \neq 0$ the condition

$$(8.16) \quad \frac{2}{\sqrt{|\alpha_s|}} \frac{\tanh \frac{1}{2} \sqrt{|\alpha_s|} \tau}{\tan \frac{1}{2} \sqrt{|\alpha_s|} \tau} = \tau - \frac{1}{\tau}.$$

For hard bodies there is exactly one root $\tilde{\tau}$, and it lies in the range

$$(8.17) \quad 1 < \tilde{\tau} < \frac{1 + \sqrt{1 + \alpha_s}}{\sqrt{\alpha_s}}.$$

From (8.14)₃ it is plain that

$$(8.18) \quad s \sim -s_{xy} \text{ as } \tau \rightarrow \infty.$$

For soft bodies the left-hand side of (8.16) exceeds τ for $\tau > 0$ and hence can never equal the right side. Summarizing and slightly augmenting the foregoing analysis, we conclude that *in all bodies of grade one, the magnitude of the shear stress s on the slanted faces falls away from the magnitude of the shear stress s_{xy} on the shearing planes; for soft bodies, it remains of the same sign as s_{xy} , reaching the value*

$$(8.19) \quad s = \frac{\pi(4 - \pi) - 4\alpha_s}{\pi^2 - 4\alpha_s} = \frac{1 + \pi(1 - \frac{1}{4}\pi)s_s^2}{1 + \frac{1}{4}\pi^2 s_s^2}$$

at the ultimate shear angle; for neutral bodies, the magnitude of s decreases steadily to 0 as $\tau \rightarrow \infty$; for hard bodies, the magnitude of s decreases until a certain angle $\tilde{\theta}$, always greater than $\frac{1}{4}\pi$ and limited to the range (8.17), is attained; at this angle, s changes sign, and for very large angles of shear s becomes equal and opposite to s_{xy} .

In Figure 8.2 appears the curve $s = 0$, showing the value of the angle $\tilde{\theta}$ as a function of α_s .

FIGURE 8.2

The angle $\tilde{\theta}$ at which the shear stress s on the slanted faces of a hard body of shear stiffness α_s changes sign.

For the normal force n on the slanted faces we get from (8.5)₅ when $\alpha_s \neq 0$

$$\begin{aligned}
 n &= \frac{-\frac{2\tau}{\sqrt{|\alpha_s|}} \frac{\sinh}{\sin} \sqrt{|\alpha_s|} \tau + \frac{4}{|\alpha_s|} [1 + (\alpha_3 + \frac{1}{2}\alpha_4)(1 + \tau^2)] \frac{\sinh^2}{\sin^2} \frac{1}{2} \sqrt{|\alpha_s|} \tau}{1 + \tau^2} \\
 (8.20) \quad &= \frac{-2\tau s_{xy} + \frac{2}{\alpha_s} [1 + (\alpha_3 + \frac{1}{2}\alpha_4)(1 + \tau^2)] [\sqrt{1 + \alpha_s s_{xy}^2} - 1]}{1 + \tau^2}, \\
 &= \left\{ -\tau + \frac{1}{\sqrt{|\alpha_s|}} \left[1 + (\alpha_3 + \frac{1}{2}\alpha_4)(1 + \tau^2) \frac{\tanh}{\tan} \frac{1}{2} \sqrt{|\alpha_s|} \tau \right] \right\} \frac{2s_{xy}}{1 + \tau^2}, \\
 &= (\alpha_3 + \frac{1}{2}\alpha_4 - 1)\tau^2 + \frac{5}{8}\alpha_s(\alpha_3 + \frac{1}{2}\alpha_4)\tau^4 + O(\tau^6) \text{ as } \tau \rightarrow 0, \\
 &= (\alpha_3 + \frac{1}{2}\alpha_4 - 1)s_{xy}^2 + O(\tau^4);
 \end{aligned}$$

while if $\alpha_s = 0$

$$(8.21) \quad n = -\sin^2 \theta + (\alpha_3 + \frac{1}{2}\alpha_4) \tan^2 \theta.$$

From (8.20)₄, (8.21), and (8.12)₂ we conclude that as $\tau \rightarrow 0$

$$\begin{aligned}
 (8.22) \quad s_{yy} - n &= \tau^2 + O(\tau^4) \\
 &= s_{xy}^2 + O(\tau^4).
 \end{aligned}$$

This relation, being independent of the values of the α 's, must hold for all bodies of grades zero and one. Since it connects three measurable quantities and no empirical constants,² it may serve as the basis of an experimental test for the theory. From (8.20)₄ and (8.12) follows also that for small shears in a body such that $\alpha_3 + \frac{1}{2}\alpha_4 \neq 1$, n is a pressure or a tension according as

$$(8.23) \quad \alpha_3 + \frac{1}{2}\alpha_4 \lessgtr 1.$$

Thus the magnitude of the shear tenseness, when it is not 1, is the determining factor. When $\alpha_3 + \frac{1}{2}\alpha_4 = 1$, however, at small shears n is a tension in hard or neutral bodies but a pressure in soft bodies.

To see if n can change sign, we must solve the equation $n = 0$, which by (8.20)₃ is equivalent to

$$(8.24) \quad \frac{1}{\sqrt{|\alpha_s|}} \frac{\tanh \frac{1}{2}\sqrt{|\alpha_s|}}{\tan \frac{1}{2}\sqrt{|\alpha_s|}} \tau = \frac{\tau}{1 + (\alpha_3 + \frac{1}{2}\alpha_4)(1 + \tau^2)}$$

in the case $\alpha_s \neq 0$. Analysis of the functions on the two sides of this equation shows easily that for soft bodies there is at most one positive root in the principal branch of the tangent if $|\alpha_3 + \frac{1}{2}\alpha_4| < 1$, no such root if $|\alpha_3 + \frac{1}{2}\alpha_4| \geq 1$. To see whether such a root exists and, if so, whether it occurs before the ultimate shear angle, it suffices to calculate the value of n at that angle and compare its sign with that given by (8.23). In fact we have

$$(8.25) \quad n_s = \frac{2}{|\alpha_s|} (\alpha_3 + \frac{1}{2}\alpha_4) - \frac{\pi - 2}{|\alpha_s| + \frac{1}{4}\pi^2};$$

this is a non-zero pressure, zero, or a non-zero tension according as

$$(8.26) \quad \alpha_3 + \frac{1}{2}\alpha_4 \lessgtr \frac{1}{2} \frac{\pi - 2}{1 + \frac{\pi^2}{4|\alpha_s|}} < 1.$$

By the remark just foregoing and (8.23) we conclude that for soft bodies, n is always a pressure if

$$(8.27) \quad \alpha_3 + \frac{1}{2}\alpha_4 \leq \frac{1}{2} \frac{\pi - 2}{1 + \frac{\pi^2}{4|\alpha_s|}};$$

² Except of course the ordinary shear modulus μ , since in terms of the usual stress t_{ij} it reads $2\mu(t_{yy} - t_n) = t_{xy}^2 + O(\tau^4)$.

always a tension if

$$(8.28) \quad \alpha_3 + \frac{1}{2}\alpha_4 \geq 1;$$

while if

$$(8.29) \quad \frac{1}{2} \frac{\pi - 2}{1 + \frac{\pi^2}{4|\alpha_s|}} < \alpha_3 + \frac{1}{2}\alpha_4 < 1,$$

n changes from pressure to tension, passing through $n = 0$ at the angle satisfying (8.24); in all cases, (8.25) gives the value of n at the ultimate shear angle.

For a neutral body (8.21) yields $n = 0$ when

$$(8.30) \quad \cos^2 \theta = \alpha_3 + \frac{1}{2}\alpha_4.$$

An angle θ other than 0 or $\frac{1}{2}\pi$ satisfying this condition exists if and only if $0 < \alpha_3 + \frac{1}{2}\alpha_4 < 1$.

For a hard body we obtain from (8.20)₃

$$(8.31) \quad n \sim \frac{2(\alpha_3 + \frac{1}{2}\alpha_4)}{\sqrt{\alpha_s}} s_{xy} \sim s_{yy}$$

as $\tau \rightarrow \infty$, provided only $\alpha_3 + \frac{1}{2}\alpha_4 \neq 0$. If $\alpha_3 + \frac{1}{2}\alpha_4 = 0$, we get instead

$$(8.32) \quad n\tau \sim -2s_{xy}.$$

Thus for very large shears n is a pressure or tension according as $\alpha_3 + \frac{1}{2}\alpha_4 \leq 0$ or $\alpha_3 + \frac{1}{2}\alpha_4 > 0$. Moreover, (8.31) and (8.32) show that *for all hard bodies, in very large shear the normal forces on the shear planes and on the slanted faces become equal*.

Analysis of the case $\alpha_s \geq 0$ is not entirely trivial. If $0 < \alpha_3 + \frac{1}{2}\alpha_4 < 1$, the equation (8.24) possesses a single real non-zero root $\hat{\tau} = \tan \hat{\theta}$ such that

$$(8.33) \quad \text{Arccos } \sqrt{\alpha_3 + \frac{1}{2}\alpha_4} \leq \hat{\theta} < \frac{1}{2}\pi,$$

and for a fixed value of $\alpha_3 + \frac{1}{2}\alpha_4$ this root $\hat{\theta}$ is a monotone increasing function of α_s . By (8.21), the lower bound in (8.33) is achieved in the case of the neutral body, $\alpha_s = 0$. If $\alpha_3 + \frac{1}{2}\alpha_4 = 1$, there is a single real non-zero root if $\alpha_s > 6$, no such root if $\alpha_s \leq 6$. The case $\alpha_3 + \frac{1}{2}\alpha_4 > 1$ is more elaborate: There may be two, one, or no non-zero real roots. In fact, let $\hat{\alpha}_s$ be the single positive root of the system

$$(8.34) \quad \frac{1}{\sqrt{\hat{\alpha}_s}} \tanh \frac{1}{2} \sqrt{\hat{\alpha}_s} \tau = \frac{\tau}{1 + k(1 + \tau^2)},$$

$$\tau^2 = \frac{\alpha_s - 2k(k+2) - \sqrt{20k^2 + 16k^3 + \alpha_s^2 - 4\alpha_s k(k+2)}}{2k^2},$$

$$k \equiv \alpha_3 + \frac{1}{2}\alpha_4 > 1,$$

FIGURE 8.3

$\hat{\alpha}_s$, the value of α_s for which the normal tension n on the slanted faces in a body of given shear tenseness experiences a double root.

regarded as an equation for $\hat{\alpha}_s$ as a function of $\alpha_3 + \frac{1}{2}\alpha_4$. Then there are two, one, or no zeros of n according as $\alpha_s \geq \hat{\alpha}_s$. A graph of $\hat{\alpha}_s$, which satisfies the inequalities

$$(8.35) \quad 2k[\sqrt{k^2 - 1} + k + 2] \leq \hat{\alpha}_s < 4k(1 + k),$$

so that $\hat{\alpha}_s \lesssim 4k(1 + k)$ as $k \rightarrow \infty$, is shown in Figure 8.3. When $\alpha_s > \hat{\alpha}_s$, at least one of the two roots satisfies

$$(8.36) \quad \hat{\tau} < \sqrt{\frac{1+k}{k}},$$

while as $\alpha_s \rightarrow \infty$ we have for the two roots $\hat{\tau}_1$ and $\hat{\tau}_2$ the asymptotic forms

$$(8.37) \quad \begin{aligned} 2\tilde{\tau}_1 &\lesssim \sqrt{\alpha_s} - \sqrt{\alpha_s - 4k(k+1)}, \\ 2\tilde{\tau}_2 &\lesssim \sqrt{\alpha_s} + \sqrt{\alpha_s - 4k(k+1)}; \end{aligned}$$

also $\tilde{\tau}_1 \searrow 0$, $\tilde{\tau}_2 \nearrow \infty$, both being monotone functions of α_s .

Comparison of these results and further ones of a like kind with those in the paragraph preceding and with (8.23) shows that for *hard or neutral bodies*, n is

always a pressure if $\alpha_3 + \frac{1}{2}\alpha_4 \leq 0$; when $0 < \alpha_3 + \frac{1}{2}\alpha_4 < 1$, n changes from pressure to tension at the angle given by the single root (8.24), this angle being restricted to the range (8.33); when $\alpha_3 + \frac{1}{2}\alpha_4 = 1$, n remains always a tension if $\alpha_s \geq 6$ but changes from pressure to tension if $\alpha_s < 6$; when $\alpha_3 + \frac{1}{2}\alpha_4 > 1$, n remains always a tension if $\alpha_s < \hat{\alpha}_s$, where α_s is determined by (8.34) and satisfies (8.35); if $\alpha_3 + \frac{1}{2}\alpha_4 > 1$ and $\alpha_s = \hat{\alpha}_s$, n remains a tension except at the single root of (8.24), where it vanishes, having previously passed through a maximum and being thereafter to in-

FIGURE 8.4

Level curves on the surface $n = 0$, showing the angle(s) $\hat{\theta}$ at which the normal tension n on the slanted faces of a body of given shear stiffness α_s and shear tenseness $\alpha_3 + \frac{1}{2}\alpha_4$ changes sign.

crease again; while, finally, if $\alpha_3 + \frac{1}{2}\alpha_4 > 1$ and $\alpha_s > \hat{\alpha}_s$, n begins as a tension, experiences a maximum, changes to a pressure, experiences a maximum, and changes to tension again, the changes of sign occurring at the roots of (8.24).

In Figure 8.4 appears the surface $n = 0$, showing the value(s) of the angle $\hat{\theta}$ as a function of α_s and $\alpha_3 + \frac{1}{2}\alpha_4$.

Our results show that for a body of grade one measurement of s_{zz} and s_{yy} at one angle of shear of any magnitude combined with measurement of s_{xy} at

one moderately large angle of shear will serve to determine α_3 , $\alpha_3 + \frac{1}{2}\alpha_4$, and α_s , respectively. Hence follows the value of α_1 :

$$(8.38) \quad \alpha_1 = \frac{\alpha_s + 2\alpha_3(1 + 3\alpha_3) - 2(1 + \alpha_3)(2\alpha_3 + \alpha_4) - (2\alpha_3 + \alpha_4)^2}{2[1 + \alpha_3 + (2\alpha_3 + \alpha_4)]}.$$

Measurements of these same quantities at other shear angles and measurements of such other quantities as n and s can then test the experimental self-consistency of the theory for simple shear in initially unstressed materials.

On the basis of the foregoing theorems we see that there are 42 different types of response which bodies of grade one may exhibit in shear, determined by the values of the parameters α_1 , α_3 , α_4 . This classification exhausts all compatible combinations of the following possibilities:

1. The body is soft, neutral, or hard. (These may be replaced, if we prefer, by the following mutually exclusive possibilities: s_{xy} has a theoretical maximum, s_{xy} is monotone and s is positive, s_{xy} is monotone and s changes sign.)
2. s_{zz} is positive, identically zero, or negative.
3. s_{yy} is positive, identically zero, or negative.
4. n experiences 0, 1, 2 coincident, or 2 distinct roots.

The stress system in a typical example of each of these 42 types is drawn accurately and to a common scale in Figure 8.5. The abscissa is θ , the angle of shear, allowed to vary from 0 to $\frac{1}{2}\pi$. The ticks on the ordinate scale indicate units.³ Types 3, 7, 10, 15, 19, 23, 27, 31, 35 differ qualitatively from their immediate predecessors only in that for them n changes sign. In types 37, 39, and 41 n experiences a double root; in types 38, 40, and 42 n experiences two changes of sign.

The first twelve types are soft bodies. For all cases drawn, $\alpha_s = -1$. Thus all twelve have shear stresses s_{xy} and s , the classical objects of measurement, in common, and in all cases $\theta_s = \text{Arctan } \frac{1}{2}\pi$, $s_s = 1$. A small part of the theoretical curve for s_{xy} past the ultimate shear angle is shown dashed. A glance at these remarkable curves shows how much nearer to experience they are than might have been expected from the bare formulae (8.6). When the shear angle θ is taken as independent variable, as it is here, the at first thought rather disconcerting sinusoidal nature of the shear curve becomes imperceptible. Instead, the shear curve is very nearly a straight line through the entire range $0 \leq \theta < \theta_s$, while at $\theta = \theta_s$ it suddenly turns down! Thus, as far as shear stress is concerned, the soft bodies exhibit very nearly the classical linear behavior right up to yield in shear. We must add two qualifications: first, the tangent line at $\theta = 0$, which is the prediction of classical elasticity, is not so good an approximation to the hypo-elastic result as would be a line of slightly lesser slope; second, the range of approximate linearity is adjustable through the value of $|\alpha_s|$, in the cases

³ *I.e.*, units of s_{ij} . In ordinary measures, each tick therefore denotes the end of an interval of length 2μ .

FIGURE 8.5, Types 1-6

The stress systems in typical soft bodies subject to simple shearing. (In all cases, $\alpha_s = -1$, so all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 7-12

The stress systems in typical soft bodies subject to simple shearing. (In all cases, $\alpha_s = -1$, so all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 13-18

The stress systems in typical neutral bodies subject to simple shearing. (Since $\alpha_s = 0$, all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 19-24

The stress systems in typical neutral bodies subject to simple shearing. (Since $\alpha_s = 0$, all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 25-30

The stress systems in typical hard bodies subject to simple shearing. (In all cases, $\alpha_s = 1$, so all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 31-36

The stress systems in typical hard bodies subject to simple shearing. (In all cases, $\alpha_s = 1$, so all the curves for s_{xy} and s are alike.)

FIGURE 8.5, Types 37-42

The stress systems in typical hard bodies subject to simple shearing. (In these cases n experiences a minimum. In all examples drawn, $\alpha_3 + \frac{1}{2}\alpha_4 = \frac{3}{2}$, so that $\frac{1}{2}\alpha_s = 7.227$. For the three cases shown at the left, $\alpha_s = \alpha_s$; for the three shown at the right, $\alpha_s = 8$.)

drawn here chosen small enough ($|\alpha_s| = 1$) to make this range strikingly large, greater than 45° of shear.

The twelve different cases of soft bodies show the great variety of non-linear normal stresses which are compatible with a single, virtually classical linear shear stress. An experimenter guided in his expectation by the classical theory might fail to distinguish these different possibilities and might observe only the linearly increasing shear stress and sudden yield.

Types 13–24 are neutral bodies, the body of grade zero being Type 13. The curves of the shear stresses s_{xy} and s , of course, are the same for all neutral bodies but different from one another. The shear s_{xy} on the shear planes is virtually linear for the first 45° of shear, after which the body stiffens markedly. The shear stress s on the slanted faces rather quickly falls away from s_{xy} . The possible behaviors of the normal stresses are again extremely various.

Types 25–36 are hard bodies. As far as these twelve types are concerned, the distinction between neutral and hard bodies when θ instead of τ is used as argument is rather one of quantity than of quality, except in respect to s , which now changes sign and becomes negatively infinite in all cases. The hard bodies stiffen somewhat sooner than the neutral ones. In the cases illustrated, $\alpha_s = 1$, and hence the curves of s_{xy} and s are common to all. For about the first 30° of shear these two curves are virtually identical and virtually linear.

Types 37–42 are those in which the normal tension n on the slanted faces experiences two roots, whether coincident or distinct. In all cases drawn, $\alpha_3 + \frac{1}{2}\alpha_4 = \frac{3}{2}$, so that the curves of s_{xx} and s_{yy} are common to all six. The values of $\frac{1}{2}\alpha_s$ are $\frac{1}{2}\hat{\alpha}_s = 7.227$ for cases 37, 39, 41; 8 for cases 38, 40, and 42. Hence the curves of s_{xy} and s for the former three differ but little from those for the latter. There are two possibilities for n , three for t_{zz} . As was proved above, the physically rather unexpected behavior of n in these six types cannot occur except for very hard bodies in which $\alpha_s > 6$. The rapid stiffening in shear which must accompany so great a value of α_s is hardly more to be expected than the minimum of n , so that it is not likely that these last six types correspond to any physical material.

9. Simple shear of incompressible bodies. To obtain the equations for an incompressible body we may set $g_3 = g_7 = 0$ and replace $\mathbf{s}$ by $\mathbf{s} - p\mathbf{1}$. To do this in (8.3) for a body of grade one is equivalent to making the same substitution in the solutions (8.6). From (8.6)₄ it follows that $s_{zz} - p = S_{zz} - P$; since p is arbitrary, we may take $p = s_{zz}$. For an initially unstressed block, the remaining equations (8.6) then become

$$(9.1) \quad \begin{aligned} s_{xy} &= \frac{1}{\sqrt{|\alpha_s|}} \frac{\sinh}{\sin} \sqrt{|\alpha_s|} \tau, \\ \frac{1}{2}\alpha_s &\equiv \alpha_1 + \alpha_4 + \alpha_1\alpha_4 + \frac{1}{2}\alpha_4^2, \\ \frac{s_{yy} - s_{zz}}{\alpha_4} &= \frac{s_{xx} - s_{zz}}{2 + \alpha_4} = \frac{s_{xx} - s_{yy}}{2} = \frac{2}{|\alpha_s|} \frac{\sinh^2}{\sin^2} \frac{1}{2} \sqrt{|\alpha_s|} \tau \end{aligned}$$

if $\alpha_s \neq 0$, while if $\alpha_s = 0$

$$(9.2) \quad s_{xy} = \tau, \quad \frac{s_{yy} - s_{zz}}{\alpha_4} = \dots = \frac{1}{2}\tau^2.$$

It is now possible to assign to any one of the normal stresses s_{xx} , s_{yy} , s_{zz} , n an arbitrary value; the rest are then determined. For example, we may take $s_{zz} = 0$.

Results much like those for shear of compressible bodies but simpler in detail may be worked out. We do not pause to obtain them. It is interesting to note that (8.22) remains valid for incompressible bodies, expressing in fact a truly universal property of bodies of grades zero and one.

10. Simple extension of compressible bodies. Elsewhere¹ I have analyzed the special case of the linear velocity field (6.1) in which

$$(10.1) \quad \mathbf{a} = \mathbf{d} = k \begin{vmatrix} -\sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mathbf{b} = 0,$$

$$k = k(t) = \frac{k_0}{1 + k_0 t}, \quad \sigma = \sigma(t) = \sigma_0 \frac{1 + k_0 t}{1 - k_0 \sigma_0 t}.$$

Like the motions treated in §§7–9, this one also is devoid of acceleration, so that its theory in hypo-elasticity falls within the case of §6. For the stress field the form

$$(10.2) \quad \mathbf{s} = \begin{vmatrix} -Q & 0 & 0 \\ 0 & -Q & 0 \\ 0 & 0 & T \end{vmatrix}$$

will be general enough for our purpose, since in order to produce the motion (10.1) a stress system initially satisfying (10.2) must continue ever to do so. Then

$$(10.3) \quad \begin{aligned} I &= T - 2Q, & II &= Q(Q - 2T), & III &= Q^2 T, \\ M &= k(T + 2\sigma Q), & N &= k(T^2 - 2\sigma Q^2), \end{aligned}$$

and the constitutive equations (2.3) reduce to

¹ In the note cited in footnote 1 of §1.

$$\begin{aligned}
(10.4) \quad \frac{1}{k} \frac{dT}{dt} &= (1 + 2\sigma)T + (1 - 2\sigma)(g_0 + g_2 T + g_6 T^2) \\
&\quad + g_1 + g_4 T + g_8 T^2 + (T + 2\sigma Q)(g_3 + g_6 T + g_9 T^2) \\
&\quad + (T^2 - 2\sigma Q^2)(g_7 + g_{10} T + g_{11} T^2) \equiv f(T, \sigma, Q), \\
\frac{1}{k} \frac{dQ}{dt} &= -Q - (1 - 2\sigma)(g_0 - g_2 Q + g_5 Q^2) \\
&\quad + \sigma(g_1 - g_4 Q + g_8 Q^2) - (T + 2\sigma Q)(g_3 - g_6 H + g_9 Q^2) \\
&\quad - (T^2 - 2\sigma Q^2)(g_7 - g_{10} Q + g_{11} Q^2).
\end{aligned}$$

Now in order for a solution of this system to satisfy the dynamical equations also it is necessary and sufficient that $k(t)$ and $\sigma(t)$ be given by (10.1)₅ and (10.1)₇. If we choose to overlook this fact (*i.e.* to neglect the effect of the inertia of the material), we may leave k and σ arbitrary and put $Q = 0$ for all t . Then $II = III = 0$, so that the g_r become functions of T only, and (10.4)₂ yields

$$(10.5) \quad \sigma = \frac{g_0 + g_8 T + g_7 T^2}{2g_0 + g_1} \equiv \hat{\sigma}(T),$$

while, since $k = \frac{\dot{z}}{\log z}$, from (10.4)₁ follows

$$(10.6) \quad \log \frac{z}{Z} = \int_{T_0}^T \frac{d\xi}{f(\xi, \hat{\sigma}(\xi), 0)},$$

where $\hat{\sigma}(T)$ is defined by (10.5) and $f(T, \sigma, Q)$ is defined by (10.4)₁. According to this quasi-static treatment, then, the extension is a definite function of the tensile stress T , which in turn, however it may be applied in time, determines the lateral contraction ratio through the relation (10.5): $\sigma(t) = \hat{\sigma}(T(t))$. For example, for a body of grade zero or one (10.5) yields

$$(10.7) \quad \hat{\sigma}(T) = \frac{\nu + (1 - 2\nu)(\alpha_0 + \alpha_3)T}{1 + (1 - 2\nu)(2\alpha_0 + \alpha_1)T}$$

where ν is the usual Poisson modulus. For a body of grade zero this result reduces to the classical $\hat{\sigma}(T) = \nu$. For a body of grade one (10.7) implies that the contraction ratio σ for very large tensile or compressive stress has the value $(\alpha_0 + \alpha_3)/(2\alpha_0 + \alpha_1)$, independently of its value ν for small stresses.

Attractive as are the foregoing products of the quasi-static approach, they cannot be correct.² For in order that the motion assumed be dynamically possible it is not only sufficient but also necessary that (10.1)₅ and (10.1)₇ hold; then (10.5) assumes the form $\hat{\sigma}(T) = \sigma(t)$, yielding a determinate tension $T = T_1(t)$.

² The error in this class of results was first noted by ERICKSEN, *Math. rev.* **15**, 178 (1954).

But (10.6) also yields a determinate tension $T = T_2(t)$, and there is no reason to expect that these two determinations will agree. In fact, we may write (10.6) and the time derivative of (10.5) in the respective forms,

$$\begin{aligned} \frac{dT}{d\tau} &= \frac{1}{1+\tau} \left[\frac{1+2\sigma_0+\sigma_0\tau}{1-\sigma_0\tau} T + \frac{1-2\sigma_0-3\sigma_0\tau}{1-\sigma_0\tau} A(T) + B(T) \right], \\ \frac{dT}{d\tau} &= \frac{\sigma_0(1+\sigma_0)C(T)}{(1-\sigma_0\tau)^2 D(T)}, \\ \tau &\equiv k_0 t, \quad A(T) \equiv g_0 + g_2 T + g_5 T^2, \\ (10.8) \quad B(T) &\equiv g_1 + (g_3 + g_4)T + (g_6 + g_7 + g_8)T^2 + (g_9 + g_{10})T^3 + g_{11}T^4, \\ C(T) &\equiv (2g_0 + g_1)^2, \\ D(T) &\equiv (2g_0 + g_1)(g_3 + g_3' T + 2g_7 T + g_7' T^2) \\ &\quad - (2g_0' + g_1')(g_3 T + g_7 T^2) + g_1 g_0' - g_0 g_1', \end{aligned}$$

where a prime indicates differentiation with respect to T . In order that these two formulae for $dT/d\tau$ be alike it is necessary and sufficient that

$$\begin{aligned} D\{(1+2\sigma_0)T + (1-2\sigma_0)A + B\} &= C\sigma_0(1+\sigma_0), \\ (10.9) \quad \sigma_0 D\{T - 3A - B - (1+2\sigma_0)T - (1-2\sigma_0)A - B\} &= C\sigma_0(1+\sigma_0), \\ \sigma_0 D\{T - 3A - B\} &= 0. \end{aligned}$$

The solution $D \equiv 0$, $C \equiv 0$ is excluded by the requirement (4.1) for consistency with the classical linear theory in first approximation. Hence (10.9)₃ yields $T = 3A + B$. Then (10.9)₂ and (10.9)₁ become inconsistent unless $\sigma_0 = -1$, the hydrostatic case, treated in §7. Thus we have shown that *in the isotropic hypo-elastic body biaxial extension can never be produced by uniaxial stress*.

This fact does not contradict the consistency of the hypo-elastic theory with classical elasticity theory when the stress is small. Indeed, if at $t = 0$ we put $Q = 0$, $T = 0$ in (10.4)₂ we get

$$(10.10) \quad \sigma_0 = \frac{1}{2 + \frac{g_1(0, 0, 0)}{g_0(0, 0, 0)}} = \frac{1}{2 + \frac{2\mu}{\lambda}} = \nu$$

as necessary and sufficient that $dQ/dt = 0$ when $t = 0$. For arbitrary σ_0 the hypo-elastic theory yields solutions, and in general both Q and T will rise at about the same rates; if we choose $\sigma_0 = \nu$, however, Q rises much more slowly at first. That a cross-stress Q is necessary to produce simple extension, the exact solutions in other parts of non-linear continuum mechanics might well have led us to expect. If this cross-stress is wanting, we may expect that the body when

pulled or pressed will tend to bulge or thin, as is the case in experience but not in the classical linear theory of elasticity.

The quasi-static solutions according to a somewhat less general theory have been discussed in my earlier work.³ In a preliminary note⁴ I have obtained the rigorous solution for bodies of grade zero, and I have shown how the quasi-static solution in this case becomes completely wrong as soon as the strain is large enough to require abandoning the classical linear theory. In the remainder of this section I shall therefore limit attention to the rigorous solutions.

The dimensionless variable $\tau \equiv k_0 t$ may be taken as a measure of the *extension*, since in fact $\tau = (z/Z) - 1$, where Z is the initial co-ordinate of the particle which at time t occupies the place z . In a motion of stretching with contraction ($k_0 > 0$, $\sigma_0 > 0$) the maximum extension is $\tau = \tau_m = 1/\sigma_0$, corresponding to volume zero. We consider here only the case $\sigma_0 = \nu$. Recall that the singularities of linear motions are determined by their initial velocity gradients (§6). In the present example, we must therefore expect yield or collapse at or before the extensional strain $1/\nu$, where ν is the ordinary Poisson modulus of the linear theory of elasticity. That this is so follows from the degeneracy of the motion itself at $\tau = 1/\nu$.

This *limit of extension*, at which (change in length)/(initial length) = $1/\nu$, should not be confused with tensile strength. It is appropriate only to the special class of extensile motions considered; in order to produce these special accelerationless motions, increasingly large stresses must be applied. If in a test we try to produce such a motion, we must expect failure before, perhaps long before, the strain reaches this very large value $1/\nu$. In this respect the limit of extension is like the ultimate hydrostatic tension or pressure (§7), but different from the shear strength (§8). It is not a limitation of the theory I am proposing, but rather the result of my inability to solve its equations in a more realistic case, that only accelerationless motions are included in the present study of extension. Nevertheless this study retains its value in that the solutions given are *exact*, so that it is at least reasonable to try to test them by experiment.

If we put $\sigma_0 = \nu$ and $\omega = \log(1 + \tau)$, the system (10.4) assumes the form:

$$\begin{aligned} \frac{dT}{d\omega} &= (\sigma A_{\mathbf{I}\mathbf{J}} + B_{\mathbf{I}\mathbf{J}}) T^{\mathbf{I}} Q^{\mathbf{J}}, \\ \frac{dQ}{d\omega} &= (\sigma C_{\mathbf{I}\mathbf{J}} + D_{\mathbf{I}\mathbf{J}}) T^{\mathbf{I}} Q^{\mathbf{J}}, \\ \sigma &= \frac{\nu}{(1 + \nu)e^{-\omega} - \nu}, \end{aligned} \tag{10.11}$$

³ See §58 of the *Mechanical foundations* as rewritten in the additions.

⁴ The second work cited in footnote 1 of §1.

where the dimensionless coefficients A , B , C , D are material constants. For bodies of grades zero, one, and two, these coefficients are expressed in terms of the defining constants (4.4) through

$$\begin{aligned}
 \frac{1}{2}A_{00} &= -\frac{\nu}{1-2\nu}, & B_{00} &= \frac{1-\nu}{1-2\nu}, \\
 C_{00} &= \frac{1}{1-2\nu}, & D_{00} &= -\frac{\nu}{1-2\nu}, \\
 \\
 \frac{1}{2}A_{10} &= 1 - \alpha_1 - \alpha_2, & B_{10} &= 1 + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
 C_{10} &= 2\alpha_0 + \alpha_1, & D_{10} &= -\alpha_0 - \alpha_3, \\
 \\
 \frac{1}{2}A_{01} &= 2\alpha_0 + \alpha_3, & B_{01} &= -2\alpha_0 - 2\alpha_1, \\
 C_{01} &= -4\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4, & D_{01} &= -1 + 2\alpha_0 + \alpha_2, \\
 \\
 \frac{1}{2}A_{20} &= -\beta_0 - \beta_2 - \beta_5, & B_{20} &= \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 \\
 & & & + \beta_5 + \beta_6 + \beta_7 + \beta_8, \\
 C_{20} &= 2\beta_0 + \beta_1, & D_{20} &= -\beta_0 - \beta_3 - \beta_7, \\
 \\
 \frac{1}{2}A_{11} &= 4\beta_0 + 2\beta'_0 + 2\beta_2 + \beta_3 + \beta_6, & B_{11} &= -4\beta_0 - 2\beta'_0 - 4\beta_1 - 2\beta'_1 \\
 & & & - 2\beta_2 - 2\beta_3 - 2\beta_4, \\
 C_{11} &= -8\beta_0 - 4\beta'_0 - 4\beta_1 - 2\beta'_1 - 2\beta_2 - 2\beta_3 - \beta_4, & D_{11} &= 4\beta_0 + 2\beta'_0 + \beta_2 + 2\beta_3 + \beta_6, \\
 \\
 \frac{1}{2}A_{02} &= -4\beta_0 - \beta'_0 - 2\beta_3 - \beta_7, & B_{02} &= 4\beta_0 + \beta'_0 + 4\beta_1 + \beta'_1, \\
 C_{02} &= 8\beta_0 + 2\beta'_0 + 4\beta_1 + \beta'_1 + 4\beta_2 + 4\beta_3 + 2\beta_4 + 2\beta_5 + 2\beta_6 + 2\beta_7 + \beta_8, & D_{02} &= -4\beta_0 - \beta'_0 - 2\beta_2 - \beta_5.
 \end{aligned}
 \tag{10.12}$$

These coefficients are not mere arbitrary constants. The 4 for which the sum of the suffices is 0 are all determined by ν . The 8 whose suffices add up to one are restricted by the 3 conditions

$$\begin{aligned}
 A_{01} - B_{01} - 2C_{10} + 2D_{10} &= 0, \\
 \frac{1}{2}A_{01} - \frac{1}{2}A_{10} + D_{10} - D_{01} &= 0, \\
 B_{10} - B_{01} + C_{01} - C_{10} + D_{01} - D_{10} &= 0.
 \end{aligned}
 \tag{10.13}$$

The 12 whose suffices add up to 2 are restricted by 1 linear relation. Thus the hypo-elastic theory of extension, while far more general than the elastic theory, does not result in mere arbitrariness but predicts a definite class of stress-strain relations.

For the case of a body of grade one, it can be shown that for each choice of the coefficients A, B, C, D it is possible either to integrate (10.11) by quadratures or to reduce it to a single linear differential equation. This is rather a matter of analysis than mechanics and will not be presented here,⁵ where we shall rest content with the general solution for the body of grade zero, which has been given and discussed in my preliminary work on this subject.

11. The kinematics and dynamics of torsion of a circular cylinder. A class of motions representing torsion of a circular cylinder is given in cylindrical co-ordinates r, θ, z by the contravariant velocity field

$$(11.1) \quad v^r = r\mathfrak{R}(t), \quad v^\theta = z\mathfrak{A}(t), \quad v^z = z\mathfrak{Z}(t),$$

where $\mathfrak{A} \equiv 0$. For the physical components of $\mathbf{d}$ we have

$$(11.2) \quad \mathbf{d} = \begin{vmatrix} \mathfrak{R} & 0 & 0 \\ \cdot & \mathfrak{R} & \frac{1}{2}r\mathfrak{A} \\ \cdot & \cdot & \mathfrak{Z} \end{vmatrix}.$$

Hence $\delta = 2\mathfrak{R} + \mathfrak{Z}$, so that the motion is isochoric if and only if $2\mathfrak{R} + \mathfrak{Z} = 0$, while more generally

$$(11.3) \quad \rho = \rho_0 \mathfrak{S}(t), \quad \mathfrak{S}(t) \equiv \exp \left[- \int (2\mathfrak{R} + \mathfrak{Z}) dt \right].$$

The acceleration is given by

$$(11.4) \quad v^r = r[\dot{\mathfrak{R}} + \mathfrak{R}^2 - z^2\mathfrak{A}^2], \quad v^\theta \equiv z[\dot{\mathfrak{A}} + \mathfrak{A}(2\mathfrak{R} + \mathfrak{Z})], \quad v^z = z[\dot{\mathfrak{Z}} + \mathfrak{Z}^2].$$

The physical components $\widehat{ij}$ of $\tilde{\mathbf{s}}$ are easily obtained by (5.1):

$$(11.5) \quad \begin{aligned} \|\widehat{ij}\| &= \mathbf{D} \|\widehat{ij}\| + (2\mathfrak{R} + \mathfrak{Z}) \|\widehat{ij}\| \\ &- \begin{vmatrix} 2\mathfrak{R} & 2\mathfrak{R} & \mathfrak{R} + \mathfrak{Z} \\ \cdot & 2\mathfrak{R} & \mathfrak{R} + \mathfrak{Z} \\ \cdot & \cdot & 2\mathfrak{Z} \end{vmatrix} \cdot \widehat{ij} - \begin{vmatrix} 0 & r\mathfrak{A}r_z & 0 \\ \cdot & 2r\mathfrak{A}\theta_z & r\mathfrak{A}z_z \\ \cdot & \cdot & 0 \end{vmatrix}, \end{aligned}$$

where the quantities $\widehat{ij}$ are the physical components of $\mathbf{s}$ and where

$$(11.6) \quad \mathbf{D} \equiv \partial_t + z\mathfrak{A}\partial_\theta + z\mathfrak{Z}\partial_z + r\mathfrak{R}\partial_r.$$

⁵ See a forthcoming work, "The solution of linear differential equations by quadratures," by B. BERNSTEIN & C. TRUESDELL.

Later we shall wish to use the general solution of $\mathbf{D}u = 0$ in the case $\partial_\theta = 0$, namely

$$(11.7) \quad u = f(re^{-\int \mathfrak{R} dt}, ze^{-\int \mathfrak{Z} dt}).$$

If we try to fit to the motion (11.1) a stress system in which

$$(11.8) \quad \partial_\theta \widehat{ij} = 0, \quad \widehat{rz} = 0,$$

the dynamical equations (2.7) in the case when $f^i = 0$ become

$$(11.9) \quad \begin{aligned} 2\mu[\partial_r \widehat{rr} + (\widehat{rr} - \widehat{\theta\theta})/r] &= \rho_0 \mathfrak{S}r[\mathfrak{R} + \mathfrak{R}^2 - z^2 \mathfrak{A}^2], \\ 2\mu[\partial_r \widehat{r\theta} + \partial_z \widehat{\theta z} + 2\widehat{r\theta}/r] &= \rho_0 \mathfrak{S}rz[\mathfrak{A} + \mathfrak{A}(2\mathfrak{R} + \mathfrak{Z})], \\ 2\mu[\partial_z \widehat{zz}] &= \rho_0 \mathfrak{S}z[\mathfrak{Z} + \mathfrak{Z}^2]. \end{aligned}$$

Since all the German letters stand for functions of time only, they may be regarded as constants in the integration, which yields the general solution

$$(11.10) \quad \begin{aligned} \widehat{\theta z} &= -\partial_r(r^2 \int \widehat{r\theta} dz)/r^2 + \tfrac{1}{2}\bar{\rho} rz^2 \mathfrak{S}[\mathfrak{A} + \mathfrak{A}(2\mathfrak{R} + \mathfrak{Z})] + A, \\ \widehat{\theta\theta} &= \partial_r(r\widehat{rr}) + \bar{\rho} r^2 \mathfrak{S}[z^2 \mathfrak{A}^2 - \mathfrak{R} - \mathfrak{R}^2], \\ \widehat{zz} &= \tfrac{1}{2}\bar{\rho} z^2 \mathfrak{S}[\mathfrak{Z} + \mathfrak{Z}^2] + B, \quad \bar{\rho} \equiv \rho_0/2\mu \end{aligned}$$

where $\widehat{r\theta}$ and $\widehat{rr}$ are arbitrary functions of r, x, t and where A and B are arbitrary functions of r, t . The terms containing $\bar{\rho}$ arise from the inertia of the material.

This general solution will be useful in the attempt to treat torsional motions in any field of the mechanics of continua.

12. Remarks on torsion of compressible hypo-elastic bodies of grade one.

In order that $\widehat{rz} = 0$ in the motion (11.1) subject to the stress-system (11.6) and the restrictions (4.3), from (2.3) it is plain that we must have

$$(12.1) \quad \alpha_4 \mathfrak{A}\widehat{r\theta} = 0.$$

Since $\mathfrak{A} \neq 0$, we must have either $\alpha_4 = 0$ or $\widehat{r\theta} = 0$. From (2.3) follows an equation for $\widehat{r\theta}$:

$$(12.2) \quad \mathbf{D}\widehat{r\theta} = \widehat{r\theta}[(2\alpha_2 + \alpha_4)\mathfrak{R} + (-1 + \alpha_2)\mathfrak{Z}],$$

where the operator $\mathbf{D}$ is defined by (11.6) with $\partial_\theta = 0$. From (11.7) it is easy to see that the general solution of this equation is

$$(12.3) \quad \widehat{r\theta} = e^{\int \mathfrak{U} dt} f(ze^{-\int \mathfrak{Z} dt}, re^{-\int \mathfrak{R} dt}),$$

where

$$(12.4) \quad \mathfrak{U}(t) \equiv (2\alpha_2 + \alpha_4)\mathfrak{R} + (-1 + \alpha_2)\mathfrak{Z}.$$

Hence if $\mathfrak{R} \equiv 0$ the only solutions satisfying the boundary condition $\widehat{r\theta} = 0$ when $r = a$, a being the radius of the cylinder, is $\widehat{r\theta} \equiv 0$. Thus we have shown that $\widehat{r\theta} \equiv 0$, except possibly in the case $\alpha_4 = \mathfrak{R} \equiv 0$. It is possible that there are certain solutions in which $\mathfrak{R} \equiv 0$, $\widehat{r\theta} \equiv 0$, but we shall not pursue them here. Henceforth we put $\widehat{r\theta} \equiv 0$.

From (2.3) we see that the remaining constitutive equations are

$$\begin{aligned}
 \mathbf{D}\widehat{\theta z} &= \tfrac{1}{2}r\mathfrak{A}[1 + \alpha_1\widehat{r\overline{r}} + (\alpha_1 + \tfrac{1}{2}\alpha_4)\widehat{\theta\overline{\theta}} + (2 + \alpha_1 + \tfrac{1}{2}\alpha_4)\widehat{z\overline{z}}] \\
 &\quad + \widehat{\theta z}[(-1 + 2\alpha_2 + \tfrac{1}{2}\alpha_4)\mathfrak{R} + (\alpha_2 + \tfrac{1}{2}\alpha_4)\mathfrak{Z}], \\
 \mathbf{D}\widehat{r\overline{r}} &= \frac{\nu}{1 - 2\nu} (2\mathfrak{R} + \mathfrak{Z}) + \mathfrak{R} \\
 &\quad + \widehat{r\overline{r}}[(-1 + \alpha_0 + \alpha_2)\mathfrak{Z} + (2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)\mathfrak{R}] \\
 &\quad + \widehat{\theta\overline{\theta}}[(2\alpha_0 + \alpha_1 + \alpha_3)\mathfrak{R} + \alpha_0\mathfrak{Z}] \\
 &\quad + \widehat{z\overline{z}}[(2\alpha_0 + \alpha_1)\mathfrak{R} + (\alpha_0 + \alpha_3)\mathfrak{Z}] \\
 (12.5) \quad &\quad + \widehat{\theta z}[\alpha_3\mathfrak{R}], \\
 \mathbf{D}\widehat{\theta\overline{\theta}} &= \frac{\nu}{1 - 2\nu} (2\mathfrak{R} + \mathfrak{Z}) + \mathfrak{R} \\
 &\quad + \widehat{\theta\overline{\theta}}[(2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)\mathfrak{R} + (1 + \alpha_0 + \alpha_2)\mathfrak{Z}] \\
 &\quad + \widehat{z\overline{z}}[(2\alpha_0 + \alpha_1)\mathfrak{R} + (\alpha_0 + \alpha_3)\mathfrak{Z}] \\
 &\quad + \widehat{r\overline{r}}[(2\alpha_0 + \alpha_1 + \alpha_3)\mathfrak{R} + \alpha_0\mathfrak{Z}] \\
 &\quad + \widehat{\theta z}[(2 + \alpha_3)r\mathfrak{A}], \\
 \mathbf{D}\widehat{z\overline{z}} &= \frac{\nu}{1 - 2\nu} (2\mathfrak{R} + \mathfrak{Z}) + \mathfrak{Z} \\
 &\quad + \widehat{z\overline{z}}[(-2 + 2\alpha_0 + 2\alpha_2)\mathfrak{R} + (1 + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\mathfrak{Z}] \\
 &\quad + \widehat{r\overline{r}}[(2\alpha_0 + \alpha_3)\mathfrak{R} + (\alpha_0 + \alpha_1)\mathfrak{Z}] \\
 &\quad + \widehat{\theta\overline{\theta}}[(2\alpha_0 + \alpha_3)\mathfrak{R} + (\alpha_0 + \alpha_1)\mathfrak{Z}] \\
 &\quad + \widehat{\theta z}[(\alpha_3 + \tfrac{1}{2}\alpha_4)r\mathfrak{A}].
 \end{aligned}$$

I have obtained solutions of these equations but only for special values of the α 's; there is no appropriate solution when their values are assigned arbitrarily. Here I reproduce only the proof that for the case of a compressible body of grade zero no such solution exists. For a body of grade zero, the integral of (12.5)₂ is

$$(12.6) \quad \widehat{r\overline{r}} = e^{-\int \mathfrak{Z} dt} \left[F(ze^{-\int \mathfrak{Z} dt}, re^{-\int \mathfrak{R} dt}) + \int \mathfrak{C} e^{\int \mathfrak{Z} dt} dt \right],$$

where

$$(12.7) \quad \mathfrak{G}(t) = \frac{\nu}{1-2\nu} (2\mathfrak{R} + \mathfrak{J}) + \mathfrak{R}.$$

The condition $\widehat{rr} = 0$ when $r = a$ cannot be satisfied unless $\mathfrak{R} = \mathfrak{J} = 0$ and $\widehat{rr} \equiv 0$. From (11.10)₁ we get $\widehat{\theta z} = \frac{1}{2}\bar{\rho} r z^2 \mathfrak{A} + A$. But (12.5)₃ becomes $\partial_t \widehat{\theta\theta} = 2r\mathfrak{A} \widehat{\theta z}$, whence by (11.10)₂ we get

$$(12.8) \quad 2\bar{\rho} r^2 z^2 \mathfrak{A} \mathfrak{A} = 2r\mathfrak{A}[\frac{1}{2}\bar{\rho} r z^2 \mathfrak{A} + A],$$

impossible unless $\mathfrak{A} = 0$, $A = 0$, a case quickly seen to imply a state of rest. Thus we cannot find a simple solution of the type proposed for the problem of torsion of a circular cylinder of compressible material.

13. Torsion of an incompressible hypo-elastic body of grade zero. Disappointed but by no means discouraged by the failure in the last section, we turn now to the incompressible case, where the extra freedom resulting from the constraint of incompressibility gives better chances of a lucky guess at an appropriate solution.

The constitutive equations for the shearing stresses are unaltered; hence we take $\widehat{r\theta} \equiv 0$ again, and the basic solution (11.10) reduces to the form

$$(13.1) \quad \begin{aligned} \widehat{\theta z} &= \frac{1}{2}r[\bar{\rho} z^2 \mathfrak{A} + A], \\ \widehat{\theta\theta} &= \partial_r(r\widehat{rr}) + \bar{\rho} r^2[z^2 \mathfrak{A}^2 - \mathfrak{R} - \mathfrak{R}^2], \\ \widehat{zz} &= \bar{\rho} z^2[-\mathfrak{R} + 2\mathfrak{R}^2] + B. \end{aligned}$$

If we put these formulae into (12.5)₁ and equate to zero the coefficients of the resulting vanishing linear function of z^2 , we get

$$(13.2) \quad \begin{aligned} \frac{1}{2}\ddot{\mathfrak{A}} - \mathfrak{R}\mathfrak{A} + (\mathfrak{R} - 2\mathfrak{R}^2)\mathfrak{A} &= 0, \\ (\partial_t + r\mathfrak{R}\partial_r + \mathfrak{R})A &= \mathfrak{A}(1 + 2B). \end{aligned}$$

At this point there are several possibilities. For example, one might try to find a solution in which the only resultant load on the plane faces $z = \text{const.}$ is a torsional couple. Instead, guided by the analogous problem in finite elasticity theory, we choose the easiest case and find the stress $\widehat{zz}$ necessary in order to keep the ends of the bar plane and unextended. Hence we put $\mathfrak{R} = 0$ in (13.2), obtaining

$$(13.3) \quad \mathfrak{A} = 0, \quad B = \frac{1}{2} \left[\frac{\partial_t A}{\mathfrak{A}} - 1 \right].$$

There are still three constitutive equations to solve. By using (2.5) to modify (12.5), we see that in the present case these reduce to

$$(13.4) \quad \partial_t(\widehat{rr} + p) = 0, \quad \partial_t(\widehat{zz} + p) = 0, \quad \partial_t(\widehat{\theta\theta} + p) = 2r\mathfrak{A}\widehat{\theta z}.$$

From the first two and (13.1)₃ we get

$$(13.5) \quad \widehat{rr} = \widehat{zz} + f(r, z) = B(r, t) + f(r, z).$$

In order to satisfy the condition $\widehat{rr} = 0$ when $r = a$, this result must reduce to

$$(13.6) \quad \widehat{rr} = \widehat{zz} = B(r, t), \quad B(a, t) = 0.$$

We can now eliminate $\widehat{\theta\theta}$ between (13.4)₃ and (13.1)₂, making use of (13.6) and (13.1)₁. The result is

$$(13.7) \quad \partial_r \partial_t B - r\mathfrak{A}A = -\rho z^2 r\mathfrak{A}\mathfrak{A}.$$

Since the left-hand side is independent of z , each side must vanish separately. The vanishing of the right-hand side yields

$$(13.8) \quad \mathfrak{A} = \text{const.},$$

a result consistent with (13.3)₁. On the left-hand side, we eliminate B by (13.3) and use (13.8). There results the definitive partial differential equation of our problem:

$$(13.9) \quad \partial_r \partial_t^2 A = 2r\mathfrak{A}^2 A.$$

The constant $\mathfrak{A}$, the rate of twist, remains arbitrary. With A a suitable solution of (13.9), we then obtain from (13.1), (13.3), and (13.6) the following system of stresses:

$$(13.10) \quad \begin{aligned} \widehat{z} &= \tfrac{1}{2}rA, \\ \widehat{rr} = \widehat{zz} &= B = \tfrac{1}{2} \left[\frac{\partial_t A}{\mathfrak{A}} - 1 \right], \\ \widehat{\theta\theta} &= \tfrac{1}{2} \frac{r}{\mathfrak{A}} \partial_r \partial_t A + \tfrac{1}{2} \left[\frac{\partial_t A}{\mathfrak{A}} - 1 \right] + \bar{\rho} r^2 z^2 \mathfrak{A}^2. \end{aligned}$$

From the last equation it appears that the class of solutions we have obtained is not general enough to represent the torsion of a cylinder initially without hoop stress. However, all other physically natural boundary conditions and initial conditions can be satisfied. For example, if $\widehat{rr} = \widehat{zz} = 0$ initially, while $\widehat{rr} = 0$ when $r = a$, we need only satisfy

$$(13.11) \quad \partial_t A = \mathfrak{A} \quad \text{when} \quad \begin{cases} t = 0, 0 \leq r \leq a, \\ r = a, t \geq 0. \end{cases}$$

From (13.11) and (13.9) follows not only $A = 0$ but also $\partial_r \partial_t A = 0$ when

$t = 0, 0 \leq r \leq a$. Hence (13.10)₅ yields

$$(13.12) \quad \widehat{\theta\theta} = \bar{\rho} r^2 z^2 \mathfrak{A}^2 \quad \text{when } t = 0, 0 \leq r \leq a.$$

Thus the initial hoop stress is small (in a precise and obvious sense) near the axis and near the plane of no twist. If the inertia of the material is relatively small¹ the initial hoop stress is small throughout the cylinder.

The solution from classical elasticity theory is $A = \mathfrak{A}t$. This solution satisfies all our boundary conditions and initial conditions, but it does not satisfy the partial differential equation (13.9). This is only to be expected. In the sense of the present theory, the equations of the classical theory are only approximate. In particular, our equation (13.9) results directly from consideration of the *inertia of the material*. Had we chosen to neglect inertia, we could not have derived (13.8); we should indeed have gotten (13.9), but with $\mathfrak{A}$ an arbitrary function of time. Thus in the torsion problem as in the theory of extension, a quasi-static "approximation" in the equations would have permitted a gross over-determination of the solution.

We return to the solution of (13.9) subject to the conditions (13.11) and (13.12). It is easy to see that this solution is given by the complex integral²

$$(13.13) \quad A(r, t) = \frac{\mathfrak{A}}{2\pi i} \oint \frac{1}{\xi^2} \exp \left[\xi t + \frac{(r^2 - a^2)\mathfrak{A}^2}{\xi^2} \right] d\xi,$$

where the contour of integration is the unit circle. Hence

$$(13.14) \quad \begin{aligned} A &= \sum_{p=0}^{\infty} \frac{(\mathfrak{A}t)^{2p+1} (r^2 - a^2)^p}{(2p+1)! p!}, \\ &= \mathfrak{A}t {}_0F_2\left(1, \frac{3}{2}; \frac{1}{2}\mathfrak{A}^2 t^2 (r^2 - a^2)\right). \end{aligned}$$

The classical solution $A \equiv \mathfrak{A}t$ is the leading term in this series.

The resultant torque per unit height is given by

$$(13.15) \quad \begin{aligned} L &= 2\mu \int_0^a \widehat{r z} \cdot 2\pi r dr = 2\pi\mu \int_0^a r^3 A dr, \\ &= \pi\mu a^3 \sum_{p=0}^{\infty} \frac{(a\mathfrak{A}t)^{2p+1}}{(2p+1)! (p+2)!}. \end{aligned}$$

¹ In a well defined sense which is obvious from (13.10) and (13.12). Note that this statement is of a type entirely different from those current regarding inertial effects. We do not, as is usual, drop small inertial terms in a differential equation and then hope that the resulting error is small. Rather, we note *precisely* the effect of inertia on the *solution* to a particular problem.

² This solution is unique within an appropriate class, as may be seen, for example, by deriving it by BATEMAN's method ("Laplace transforms").

The resultant normal traction on the plane ends is given by

$$\begin{aligned}
 (13.16) \quad N &= 4\pi\mu \int_0^a \widehat{z}zr \, dr = 2\pi\mu \int_0^a \left[\frac{\partial_t A}{\mathfrak{A}} - 1 \right] r \, dr, \\
 &= \pi\mu a^2 \sum_{p=1}^{\infty} \frac{(-1)^p (a\mathfrak{A}t)^{2p}}{(2p)! (p+1)!},
 \end{aligned}$$

expressed in the usual units. For small twists, N is a pressure proportional to the square of the angle of twist. L and N may be expressed in terms of ${}_0F_2$ if desired.

Thus our solution for the torsional motion of an incompressible cylinder is complete. All series written down are convergent for all t and very rapidly convergent for small values of $a\mathfrak{A}t$. This solution shows several points of similarity with that for static torsion in the classical theory of finite strain.

14. Concluding remarks. This paper presents the beginnings of a new theory of the large deformation of materials which are perfectly elastic in small strain. Confined to simple problems in torsion and homogeneous stress, the present study is obviously far less than a patient unfolding of the potentials of this new theory. The rapid flow of fundamental research on continuum mechanics coming into print almost daily makes it better to publish what I have done now so that others need not waste their labor on the same ground. The only reason that I do not label this paper "Part I" is that extensive commitments make unlikely the effort which Part II would require until after a lapse of years, in which the face of continuum mechanics may be changed and these simple thoughts quite overridden. Thus, incomplete as are the foregoing notes, I venture now to make certain comments.

While at first sight the results might suggest that the hypo-elastic theory rests upon stress-strain relations, that is not the case. Such relations as (7.6) or (8.4), for example, can result only from definite assumptions regarding the *motion*. If a body of grade zero is compressed or expanded in such a way that the particles suffer non-zero accelerations, the pressure cannot satisfy (7.6); if in shearing a body of grade zero we move the shear planes at non-uniform speed the shear stress cannot satisfy (8.4). A relation between stress and strain is thus the *outcome*, not the assumption, of our theory, and the form of this relation depends on the manner in which the stress is applied in time.

However, the time enters the various results in a dimensionless product $k_0 t$, where k_0 is an *arbitrary* constant of dimension $\mathbf{T}^{-1}$. Indeed, our constitutive equations (1.6) contain no modulus of the dimension of time, and thus such a conclusion must be perfectly general.¹ While in each case the relation between stress and strain will depend on the *manner* in which the motion takes place in time, it cannot depend on the *speed*. In particular, to expect that in a motion

¹ Since this statement was written, Dr. NOLL has proved a theorem substantiating it. See §14 of his paper cited in §2 above.

which is slow enough it will be permissible to neglect the inertia of the material can hardly lead to anything else than serious error. In the case of shear, for example, the correctness of our results as solutions of the basic equations has nothing to do with the speed at which the motion occurs: at any uniform speed, however large or small, they are solutions, and at any non-uniform speed they are not. As we have seen, neglecting the inertia of the material allows gross over-determination of the problem. Whether the true solution or another more or less in accord with it is chanced upon in the exercise of this illicit license depends only on "physical intuition" or other psychological factors.

At the risk of boring mathematicians and engineers equally but for the opposite reason, I recall that every mathematician knows that it is possible for a small term in a differential equation to have a large effect on the solutions, while engineers or specialists in perturbation processes are inclined to regard the mathematicians' examples as "pathological" and to rely on personal taste to give them confidence that the terms which they choose to neglect are harmless. Inertia is a case in point. There is a school of plasticity which neglects the inertia of the material, often without even mentioning it, and in dynamic elasticity it is customary to neglect the non-linear part of the inertia. Now in §13 we have seen an example of the danger in such neglect. *However small are the inertial terms in the differential equations there studied, they alone yield the definitive condition to be satisfied.* If they are neglected, an infinite number of solutions becomes possible. The correct one is included, but so also are infinitely many incorrect ones.² In the single exact solution, there is a term depending on the inertia. This term in many applications will be small and can justly be neglected. The result of neglecting this term in the *solution* is in general entirely different from the result of neglecting the inertial terms in the *differential equations*.

A remarkable product of our study has been the discovery of phenomena indicative of yield or rupture. I say "discovery," since our theory is put intentionally in terms of analytic differential equations: its postulates are entirely free from statements regarding yield or rupture or any sort of discontinuity, so that such phenomena, if they are to appear at all, must be predicted in the course of solution of particular problems. The results are of two types. In some cases, indefinitely increasing stress is required to produce a certain finite strain; this outcome I regard as indicating that a somewhat lesser stress must rupture the body and thus deprive the theory of its continued validity. In other cases, as a given stress is approached the strain increases indefinitely; this I regard as yield. In still others, increased stress theoretically reduces the strain; this is more difficult to interpret, but certainly the theory cannot continue to describe actual materials past the maximum of the strain. Tentatively I regard this last effect as a prediction that if rupture does not occur at a lower strain, then the

² This situation recalls the problem of stress in the classical theory, where neglect of the conditions of compatibility gave rise to various "approximate" theories about half a century ago.

material must yield in a sense distinct from that described above: the material suffers a structural change resulting in a different mechanical response and the consequent need for a different theory at greater strains.

It is not the least in importance to recall that our theory really embodies no new ideas of mechanical behavior beyond those of classical linear elasticity. Rather, it simply extends the classical view of elastic response by formulating it in terms of rates. That these ideas in some cases predict yield without assuming a yield condition indicates their strength and potential relevance.

Finally, some of the unexplored possibilities of this theory should be mentioned. First, there is the more liberal field for initial stresses, a field which was noted but not investigated. In particular, it is possible that there are solutions showing stresses which cannot be relaxed without destroying the continuity of the body, thus leading to a continuous model for phenomena usually treated by the theory of dislocations. Second, this paper contains no study of the effect of reversing a deformation and no attempt to fit together one solution for loading and another for unloading. Third, energy relations and the thermodynamics of this theory remain untouched.