*On the Continuity of the Solid and Fluid States**

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A. Introduction. MAXWELL [4] expressed the opinion that the phenomenon of viscosity should be explained by the combination of an effect of elasticity with an effect of relaxation. Treating only a one-dimensional model, he proposed a relation of the form

$$(A.1) \quad \frac{ds}{dt} = \mu \frac{da}{dt} - \frac{s}{\tau},$$

where s = stress, a = “strain”, μ = elastic modulus, τ = relaxation time. For $\tau \rightarrow \infty$, (A.1) becomes

$$(A.2) \quad \frac{ds}{dt} = \mu \frac{da}{dt},$$

which is the differentiated form of the stress-strain relation

$$(A.3) \quad s = \mu a$$

characterizing an elastic solid. If we put $\eta = \mu\tau$, where η is a viscosity constant, we can write (A.1) in the form

$$(A.4) \quad s = \eta \frac{da}{dt} - \tau \frac{ds}{dt}.$$

For $\tau \rightarrow 0$, η fixed, this becomes a relation between stress and “rate of deformation”

$$(A.5) \quad s = \eta \frac{da}{dt}$$

which characterizes a viscous fluid.

The equation (A.1) describes a class of materials including both solids and fluids. Materials of this general type shall be called *hygrosteric*. The first rigorous three-dimensional generalization of MAXWELL’s equation (A.1) was given by ZAREMBA [9]. He introduced a special case of what I call below the linear fluent body (see §12). He considered his theory to be a generalization of the classical theory of viscous fluids rather than a general theory representing the continuity of the fluid and solid states, and he did not establish any connection with the theory of finite elasticity. There is a large literature on this subject, generally called “visco-elasticity”. Since all this literature, to the best of my knowledge, concerns only various linearizations and “approximations” and in generality does not go beyond ZAREMBA’s work¹, it will not be considered here.

Following the scheme suggested by (A.1)–(A.5), in this paper I give a general three-dimensional realization of MAXWELL’s one-dimensional model. This is not so easy as might seem at first glance. The equations (A.1)–(A.5) cannot be ex-

¹ It seems that ZAREMBA’s papers remained almost completely unnoticed until mentioned in [8].

tended mechanically to three dimensions. The constitutive equations which I propose below are the most general which can fall under MAXWELL's scheme²; they are of course in general non-linear. Their form is restricted by certain invariance requirements, which become relevant only in the case of three dimensions. The investigation of these requirements is essential; it will be a major concern below. Furthermore, which of the various kinematical quantities defined in §2 are to be taken as corresponding to the "strain" a and the "rate of deformation" da/dt is not clear without some thought.

The general type of hygrosteric materials corresponding to (A.1) will be defined in chapter II. In chapter III the connection with elastic solids and with viscous fluids will be established. (A.2) corresponds to the theory of hypo-elastic bodies (see §14), which has been introduced very recently by C. TRUESDELL [7] as the final step in a series of earlier attempts (see [8], Ch. IV D, p. 217). (A.3) corresponds to the classical theory of finite strain (see §15, §16). The relation of this latter theory to the theory of hypo-elastic bodies is not obvious, but it is shown in §15b that indeed the theory of hypo-elastic materials includes the theory of finite strain in the case of isotropic bodies. (A.5) corresponds to the general theories of Reiner-Rivlin fluids and Stokesian fluids (see [8], Chs. V B, V C, p. 232). Their relation to the general class of hygrosteric materials is discussed in §13. The connections of the various materials are shown on a diagram in §17.

Chapter I contains a number of definitions for later use and a summary of the general principles of continuum mechanics. In §4b I state a general invariance requirement, called the *principle of isotropy of space*, which any constitutive equation defining an ideal material must satisfy. CAUCHY's flux principle (see [8], p. 604) follows as a special case from this general principle. In many other special cases, various authors have given intuitive reasons to justify the form of the constitutive equations they proposed. These reasonings can be replaced by rigorous deductions from the principle of isotropy of space. Such deductions are given here for all constitutive equations considered. It turns out that in the case of hyperelastic bodies (elastic bodies according to GREEN's definition (see [8], p. 173), the principle of isotropy of space is equivalent to the moment of momentum principle (see §16b, corollary to theorem I). In general, however, the principle of isotropy of space is an independent axiom: it must be satisfied for all materials, isotropic or not. Isotropy of space and isotropy of materials are not clearly separated in most of the literature. In §4c a general definition of an isotropic material is given. From this definition the form of the constitutive equations for isotropic elastic and hyperelastic bodies is derived in a rigorous way (see §15a, §16c). It should be pointed out that the clear distinction between material points (as elements of a measure space), their reference positions, and their initial positions (see §1) is essential for the understanding of both the definition of isotropic materials and the principle of isotropy of space.

² Of course, more general bodies can be defined by differential equations involving higher derivatives or by constitutive equations which are not differential equations at all.

It turns out that the hygrosteric materials and the fluids defined in §6 and §13, respectively, are automatically isotropic because their constitutive equations do not depend on the choice of the reference position (see §4). *E.g.*, a fluid with a defining equation

$$(A.6) \quad S = \mathfrak{G}(Q)$$

(see §5 and (2.9)) is necessarily isotropic. It follows from the principle of isotropy of space, and not from the isotropy of the material, that (A.6) must reduce to the special form

$$(A.7) \quad S = \mathfrak{G}(D)$$

where $\mathfrak{G}$ is a hemitropic function of D (see §Cf). An equation of the form (A.7), where $\mathfrak{G}$ is not hemitropic, is not admissible¹.

At the end of §5 is indicated how the theory of hygrosteric materials considered in this paper could be generalized to include anisotropic materials also. I believe that it is possible to show that such a more general theory would include anisotropic elastic bodies. However, considerable mathematical difficulties would have to be overcome before this generalization could be realized.

In chapter IV some exact solutions in special cases, mainly for incompressible linear fluent bodies, are obtained. It turns out that, if the motion is “slow” and if the material time constants are “small”, a linear fluent body behaves like a classical viscous fluid, as can be expected from §13. However, the results of §21 indicate that the deviation will be considerable when the rate of shear is large, which is often the case at the boundaries of a flow. It is interesting that, for one type of material, a steady Poiseuille flow is possible only up to a certain critical velocity depending on the value of a certain material time constant (see §22b β). It seems to me to be not impossible that this result has something to do with the occurrence of turbulence. ZAREMBA [9] also gave some special solutions for his special case of a linear fluent body. However, except in the case of pure stress relaxation (see §19b), he neglected just the most interesting terms, so that his solutions are actually valid only for classical viscous fluids.

Chapter IV can be read independently from chapters I–III.

In this paper no terms are ever neglected, and all solutions given in chapter IV are exact. All constitutive equations occurring in chapters II and III are valid for finite motions and not only for small displacements or small rotations. This does not mean that they necessarily represent accurately the behavior of real materials; but the least that can be expected from a mathematical theory is that it is consistent, exact and rigorous. Although I believe that this paper is a small step towards an axiomatic foundation of continuum mechanics, there is still much to be done in this direction. The final form of such an axiomatic founda-

¹ In the literature of hydrodynamics such equations have been proposed occasionally to represent anisotropic fluids. They can correspond to a physical situation, if at all, only approximately for motions with small rotation.

tion should include also thermodynamic principles, which are left out of consideration here.

Modern linear algebra (as presented for instance in [1]) is used as the mathematical tool in this work. In particular, points, vectors and linear transformations are used instead of coordinates, components and matrices (except, of course, for most of the special solutions). Not only do I believe that in this way the mathematical deductions become shorter and more elegant, but I think also that such a treatment is more intuitive.

B. Notation.

a. We give here a list which indicates how the various letters will be used:
 t : time or other real parameter.

P : material point (element of a measure space).

$x, y, X, Y, \dots$: points in three dimensional point space $\mathcal{E}$.

$f, g, \dots$: functions with values in $\mathcal{E}$.

Bold face Latin letters (*e.g.* $\mathbf{y}, \mathbf{Y}, \mathbf{s}, \mathbf{n}, \mathbf{v}, \dots$): vectors of the three dimensional vector space $\mathcal{V}$ connected with $\mathcal{E}$ or functions with values in $\mathcal{V}$.

Greek and Latin minuscules, except x, y, f, g : real numbers or scalar (*i.e.* real-valued) functions.

Latin majuscules, except P, X, Y (*e.g.* $A, B, C, I, R, Q, D, W, Z$): linear transformations of $\mathcal{V}$, or functions whose values are such transformations. The term “*tensor*” is used here as a synonym for “*linear transformation*”.

I : identity transformation.

$R, R^*, \dots$: rotations, *i.e.* tensors which belong to the group of proper orthogonal transformations.

We denote the 9-dimensional vector space of all tensors by $\mathfrak{I}$ and its 6-dimensional subspace of all symmetric tensors by $\mathfrak{S}$.

German majuscules (*e.g.* $\mathfrak{F}, \mathfrak{G}, \mathfrak{S}$): functions with range and values in $\mathfrak{I}$ or $\mathfrak{S}$.

Gothic majuscules: physical dimensions, *i.e.*

$$\underline{\mathbf{T}} = \text{time}, \quad \underline{\mathbf{L}} = \text{length}, \quad \mathbf{M} = \text{mass}, \quad \mathbf{I} = \text{no dimension.}$$

Greek majuscules (Φ, Ψ): transformations of a special nature.

Superposed dot: derivative with respect to t . A superposed bar is used as a parenthesis to indicate the extent of the symbol on which the dot operates (*e.g.* $\dot{\bar{R}}^{-1}$).

Superposed -1 : inverse function (mapping), *e.g.* $\bar{f}^{-1}$ is the inverse mapping of f .

b. We denote by $x - y$ the vector in $\mathcal{V}$ determined by the points x, y in $\mathcal{E}$ (see Figure 1). We denote by $x + \mathbf{y}$ the point in $\mathcal{E}$ determined by the point $x \in \mathcal{E}$ and the vector $\mathbf{y}$ in $\mathcal{V}$ (see Figure 2).

The value of the transformation A for the vector $\mathbf{y} \in \mathcal{V}$ is denoted by $A\mathbf{y}$. It is again a vector in $\mathcal{V}$.

For operations on tensors in $\mathfrak{I}$ we use the following notation:

$A \cdot B$: *product* of the two transformations A and B ,

A' : *transpose* of the transformation A ,
 A^{-1} : *inverse* of A ,
 $\text{tr } A$: *trace* of A ,
 $\det A$: *determinant* of A .

c. For the convenience of readers accustomed to *coordinate notation* we repeat some of the formulas in this notation. *E.g.*, the formula (1.3c) is the counterpart of (1.3) in coordinate notation. Our coordinate notation is essentially that introduced in [8], except that we write s^{ik} instead of t^{ik} , $c^{\alpha\beta}$ instead of $C^{\alpha\beta}$, and b^{ik} instead of $(c^{-1})^{ik}$. Generally we use the same letter for a tensor and its components, but we change from upper case to lower case. Formulas with Latin indices only are valid for arbitrary curvilinear coordinates. If Greek and Latin indices occur, Cartesian coordinates must be used.

In chapter IV we use the notation of [2].

FIGURE 1

FIGURE 2

C. Mathematical preliminaries. We list here some of the definitions and theorems on vectors and linear transformations which are frequently used in this paper.

a. *Polar decomposition.* Any tensor A with $\det A > 0$ has unique decompositions

$$(C.1) \quad A = RV = UR,$$

where R is a rotation and V , U are positive definite symmetric tensors. We have

$$(C.2) \quad U = RVR^{-1}, \quad V^2 = A^t A, \quad U^2 = AA^t.$$

b. *Propositions concerning the trace.* We have

$$(C.3) \quad \text{tr } A = \text{tr } A^t,$$

$$(C.4) \quad \text{tr } (A_1 A_2 \cdots A_n) = \text{tr } (A_{i+1} A_{i+2} \cdots A_n A_1 A_2 \cdots A_i),$$

$$i = 1, 2, \dots, n - 1.$$

The equation

$$(C.5) \quad \text{tr } (AZ) = 0$$

holds for all tensors Z if and only if $A = 0$. The equation

$$(C.6) \quad \text{tr}(AW) = 0$$

holds for all antisymmetric tensors W if and only if A is symmetric.

c. Differentiation of point functions. Let f be a function with range in $\mathcal{E}$ and values in $\mathcal{E}$ or $\mathcal{V}$ (see §Ba). Assume that the limit

$$(C.7) \quad \lim_{\delta \rightarrow 0} \frac{f(x + \delta \mathbf{y}) - f(x)}{\delta} = \nabla_x f(x)[\mathbf{y}]$$

exists for x in a region of $\mathcal{E}$ and all vectors $\mathbf{y} \in \mathcal{V}$ and that it is continuous in x and $\mathbf{y}$. Then we shall say that f is *continuously differentiable* in the region considered, and we shall call $\nabla_x f(x)[\mathbf{y}]$ the *derivative of f at x in the direction of $\mathbf{y}$* . The derivative is a vector in $\mathcal{V}$. It can be shown that $\nabla_x f(x)[\mathbf{y}]$ is linear and homogeneous in $\mathbf{y}$. Therefore, $\nabla_x f(x)$ is a linear transformation of $\mathcal{V}$, *i.e.* it is a tensor in $\mathfrak{J}$; we call it the *differential of f at x* . The notation of §Bb enables us to write

$$(C.8) \quad \nabla_x f(x)[\mathbf{y}] = (\nabla_x f(x))\mathbf{y}.$$

If we have an arbitrary curvilinear coordinate system in $\mathcal{E}$, then (C.7) corresponds to the covariant differentiation, the i -th contravariant component of the vector $\nabla_x f(x)[\mathbf{y}]$ being equal to

$$(C.8c) \quad \nabla_i f^k(x^j)y^i = f^k_{,i} y^i.$$

The definition (C.7) is valid also for functions which map $\mathcal{E}$ into any other vector space. It is valid in particular for scalar functions ϕ . In this case the differential is a linear functional $\nabla_x \phi(x)$, which maps $\mathcal{V}$ into the reals. Any such linear functional can be represented by an inner product, and therefore it can be identified with a vector in $\mathcal{V}$, which is called the *gradient of ϕ* :

$$(C.9) \quad \nabla_x \phi(x)[\mathbf{y}] = (\text{grad } \phi(x)) \cdot \mathbf{y}.$$

In a similar way, the divergence $\text{div } \mathbf{v}(x)$ of a vector function $\mathbf{v}$ and the divergence $\text{Div } T(x)$ of a tensor function $T(x)$ can be defined.

d. Differentiation of general functions. The definition (C.7) can be extended in an obvious way for functions which map any vector space into any other vector space. In particular, if we consider a function $\mathfrak{F}$ mapping the space $\mathfrak{J}$ (or $\mathfrak{S}$) into $\mathfrak{J}$ (or $\mathfrak{S}$), we have

$$(C.10) \quad \nabla_A \mathfrak{F}(A)[Z] = \lim_{\delta \rightarrow 0} \frac{\mathfrak{F}(A + \delta Z) - \mathfrak{F}(A)}{\delta}, \quad A, Z \in \mathfrak{J} \quad (\text{or } A, Z \in \mathfrak{S}).$$

$\nabla_A \mathfrak{F}(A)$ is a linear transformation of the space $\mathfrak{J}$ (or $\mathfrak{S}$) into $\mathfrak{J}$ (or $\mathfrak{S}$).

Let $\zeta(A)$ be a scalar function defined and continuously differentiable in a region of $\mathfrak{J}$ (or $\mathfrak{S}$). Then there exists a unique function $\zeta_A(A)$ with values in $\mathfrak{J}$ (or $\mathfrak{S}$) such that

$$(C.11) \quad \nabla_A \zeta(A)[Z] = \text{tr} (\zeta_A(A)Z), \quad A, Z \in \mathfrak{J} \quad (\text{or } A, Z \in \mathfrak{S}).$$

When using coordinates, we have

$$(C.11c) \quad \nabla_A \zeta(A)[Z] = \frac{\partial \zeta}{\partial a^{ik}} z^{ik}.$$

Thus $\zeta_A(A)$ corresponds to the transpose of the matrix $\| \partial \zeta / \partial a^{ik} \|$.

e. Differentiation formulas. The usual rules hold for the differentiation defined in **c** and **d**. In particular, for two functions $\mathfrak{F}(A)$ and $\mathfrak{G}(B)$ such that the range space of $\mathfrak{F}$ coincides with the value space of $\mathfrak{G}$, we have the *chain rule*:

$$(C.12) \quad \nabla_B \mathfrak{F}(\mathfrak{G}(B))[Z] = \nabla_A \mathfrak{F}(\mathfrak{G}(B))[\nabla_B \mathfrak{G}(B)[Z]].$$

If $\mathfrak{F}(A)$ and $\mathfrak{G}(A)$ are two functions mapping both $\mathfrak{V}_1$ into $\mathfrak{V}_2$ such that

$$(C.13) \quad \nabla_A \mathfrak{F}(A) = \nabla_A \mathfrak{G}(A) \quad \text{for all } A,$$

then we have

$$(C.14) \quad \mathfrak{F}(A) = \mathfrak{G}(A) + C,$$

where $C \in \mathfrak{V}_2$ is constant, *i.e.* independent of A .

For two functions $\mathfrak{F}(A)$ and $\mathfrak{G}(A)$ both mapping $\mathfrak{J}$ into itself, we have the *product rule*:

$$(C.15) \quad \nabla_A \{ \mathfrak{F}(A) \cdot \mathfrak{G}(A) \}[Z] = \nabla_A \mathfrak{F}(A)[Z] \cdot \mathfrak{G}(A) + \mathfrak{F}(A) \cdot \nabla_A \mathfrak{G}(A)[Z].$$

Special formulas. For A and Z in $\mathfrak{J}$ we have

$$(C.16) \quad \nabla_A(A)[Z] = Z, \quad \nabla_A(A^t)[Z] = Z^t,$$

$$(C.17) \quad \nabla_A(A^{-1})[Z] = -A^{-1}ZA^{-1},$$

$$(C.18) \quad \text{tr}_A(A) = I,$$

$$(C.19) \quad \det_A(A) = (\det A) A^{-1}.$$

f. Hemitropic functions. A function $\mathfrak{F}(A_1, \dots, A_n)$, $A_i \in \mathfrak{J}$ (or $A_i \in \mathfrak{S}$) with values in $\mathfrak{J}$ (or $\mathfrak{S}$) is called *hemitropic* if the functional equation

$$(C.20) \quad R\mathfrak{F}(A_1, \dots, A_n)R^{-1} = \mathfrak{F}(RA_1R^{-1}, \dots, RA_nR^{-1})$$

is satisfied for any rotation R .

A scalar function $\zeta(A_1, \dots, A_n)$, $A_i \in \mathfrak{J}$ (or $A_i \in \mathfrak{S}$) is called *hemitropic* if the tensor function ζI is hemitropic, *i.e.* if

$$(C.21) \quad \zeta(A_1, \dots, A_n) = \zeta(RAR^{-1}, \dots, RA_nR^{-1})$$

for any rotation R .

If a scalar hemitropic function ζ is defined for symmetric tensors ($A_i \in \mathfrak{S}$), then it can be expressed as a function of a complete set of simultaneous invariants of the A_i .

I. BASIC CONCEPTS OF CONTINUUM MECHANICS

1. Definition of a continuous medium. By a continuous medium we understand a set of points, called *material points* P , with a measure, called *mass* m . At each time t these points have positions

$$(1.1) \quad x = g(P, t)$$

in space and occupy a certain region. The mapping g is assumed to be one-to-one (impenetrability of matter). The mass m induces at each time t in space a measure which is assumed to be absolutely continuous. Therefore it has a *density*

$$\rho = \rho(x, t).$$

Let

$$(1.2) \quad X = g_0(P), \quad P = g_0^{-1}(X)$$

define a *possible position* of the body. *E.g.*, X may be chosen as the position of P at the time $t = 0$:

$$X = g_0(P) = g(P, 0),$$

but it is not necessary that the position $X = g_0(P)$ is ever occupied by P during the motion considered. If g_0 is fixed once and for all, we may identify P with $X = g_0(P)$ and write

$$(1.3) \quad x = f(X, t) = g(g_0^{-1}(X), t)$$

$$(1.3c) \quad x^i = f^i(X^\alpha, t)$$

as the equation describing the motion of the body.

Since g_0 and f are one-to-one, any function $h(P, t)$ may be expressed as a function either of X and t or of x and t :

$$(1.4) \quad \begin{aligned} h(P, t) &= h(g_0^{-1}(X), t) = h^*(X, t) \\ &= h(g^{-1}(x, t), t) = h^{**}(x, t) = h^*(f^{-1}(x, t), t). \end{aligned}$$

Although h^* and h^{**} are different functions, for simplicity we shall use the same symbol h and write

$$(1.5) \quad \begin{aligned} \dot{h} &= \frac{\partial}{\partial t} h^*(X, t), & \frac{\partial h}{\partial t} &= \frac{\partial}{\partial t} h^{**}(x, t), \\ \nabla_X h &= \nabla_X h^*(X, t), & \nabla_x h &= \nabla_x h^{**}(x, t). \end{aligned}$$

$\dot{h}$ and $\nabla_X h$ are called the *material derivatives* of h ; $\partial h/\partial t$ and $\nabla_x h$, the *spatial derivatives* of h .

We note that $\dot{h}$, $\partial h/\partial t$ and $\nabla_x h$, when expressed as functions of x and t , are independent of the choice of the reference position $X = g_0(P)$, but $\nabla_X h$ is not. The following relations are simple consequences of the chain rule (see (C.12)).

$$(1.6) \quad \dot{h} = \frac{\partial h}{\partial t} + (\nabla_x h) f_t \quad (f_t = \frac{\partial}{\partial t} f(X, t)),$$

$$(1.7) \quad \nabla_X h = (\nabla_x h) \cdot (\nabla_x f).$$

FIGURE 3

The density ρ is an example of such a function h . Not only scalar, but also vector or tensor functions may be examples of h . All functions occurring in our considerations are assumed to be smooth enough so that all derivatives which will be written down exist and are continuous. In particular, f and f^{-1} are supposed to be twice continuously differentiable.

2. Kinematics. a. Displacement tensor. Let X and $X + \delta Y$ (δ real) be the positions of two material points in the reference state $X = g_0(P)$, and let x and $x + y_\delta$ be their positions at time t (see Figure 3). By (1.3) we have

$$x = f(X, t), \quad x + y_\delta = f(X + \delta Y, t);$$

hence

$$y_\delta = f(X + \delta Y, t) - f(X, t).$$

For $\delta \rightarrow 0$ we get (see (C.7))

$$\mathbf{y} = \lim_{\delta \rightarrow 0} \frac{\mathbf{y}}{\delta} = \nabla_x f(X, t)[Y].$$

Hence the *displacement tensor*

$$(2.1) \quad A = \nabla_x f(X, t),$$

$$(2.1c) \quad a^i{}_\alpha = x^i{}_{,\alpha} = \nabla_\alpha f^i(X^\beta, t),$$

describes the change which the displacement from the reference state to the state at the time t induces in the relative position of any two points as these two points draw together. Since f is always one-to-one, it follows that $\det A$ never vanishes and therefore, by continuity, always has the same sign. We admit only reference positions such that

$$(2.2) \quad \det A > 0.$$

If the reference position is the position at time $t = 0$, we have

$$(2.3) \quad X = f(X, 0), \quad A_0 = \nabla_x f(X, 0) = I.$$

In this case (2.2) is automatically satisfied.

b. Deformation tensors. By means of the theorem on polar decomposition (see (C.1), (C.2)) we can decompose the displacement tensor A :

$$(2.4) \quad A = RV = UR, \quad U = RVR^{-1},$$

where R is a rotation and U and V are positive definite symmetric transformations. R corresponds to a local rotation and V and U correspond to local pure deformations. The equation (2.4) states that the displacement A can be produced either by first applying a pure deformation V and then a rotation R or by first applying a rotation R and then a pure deformation U . Instead of using U and V as measures of deformation, it is more convenient to use

$$(2.5) \quad C = V^2 = A^t A,$$

$$(2.5c) \quad c^{\alpha\beta} = x^{i,\alpha} x_i{}^{,\beta},$$

$$(2.6) \quad B = U^2 = AA^t = RCR^{-1},$$

$$(2.6c) \quad b^{ik} = x^i{}_{,\alpha} x^{k,\alpha},$$

since C and B can be expressed in terms of A without introducing square roots of linear transformations.

c. Velocity gradient. The velocity of a material point is given by

$$(2.7) \quad \mathbf{v}(x, t) = f_t(\bar{f}^{-1}(x, t), t),$$

$$(2.7c) \quad v^i(x^j, t) = \dot{x}^i = f_t^i(\bar{f}^{-1\beta}(x^j, t), t).$$

Thus by (1.6) we have for any function h

$$(2.8) \quad \dot{h} = \frac{\partial h}{\partial t} + (\nabla_x h) \cdot \mathbf{v},$$

$$(2.8c) \quad \dot{h} = \frac{\partial h}{\partial t} + h_{,i} v^i.$$

If we substitute $\mathbf{v}$ for h in (1.7), we see that the *velocity gradient*

$$(2.9) \quad Q = \nabla_x \mathbf{v},$$

$$(2.9c) \quad q_{ik} = v_{i,k},$$

is connected with the displacement tensor (2.1) by the equation

$$(2.10) \quad \dot{A} = QA,$$

$$(2.10c) \quad \dot{x}^i_{,\alpha} = v^i_{,k} x^k_{,\alpha}.$$

d. Rate of deformation, vorticity. Material differentiation of $A = RV$ with respect to time (see (2.4)) yields

$$(2.11) \quad \dot{A} = \dot{R}V + R\dot{V} = QA.$$

Differentiation of $A^t = VR^{-1}$ yields (see (C.17))

$$(2.12) \quad \dot{A}^t = \dot{V}R^{-1} + \overline{VR^{-1}} = \dot{V}R^{-1} - VR^{-1}RR^{-1} = A^t Q^t.$$

If we choose the initial position as reference, we have by (2.3)

$$A_0 = R_0 = V_0 = I.$$

(2.11) and (2.12) become for $t = 0$

$$Q_0 = \dot{V}_0 + R_0, \quad Q_0^t = \dot{V}_0 - \dot{R}_0;$$

hence

$$\dot{V}_0 = \tfrac{1}{2}(Q_0 + Q_0^t),$$

$$\dot{R}_0 = \tfrac{1}{2}(Q_0 - Q_0^t).$$

I.e. the symmetric part of Q_0 is the rate of deformation and the antisymmetric part the rate of rotation at the instant considered. At any instant t , we call

$$(2.13) \quad D = \tfrac{1}{2}(Q + Q^t),$$

$$(2.13c) \quad d_{ik} = \tfrac{1}{2}(v_{i,k} + v_{k,i}),$$

the *rate of deformation tensor* and

$$(2.14) \quad W = \frac{1}{2}(Q - Q'),$$

$$(2.14c) \quad w_{ik} = \frac{1}{2}(v_{i,k} - v_{k,i}),$$

the *vorticity tensor*. We can also start out with $A = UR$ and get

$$\dot{U}_0 = \frac{1}{2}(Q_0 + Q'_0) = \dot{V}_0.$$

Thus the instantaneous rate of the left deformation is equal to that of the right deformation.

3. Principles of mechanics. a. Stress tensor. The Euler-Cauchy stress principle postulates the existence of a stress vector

$$(3.1) \quad \mathbf{s} = \mathbf{s}(x, t, \mathbf{n}),$$

a force per unit area acting through a surface element with unit normal $\mathbf{n}$ at the point x and at time t such that the system of stress vectors acting on any closed surface is equipollent to the action of the material outside upon that inside, $\mathbf{n}$ being directed toward the outside. It can be deduced from the momentum principle that (3.1) is linear in $\mathbf{n}$; i.e., $\mathbf{s}$ can be written in the form

$$(3.2) \quad \mathbf{s} = S(x, t)\mathbf{n},$$

where S is a linear transformation, called the *stress tensor*.

b. Conservation of mass. This principle states that the mass of any part $\mathfrak{M}$ of the continuum is preserved in time. If $\mathfrak{M}$ occupies the volume $\mathfrak{V}_0$ at the reference state and $\mathfrak{V}_t$ at the time t , we get

$$\int_{\mathfrak{M}} dm = \int_{\mathfrak{V}_0} \rho_0(X) dX = \int_{\mathfrak{V}_t} \rho(x, t) dx,$$

where $\rho_0(X)$ is the density distribution of the reference state. Taking (1.3) as a transformation of variables for the last integral, we get

$$\int_{\mathfrak{V}_0} \rho_0(X) dX = \int_{\mathfrak{V}_0} \rho(x, t) \det(\nabla_x f) dX.$$

Since this must hold for all volumes $\mathfrak{V}_0$, we obtain the *density equation*

$$(3.3) \quad \rho = \rho_0(\det A)^{-1}.$$

Material time differentiation yields (see (C.11), (C.19))

$$\dot{\rho} \det A + \rho (\det A) \operatorname{tr} (A^{-1} \dot{A}) = 0.$$

Observing (2.2) and (2.10), since by (2.9) and (2.13)

$$(3.4) \quad \text{tr } Q = \text{tr } D = \text{div } \mathbf{v}$$

holds, we get the *continuity equation*

$$(3.5) \quad \dot{\rho} + \rho \text{ div } \mathbf{v} = 0,$$

$$(3.5c) \quad \dot{\rho} + \rho v^i_{,i} = 0.$$

c. Momentum principle. The principle of conservation of linear momentum yields the *dynamical equation*

$$(3.6) \quad \text{Div } S + \rho \mathbf{g} = \rho \dot{\mathbf{v}},$$

$$(3.6c) \quad s^{ik}_{,k} + \rho g^i = \rho \dot{v}^i,$$

where $\mathbf{g}$ is the extraneous force field (force per unit mass). Furthermore, if there is no extraneous torque, it follows from the moment of momentum principle that S is a *symmetric* transformation. We shall always assume this to be true.

4. Invariance requirements for constitutive equations. a. Definition. The equations (3.5) and (3.6) are not sufficient to determine the motions of continuous media. We must have in addition certain *constitutive equations* defining the particular ideal material which we wish to study. These equations will be functional relations (*e.g.* differential or integral equations) between the stress $S(x, t)$, the density $\rho(x, t)$ and the motion $f(x, t)$. While in many cases thermodynamic variables and principles have to be taken into account, in this paper we shall confine our attention to mechanics alone.

Any admissible constitutive equation must satisfy certain invariance requirements. These we shall now discuss.

b. Isotropy of Space. We wish to give a precise form to the statement that the physical space is homogeneous and isotropic. This means that in space there are no distinguished positions or directions. Therefore the constitutive equations of a material can involve only the relative positions of the material points and not their absolute position in space. Therefore, if for a given body in one configuration the constitutive equation is satisfied, and if we consider another configuration which is identical with the first except for position and orientation in space at each instant, then the constitutive equation must be satisfied also for the second configuration. The second configuration can be obtained from the first by a rigid displacement:

$$(4.1) \quad x' = y(t) + R(t)(x - y(t)),$$

where $y(t)$ is a point and $R(t)$ a rotation, both variable in time.

(4.1) actually represents a family of rigid displacements with time as a parameter.

Scalar functions of the material point X and the time t are not affected by a rigid displacement.

Let us fix the time t and consider a vector field $\mathbf{u}(x)$. It is carried by the rigid displacement (see Figure 4) into another vector field

$$(4.2) \quad \mathbf{u}'(x') = R\mathbf{u}(x).$$

Hence if a stress configuration at x (see Figure 5) is given by

$$(4.3) \quad \mathbf{s}(x) = S(x)\mathbf{n}(x),$$

FIGURE 4

FIGURE 5

then, by (4.2), the corresponding configuration after the rigid displacement is

$$(4.4) \quad \mathbf{s}'(x') = S'(x')\mathbf{n}'(x'), \quad \mathbf{s}'(x') = R\mathbf{s}(x), \quad \mathbf{n}'(x') = R\mathbf{n}(x).$$

Comparing (4.3) and (4.4) yields

$$S'(x', t) = R(t)S(x, t)R^{-1}(t).$$

A motion

$$x = f(X, t)$$

will be carried by the rigid displacements (4.1) into another motion

$$x' = f'(X, t) = y(t) + R(t)\{f(X, t) - y(t)\}.$$

Thus to any family of displacements (4.1) corresponds a transformation Φ of all quantities involved in the constitutive equation. This equation has to be invariant under any such Φ . Hence we can state the

Principle of isotropy of space. *For any point $y(t)$ and any rotation $R(t)$, the transformation $\Phi\{y(t), R(t)\}$ defined by*

$$\begin{aligned}
 x &\rightarrow x' = y(t) + R(t)[x - y(t)], \\
 f(X, t) &\rightarrow f'(X, t) = y(t) + R(t)\{f(X, t) - y(t)\}, \\
 S(x, t) &\rightarrow S'(x', t) = R(t)S(x, t)R^{-1}(t), \\
 t, X, \rho(X, t) &\quad \text{unchanged}
 \end{aligned}
 \tag{4.5}$$

leaves the constitutive equation invariant¹.

Generally it is not hard to see how the various functions derived from $f(X, t)$ and $S(x, t)$ transform under Φ . In the defining equations of these functions, we replace $x, f, S(x, t)$ by $x', f', S'(x', t)$ and express these according to (4.5) by $x, f, S(x, t), y(t)$ and $R(t)$. Applying simple differentiation rules, we thus obtain simple formulas for the transformed functions, distinguished from the original functions by accents. Following are some examples.

α . Displacement and deformation tensors. Applying (4.5) to (2.1), (2.5) and (2.6) yields

$$A' = \nabla_x f'(X, t) = R \nabla_x f(X, t) = RA, \tag{4.6}$$

$$C' = A'^t A' = A^t R^{-1} R A = A^t A = C, \tag{4.7}$$

$$B' = A' A'^t = R A A^t R^{-1} = R B R^{-1}. \tag{4.8}$$

We note that C is invariant under Φ .

β . Velocity gradient, rate of deformation, vorticity. Applying the transformation (4.5) to (2.7) and (2.9) yields

$$\begin{aligned}
 \mathbf{v}' &= f'_t(X, t) = \dot{y}(t) + \dot{R}(t)[f(X, t) - y(t)] + R(t)[f_t(X, t) - \dot{y}(t)] \\
 &= \dot{y} + \dot{R}(x - y) + R(\mathbf{v} - \dot{y});
 \end{aligned}
 \tag{4.9}$$

$$\begin{aligned}
 Q' &= \nabla_{x'} \mathbf{v}' = (\dot{R} + R \cdot \nabla_x v) R^{-1} \\
 &= (\dot{R} + RQ) R^{-1}.
 \end{aligned}
 \tag{4.10}$$

From

$$(\dot{R} R^{-1})^t = R \dot{R}^t = \overline{R \dot{R}^{-1}} = -R R^{-1} \dot{R} R^{-1} = -\dot{R} R^{-1} \tag{4.11}$$

¹ If the constitutive equation involves quantities other than those occurring in (4.5) (e.g. heat flux vector), then these have to be transformed according to (4.2).

(see (C.17)) follows that $\dot{R}R^{-1}$ is antisymmetric. Hence we get by (2.13) and (2.14)

$$\begin{aligned} D' &= RDR^{-1}, \\ W' &= RR^{-1} + RWR^{-1}. \end{aligned}$$

γ . *Material time derivative of the stress tensor.* By (4.5) we obtain

$$\dot{S}' = RSR^{-1} + R\dot{S}\bar{R}^{-1} + R\dot{S}R^{-1}.$$

Hence (see (C.17))

$$(4.12) \quad \dot{S}' = R\dot{S}R^{-1} + RSR^{-1} - RSR^{-1}\dot{R}R^{-1}.$$

Remarks. The distinction between the initial position and the reference position is essential here. The transformation Φ will in general alter the initial position, but the reference position remains unaffected by Φ . Thus, if the initial state and the reference state happen to coincide for the original motion $x = f(X, t)$, they will in general not do so for the transformed motion $x' = f'(X, t)$. Since $\dot{R}R^{-1}$ is antisymmetric, it follows from (4.10) that $\text{tr } Q' = \text{tr } Q$. Hence the continuity equation (3.5) is also invariant under transformations Φ . However, this is in general not true for the dynamical equation (3.6). This equation is invariant only for uniform extraneous force fields under linear uniform motions, *i.e.* under transformations of the form $\Phi\{y_0 + at, I\}$ (Galilean invariance). In general, if $f(X, t)$ and $S(x, t)$ are such that (3.6) is satisfied, then this will not be true for $f'(X, t)$ and $S'(x', t)$ with the same extraneous force field. An additional extraneous force field depending on $y(t)$ and $R(t)$ has to be applied to make $f'(X, t)$ and $S'(x', t)$ dynamically admissible. The additional field is equipollent to the centrifugal and Coriolis forces due to the rigid motion defined by the family of rigid displacements (4.1). The principle of isotropy of space ensures only that $f'(X, t)$ and $S'(x', t)$ are compatible with the nature of the material under the condition that $f(X, t)$ and $S(x, t)$ are.

This principle can be formulated also in the following way: In any system of reference, Galilean or not, the constitutive equations must be the same. The continuity equation is also the same in any system of reference. The dynamical equations, however, are only valid in a Galilean system. In a non-Galilean system additional terms appear which represent the centrifugal and Coriolis forces.¹

c. Homogeneous and isotropic materials. A material is *homogeneous* if it does not matter whether we refer its constitutive equation to the reference state (see (1.2))

$$(4.13) \quad X = g_0(P)$$

¹ It is just this failure of invariance of the dynamical equations that has led to the revision of classical mechanics and to the discovery of the general theory of relativity.

or if we refer it to any other reference state $\tilde{X} = \tilde{g}_0(P)$ obtained from (4.13) by translation

$$\tilde{X} = X + \mathbf{Y} = g_0(P) + \mathbf{Y} = \tilde{g}_0(P),$$

where $\mathbf{Y}$ is an arbitrary vector. By (1.1) and (1.3) we have

$$x = g(P, t) = g(\tilde{g}_0^{-1}(X), t) = f(X, t) = f(\tilde{X} - \mathbf{Y}, t) = g(\tilde{g}_0^{-1}(\tilde{X}), t) = \tilde{f}(\tilde{X}, t).$$

Thus we can state the

Definition. A body is called homogeneous with respect to a reference state $X = g_0(P)$ if and only if its constitutive equation is invariant under transformations of the form

$$\begin{aligned} X &\rightarrow \tilde{X} = X + \mathbf{Y} \\ (4.14) \quad f(X, t) &\rightarrow \tilde{f}(\tilde{X}, t) = f(\tilde{X} - \mathbf{Y}, t) \\ x, t, S(x, t) \quad \text{and} \quad \rho(x, t) &\text{ unchanged,} \end{aligned}$$

where $\mathbf{Y}$ is an arbitrary vector.

All derivatives of f are unchanged under such a transformation. For a constitutive equation which is a relation between X, x , the derivatives of f and ρ and S and its derivatives, the above invariance simply implies that this relation cannot involve X explicitly.

Many physical materials are said to be *isotropic*. This concept will now be put into precise mathematical form.

First we define isotropy for homogeneous materials only. We take a spherical part of a homogeneous and isotropic body at the position

$$X = g_0(P)$$

and rotate it by means of¹ R^{-1} (see Figure 6) about the center Y , so that afterwards it has the position

$$(4.15) \quad X^* = g_0^*(P) = Y + R^{-1}(X - Y) = Y + R^{-1}\{g_0(P) - Y\}.$$

According to the intuitive concept of isotropy this rotation has no effect on the physical properties of the body. Thus it should make no difference whether we take $g_0(P)$ or $g_0^*(P)$ as reference position. We now consider a motion (see (1.1))

$$x = g(P, t).$$

According to (1.3) and (4.15) we have for the position $g_0(P)$

$$x = g(P, t) = g(\tilde{g}_0^{-1}(X), t) = f(X, t) = f(Y + R(X^* - Y), t).$$

¹ We write R^{-1} instead of R for notational convenience.

For the position $g_0^*(P)$, we have

$$x = g(P, t) = g(\bar{g}_0^*(X^*), t) = f^*(X^*, t).$$

Hence

$$f^*(X^*, t) = f(Y + R(X^* - Y), t).$$

We can say that a homogeneous medium is isotropic if its constitutive equation remains invariant under replacement of X by X^* and $f(X, t)$ by $f^*(X^*, t)$. Functions of the actual position x and the time t are unaffected by this replacement. Thus we can state the

FIGURE 6

Definition. A homogeneous material is called isotropic with respect to its reference position $X = g_0(P)$ if and only if for any point Y and any rotation R , the transformation $\Psi(Y, R)$ defined by

$$X \rightarrow X^* = Y + R^{-1}(X - Y)$$

$$(4.16) \quad f(X, t) \rightarrow f^*(X^*, t) = f(Y + R(X^* - Y), t)$$

$$x, t, S(x, t) \quad \text{and} \quad \rho(x, t) \quad \text{unchanged}$$

leaves the constitutive equation invariant.¹

A material which is not necessarily homogeneous is called isotropic if the invariance under $\Psi(Y, R)$ holds for each point Y in the limit $X \rightarrow Y$.

We noted in §1 that derivatives of the form $\dot{h}$, $\partial h / \partial t$ and $\nabla_x h$ do not depend on the choice of the reference position. Hence they are invariant under Ψ . The

¹ For certain anisotropic materials (e.g. crystals) this invariance holds for a certain group of rotations but not for all rotations.

effect of Ψ on the displacement tensor and the deformation tensors can easily be obtained from (2.1), (2.5), and (2.6). We get by (4.16)

$$(4.17) \quad A^* = \nabla_{x^*} f^*(X^*, t) = \nabla_x f(X, t)R = AR,$$

$$(4.18) \quad C^* = A^{*t} A^* = R^{-1} A^t AR = R^{-1} CR,$$

$$(4.19) \quad B^* = A^* A^{*t} = ARR^{-1} A^t = B.$$

We note that B is invariant under Ψ .

Remarks. As for the principle of isotropy of space, here too the distinction between the reference position and the initial position is essential. In this case the reference state is transformed under Ψ , while the initial state remains unaffected. It should be pointed out that homogeneity and isotropy can be defined only with respect to a reference position. In general a body will be homogeneous or isotropic for some reference positions and not for others. *E.g.* an elastic body cannot be regarded as isotropic with respect to any stressed state.

The continuity equation (3.5) is obviously invariant under Ψ , but the density equation (3.3), from which (3.5) was derived, is invariant only if

$$\rho_0(X) = \rho_0 = \text{const.},$$

i.e. if the material has uniform density in the reference position.

We have to distinguish two types of constitutive equations:

α . For a given motion $f(X, t)$ the constitutive equation determines the stress $S(x, t)$ completely. In particular, if a body is at rest at the reference position, the stress for this position will be determined. The state of rest (as a special “motion”) and its corresponding stress are given by

$$(4.20) \quad x = f(X) = X, \quad S(x) = S(X).$$

By (4.16), the transformation Ψ transforms (4.20) into

$$x = f^*(X^*) = Y + R(X^* - Y), \quad S(x) = S(Y + R(X^* - Y)).$$

Since it does not matter whether we call the independent variable X^* or X , we can write

$$(4.21) \quad x' = f^*(X) = Y + R(X - Y), \quad S(x') = S(Y + R(X - Y)).$$

By (4.5), the transformation $\Phi\{Y, R\}$ transforms (4.20) into

$$(4.22) \quad x' = f'(X) = Y + R(X - Y) = f^*(X),$$

$$S'(x') = RS(x)R^{-1} = RS(X)R^{-1}.$$

Since the constitutive equations are always invariant under Φ (isotropy of space), it follows that $S'(x')$ is the stress determined by the “motion” $x' = f^*(X)$ (which is also a state of rest).

For isotropic bodies it follows from the definition given above and from (4.21), (4.22) that

$$\lim_{x \rightarrow Y} S'(x') = RS(Y)R^{-1} = \lim_{x \rightarrow Y} S(x') = S(Y).$$

By a simple theorem of linear algebra, this is possible for any rotation R only if

$$S(X) = p(X)I.$$

Thus, for the type of constitutive equation considered here, an isotropic body can be at rest at the reference position only if the stress is a pressure, a result which is obvious from the intuitive concept of isotropy.

If the body is also homogeneous, we obtain in a similar way from (4.14) that $p(X) = \text{constant}$, *i.e.* that we must have uniform pressure.

β. For a given motion $f(X, t)$, the constitutive equation determines the stress $S(x, t)$ only up to an arbitrary initial stress distribution $S(x, 0)$. Then the reference position is compatible with any stress distribution. Nevertheless, the constitutive equation may be invariant under Ψ .

The behavior of equations of the type α or β with respect to initial values is analogous to that of the density equation (3.3) and the continuity equation (3.5), respectively.

This shows that there are certain difficulties which can be overcome only by a rigorous axiomatic definition of the words "material" and "constitutive equation." However, such a definition will not be given in this paper. If we deal with an equation of type β invariant under Ψ , we still shall call the material homogeneous and isotropic, since the only anisotropy which may occur is due to the choice of the initial stress distribution and not to the inherent nature of the material.

II. A CLASS OF CONSTITUTIVE EQUATIONS

5. General remarks. We shall investigate constitutive equations which are partial differential equations. The most general such equation is of the form

$$(5.1) \quad \mathfrak{F}(t, x, X, \mathbf{v}, \dot{\mathbf{v}}, \dots, A, Q, \dot{Q}, \dots, S, \dot{S}, \dots, \rho, \dots) = 0,$$

$\mathfrak{F}$ being a function of a finite number of the following variables:

- α) the time t , the actual position x and a reference position X ;
- β) the various derivatives of $f(X, t)$, *e.g.* $\mathbf{v}, \dot{\mathbf{v}}, A, Q, \dot{Q}$ (see (2.1), (2.7), and (2.9));
- γ) the stress and its derivatives;
- δ) the density and its derivatives.

We can assume that no spatial time derivatives occur since these could be eliminated by (2.8). By the continuity equation (3.5), the material time derivatives of ρ can be expressed in terms of $\rho, Q, \dot{Q}, \ddot{Q}, \dots$. Hence we can assume that derivatives of ρ with respect to time do not occur in (5.1).

For physical relevance we should also assume a dependence on thermodynamic variables (*e.g.* temperature), but such a dependence will not be considered in this paper.

For an equation of the form (5.1) to be admissible, it must be invariant under transformations $\Phi\{y(t), R(t)\}$ according to the principle of isotropy of space. Taking $R(t) = I, y(t) = y_0 = \text{const.}$ yields that $\mathfrak{F}$ must be independent of x . By putting $R(t) = I, y(t) = y_0 + \mathbf{a}t$ ($\mathbf{a} = \text{const.}$), we can then conclude that $\mathfrak{F}$ must also be independent of $\mathbf{v}$. It can be easily shown by induction that the material time derivatives of $\mathbf{v}$ of any order cannot occur as variables of $\mathfrak{F}$. The further consequences of the principle of isotropy of space will not be discussed here in general.¹

If the mechanical properties of the material do not change in time, then $\mathfrak{F}$ must be independent of t . If the material is also homogeneous (see (4.14)), then $\mathfrak{F}$ will be independent of X . Thus (5.1) reduces to

$$(5.2) \quad \mathfrak{F}(A, Q, \dot{Q}, \dots, S, \dot{S}, \dots, \rho, \dots) = 0.$$

Simple examples of materials with an equation of the form (5.2) are elastic bodies, with

$$S = \mathfrak{G}(A),$$

and Reiner-Rivlin fluids, with

$$S = \mathfrak{G}(Q, \rho).$$

¹ Such a discussion has been given by RIVLIN & ERICKSEN ([6]) for isotropic materials with an equation of the form $S = \mathfrak{F}(A, Q, \dot{Q}, \dots, \overset{(n)}{Q})$.

In this paper we shall consider the case when $\mathfrak{F}$ involves only S , $\dot{S}$, Q , and ρ . It is clear from §4c that such a material is necessarily isotropic. An anisotropic material with similar properties would require a dependence on a rotation R determined by the decomposition $A = RV = UR$ (see (2.4)), A being the displacement with respect to some reference position. The rotation R would specify the orientation of the distinguished directions in the material at the time t relative to their orientation at some reference position. However, we shall study only certain homogeneous isotropic materials which shall now be defined.

6. Constitutive assumptions defining a type of hygrosteric materials. A body is said to be *at ease* if the time derivatives of all functions describing its state vanish. We assume:

α) For a given density ρ , the body can be at ease at a certain hydrostatic pressure $S = -pI$, where

$$(6.1) \quad p = p(\rho)$$

is a given function of ρ . The deviation from this pressure is called *extra stress*:

$$(6.2) \quad T = S + pI.$$

β) The constitutive equation has the form

$$(6.3) \quad \dot{T} = \mathfrak{F}(Q, T, \rho);$$

i.e. the material time derivative of the extra stress is assumed to be a function of the velocity gradient, the extra stress and the density. $\mathfrak{F}$ is assumed to be defined and continuously differentiable with respect to all its variables in a suitable neighborhood of $Q = 0$, $T = 0$, $\rho = \rho_0$.

It follows from α) that

$$(6.4) \quad \mathfrak{F}(0, 0, \rho) = 0.$$

Remarks. As we shall see in the following parts of this paper, the assumption (6.3) leads to a class of materials which includes materials with properties of a solid as well as materials with properties of a fluid. Materials belonging to such a class shall be called *hygrosteric*,¹ so that (6.3) represents a certain type of hygrosteric materials.

The idea that solids and fluids are only limit cases of a general type of material is due to MAXWELL [4]. However, he considered only the one-dimensional case, (A.1), of an equation of the form (6.3). Special cases of three-dimensional constitutive equations of the form (6.3) have been introduced by various authors. The first of these was NATANSON in 1903; his work was faulty and was corrected by ZAREMBA [9] (see footnote to §12 below).

¹ From *ὑγρός*, “fluid”, and *στερεός*, “solid”.

7. Invariance requirements. In order that an equation of the form (6.3) be admissible, it is necessary that it satisfy the principle of isotropy of space, *i.e.* that it be invariant under transformations Φ . Since $p(\rho)$ is invariant under Φ , we have

$$T' = S' + pI, \quad \dot{T}' = \dot{S}' + \dot{p}I.$$

Hence by applying Φ to (6.3) and by using (4.5), (4.10) and (4.12) we get

$$(7.1) \quad R\dot{T}R^{-1} + \dot{R}TR^{-1} - RTR^{-1}\dot{R}R^{-1} = \mathfrak{F}((\dot{R} + RQ)R^{-1}, RTR^{-1}, \rho).$$

It can easily be seen that the function $R(t)$ can be chosen so that at a particular time t we have $R(t) = I$ and $\dot{R}(t) =$ any antisymmetric transformation given *a priori*. Choosing R in (7.1) so that at the time t

$$R = I \quad \text{and} \quad \dot{R} = -W$$

hold, we obtain

$$\dot{T} + TW - WT = \mathfrak{F}(Q - W, T, \rho) = \mathfrak{F}(D, T, \rho).$$

From this result and (6.3) we get the following functional equation for $\mathfrak{F}$:

$$\mathfrak{F}(Q, T, \rho) + TW - WT = \mathfrak{F}(D, T, \rho).$$

Hence $\mathfrak{F}$ is completely determined for all tensors Q if it is given for all symmetric tensors D . Thus we may consider $\mathfrak{F}$ as a function of symmetric tensors D only and write the constitutive equation in the form

$$(7.2) \quad \dot{T} + TW - WT = \mathfrak{F}(D, T, \rho),$$

$$(7.2c) \quad \dot{t}_k^i + t^{ij} w_{jk} - w^{ij} t_{jk} = \mathfrak{f}_k^i(d^{lm}, t^{np}, \rho).$$

If we choose R in (7.1) to be arbitrary but independent of t and take $Q = D$ to be symmetric, we get by (6.3)

$$(7.3) \quad R\mathfrak{F}(D, T, \rho)R^{-1} = \mathfrak{F}(RDR^{-1}, RTR^{-1}, \rho);$$

i.e. $\mathfrak{F}$ is a hemitropic function of D and T (see (C.20)). According to a result of RIVLIN & ERICKSEN ([6] app. I.), $\mathfrak{F}$ then admits a representation of the form

$$(7.4) \quad \begin{aligned} \mathfrak{F}(D, T, \rho) = & a_1 I + a_2 D + a_3 D^2 + a_4 T + a_5 T^2 \\ & + a_6(TD + DT) + a_7(T^2 D + DT^2) + a_8(DT^2 + T^2 D), \end{aligned}$$

where $a_1, \dots, a_8$ are continuous scalar functions of ρ and of a complete set of simultaneous invariants of D and T , *e.g.*

$$(7.5) \quad \text{tr } T, \text{tr } T^2, \text{tr } T^3, \text{tr } D, \text{tr } D^2, \text{tr } D^3, \text{tr } (DT), \text{tr } (DT^2), \text{tr } (D^2 T).$$

From (6.4) follows

$$(7.6) \quad a_1 = 0 \quad \text{for} \quad T = D = 0, \quad \rho \text{ arbitrary.}$$

Conversely, given a constitutive equation of the form (7.2) with a hemitropic function $\mathfrak{F}$, it can readily be seen that (7.1) is satisfied, *i.e.* that such a constitutive equation is invariant under any Φ .

The equation (7.2), where (7.4) is substituted for $\mathfrak{F}(D, T, \rho)$, is the *final form of the constitutive equation for our class of hygrosteric materials*.

It should be noted that the laws of thermodynamics will impose further restrictions on the possible functional form of $\mathfrak{F}$ in (7.2). The investigation of these restrictions in general is rather difficult, and we shall not attempt it. In special cases we shall see what restrictions are to be made.

8. Initial value problems. Given a motion $f(X, t)$, the density function $\rho(X, t)$ is determined by the density equation (3.3). Then, for every X , the constitutive equation (7.2) becomes an ordinary first order differential equation for the stress S as a function of t . By the existence theorem for ordinary differential equations, S is uniquely determined for all times t ($0 \leq t \leq t^*$) if the initial stress distribution $S(x, 0)$ is given. Thus our equation is of the type $\mathfrak{B}$ discussed in the remarks of §4c.

For a motion of our medium to be possible, it is necessary that the continuity equation (3.5), the dynamical equation (3.6) and the constitutive equation (7.2) be satisfied. By means of (2.8), these equations can be written in the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\mathbf{v} \cdot \text{grad } \rho - \rho \text{ div } \mathbf{v}, \\ (8.1) \quad \frac{\partial \mathbf{v}}{\partial t} &= -Q\mathbf{v} + \mathbf{g} + \frac{1}{\rho} (\text{Div } T - \text{grad } p), \\ \frac{\partial T}{\partial t} &= (\nabla_x T)\mathbf{v} - TW + WT + \mathfrak{F}(D, T, \rho). \end{aligned}$$

This is a system of partial differential equations for $\rho(x, t)$, $\mathbf{v}(x, t)$ and $T(x, t)$, solved for the time derivatives. Written in coordinates, (8.1) forms a system of 10 equations for 10 unknowns. We can expect from the form of this system that it has a unique solution for $0 \leq t \leq \bar{t}$ if the initial values $\rho(x, 0)$, $\mathbf{v}(x, 0)$, and $T(x, 0)$ are given.

9. Classification: fluent and resilient bodies. We distinguish two classes of materials with constitutive equations of the form (7.2) according to their static behavior, *i.e.* according to their possible states when at ease ($D = W = \dot{S} = 0$). We note that a medium can be at rest without being at ease. This is the case, for instance, if $\mathbf{v} = 0$ but $\dot{S} \neq 0$. "Static behavior" is understood here in a generalized sense.

α . Fluent bodies. A medium shall be called a *fluent body* if it can be at ease only when the stress is equal to a certain hydrostatic pressure, depending on the density:

$$S = -p(\rho)I, \quad p > 0.$$

We then have

$$(9.1) \quad \mathfrak{F}(0, 0, \rho) = 0,$$

but

$$(9.2) \quad \mathfrak{F}(0, T, \rho) \neq 0 \quad \text{for} \quad T \neq 0, \quad T = S + p(\rho)I.$$

§. *Resilient bodies.* A medium shall be called a *resilient body* if it can be at ease for arbitrary stresses within a certain range. Then the pressure function (6.1) has no physical meaning and we may put $p(\rho) = 0$, $S = T$. Hence (7.2) becomes in this case

$$(9.3) \quad \dot{S} + SW - WS = \mathfrak{F}(D, S, \rho).$$

We have

$$(9.4) \quad \mathfrak{F}(0, S, \rho) = 0$$

for a certain range of S and ρ .

It may be that the body can be at ease only if a certain relation between ρ and S holds, *e.g.* for a given stress the state of ease may be possible only at a certain density $\rho = \varphi(S)$. In this case (9.4) must be replaced by

$$(9.5) \quad \mathfrak{F}(0, S, \varphi(S)) = 0, \quad \mathfrak{F}(0, S, \rho) \neq 0 \quad \text{for} \quad \rho \neq \varphi(S).$$

This means that if we have a body in a certain state of ease and two motions carrying the body into two states of ease with the same stress, then the density must also be the same in these two states. In other words there is no hysteresis for density (but there may be hysteresis for strain deviation).

We note that it is immaterial that we have chosen $p(\rho) = 0$. We could take $p(\rho)$ to be an arbitrary function of ρ . If we then write (7.2) with $T = S + p(\rho)I$, we get by (3.5)

$$\dot{S} + SW - WS = \mathfrak{F}(D, S + p(\rho)I, \rho) + p'(\rho)\rho \operatorname{tr} D.$$

This is again of the form (9.3).

10. Incompressible bodies. A material is called *incompressible* if it is susceptible only of motions in which the density at each material point is constant in time. Assuming also homogeneity, we have $\rho = \rho_0 = \text{const}$. In this case, the pressure (6.1) is no longer a function of ρ . It has to be considered as a primitive unknown function $p = p(x, t)$. With

$$S = T - pI$$

the basic equations (3.5), (3.6) and (7.2) go over into

$$(10.1) \quad \begin{aligned} \operatorname{div} \mathbf{v} &= \operatorname{tr} D = 0, \\ \operatorname{Div} T - \operatorname{grad} p + \rho_0 \mathbf{g} &= \rho_0 \dot{\mathbf{v}}, \\ \dot{T} + TW - WT &= \mathfrak{F}(D, T). \end{aligned}$$

Written in coordinates, these form a system of 10 partial differential equations for the 10 unknowns corresponding to $p(x, t)$, $\mathbf{v}(x, t)$, and $T(x, t)$. We observe from (10.1) that adding an arbitrary hydrostatic pressure depending only on time does not change our equations. Hence for an incompressible body the stress is determined only up to a uniform time-dependent pressure.

For a fluent body we have, according to (9.2),

$$(10.2) \quad \mathfrak{F}(0, 0) = 0, \quad \mathfrak{F}(0, T) \neq 0 \quad \text{for } T \neq 0.$$

For a resilient body we have

$$(10.3) \quad \mathfrak{F}(0, T) = 0 \quad \text{for all } T \text{ in a certain range.}$$

11. Dimensional analysis. We assume now that $\mathfrak{F}$ in (7.2) depends on ρ not explicitly but only through the pressure function $p(\rho)$, *i.e.* we replace (7.2) by

$$(11.1) \quad \dot{T} + TW - WT = \mathfrak{F}(D, T, p), \quad T = S + p(\rho)I.$$

T, p, S have the physical dimension of stress:

$$\dim(T) = \dim(S) = \dim(p) = \frac{\mathbf{M}}{\mathbf{T}^2 \mathbf{L}}.$$

W and D have the dimension of the reciprocal of time:

$$\dim(W) = \dim(D) = \frac{1}{\mathbf{T}}.$$

With a material constant τ of the dimension of time and a constant μ of the dimension of stress, (11.1) can be written in the form

$$(11.2) \quad \dot{T} + TW - WT = \frac{\mu}{\tau} \mathfrak{F}_0(D\tau, T\mu^{-1}, p\mu^{-1}),$$

$$\dim(\mu) = \frac{\mathbf{M}}{\mathbf{T}^2 \mathbf{L}}, \quad \dim(\tau) = \mathbf{T},$$

where $\mathfrak{F}_0$ is a dimensionless function.

We see that a body with a constitutive equation of the form (11.2) has only two dimensionally independent moduli. Instead of μ we can introduce the viscosity

$$\eta = \tau\mu, \quad \dim(\eta) = \frac{\mathbf{M}}{\mathbf{T} \mathbf{L}}.$$

Then (11.2) becomes

$$(11.3) \quad \dot{T} + TW - WT = \frac{\eta}{\tau^2} \mathfrak{F}_0(D\tau, T\eta^{-1}\tau, p\eta^{-1}\tau).$$

III. SPECIAL CASES

In this chapter we shall discuss a number of special constitutive equations of the form (7.2) corresponding to special forms for the function $\mathfrak{F}$.

12. Linear fluent bodies. We assume that $\mathfrak{F}(D, T, \rho)$ is independent of ρ and that it is linear (inhomogeneous) in D for fixed T and linear (inhomogeneous) in T for fixed D . It can be seen from (7.4) that in this case $\mathfrak{F}$ must have the form

$$(12.1) \quad \mathfrak{F}(D, T) = a_1 I + a_2 D + a_4 T + a_6(TD + DT),$$

where

$$\begin{aligned} a_1 &= \mu_2 \operatorname{tr} D + \lambda_2 \operatorname{tr} T + \alpha_1 \operatorname{tr}(DT) + \alpha_4(\operatorname{tr} T)(\operatorname{tr} D), \\ a_2 &= \mu_1 + \alpha_2 \operatorname{tr} T, \quad a_4 = \lambda_1 + \alpha_5 \operatorname{tr} D, \quad a_6 = \alpha_3, \end{aligned}$$

λ_i, μ_i and α_i being constants. A fluent body with a function $\mathfrak{F}$ of the form (12.1) shall be called a *linear fluent body*. Putting (12.1) into (7.2) we see that it has a constitutive equation of the form¹:

$$\begin{aligned} \dot{T} + TW - WT &= \mu_1 D + \mu_2(\operatorname{tr} D)I + \lambda_1 T + \lambda_2(\operatorname{tr} T)I \\ (12.2) \quad &+ \alpha_1 \operatorname{tr}(DT)I + \alpha_2(\operatorname{tr} T)D + \alpha_3(TD + DT) \\ &+ \alpha_4(\operatorname{tr} T)(\operatorname{tr} D)I + \alpha_5(\operatorname{tr} D)T, \end{aligned}$$

$$\begin{aligned} (12.2c) \quad \dot{t}^i_k + t^{ij} w_{jk} - w^{ij} t_{jk} &= (\mu_1 + \alpha_2 t^j_j) d^i_k + (\lambda_1 + \alpha_5 d^j_j) t^i_k \\ &+ \alpha_3(t^i_j d^j_k + d^i_j t^j_k) + (\mu_2 d^j_j + \lambda_2 t^j_j + \alpha_1 d^j_l t^l_j + \alpha_4 d^j_j t^l_l) \delta^i_k. \end{aligned}$$

We have

$$\dim(\mu_i) = \frac{\mathbf{M}}{\mathbf{T}^2 \mathbf{L}} (\text{stress}), \quad \dim(\lambda_i) = \frac{1}{\mathbf{T}}, \quad \dim(\alpha_i) = \mathbf{1}.$$

Since the body is assumed to be fluent, we have by (9.2)

$$(12.3) \quad \lambda_2(\operatorname{tr} T)I + \lambda_1 T \neq 0 \quad \text{for} \quad T \neq 0.$$

The relation $\lambda_2(\operatorname{tr} T)I + \lambda_1 T = 0$ would be possible for $\lambda_1 \neq 0$ only if $T = qI$, in which case we should have $(3\lambda_2 + \lambda_1)q = 0$. This can hold for $q \neq 0$ only if $3\lambda_2 + \lambda_1 = 0$. We conclude that (12.3) implies

$$(12.4) \quad \lambda_1 \neq 0, \quad 3\lambda_2 + \lambda_1 \neq 0.$$

¹ The special case $\alpha_i = 0$ was introduced by ZAREMBA [9].

For an incompressible body we have $\text{tr } D = 0$; hence (12.2) reduces then to

$$(12.5) \quad \begin{aligned} \dot{T} + TW - WT &= \mu_1 D + \lambda_1 T + \lambda_2 (\text{tr } T)I + \alpha_1 \text{tr } (DT)I \\ &\quad + \alpha_2 (\text{tr } T)D + \alpha_3 (TD + DT), \end{aligned}$$

$$(12.5c) \quad \begin{aligned} \dot{t}^i_k + t^{ij} w_{jk} - w^{ij} t_{jk} &= (\mu_1 + \alpha_2 t^j_j) d^i_k + \lambda_1 t^i_k \\ &\quad + \alpha_3 (t^i_j d^j_k + d^i_j t^j_k) + (\lambda_2 t^j_j + \alpha_1 d^j_l t^l_j) \delta^i_k. \end{aligned}$$

13. Viscous fluids as special fluent bodies. *α . General case.* The equation (11.1) can be written in the dimensionless form (11.3). But for a fluent body ($p > 0$) we can write it also in the form

$$(13.1) \quad \dot{T} + TW - WT = \frac{\eta}{\tau_1 \tau_2} \mathfrak{F}_0^* \left(D\tau_1, \frac{T}{p}, \frac{p}{\eta} \tau_1 \right),$$

where $\mathfrak{F}_0^*$ is again a dimensionless function, and τ_1, τ_2 are two material constants of the dimension of time. For a fixed function $\mathfrak{F}_0^*$, (13.1) describes a whole class of materials, the members of this class being distinguished by different values of τ_1, τ_2 and η .

If we multiply (13.1) by τ_2 and then let τ_2 go to 0, holding everything else fixed, we get:

$$(13.2) \quad \mathfrak{F}_0^* \left(D\tau_1, \frac{T}{p}, \frac{p}{\eta} \tau_1 \right) = 0.$$

For fluent bodies, since by (9.2) we have

$$\mathfrak{F}_0^* \left(0, \frac{T}{p}, \frac{p}{\eta} \tau_1 \right) \neq 0 \quad \text{for } T \neq 0,$$

we shall in general be able to solve (13.2) for T , obtaining

$$(13.3) \quad T = p \mathfrak{G}_0 \left(D\tau_1, \frac{p}{\eta} \tau_1 \right).$$

This is the constitutive equation for a *Reiner-Rivlin fluid*¹. Thus, for a particular $\mathfrak{F}_0^*$, the class of materials described by (13.1) contains a class of Reiner-Rivlin fluids as the sub-class characterized by $\tau_2 = 0$. Furthermore, if we also let τ_1 go to 0 in (13.3), we get a *Stokesian fluid* with an equation of the form

$$T = p \mathfrak{S}_0 \left(\frac{D\eta}{p} \right).$$

Remarks. It is clear from the derivation of (13.3) that $\mathfrak{G}_0$ is a hemitropic function of D . This follows also from a direct definition of Reiner-Rivlin fluids, which is given by the assumptions

¹ See [8], p. 236.

$$(13.4) \quad T = \mathfrak{G}(Q, p), \quad 0 = \mathfrak{G}(0, p), \quad T = S + pI,$$

where Q is the velocity gradient (2.9) and $p = p(\rho)$ is a given function of the density. Since Q does not depend on the choice of the reference position, the assumption (13.4) necessarily defines an isotropic material (see §4c). By a procedure similar to the one in §7, we can infer from the principle of isotropy of space that

$$\mathfrak{G}(Q, p) = \mathfrak{G}(D, p), \quad Q = W + D$$

and that $\mathfrak{G}$ must be a hemitropic function of D in order that the constitutive equation (13.4) be admissible.

A generalization of (13.4) to anisotropic fluids would be given by an assumption of the form

$$T = \mathfrak{G}(Q, R, p),$$

where R is defined by the polar decomposition $A = RV = UR$, A being the displacement tensor taken with respect to some reference position. The rotation R determines the orientation of the distinguished material directions. In this case the principle of isotropy of space implies

$$\mathfrak{G}(Q, R, p) = \mathfrak{G}(D, R, p), \quad Q = W + D$$

and the validity of the functional equation

$$\tilde{R}\mathfrak{G}(D, R, p)\tilde{R}^{-1} = \mathfrak{G}(\tilde{R}D\tilde{R}^{-1}, \tilde{R}R, p),$$

which is to hold for any symmetric D and any two rotations R and $\tilde{R}$.

§. *Case of a linear fluent body.* For linear fluent bodies with an equation (12.2) we shall now carry out explicitly the limiting process indicated in α . If we write

$$\mu_i = \frac{\eta_i}{\tau_2}, \quad \lambda_2 = -\frac{\kappa}{\tau_2}, \quad \lambda_1 = -\frac{1}{\tau_2}, \quad \alpha_i = \alpha'_i \frac{\tau_1}{\tau_2},$$

then (12.2) goes over into the form (13.1). Multiplying by τ_2 and letting $\tau_2 \rightarrow 0$ yields:

$$(13.5) \quad \begin{aligned} 0 = & \eta_1 D + \eta_2(\text{tr } D)I - T - \kappa(\text{tr } T)I \\ & + \tau_1\{\alpha'_1 \text{tr } (DT)I + \alpha'_2(\text{tr } T)D + \alpha'_3(TD + DT) \\ & + \alpha'_4(\text{tr } T)(\text{tr } D)I + \alpha'_5(\text{tr } D)T\}. \end{aligned}$$

To solve this equation for T requires a rather cumbersome calculation, omitted here. But it is clear that the result would be a non-linear equation of the form (13.3).

If we also let $\tau_1 \rightarrow 0$, then we get from (13.5)¹:

$$(13.6) \quad 0 = \eta_1 D + \eta_2 (\text{tr } D)I - T - \kappa (\text{tr } T)I.$$

Taking the trace of (13.6) yields

$$0 = (3\eta_2 + \eta_1) \text{tr } D - (3\kappa + 1) \text{tr } T.$$

(12.4) ensures that $3\kappa + 1 \neq 0$. Hence we can solve for $\text{tr } T$ and put it back into (13.6). We thus obtain the linear Newton-Cauchy-Poisson law

$$(13.7) \quad T = \eta_1 D + \frac{\eta_2 - \kappa\eta_1}{3\kappa + 1} (\text{tr } D)I,$$

which defines the *classical theory of viscous fluids*.

14. Hypo-elastic materials². Let us consider a resilient material with an equation (9.3) and assume first that $\mathfrak{F}$ does not depend on ρ :

$$(14.1) \quad \dot{S} + SW - WS = \mathfrak{F}(D, S).$$

By (11.2), this can be written in the dimensionless form

$$(14.2) \quad \dot{S} + SW - WS = \frac{\mu}{\tau} \mathfrak{F}_0(D\tau, S\mu^{-1}).$$

If the equation (14.2) does not contain a material constant of the dimension of time, the material is called *hypo-elastic*. This is possible only if τ cancels out in the right-hand side of (14.2). This implies that $\mathfrak{F}$ is homogeneous of degree one in D . Since $\mathfrak{F}$ was assumed to be continuously differentiable in a neighborhood of $D = 0$, it follows that $\mathfrak{F}(D, S)$ must be linear in D . By (7.4), the equation (14.1) reduces then to

$$(14.3) \quad \begin{aligned} \dot{S} + SW - WS = & b_1 D + b_2(SD + DS) + b_3(S^2 D + DS^2) \\ & + [b_4 \text{tr } D + b_5 \text{tr } (DS) + b_6 \text{tr } (DS^2)]I \\ & + [b_7 \text{tr } D + b_8 \text{tr } (DS) + b_9 \text{tr } (DS^2)]S \\ & + [b_{10} \text{tr } D + b_{11} \text{tr } (DS) + b_{12} \text{tr } (DS^2)]S^2, \end{aligned}$$

where the b_i are functions of the three invariants of S alone.

In the case of an incompressible body, S has to be replaced by $T = S + pI$ (see §10).

We note that hypo-elastic materials are automatically resilient bodies because (9.4) holds for any $\mathfrak{F}(D, S)$ homogeneous in D . They can be characterized by the following property.

¹ The following analysis was given essentially by ZAREMBA [9], p. 14.

² These materials were introduced and discussed by C. TRUESDELL [7].

Theorem. *A material with a constitutive equation of the form (14.1) is hypo-elastic if and only if, for a given initial stress, the stress at a final state depends only on the paths by which the material points reach the final state and not upon the rate at which they traverse these paths.*

For the proof it suffices to remark that replacing the time t by $\chi(t)$, where $\chi(t)$ is any function of t , corresponds to replacing $\dot{S}$, W , and D by $\dot{\chi}S$, $\dot{\chi}W$, and $\dot{\chi}D$ respectively. So if (14.1) holds, then

$$\dot{\chi}S + S(\dot{\chi}W) - (\dot{\chi}W)S = \dot{\chi}(S + SW - WS) = \mathfrak{F}(\dot{\chi}D, S)$$

also holds if and only if

$$\mathfrak{F}(\dot{\chi}D, S) = \dot{\chi}\mathfrak{F}(D, S).$$

This equation is valid for any $\dot{\chi}$ if and only if $\mathfrak{F}(D, S)$ is homogeneous in D .

If $\mathfrak{F}(D, S)$ is also linear (inhomogeneous) in S for fixed D , we call the material *linear hypo-elastic*¹. In this case (14.3) reduces to

$$(14.4) \quad \begin{aligned} \dot{S} + SW - WS &= \mu_1 D + \mu_2(\text{tr } D)I + \alpha_1 \text{tr } (DS)I \\ &+ \alpha_2(\text{tr } S)D + \alpha_3(SD + DS) + \alpha_4(\text{tr } S)(\text{tr } D)I + \alpha_5(\text{tr } D)S \end{aligned}$$

$$(14.4c) \quad \begin{aligned} \dot{s}^i_k + s^{ij}w_{jk} - w^{ij}s_{jk} &= (\mu_1 + \alpha_2 s^j_j)d^i_k + \alpha_5 d^j_j s^i_k \\ &+ \alpha_3(s^i_j d^j_k + d^i_j s^j_k) + (\mu_2 d^j_j + \alpha_1 d^j_l s^l_j + \alpha_4 d^j_j s^l_l)\delta^i_k, \end{aligned}$$

$$\dim(\mu_i) = \frac{\mathbf{M}}{\mathbf{T}^2 \mathbf{L}} (\text{stress}), \quad \dim(\alpha_i) = \mathbf{1}.$$

For incompressible linear hypo-elastic bodies, (14.4) goes over into ($T = S + pI$):

$$(14.5) \quad \dot{T} + TW - WT = \mu_1 D + \alpha_1 \text{tr } (DT)I + \alpha_2(\text{tr } T)D + \alpha_3(TD + DT),$$

$$(14.5c) \quad \begin{aligned} \dot{t}^i_k + t^{ij}w_{jk} - w^{ij}t_{jk} &= (\mu_1 + \alpha_2 t^j_j)d^i_k + \alpha_3(t^i_j d^j_k + d^i_j t^j_k) \\ &+ \alpha_1 t^j_l d^l_j \delta^i_k. \end{aligned}$$

15. Elastic Materials. a. Definition. A body is said to be *elastic* if it has the following properties. First, it possesses a certain distinguished position $X = g_0(p)$, called its natural state. This state is assumed to be unstressed and to have homogeneous density ρ_0 . Second, the stress at any other position is a function only of the displacement tensor, taken with respect to the natural state (see §2a); in formulas,

$$(15.1) \quad S = \mathfrak{G}(A), \quad \mathfrak{G}(I) = 0, \quad A = \nabla_{\mathbf{x}} f(X, t).$$

¹ These materials are called *hypo-elastic bodies of grade one* by TRUESDELL [7].

The function $\mathfrak{G}$ is assumed to be defined and twice continuously differentiable in a neighborhood of $A = I$.

By the principle of isotropy of space, $\mathfrak{G}$ is admissible only if (15.1) is invariant under transformations Φ (see §4b). By (4.5) and (4.6), this implies the functional equation

$$(15.2) \quad \mathfrak{G}(RA) = R\mathfrak{G}(A)R^{-1},$$

which is to be satisfied by any A and any rotation R . Replacing A by V where V is defined by the polar decomposition (2.4), we get¹

$$(15.3) \quad S = \mathfrak{G}(A) = R\mathfrak{G}(V)R^{-1}, \quad A = RV.$$

For symmetric positive definite C , the root $\sqrt{C}$ is uniquely determined. We therefore can define

$$(15.4) \quad \mathfrak{S}(C) = \mathfrak{G}(\sqrt{C}).$$

Then (15.3) becomes (see (2.5)):

$$(15.5) \quad S = R\mathfrak{S}(C)R^{-1}, \quad C = A^t A, \quad R = AC^{-\frac{1}{2}}.$$

This relation holds for all elastic bodies, whether or not they are isotropic. For homogeneous isotropic materials, (15.1) must be invariant also under transformations Ψ (see §4c). By (4.17), this implies

$$(15.6) \quad \mathfrak{G}(A) = \mathfrak{G}(AR),$$

which is another functional equation for $\mathfrak{G}$. Replacing A by U , where U is defined by (2.4), we get

$$S = \mathfrak{G}(U), \quad A = UR.$$

Observing (2.6), by (14.4) we can write the *stress-strain relation for isotropic elastic bodies* in the form

$$(15.7) \quad S = \mathfrak{S}(B), \quad B = AA^t,$$

$$(15.7c) \quad s^{ik} = \mathfrak{S}^{ik}(b^{lm}), \quad b^{lm} = x^l_{,\alpha} x^{m,\alpha}.$$

From $B = RCR^{-1}$ (see (2.6)), together with (15.5), follows

$$S = \mathfrak{S}(RCR^{-1}) = R\mathfrak{S}(C)R^{-1}.$$

This is to be true for arbitrary A , *i.e.* for arbitrary rotations R and arbitrary symmetric positive definite C . Hence $\mathfrak{S}$ satisfies the functional equation (writing B instead of C):

¹ See [5], p. 68, (3.6).

² This formula can be regarded as a generalization to elastic bodies not necessarily having a strain energy (CAUCHY's definition) of a result due to CELLERIER (see [8], p. 175, §37) for hyperelastic bodies (GREEN's definition).

$$(15.8) \quad \mathfrak{S}(RBR^{-1}) = R\mathfrak{S}(B)R^{-1};$$

i.e. $\mathfrak{S}$ is a hemitropic function of B (see (C.20)).

$\mathfrak{S}$ is a function which maps a neighborhood of $B = I$ in the space of all symmetric tensors into a neighborhood of $S = 0$ in that same space. We assume that this mapping is one-to-one in a neighborhood of $B = I$. A sufficient condition for this to be true is that the differential $\nabla_B \mathfrak{S}(I)$ (see (C.10)), being a linear transformation of the space of symmetric tensors, have an inverse. This is true if its determinant is not zero. (In coordinates, this is a determinant of a matrix with 6 rows.) Under this assumption we have also a *strain-stress relation*:

$$(15.9) \quad B = \bar{\mathfrak{S}}^{-1}(S).$$

It is clear that $\bar{\mathfrak{S}}^{-1}$ is a hemitropic function of S , since $\mathfrak{S}$ is a hemitropic function of B by (15.8).

b. Isotropic elastic bodies as special hypo-elastic bodies. Material time differentiation of (15.7) yields (see (C.10)):

$$(15.10) \quad \dot{S} = \nabla_B \mathfrak{S}(B)[\dot{B}].$$

We have by (2.10), (2.13) and (2.14)

$$\begin{aligned} \dot{B} &= \dot{\bar{A}}\bar{A}^t = \dot{A}A^t + \dot{A}A^t = AA^t Q^t + QAA^t, \\ &= BQ^t + QB = (BD + DB) + (WB - BW). \end{aligned}$$

Hence (15.10) becomes

$$(15.11) \quad \dot{S} = \nabla_B \mathfrak{S}(B) [BD + DB] + \nabla_B \mathfrak{S}(B) [WB - BW].$$

Since the functional equation (15.8) holds for arbitrary B and R , we may fix B and choose $R = R(t)$ as a function of a parameter t such that $R(0) = I, \dot{R}(0) = W$. If we then differentiate (15.8) with respect to t , we get (see §Cd and §Ce)

$$\nabla_B \mathfrak{S}(RBR^{-1}) [\dot{R}BR^{-1} - RBR^{-1} \dot{R}R^{-1}] = \dot{R}\mathfrak{S}(B)R^{-1} - R\mathfrak{S}(B)R^{-1} \dot{R}R^{-1}.$$

For $t = 0$, this becomes

$$\nabla_B \mathfrak{S}(B) [WB - BW] = W\mathfrak{S}(B) - \mathfrak{S}(B)W.$$

Putting this equation into (15.11), by means of (15.7) we get

$$\dot{S} + SW - WS = \nabla_B \mathfrak{S}(B) [BD + DB];$$

hence by (15.9) follows

$$(15.12) \quad \dot{S} + SW - WS = \nabla_B \mathfrak{S}(\bar{\mathfrak{S}}^{-1}(S)) [\bar{\mathfrak{S}}^{-1}(S)D + D\bar{\mathfrak{S}}^{-1}(S)],$$

$$(15.12c) \quad \dot{s}^i_k + s^{ij}W_{jk} - W^{ij}s_{jk} = \frac{\partial \mathfrak{h}^{ik}}{\partial b_{lm}} (b^{ln}d^m_n + d^{ln}b^m_n), \quad b^{ln} = \bar{\mathfrak{h}}^{ln} s^{ik}.$$

The right-hand side of this equation is a function of S and D , linear and homogeneous in D .

Differentiation of equation (15.8) with respect to B yields (see §Ce)

$$\nabla_B \mathfrak{S}(RBR^{-1}) [RZR^{-1}] = R \nabla_B \mathfrak{S}(B) [Z] R^{-1},$$

where Z is an arbitrary symmetric tensor. Putting $Z = BD + DB$, we get

$$\begin{aligned} \nabla_B \mathfrak{S}(RBR^{-1}) [(RBR^{-1})(RDR^{-1}) + (RDR^{-1})(RBR^{-1})] \\ = R \nabla_B \mathfrak{S}(B) [BD + DB] R^{-1}. \end{aligned}$$

Hence $\nabla_B \mathfrak{S}(B) [BD + DB]$ is a hemitropic function of B and D (see (C.20)). Since $\mathfrak{S}^{-1}(S)$ is a hemitropic function of S , it follows that the right side of (15.12) is a hemitropic function of S and D . Since it is also linear and homogeneous in D it follows according to §14 that (15.12) represents a hypo-elastic body.

Thus we have proved that *an isotropic elastic material is a special case of a hypo-elastic material*.

c. Elastic relief, compatibility. The stress-strain relation (15.7) is not exactly equivalent to the equation (15.12), which was obtained from (15.7) by differentiation.

In considering a motion we choose as reference position the initial position $\bar{x}$ at time $t = 0$ (in general different from the natural state X):

$$(15.13) \quad x = \bar{f}(\bar{x}, t), \quad \bar{x} = \bar{f}(\bar{x}, 0).$$

Then, for each $\bar{x}$, (15.12) is an ordinary differential equation for the stress $S = \bar{S}(\bar{x}, t)$ as a function of t . If the initial values

$$(15.14) \quad \bar{S}(\bar{x}, 0) = S_0(\bar{x})$$

are given, then $\bar{S}(\bar{x}, t)$ is uniquely determined (see also §8). If the body has a natural state X as a possible position, let it be connected with the initial position by $\bar{x} = f_0(X)$. When referred to the natural state, our motion is represented by

$$(15.15) \quad x = \bar{f}(f_0(X), t) = f(X, t).$$

If the stress-strain relation (15.7) holds, we get for the stress $S = \bar{S}(X, t)$ as a function of X and t :

$$(15.16) \quad \bar{S}(X, t) = \mathfrak{S}(B(X, t)),$$

where

$$B(X, t) = A(X, t)A^t(X, t), \quad A(X, t) = \nabla_x f(X, t).$$

Differentiating (15.15) with respect to X at $t = 0$, since by (15.13) $\nabla_{\bar{x}} \bar{f}(\bar{x}, 0) = I$ holds we get with the help of the chain rule (C.12)

$$A(X, 0) = \nabla_x f_0(X) = A_0(X);$$

hence by (15.16),

$$\tilde{S}(X, 0) = \mathfrak{S}(B_0(X)), \quad B_0(X) = A_0(X)A_0^t(X).$$

It is clear from the derivation of the equation (15.12) that $S = \tilde{S}(X, t)$ satisfies this equation. By the uniqueness of solution, we have

$$\tilde{S}(X, t) = \tilde{S}(\bar{x}, t), \quad \bar{x} = f_0(X)$$

if and only if the initial values are the same, *i.e.* if and only if (see (15.14) and (15.16))

$$(15.17) \quad S_0(\bar{x}) = \tilde{S}(\bar{x}, 0) = \tilde{S}(X, 0) = \mathfrak{S}(B_0(X)), \quad \bar{x} = f_0(X).$$

For a given initial stress $S_0(\bar{x})$ it may or may not be possible to find a mapping $\bar{x} = f_0(X)$ such that (15.17) holds. If it is possible, we say that $S_0(\bar{x})$ is susceptible of elastic relief. In this case $X = \bar{f}_0^{-1}(\bar{x})$ determines the natural state in relation to the initial position. Thus we have the

Definition. *The stress distribution $S = S_0(\bar{x})$ is called elastically relievable with the stress-strain relation $S = \mathfrak{S}(B)$ if there is a continuously differentiable one-to-one mapping*

$$(15.18) \quad \bar{x} = f_0(X)$$

such that

$$(15.19) \quad S_0(\bar{x}) = \mathfrak{S}(B_0(X)),$$

where

$$(15.20) \quad B_0(X) = A_0(X)A_0^t(X), \quad A_0(X) = \nabla_x f_0(X).$$

The equations (15.7) and (15.12) are equivalent only for relievable initial stress distributions $S_0(\bar{x})$.

d. Initial stresses, dislocation. It is physically natural to consider problems with initial stresses $S_0(\bar{x})$ which are not elastically relievable. In an elastic body there may be initial or residual stresses which cannot be relieved by means of a continuous displacement $X = \bar{f}_0^{-1}(\bar{x})^1$. In this case it is no longer adequate to describe the body by the stress-strain relation (15.7), but it is adequate to describe it by (15.12). In this sense, (15.12) is a generalization of (15.7), since it permits the study of isotropic elastic bodies subject to arbitrary initial stresses.

Sometimes it is possible to apply a finite number of cuts to the body in such a way that the relievability condition is satisfied for the resulting pieces. In other words, it may not be possible by a continuous mapping, to relieve the body

¹ See [8], p. 217-218.

as a whole into the natural state but such relief may be made possible by applying appropriate cuts. In this case we shall call the body *dislocatable*¹. *E.g.*, a disk may be dislocatable by a cut as indicated in Figure 7. The relaxed state may have the form shown in Figure 8.

The relievability condition (15.19), (15.20) is a partial differential equation for the function $f_0(X)$. The integrability condition for this equation yields a dif-

FIGURE 7

FIGURE 8

ferential equation for $S_0(\bar{x})$, which is called the *compatibility condition*. It is necessary that $S_0(\bar{x})$ satisfies this differential equation in order to be relievable. But the fact that $S_0(\bar{x})$ satisfies the compatibility condition is sufficient for the relievability of $S_0(\bar{x})$ only if the body is simply connected. If the body is multiply connected, then the validity of the compatibility condition ensures only that the body is dislocatable by any cuts which make it simply connected.

¹ See [3], p. 221–228.

e. Incompressible case. For a homogeneous incompressible elastic body we have

$$\rho = \rho_0 = \text{const.};$$

i.e., by (3.3)

$$\det A = 1, \quad \det B = (\det A)^2 = 1.$$

Hence only deformations with $\det B = 1$ are admissible. But since we still want to admit all possible stresses S in a neighborhood of $S = 0$, we cannot expect the strain-stress relation (15.9) to be one-to-one any more. We assume that the addition of an arbitrary pressure pI to S does not change the deformation; *i.e.*

$$(15.21) \quad B = \bar{\mathfrak{S}}^{-1}(S) = \bar{\mathfrak{S}}^{-1}(S + pI), \quad \det B = 1.$$

Now $\bar{\mathfrak{S}}^{-1}(S)$ is not the unique inverse of a function $\mathfrak{S}(B)$, but we can always find a hemitropic function $\mathfrak{S}(B)$ such that $\bar{\mathfrak{S}}^{-1}(\mathfrak{S}(B)) = B$ for $\det B = 1$. $\mathfrak{S}(B)$ is determined only up to an arbitrary term of the form $h(B)I$, where $h(B)$ is a hemitropic scalar function of B . By (15.21), the stress-strain relation then becomes

$$(15.22) \quad T = S + pI = \mathfrak{S}(B), \quad B = \bar{\mathfrak{S}}^{-1}(T).$$

We can now apply the same consideration as in **b**, provided we take the extra stress T instead of S . Thus we get in place of (15.12) for incompressible elastic bodies:

$$(15.23) \quad \dot{T} + TW - WT = \nabla_B \mathfrak{S}(\bar{\mathfrak{S}}^{-1}(T)) [\bar{\mathfrak{S}}^{-1}(T)D + D\bar{\mathfrak{S}}^{-1}(T)].$$

16. Hyperelastic materials¹. a. Definition. Let ϵ be the internal energy density² for a continuous medium. For adiabatic³ motions $x = f(X, t)$, from the principle of conservation of energy and from the linear momentum principle (3.6) can be derived the equation⁴

$$(16.1) \quad \rho \dot{\epsilon} = \text{tr} (Q^t S), \quad Q = \nabla_x v, \quad \rho = \text{density}.$$

We assume now that we have a body for which ϵ depends only on the displacement tensor $A = \nabla_x f(X, t)$, taken with respect to a natural state X , and for which (16.1) holds for any motion. Putting $\epsilon = \epsilon(A)$, which we assume to be a

¹ We use the term "elastic" for CAUCHY's definition and the term "hyperelastic" for GREEN's (see [8], p. 173).

² *I.e.*, energy per unit mass.

³ *I.e.*, motions in which the heat flux is zero.

⁴ See [8], p. 160, (27.2).

continuously differentiable function of A , into (16.1) and observing (2.10) we get (see (C.11), (C.3), and (C.4)):

$$\rho \operatorname{tr} (\epsilon_A(A) \dot{A}) = \operatorname{tr} \{ (A^t)^{-1} \dot{A}^t S \} = \operatorname{tr} \{ S (A^t)^{-1t} \dot{A}^t \} = \rho \operatorname{tr} (\epsilon_A^t(A) \dot{A}^t).$$

Since this is to be true for arbitrary $\dot{A}^t$, we conclude by (C.5) that

$$\rho \epsilon_A^t(A) = S (A^t)^{-1}.$$

Hence we have

$$(16.2) \quad S = \rho(A) \epsilon_A^t(A) A^t, \quad \rho(A) = \rho_0 (\det A)^{-1},$$

the density being given by (3.3).

Since S is symmetric, it follows that

$$(16.3) \quad A \epsilon_A(A) = \epsilon_A^t(A) A^t.$$

Hence the stress-strain relation (16.2) can be written in the form¹

$$(16.4) \quad S = \rho A \epsilon_A(A), \quad \rho = \rho_0 (\det A)^{-1}.$$

A body with a constitutive equation of the form (16.4) shall be called *hyperelastic*.

b. Functional equations for the strain energy function. We shall give here certain facts concerning the strain energy function $\epsilon(A)$.

Theorem I. *If $\epsilon(A)$ satisfies any one of the following three functional equations:*

$$(16.5) \quad A \epsilon_A(A) = \epsilon_A^t(A) A^t \quad \text{for all } A,$$

$$(16.6) \quad \epsilon_A(A) = \epsilon_A(RA)R \quad \text{for all } A \text{ and all rotations } R,$$

$$(16.7) \quad \epsilon(A) = \epsilon(RA) \quad \text{for all } A \text{ and all rotations } R,$$

then it satisfies also the two others.

Proof. We show first that (16.5) is equivalent to (16.7). Consider

$$\varphi(t) = \epsilon(R(t)A),$$

where $R(t)$ is a rotation depending on a parameter t such that $R(0) = I$. Differentiation yields (see (C.11))

$$\dot{\varphi}(t) = \operatorname{tr} (\epsilon_A(RA) \dot{R}A) = \operatorname{tr} (RA \epsilon_A(RA) \dot{R}R^{-1}).$$

$\dot{R}R^{-1}$ is antisymmetric (see (4.11)). If now (16.5) holds, then $RA \epsilon_A(RA)$ is symmetric, hence $\dot{\varphi}(t) = 0$ (see (C.6)). Therefore we have

$$\varphi(0) = \epsilon(A) = \varphi(t) = \epsilon(R(t)A).$$

Since we can choose $R(t)$ so that, for some particular $t > 0$, it is equal to an arbitrary rotation given *a priori*, the equation (16.7) follows. Conversely, if (16.7) holds, then $\varphi(t) = \epsilon(A)$, *i.e.* $\dot{\varphi}(t) = 0$; in particular,

$$(16.8) \quad \dot{\varphi}(0) = \operatorname{tr} (A \epsilon_A(A) \dot{R}(0)) = 0.$$

¹ See [8], p. 177, (39.6).

Since we can choose $R(t)$ so that $\dot{R}(0)$ is an arbitrary antisymmetric tensor, (16.8) implies that $A\epsilon_A(A)$ is symmetric; *i.e.* it follows that (16.5) holds (see (C.6)).

We show now that (16.6) is equivalent to (16.7). If we differentiate (16.7) with respect to A we get

$$\epsilon_A(A) = \epsilon_A(RA)R,$$

i.e. (16.6). Conversely, (16.6) states that $\epsilon^*(A) = \epsilon(RA)$ and $\epsilon(A)$ have the same differential with respect to A :

$$\epsilon_A^*(A) = \epsilon_A(A).$$

Hence the difference between $\epsilon^*(A)$ and $\epsilon(A)$ must be independent of A (see (C.13), (C.14)) and can therefore depend only on R :

$$(16.9) \quad \epsilon^*(A) = \epsilon(A) + \psi(R) = \epsilon(RA).$$

Replacing A by R^*A , where R^* is another rotation, we get

$$\epsilon(RR^*A) = \epsilon(R^*A) + \psi(R).$$

Applying (16.9) again, we derive

$$\epsilon(A) + \psi(RR^*) = \epsilon(A) + \psi(R^*) + \psi(R);$$

hence

$$\psi(RR^*) = \psi(R) + \psi(R^*)$$

for any two rotations R and R^* . This implies

$$(16.10) \quad n\psi(R) = \psi(R^n).$$

Since $\epsilon(A)$ was assumed to be continuously differentiable, it follows from (16.9) that $\psi(R)$ is continuous. Now, the group of all rotations is compact. Hence $\psi(R)$ must be bounded. Thus (16.10) implies

$$n |\psi(R)| = |\psi(R^n)| < M$$

which is valid for all n , M being independent of n . This is possible only if $\psi(R) = 0$. Hence (16.9) implies (16.7). Q.E.D.

By (16.3) the validity of the relation (16.5) is equivalent to the symmetry of the stress tensor S , *i.e.* to the moment of momentum principle (see §3c).

The principle of isotropy of space applied to elastic bodies is equivalent to (15.2). In the special case of a stress-strain relation of the form (16.2), this yields

$$\rho(RA)\epsilon_A^t(RA) \cdot (RA)^t = R\rho(A)\epsilon_A^t(A)A^t R^{-1}.$$

It can easily be seen that this is equivalent to the functional relation (16.6).

For symmetric positive definite tensors C we define

$$(16.11) \quad \eta(C) = \epsilon(\sqrt{C}).$$

Using the polar decomposition $A = RU$ (see (2.4)), we see that (16.7) is equivalent to

$$\epsilon(A) = \epsilon(RU) = \epsilon(U) = \epsilon(\sqrt{C}) = \eta(C), \quad C = U^2 = A^t A.$$

Thus we have the following

Corollary to Theorem I. *For hyperelastic bodies with strain energy $\epsilon(A)$ the moment of momentum principle, the principle of isotropy of space, and the validity of the relation*

$$(16.12) \quad \epsilon(A) = \epsilon(\sqrt{C}) = \eta(C), \quad C = A^t A$$

*are equivalent to one another*¹.

Differentiation of (16.12) with respect to A yields (see §Cd and §Ce)

$$\begin{aligned} \text{tr} (\epsilon_A(A)Z) &= \nabla_A \eta(A^t A)[Z] \\ &= \text{tr} \{ \eta_C(A^t A)(Z^t A + A^t Z) \} \\ &= 2 \text{tr} (\eta_C(C)A^t Z), \end{aligned}$$

which is valid for arbitrary tensors Z . Hence we obtain by (C.5)

$$\epsilon_A(A) = 2\eta_C(C)A^t.$$

Thus the stress-strain relation (16.4) can be written in the form²

$$(16.13) \quad S = 2\rho A \eta_C(C)A^t, \quad C = A^t A.$$

c. Isotropic hyperelastic bodies. First we state

Theorem II. *If $\epsilon(A)$ satisfies either one of the following two functional equations:*

$$(16.14) \quad \epsilon_A(A) = R \epsilon_A(AR) \quad \text{for all } A \text{ and all rotations } R,$$

$$(16.15) \quad \epsilon(A) = \epsilon(AR) \quad \text{for all } A \text{ and all rotations } R,$$

then it satisfies also the other.

The proof is completely analogous to the second part of the proof of Theorem I.

The isotropy of an elastic body is equivalent to the validity of the relation (15.6). In the special case of a stress-strain relation of the form (16.2), this yields

$$\rho(AR) \epsilon_A^t(AR) \cdot (AR)^t = \rho(A) \epsilon_A^t(A) A^t.$$

It is not hard to see that this is equivalent to the functional relation (16.14).

¹ The statement that the moment of momentum principle implies the validity of (16.12) is the content of the theorem of CELLERIER (see [8], p. 175.)

² See [8], p. 176, (39.2).

Using the polar decomposition $A = VR$ (see (2.4)) and the definition (16.11), we see that (16.15) is equivalent to

$$\epsilon(A) = \epsilon(VR) = \epsilon(V) = \epsilon(\sqrt{B}) = \eta(B), \quad B = V^2 = AA^t.$$

Thus we have the following

Corollary to Theorem II. *A hyperelastic body with strain energy $\epsilon(A)$ is isotropic if and only if*

$$(16.16) \quad \epsilon(A) = \epsilon(\sqrt{B}) = \eta(B), \quad B = AA^t.$$

By (16.12), (16.16), and (2.6), in the case of isotropic bodies we have

$$\epsilon(A) = \eta(C) = \eta(B) = \eta(R^{-1}BR).$$

This is valid for any A , *i.e.* for any positive definite symmetric B and any rotation R . Hence $\eta(B)$ is a hemitropic function of B and therefore depends only on the three invariants of B (see §Cf)¹.

Differentiation of (16.16) with respect to A yields (see §Cd, §Ce)

$$\text{tr}(\epsilon_A(A)Z) = \text{tr}\{\eta_B(B)(ZA^t + AZ^t)\} = 2 \text{tr}(A^t \eta_B(B)Z),$$

which is valid for any tensor Z , whence by (C.5)

$$\epsilon_A(A) = 2A^t \eta_B(B).$$

Thus by (16.4) the *stress-strain relation for isotropic hyperelastic bodies* can be written in the form

$$(16.17) \quad S = 2\rho(B)B\eta_B(B), \quad \rho(B) = \rho_0 (\det B)^{-\frac{1}{2}}, \quad B = AA^t,$$

where $\eta(B)$ is a hemitropic function of B , *i.e.*

$$(16.18) \quad \eta(RBR^{-1}) = \eta(B).^2$$

Since S is symmetric, it follows that

$$(16.19) \quad B\eta_B(B) = \eta_B(B)B.$$

By the corollaries to Theorems I and II, (16.19) is an automatic consequence of (16.18).

e. Isotropic hyperelastic bodies as special hypo-elastic bodies. We want to find the form of the equation (15.12) in the special case when the stress-strain relation (16.17) holds, *i.e.* for

$$(16.20) \quad S = \mathfrak{S}(B) = 2\rho(B)B \cdot \eta_B(B).$$

We have for arbitrary symmetric Z (see (16.19) and §Cd, §Ce)

$$(16.21) \quad \begin{aligned} \nabla_B(B\eta_B(B))[Z] &= Z\eta_B(B) + B\nabla_B \eta_B(B)[Z] \\ &= \nabla_B(\eta_B(B)B)[Z] = \eta_B(B)Z + \nabla_B \eta_B(B)[Z]B. \end{aligned}$$

¹ See [8], p. 175, §38.

² From (16.17) and (16.18) the relations of FINGER (see [8], p. 180, (41.6)) follow immediately.

Putting $Z = D$, we get

$$(16.22) \quad D\eta_B + B\nabla_B \eta_B[D] = \eta_B D + \nabla_B \eta_B[D]B.$$

Putting $Z = BD + DB = E$ into (16.21), we get

$$\begin{aligned} 2\nabla_B(\eta_B(B)B)[E] &= BD\eta_B + \eta_B DB + DB\eta_B + \eta_B BD \\ &\quad + B\nabla_B \eta_B[E] + \nabla_B \eta_B[E]B. \end{aligned}$$

Using (16.19) and (16.22), we derive

$$(16.23) \quad \begin{aligned} \nabla_B(2\eta_B(B)B)[E] &= 2(DB\eta_B + \eta_B BD) - B^2 \nabla_B \eta_B[D] - \nabla_B \eta_B[D]B^2 \\ &\quad + 2B\nabla_B \eta_B[D]B + B\nabla_B \eta_B[E] + \nabla_B \eta_B[E]B. \end{aligned}$$

On the other hand, from (16.17) it follows (see (C.12), (C.19)) that

$$\rho_B(B) = -\frac{1}{2}\rho_0(\det B)^{-\frac{1}{2}}(\det B)B^{-1};$$

i.e.

$$\rho_B(B) = -\frac{1}{2}\rho B^{-1};$$

hence

$$(16.24) \quad \nabla_B \rho(B)[BD + DB] = -\rho \operatorname{tr} D.$$

Using (16.22) and (16.23), we derive for $\mathfrak{S}(B)$ of the form (16.20) the relation

$$\begin{aligned} \nabla_B \mathfrak{S}(B)[E] &= D\mathfrak{S}(B) + \mathfrak{S}(B)D - \mathfrak{S}(B) \operatorname{tr} D + \rho\{B\nabla_B \eta_B[E] + \nabla_B \eta_B[E]B \\ &\quad + 2B\nabla_B \eta_B[D]B - B^2 \nabla_B \eta_B[D] - \nabla_B \eta_B[D]B^2\}. \end{aligned}$$

Hence, in the case of an isotropic hyperelastic body, the equation (15.12) becomes

$$(16.25) \quad \dot{S} + SW - WS = SD + DS - S \operatorname{tr} D + 2\rho(S)\mathfrak{R}(D, S),$$

where

$$(16.26) \quad \rho(S) = \rho_0\{\det \mathfrak{S}^{-1}(S)\}^{-\frac{1}{2}},$$

and where $\mathfrak{R}(D, S)$ is the symmetric part of the tensor function

$$(16.27) \quad \begin{aligned} &B\nabla_B \eta_B(B)[BD + DB] + B\nabla_B \eta_B(B)[D]B - B^2 \nabla_B \eta_B(B)[D], \\ &B = \mathfrak{S}^{-1}(S), \end{aligned}$$

$\mathfrak{S}^{-1}(S)$ being obtained by solving (16.20) for B .

We have written the above equation in a form such that only terms with the second differential $\nabla_B \eta_B(B)$ occur.

Since $D + W = Q$, $D - W = Q^t$ (see (2.13), (2.14)) hold, we can also write (16.15) in the form:

$$(16.28) \quad \dot{S} - QS - SQ^t + S \operatorname{tr} Q = 2\rho(S)\mathfrak{R}(D, S),$$

$$(16.28c) \quad \dot{s}^{ik} - s^{jk} \dot{x}^i_{,j} - s^{ij} \dot{x}^k_{,j} + s^{ik} \dot{x}^j_{,j} = 2\rho \mathfrak{t}^{ik},$$

$$(16.27c) \quad \mathfrak{t}^{ik} = \frac{\partial^2 \eta}{\partial b^{(kn} \partial b^{lm}} \{ (b^{lj} d^m_j + d^l_j b^{jm}) b^{in} - d^{lm} b^n_j b^{ij} \} \\ + \frac{\partial^2 \eta}{\partial b^{lm} \partial b^{nj}} d^{lm} b^{(ij} b^{nk)}.$$

17. Diagram of special cases. The considerations of chapters II and III are summarized in the following diagram:

The diagram indicates how some of the various materials contain others as special cases.

Remarks. In the general case of an equation (7.2), an initial stress distribution

and an initial velocity distribution can be assigned arbitrarily (see §4). For elastic bodies, as described by (15.7), the initial stress distribution must be susceptible of elastic relief (see §15c). However, the equation (15.12) generalizes (15.7) in the sense that it enables us to study elastic bodies with arbitrary initial stresses (see §15d).

For Reiner-Rivlin fluids and their special cases, it is not possible to assign the initial stresses. The stresses are determined if the velocity distribution is given (see (13.3)). These fluids have to be considered as singular special cases of fluent bodies in the same sense as perfect inviscid fluids are singular special cases of viscous fluids. The order of differentiation in the corresponding equations changes in both cases and therefore the number of initial conditions or boundary conditions, respectively, has to be reduced.

The *classical linear theory of elasticity* has not been included in our diagram because it does not correspond to a special constitutive equation. It is a theory which, *a priori*, can be valid only approximately for small displacements. For such small displacements from an unstressed state, all hypo-elastic bodies can be treated by the classical linear theory¹.

¹ See [7].

IV. SPECIAL SOLUTIONS

18. Preliminaries. a. Basic equations. Any motion of a body of the general class of hygrosteric materials defined in chapter II must satisfy the continuity equation (3.5), the dynamical equation (3.6) and the constitutive equation (7.2). Hence we have the following basic equations:

$$(18.1) \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0,$$

$$(18.2) \quad \operatorname{Div} S + \rho \mathbf{g} = \rho \dot{\mathbf{v}},$$

$$(18.3) \quad \dot{T} + TW - WT = \mathfrak{F}(D, T, \rho),$$

$$(18.4) \quad T = S + pI, \quad p = p(\rho), \quad \text{a given function.}$$

We shall discuss the general hygrosteric material only for motions with homogeneous stress (§§19–20). An explicit solution will be given for hydrostatic extension of linear fluent and hypo-elastic bodies. According to (12.2) and (14.4), for these special bodies $\mathfrak{F}$ reduces to the form

$$(18.5) \quad \begin{aligned} \mathfrak{F}(D, T) = & (\lambda_1 + \alpha_5 \operatorname{tr} D)T + (\mu_1 + \alpha_2 \operatorname{tr} T)D + \alpha_3(TD + DT) \\ & + \{\mu_2 \operatorname{tr} D + \lambda_2 \operatorname{tr} T + \alpha_1 \operatorname{tr}(DT') + \alpha_4(\operatorname{tr} T)(\operatorname{tr} D)\}I, \end{aligned}$$

where

$$(18.6) \quad \lambda_1 \neq 0, \quad 3\lambda_2 + \lambda_1 \neq 0$$

for fluent bodies, and

$$(18.7) \quad \lambda_1 = \lambda_2 = 0, \quad T = S, \quad p = 0$$

for hypo-elastic bodies, according to (12.4) and (14.4).

For the remaining special solutions, we confine ourselves to incompressible linear fluent bodies, for which our basic equations reduce to (10.1) with $\mathfrak{F}(D, T)$ of the form (12.5). For the sake of convenience in the calculations we introduce the tensor

$$\tilde{T} = \dot{T} - QT - TQ^t = \dot{T} + TW - WT - (TD + DT).$$

Furthermore, we assume that the extraneous force field $\mathbf{g}$ has a single valued potential ψ :

$$(18.8) \quad \mathbf{g} = -\operatorname{grad} \psi.$$

Then, with the notation

$$(18.9) \quad \varphi = p + \rho\psi,$$

the basic equations for incompressible linear fluent bodies reduce to

$$(18.10) \quad \text{tr } Q = \text{div } \mathbf{v} = 0,$$

$$(18.11) \quad \text{Div } T - \text{grad } \varphi = \rho \dot{\mathbf{v}},$$

$$(18.12) \quad \begin{aligned} \tilde{T} = \lambda_1 T + (\mu_1 + \alpha_2 \text{tr } T)D + (\alpha_3 - 1)(TD + DT) \\ + \{\lambda_2 \text{tr } T + \alpha_1 \text{tr } (DT)\}I, \end{aligned}$$

where

$$(18.13) \quad T = S + pI, \quad Q = \nabla_x \mathbf{v}, \quad D = \frac{1}{2}(Q + Q^t), \quad \tilde{T} = \dot{T} - QT - TQ^t.$$

b. Curvilinear coordinates. When using coordinates, we have to replace the tensors and vectors occurring in (18.10)–(18.13) by their corresponding components. The differentiation of vector and tensor fields corresponds to the covariant differentiation of their components (see §Cc). With the notation of [2], we have

$$(18.14) \quad \nabla_i v^i = (\sqrt{g})^{-1} \partial_i (\sqrt{g} v^i),$$

$$(18.15) \quad \dot{v}^i = \partial_t v^i + v^j \nabla_j v^i = \partial_t v^i + v^j \partial_j v^i + \Gamma_{jk}^i v^j v^k,$$

$$(18.16) \quad q^i_k = \nabla_k v^i = \partial_k v^i + \Gamma_{kj}^i v^j$$

$$(18.17) \quad d^{ik} = \frac{1}{2}(q^{ik} + q^{ki}) = \frac{1}{2}(g^{jk} \partial_j v^i + g^{ji} \partial_j v^k) + \frac{1}{2}v^j (\Gamma_{ji}^i g^{lk} + \Gamma_{ji}^k g^{li}),$$

$$(18.18) \quad \nabla_j t^{ik} = \partial_j t^{ik} + \Gamma_{lj}^k t^{il} + \Gamma_{lj}^i t^{kl},$$

$$(18.19) \quad \nabla_j t^{ij} = \partial_j t^{ij} + (\sqrt{g})^{-1} \partial_j (\sqrt{g}) t^{ij} + \Gamma_{lj}^i t^{lj}.$$

From (18.16) and (18.18) it follows easily that

$$(18.20) \quad \begin{aligned} \tilde{t}^{ik} &= \partial_t t^{ik} + v^j \nabla_j t^{ik} - t^{jk} \nabla_j v^i - t^{ji} \nabla_j v^k, \\ &= \partial_t t^{ik} + v^j \partial_j t^{ik} - t^{jk} \partial_j v^i - t^{ji} \partial_j v^k; \end{aligned}$$

i.e. the covariant derivatives can be replaced by the partial derivatives in the formula for $\tilde{t}^{ik}$.

In coordinate form, the equations (18.10)–(18.12) become

$$(18.21) \quad \nabla_i v^i = d^i_i = q^i_i = 0,$$

$$(18.22) \quad \nabla_j t^{ij} - g^{ij} \partial_j \phi = \rho v^i,$$

$$(18.23) \quad \begin{aligned} \tilde{t}^{ik} &= \lambda_1 t^{ik} + (\mu_1 + \alpha_2 t^j_j) d^{ik} + (\alpha_3 - 1)(t^i_j d^{jk} + t^k_j d^{ij}) \\ &\quad + (\lambda_1 t^j_j + \alpha_1 d^{lj} t_{jl}) g^{ik}. \end{aligned}$$

The formulas (18.14)–(18.20) are to be used for the calculation of the quantities occurring in (18.21)–(18.23).

In the case of rectangular Cartesian coordinates we can simply use the equations (18.10)–(18.13) after replacing the tensors there by their corresponding matrices. We write $x^1 = x$, $x^2 = y$, $x^3 = z$ and replace the indices 1, 2, 3 by x, y, z , respectively.

A matrix with components a^{ik} is denoted by $\|a^{ik}\|$, where the i indicates the row and the index k the column. In the case of symmetric matrices only six components are significant; the other three will often be omitted. A vector with components a^1, a^2, a^3 will be abbreviated by $\{a^i\}$.

c. *Cylindrical coordinates.* In the case of cylindrical coordinates

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = z,$$

we write r, θ, z as indices, instead of 1, 2, 3. The equations (18.14)–(18.19) yield

$$(18.24) \quad \nabla_i v^i = \partial_i v^i + \frac{1}{r} v^r,$$

$$(18.25) \quad \{v^i\} = \{\partial_i v^i + v^j \partial_j v^i\} + \{-r(v^\theta)^2, 2r^{-1} v^\theta v^r, 0\},$$

$$(18.26) \quad \|d^{ik}\| = \frac{1}{2}\{\|g^{jk} \partial_j v^i\| + \|g^{ji} \partial_j v^k\|\} + r^{-3} v^r \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$(18.27) \quad \{\nabla_j t^{ij}\} = \{\partial_j t^{ij} + r^{-1} t^{ir}\} + \{-rt^{\theta\theta}, 2r^{-1} t^{r\theta}, 0\}.$$

19. Pure stress relaxation. a. General materials. Consider a body at rest with homogeneous density:

$$(19.1) \quad \mathbf{v} = 0, \quad \rho = \rho_0.$$

The dynamical equation (18.2) is satisfied if we assume that the stress is homogeneous and there are no extraneous forces. The constitutive equation (18.3) reduces to

$$(19.2) \quad \dot{T} = \mathfrak{F}(0, T) = \mathfrak{F}_0(T), \quad T = S + p_0 I, \quad p_0 = p(\rho_0).$$

For resilient bodies (see (9.4)), we get simply $\dot{T} = 0$, *i.e.* $T = \text{const}$. Thus a resilient body is always at ease when at rest.

For fluent bodies, (19.2) is a non-trivial differential equation for T which has a unique solution for a given initial extra-stress T_0 :

$$(19.3) \quad T = \mathfrak{T}(t; T_0), \quad T_0 = \mathfrak{T}(0; T_0).$$

The function $\mathfrak{F}_0(T) = \mathfrak{F}(0, T)$ is hemitropic (see (7.3)), and therefore has a representation (see (7.4)) of the form

$$(19.4) \quad \mathfrak{F}_0(T) = f_0 I + f_1 T + f_2 T^2,$$

where the f_i are scalar functions of the three principal invariants of T . Hence they can also be written as symmetric functions of the three proper numbers t_1, t_2, t_3 of T . It is plain from (19.4) that the principal directions of T and $\mathfrak{F}_0(T)$ are the same. This leads us to try a solution $T = \mathfrak{T}(t; T_0)$ which has the same principal directions as T_0 for all times t . Let t_1^0, t_2^0, t_3^0 be the proper numbers of T_0 . Then, by (19.4) and (19.2), the above assumption leads to the following system of three differential equations for the $\dot{t}_i$:

$$(19.5) \quad \dot{t}_i = f_0(t_1, t_2, t_3) + f_1(t_1, t_2, t_3)t_i + f_2(t_1, t_2, t_3)t_i^2, \quad i = 1, 2, 3.$$

For initial values t_i^0 it has a unique solution

$$(19.6) \quad t_i = t_i(t; t_1^0, t_2^0, t_3^0), \quad t_i^0 = t_i(0; t_1^0, t_2^0, t_3^0), \quad i = 1, 2, 3.$$

Since the solution (19.3) is unique, it follows that the principal directions of the stress do not change in time and that the principal extra stresses can be obtained by solving the differential equations (19.5). We note that a permutation of the t_i together with the corresponding permutation of the t_i^0 does not change the system (19.5) or the initial conditions (19.6). Therefore, if any one of the functions t_1, t_2, t_3 is given, then the other two are determined. Assuming, for instance,

$$(19.7) \quad t_1(t; t_1^0, t_2^0, t_3^0) = t(t; t_1^0, t_2^0, t_3^0)$$

given, we have

$$(19.8) \quad \begin{aligned} t_2(t; t_1^0, t_2^0, t_3^0) &= t(t; t_2^0, t_3^0, t_1^0), \\ t_3(t; t_1^0, t_2^0, t_3^0) &= t(t; t_3^0, t_1^0, t_2^0). \end{aligned}$$

Furthermore, interchanging t_2^0 and t_3^0 in t does not change the value of t :

$$(19.9) \quad t(t; t_1^0, t_2^0, t_3^0) = t(t; t_1^0, t_3^0, t_2^0).$$

For physical reasons we now make two *assumptions*:

α) The solution $T = \mathfrak{T}(t, T_0)$ of (19.2) exists for all initial values T_0 (in a neighborhood of $T_0 = 0$) and for all times t ($0 \leq t < \infty$).

β) We have for all T_0

$$(19.10) \quad \lim_{t \rightarrow \infty} \mathfrak{T}(t; T_0) = 0.$$

These assumptions mean that any initial extra stress T_0 relaxes to zero with time, so that only the hydrostatic pressure $S = -p_0 I$ remains permanently. The assumption β) can be replaced by the weaker one that at least one principal stress, say t_1 , approaches a limit for any initial stress t_1^0, t_2^0, t_3^0 . Then it follows from (19.8) that also the other principal stresses t_2, t_3 , and hence the total stress

T , tends to a limit. From the continuity of $\mathfrak{F}_0(T)$ it follows then by a theorem of calculus that

$$0 = \lim_{t \rightarrow \infty} \dot{T} = \mathfrak{F}_0(T_\infty) = \mathfrak{F}(0, T_\infty), \quad T_\infty = \lim_{t \rightarrow \infty} \mathfrak{T}(t; T_0).$$

By (9.2) this is possible for fluent body only if $T_\infty = 0$.

It is probable that in a theory taking into account thermodynamics the assumption β) would be a consequence of the second law (see the remark at the end of §7).

We summarize.

Theorem. *Given a fluent body at rest with homogeneous density ρ_0 and subject to a homogeneous stress S_0 at the time $t = 0$; then the body will remain at rest, the stress S will have for all times the same principal directions as the initial stress S_0 , the principal extra-stresses being given by*

$$t_1 = t(t; t_1^0, t_2^0, t_3^0),$$

$$t_2 = t(t; t_2^0, t_3^0, t_1^0),$$

$$t_3 = t(t; t_3^0, t_1^0, t_2^0),$$

where t_1^0, t_2^0, t_3^0 are the initial principal extra stresses and where t is a function of four variables determined by the system of differential equations (19.5) with initial conditions (19.6) and satisfying

$$\lim_{t \rightarrow \infty} t(t; a, b, c) = 0$$

for all values of the parameters a, b, c .

The phenomenon described here will be called *pure stress relaxation*.

b. Linear fluent bodies. In the case of a linear fluent body, (19.2) becomes by (18.5)

$$(19.11) \quad \dot{T} = \lambda_1 T + \lambda_2 (\text{tr } T) I.$$

It can readily be checked that, with the initial condition $T = T_0$ for $t = 0$, this equation has the solution¹

$$(19.12) \quad T = \exp(\lambda_1 t) \{T_0 - \frac{1}{3}(\text{tr } T_0)I\} + \exp[(3\lambda_2 + \lambda_1)t] \cdot \frac{1}{3}(\text{tr } T_0)I.$$

The first term on the right side is the traceless part (deviation) of T and the second term is the mean pressure part of T . The requirement (19.10) implies that λ_1 and $3\lambda_2 + \lambda_1$ are negative. We call

$$(19.13) \quad \tau_d = -\lambda_1^{-1}$$

the *deviation relaxation time* and

¹ It was given essentially by ZAREMBA ([9], p. 12).

$$(19.14) \quad \tau_p = -(3\lambda_2 + \lambda_1)^{-1}$$

the *pressure relaxation time* of the material. With these notations (19.12) becomes

$$(19.15) \quad T = \exp\left(-\frac{t}{\tau_d}\right) \{T_0 - \tfrac{1}{3}(\text{tr } T_0)I\} + \exp\left(-\frac{t}{\tau_p}\right) \cdot \tfrac{1}{3}(\text{tr } T_0)I.$$

This equation states that an initial deviation and initial extra pressure decrease by the factor e^{-1} after times τ_d and τ_p , respectively.

20. Homogeneous stress.¹ a. Dynamical conditions. We assume that the stress is homogeneous, *i.e.* that $S = S(t)$ does not depend on x . We then have $\nabla_x S = 0$, whence $\text{Div } S = 0$. We assume furthermore that there is no extraneous force field ($\mathbf{g} = 0$), and that the body is initially of homogeneous density, *i.e.*

$$(20.1) \quad \rho(x, 0) = \rho_0 = \text{const.}$$

Then the dynamical equation (18.2) implies (see (2.7))

$$(20.2) \quad \dot{\mathbf{v}} = f_t(X, t) = 0.$$

A motion $x = f(X, t)$ satisfies (20.2) if and only if $f(X, t)$ is linear in t . Assuming that the initial position is the reference position ($f(X, 0) = X$), we conclude that (20.2) holds if and only if

$$(20.3) \quad f(X, t) = \mathbf{u}(X)t + X,$$

where $\mathbf{u}(X)$ is an arbitrary continuously differentiable vector field such that f is one-to-one. Here we have (see (2.1), (2.7))

$$(20.4) \quad A = \nabla_x f(X, t) = \nabla_x \mathbf{u}(X)t + I,$$

$$(20.5) \quad \mathbf{v} = \mathbf{u}(X).$$

With the abbreviation

$$(20.6) \quad E = E(X) = \nabla_x \mathbf{u}(X),$$

we get by (2.9), (2.10), (2.13), (2.14)

$$(20.7) \quad A = Et + I, \quad \dot{A} = E,$$

$$(20.8) \quad Q = \dot{A}A^{-1} = E(I + Et)^{-1} = (I + Et)^{-1}E = \frac{1}{t} \{I - (Et + I^{-1})\},$$

$$(20.9) \quad D = \frac{1}{2t} \{2I - (Et + I)^{-1} - (E^t t + I)^{-1}\},$$

¹ This analysis generalizes that of TRUESDELL ([7], §6).

$$(20.10) \quad W = \frac{1}{2t} \{ (E^t t + I)^{-1} - (Et + I)^{-1} \}.$$

By (20.7) we have

$$\det A = \det (Et + I) = 1 + tI_E + t^2 II_E + t^3 III_E,$$

where $I_E = \text{tr } E$, II_E and $III_E = \det E$ are the three principal invariants of E . By (3.3) and (20.1) we thus obtain the density

$$(20.11) \quad \rho = \rho_0(1 + tI_E + t^2 II_E + t^3 III_E)^{-1}, \quad E = E(X).$$

Since the density is to be continuous, we conclude that the motion is possible only for

$$(20.12) \quad 0 \leq t \leq \{\text{Max}_X \lambda(X)\}^{-1},$$

where $-\lambda(X)$ is either the negative proper number of $E(X)$ with greatest absolute value or $\lambda(X) = 0$, if all proper numbers of $E(X)$ are non-negative. The motion is possible for all times ($0 \leq t < \infty$) only if the proper numbers of $E(X)$ are non-negative for all X .

b. Conditions imposed by the constitutive equations. The equations (20.9), (20.10) and (20.11) state that for a motion of the form (20.3) and for homogeneous initial density, the quantities D , W , and ρ depend only on E and t . Since $p = p(\rho)$ is a given function of ρ , consequently, it also depends only on E and t . Therefore, by (18.3) and (18.4), the constitutive equation can be written in the form

$$(20.13) \quad \dot{S} = -SW + WS - \dot{p}I + \mathfrak{F}(D, S - pI, \rho) = \mathfrak{F}^*(E, S, t).$$

This is an ordinary differential equation for S with E as a parameter. For given initial values S_0 , it has a unique solution

$$(20.14) \quad S = \mathfrak{S}(t; E, S_0), \quad S_0 = \mathfrak{S}(0; E, S_0).$$

Thus we have by (20.6)

$$(20.15) \quad S = \mathfrak{S}(t; \nabla_X \mathbf{u}(X), S_0).$$

This result shows that S will in general depend on X if $\nabla_X \mathbf{u}(X)$ depends on X . The hypothesis that the stress is homogeneous, *i.e.* that S does not depend on X , will impose severe restrictions on the admissible velocity distributions $\mathbf{u}(X)$ and the initial stresses S_0 . If $E = \nabla_X \mathbf{u}(X)$ is independent of X , then S is homogeneous, provided the initial stress is homogeneous.

Let E be a constant tensor; then it follows from (20.6) that

$$\mathbf{u}(X) = E(X - x_0),$$

where x_0 is some fixed point in space. By (20.3), the corresponding motion is given by

$$x = E(X - x_0)t + X.$$

By (20.8), the velocity field has the form

$$(20.16) \quad \mathbf{v} = E(Et + I)^{-1}(x - x_0).$$

A motion of this form is called a *homogeneous extension*. We can state the following

Theorem. *For the general class of hygrosteric materials, a homogeneous extension (20.16) is possible for some time interval $0 \leq t \leq t^*$ if the density ρ and the stress S are homogeneous initially. In such a motion ρ and S remain homogeneous for all times in the interval considered, ρ being given by (20.11) and S being obtained by solving the differential equation (20.13).*

From (20.15) it is plausible that motions other than homogeneous extension can yield homogeneous stress only in very exceptional cases.

c. Hydrostatic extension and compression. A homogeneous extension with a velocity field of the form

$$(20.17) \quad \mathbf{v} = h(t)(x - x_0)$$

and an extra stress of the form

$$(20.18) \quad T = q(t)I$$

is called a *hydrostatic extension* if $h(t) > 0$, a *hydrostatic compression* if $h(t) < 0$. The velocity field (20.17) is compatible with (20.16) only if $E = d^{-1}I$, $d = \text{const.}$, i.e. if

$$(20.19) \quad v = \frac{1}{d} \left(\frac{t}{d} + 1 \right)^{-1} (x - x_0) = \frac{1}{t + d} (x - x_0).$$

According to (20.12), the motion (20.19) is possible for

$$(20.20) \quad \begin{array}{lll} 0 \leq t < \infty & \text{if} & d > 0 \quad (\text{extension}) \\ 0 \leq t < |d| & \text{if} & d < 0 \quad (\text{compression}). \end{array}$$

By (20.9), (20.10) and (20.18) we have

$$(20.21) \quad \begin{aligned} D &= \frac{1}{t + d} I, \quad W = 0, \quad \text{tr } D = \frac{3}{t + d}, \\ \text{tr } T &= 3q(t), \quad \text{tr } (DT) = \frac{3q(t)}{t + d}, \quad DT + TD = \frac{2q(t)}{t + d} I. \end{aligned}$$

We confine ourselves to the case of linear fluent or hypo-elastic bodies. Substituting (20.21) into (18.5), we obtain the following differential equation for $q(t)$:

$$(20.22) \quad \dot{q} = \left\{ (3\lambda_2 + \lambda_1) + \frac{\beta}{t+d} \right\} q + \frac{\mu_p}{t+d},$$

where

$$(20.23) \quad \beta = 3(\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4) + 2\alpha_5$$

and

$$(20.24) \quad \mu_p = 3\mu_2 + \mu_1.$$

In the hypo-elastic case, we have by (18.7) $3\lambda_2 + \lambda_1 = 0$. Hence (20.22) becomes

$$(20.25) \quad \dot{q} = \frac{\beta}{t+d} q + \frac{\mu_p}{t+d}.$$

For the initial value $q(0) = q_0$, this equation has the solution

$$(20.26) \quad \begin{aligned} q &= q_0 \left(\frac{t}{d} + 1 \right)^\beta + \frac{\mu_p}{\beta} \left\{ \left(\frac{t}{d} + 1 \right)^\beta - 1 \right\}, & \beta \neq 0, \\ q &= q_0 + \mu_p \log \left(\frac{t}{d} + 1 \right), & \beta = 0. \end{aligned}$$

These solutions, as well as hydrostatic extension and compression for general hypo-elastic bodies, were discussed in detail by C. TRUESDELL [7]. He showed that for physical reasons μ_p must be positive.

In the case of a fluent body, we have by (19.14)

$$3\lambda_2 + \lambda_1 = -\tau_p^{-1} \neq 0, \quad \tau_p > 0.$$

Hence (20.22) becomes

$$(20.27) \quad \dot{q} = \left\{ -\frac{1}{\tau_p} + \frac{\beta}{t+d} \right\} q + \frac{\mu_p}{t+d}.$$

It can easily be checked that, for the initial value $q(0) = q_0$, this differential equation has the solution

$$(20.28) \quad q = q_0 \left(\frac{t}{d} + 1 \right)^\beta \exp \left(-\frac{t}{\tau_p} \right) + \frac{\mu_p}{t+d} g(t),$$

where

$$(20.29) \quad g(t) = \int_0^t \left(\frac{t+d}{u+d} \right)^{\beta+1} \exp \left(\frac{u-t}{\tau_p} \right) du.$$

This solution is valid for all times compatible with (20.20). We distinguish between the cases $d > 0$ and $d < 0$.

$\alpha)$ $d > 0$ (*extension*). For

$$t \ll \tau_p$$

we have $\exp(-t/\tau_p) \approx \exp[(u-t)/\tau_p] \approx 1$; hence (20.26) is then an approximation to (20.28). Assuming that the pressure $p(\rho)$ is small compared to the stress S , *i.e.* $S \approx T$, we conclude that in the beginning of an extension a fluent body behaves like a hypo-elastic body. The larger the pressure relaxation time τ_p , the longer this will hold true. In particular, if $\beta > 0$, the extra-stress must be increased in order to maintain the hydrostatic extension. As the time t reaches the order of magnitude of τ_p , (20.28) gives considerably smaller values for q than (20.26). For $t \rightarrow \infty$, we get the asymptotic formula

$$q = \frac{\mu_p \tau_p}{t + d} (1 + o(t)), \quad \lim_{t \rightarrow \infty} o(t) = 0,$$

as can be shown by applying the rule of Bernoulli-l'Hospital to (20.29). Hence the extra-stress must be sharply decreased for $t \gg \tau_p$. This indicates that, if the extra-stress is maintained, we shall have a tendency towards an extremely fast extension; *i.e.*, we shall have a phenomenon of *yield*.

$\beta)$ $d < 0$ (*compression*). As in the case of extension, for $t \ll \tau_p$ a fluent body acts in the same way as a hypo-elastic body. If $|d| \ll \tau_p$ we shall have no appreciable deviation for all times t , $0 \leq t < |d|$. If $|d|$ is of the order of magnitude of τ_d or larger, for times t near to $|d|$ the extra stress will be considerably smaller than it is for hypo-elastic bodies. However, the limit behavior for $t \rightarrow |d|$ is the same: An analysis of (20.29) shows that

$$(20.30) \quad \begin{aligned} \lim_{t \rightarrow |d|} q &= -\frac{\mu_p}{\beta}, & \beta > 0 \\ \lim_{t \rightarrow |d|} q &= \infty, & \beta < 0 \end{aligned}$$

holds for (20.28) as well as for (20.26). It should be noted that the pressure $p(\rho)$ cannot be assumed to be negligible for $t \rightarrow |d|$, since ρ approaches infinity for $t \rightarrow |d|$. It is reasonable to assume

$$\lim_{\rho \rightarrow \infty} p(\rho) = \infty.$$

Hence, for the total stress $q^* = q - p$, we can infer from (20.30) only that

$$\lim_{t \rightarrow |d|} q^* = -\infty \quad \text{for } \beta > 0.$$

Remark. It follows from (20.29) that

$$\lim_{\tau_p \rightarrow 0} \frac{g(t)}{\tau_p} = 1.$$

Introducing the *pressure viscosity*

$$\eta_p = \tau_p \mu_p$$

and letting $\tau_p \rightarrow 0$ in (20.28) while η_p remains fixed yields

$$\lim_{\tau_p \rightarrow 0} q = \frac{\eta_p}{t + d}.$$

The right hand side is the value for q in the case of a classical viscous fluid.

21. Steady linear flow. a. Basic formulas. In Cartesian coordinates x, y, z , a steady linear flow is defined by the condition that the velocity field be of the form

$$(21.1) \quad \mathbf{v} = \{0, v''(x), 0\}.$$

The field of extra stress is assumed to have the form

$$(21.2) \quad T = \begin{vmatrix} q_x & s & 0 \\ s & q_y & 0 \\ 0 & 0 & q_z \end{vmatrix}, \quad q_i = q_i(x), \quad s = s(x).$$

For convenience in notation we write

$$(21.3) \quad v''(x) = v, \quad \partial_x v''(x) = v', \quad \partial_x q_i(x) = q'_i, \quad \partial_x s(x) = s'.$$

In this special case, we have

$$(21.4) \quad \operatorname{div} \mathbf{v} = 0,$$

$$Q = || q_{ik} || = || \partial_k v_i || = v' \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$(21.5) \quad \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + Q\mathbf{v} = 0, \quad \operatorname{Div} T = \{q'_x, s', 0\}.$$

By (21.4), the equation (18.10) is always satisfied; *i.e.*, any steady linear flow is compatible with incompressibility. By (21.5), the dynamical equation (18.11) reduces to

$$q'_x - \partial_x \varphi = 0, \quad s' - \partial_y \varphi = 0, \quad \partial_z \varphi = 0.$$

By a simple analysis of these equations we infer that s and φ must have the form

$$(21.6) \quad s = -ax + b,$$

$$(21.7) \quad \varphi = -ay + q'_x(x) + c,$$

where a, b, c are constants.

For incompressible linear fluent bodies we have in addition to the dynamical equation the constitutive equation (18.12). In our special case, the quantities occurring in this equation reduce to

$$\begin{aligned} \tilde{T} &= -v' \begin{vmatrix} 0 & q'_x & 0 \\ & 2s & 0 \\ & & 0 \end{vmatrix}, & D &= \frac{1}{2}v' \begin{vmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{vmatrix}, \\ \text{tr } (DT) &= v's, & TD + DT &= \frac{1}{2}v' \begin{vmatrix} 2s & q_x + q_r & 0 \\ & 2s & 0 \\ & & 0 \end{vmatrix} \end{aligned}$$

Hence, with the abbreviation

$$(21.8) \quad q = q_x + q_y + q_z = \text{tr } T,$$

we obtain

$$(21.9) \quad \begin{aligned} -v' \begin{vmatrix} 0 & q_x & 0 \\ & 2s & 0 \\ & & 0 \end{vmatrix} &= \lambda_1 \begin{vmatrix} q_x & s & 0 \\ & q_y & 0 \\ & & q_z \end{vmatrix} + (\mu_1 + \alpha_2 q) \cdot \frac{1}{2}v' \begin{vmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{vmatrix} \\ &+ (\alpha_3 - 1) \cdot \frac{1}{2}v' \begin{vmatrix} 2s & q_x + q_y & 0 \\ & 2s & 0 \\ & & 0 \end{vmatrix} + (\lambda_2 q + \alpha_1 v's) \begin{vmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{vmatrix}. \end{aligned}$$

Two of the six scalar equations corresponding to (21.9) reduce to $0 = 0$. The other four are

$$(21.10) \quad 0 = \lambda_1 q_x + \lambda_2 q + (\alpha_3 + \alpha_1 - 1)v's,$$

$$(21.11) \quad 0 = \lambda_1 q_y + \lambda_2 q + (\alpha_3 + \alpha_1 + 1)v's,$$

$$(21.12) \quad 0 = \lambda_1 q_x + \lambda_2 q + \alpha_1 v's,$$

$$(21.13) \quad 0 = \lambda_1 s + \{\mu_1 + \alpha_2 q + \alpha_3(q_x + q_y) + (q_x - q_y)\} \cdot \frac{1}{2}v'.$$

Adding the first three of these yields (see (21.8))

$$(21.14) \quad 0 = (\lambda_1 + 3\lambda_2)q + (2\alpha_3 + 3\alpha_1)v's.$$

Since by (18.6) $\lambda_1 \neq 0$, we can calculate $q_x + q_y$ and $q_x - q_y$ from (21.10) and (21.11) and substitute these into (21.13). We thus obtain

$$(21.15) \quad 0 = \lambda_1 s + \{\mu_1 + \lambda_1^{-1}(\lambda_1 \alpha_2 - 2\lambda_2 \alpha_3)q - 2\lambda_1^{-1}[\alpha_3(\alpha_1 + \alpha_3) - 1]v' s\} \cdot \frac{1}{2}v'.$$

Since by (18.6) also $\lambda_1 + 3\lambda_2 \neq 0$, we can solve (21.14) for q and put it into (21.15). With the abbreviations

$$(21.16) \quad \begin{aligned} \tau_d &= -\lambda_1^{-1}, & \eta &= \tau_d \mu_1 = -\mu_1 \lambda_1^{-1} \quad (\text{viscosity}), \\ \alpha &= 1 + \frac{2\lambda_2 \alpha_3 - \lambda_1 \alpha_2}{2(\lambda_1 + 3\lambda_2)} (2\alpha_3 + 3\alpha_1) - \alpha_3(\alpha_1 + \alpha_3), \end{aligned}$$

we get

$$0 = s + (-\eta + 2\alpha\tau_d^2 v' s) \cdot \frac{1}{2}v'.$$

Solving this equation for v' or for s yields the following two basic formulas:

$$(21.17) \quad v' = \frac{4s}{\eta} \cdot \frac{1}{1 \pm \sqrt{1 - 16\alpha\tau_d^2 \eta^{-2} s^2}},$$

$$(21.18) \quad s = \frac{1}{2}\eta v' \cdot \frac{1}{1 + \alpha\tau_d^2 v'^2}.$$

For the normal extra-stresses we obtain from (21.10)–(21.12) and (21.14) the expressions

$$(21.19) \quad \begin{aligned} q_x &= \tau_d(-\alpha_0 + \alpha_3 + \alpha_1 - 1)v' s, \\ q_y &= \tau_d(-\alpha_0 + \alpha_3 + \alpha_1 + 1)v' s, \\ q_z &= \tau_d(-\alpha_0 + \alpha_1)v' s, \end{aligned}$$

where

$$\alpha_0 = \frac{\lambda_2}{\lambda_1 + 3\lambda_2} (2\alpha_3 + 3\alpha_1).$$

b. Simple shearing flow. We consider a flow between two parallel infinite plates *I* and *II* (see Figure 9). The plate *I* is assumed to be at rest while the plate *II* is moving with constant velocity V in a direction parallel to plate *I*. The material is assumed to adhere to the plates.

We assume that φ has no gradient in the direction of flow. Then we have $a = 0$ in (21.7). Hence, by (21.6), the shearing stress is constant:

$$(21.20) \quad s = b = \text{const.}$$

From (21.17) it follows then that v' is also constant, *i.e.* that $v(x)$ is linear in x . The conditions $v(0) = 0$, $v(d) = V$ (see Figure 9) imply that v has the form

$$(21.21) \quad v = \frac{V}{d} x, \quad v' = \frac{V}{d} = \text{const.}$$

We see from (21.19) that the normal extra stresses are also constant. In particular, we have $q'_x = 0$; hence, by (21.7)

$$(21.22) \quad \varphi = p + \psi = c = \text{const.}$$

If there are no extraneous forces ($\psi = 0$), we have constant pressure and therefore constant normal stresses

$$(21.23) \quad s_i = q_i - p, \quad i = x, y, z.$$

FIGURE 9

Substituting (21.21) into (21.18), we get the following relation giving the shearing stress s which is necessary to produce the plate velocity V :

$$(21.24) \quad s = \frac{1}{2}\eta \cdot \left(\frac{V}{d}\right) \cdot \frac{1}{1 + \alpha \tau_d^2 \left(\frac{V}{d}\right)^2}.$$

We analyse this equation by plotting the shearing stress s against the rate of shear V/d . We have to distinguish between the two cases $\alpha > 0$ and $\alpha < 0$. The case $\alpha = 0$ yields

$$(21.25) \quad s = \frac{1}{2}\eta \left(\frac{V}{d}\right),$$

which is the same formula as for classical viscous fluids¹.

α . The case when $\alpha > 0$. For materials of this type we introduce the material constant

$$(21.26) \quad \tau_s = \sqrt{\alpha \tau_d},$$

which will be called *shearing yield time*. Then (21.24) becomes

¹ Many textbooks use the notation 2μ for the viscosity η .

$$(21.27) \quad s = \frac{1}{2}\eta \cdot \frac{V}{d} \cdot \frac{1}{1 + \tau_s^2 \left(\frac{V}{d}\right)^2},$$

which yields the graph in Figure 10. For

$$\frac{V}{d} \leq \frac{1}{4} \tau_s^{-1}$$

FIGURE 10

the classical formula (21.25) (represented by the dotted line in Figure 10) is a good approximation to the formula (21.27), the error for $V/d = \frac{1}{4}\tau_s^{-1}$ being smaller than 6%. For

$$\frac{1}{4}\tau_s^{-1} < V/d < \tau_s^{-1}$$

the rate of shear V/d still increases with the shearing stress s , but a given stress s produces a rate of shear appreciably higher than the one predicted by the classical formula. If V/d exceeds the critical value $(V/d)_{cr} = \tau_s^{-1}$, we have a *yield-like phenomenon*: While the stress s decreases slowly, the rate of shear V/d increases very rapidly. It is rather improbable that the branch $V/d > \tau_s^{-1}$ can correspond to a physical situation. We can expect that a steady flow cannot be maintained for $V/d > \tau_s^{-1}$, and that the flow becomes unstable if the shearing stress reaches the critical value $s_{cr} = \frac{1}{4}\eta\tau_s^{-1}$.

β. *The case when $\alpha < 0$.* For materials of this type we introduce the constant

$$(21.28) \quad \tau'_s = \sqrt{|\alpha|} \tau_d,$$

which will be called *shearing limit time*. Then (21.24) becomes

FIGURE 11

$$(21.29) \quad s = \frac{1}{2}\eta \cdot \frac{V}{d} \cdot \frac{1}{1 - \tau_s'^2 \left(\frac{V}{d}\right)^2},$$

which yields the graph of Figure 11. For

$$V/d \leq \frac{1}{4}(\tau'_s)^{-1}$$

the classical formula (21.25) is again a good approximation (error smaller than 7 %). For

$$V/d > \frac{1}{4}(\tau'_s)^{-1}$$

a shearing stress appreciably greater than the one predicted by the classical formula must be applied in order to produce a given rate of shear. The material becomes more and more resistant to the shearing stress. No matter how large a stress s is applied, the rate of shear V/d will never exceed the limit value

$$\left(\frac{V}{d}\right)_{\text{lim}} = (\tau'_s)^{-1}.$$

The branch $V/d > (\tau'_s)^{-1}$ of the graph of Figure 11 cannot have any physical meaning, because a rate of shear $V/d > (\tau'_s)^{-1}$ could not be reached by a continuous motion starting from rest. Moreover, it is physically impossible that a positive shearing stress s produces a negative rate of shear V/d .

Since v' and s are constant, it follows from (21.19) that the normal extra stresses are also constant. In the case of no extraneous forces ($\psi = 0$), the pressure p is an arbitrary constant (see (21.22)), and hence the normal stresses, as given by (21.23), are also constant. We can choose

$$p = q_y,$$

so that there is no normal stress in the direction of flow. We thus have by (21.19), (21.21) and (21.23)

$$s_x = -2\tau_a s \frac{V}{d}, \quad s_y = 0, \quad s_z = -\tau_a(\alpha_3 + 1)s \frac{V}{d}.$$

In the case of a classical viscous fluid no normal stresses are necessary in order to have a steady flow. Here we have a different situation: Since $\tau_a > 0$, the stress s_x normal to the plates must be negative. This indicates that in the absence of this stress the plates will tend to spread apart. The stress s_z normal to the direction of flow and parallel to the plates is also non-zero, its sign depending on the sign of $\alpha_3 + 1$.

For a slow flow, *i.e.* for $V/d \ll \tau_a^{-1}$, the normal stresses will be small compared to the shearing stress (assuming for s_z that α_3 is of the order of magnitude of 1).

c. Flow between two fixed plates. For a flow between two plates at rest (Figure 12) we have to take $a \neq 0$ in (21.6) and (21.7). It is sufficient to consider the case $a > 0$, which will yield a flow in the positive y -direction. If we choose the origin as in Figure 12, it follows for reasons of symmetry that $b = 0$ in (21.6); hence

$$(21.30) \quad s = -ax, \quad a > 0.$$

The basic formula (21.17) becomes

$$(21.31) \quad v' = -\frac{4a}{\eta} \frac{x}{1 \pm \sqrt{1 - \kappa a^2 x^2}}, \quad \kappa = 16\alpha\tau_d^2 \eta^{-2}.$$

If we put

$$(21.32) \quad u = 1 \pm \sqrt{1 - \kappa a^2 x^2},$$

the integration of (21.31) yields

$$(21.33) \quad v = \frac{4}{\kappa\eta a} \int \frac{u-1}{u} du = \frac{4}{\kappa\eta a} (u - \log u) + \text{const.}$$

FIGURE 12

Since we must require the velocity to be continuous for $x = 0$, we have to take the positive square root in (21.32). The condition that the material adheres to the plates ($v(\pm d) = 0$, see Figure 12) determines the constant in (21.33). We finally obtain

$$(21.34) \quad v = \frac{4}{\kappa\eta a} \left\{ (u - u_d) + \log \frac{u}{u_d} \right\},$$

$$u = 1 + \sqrt{1 - \kappa a^2 x^2}, \quad u_d = 1 + \sqrt{1 - \kappa a^2 d^2}, \quad \kappa = 16\alpha\tau_d^2 \eta^{-2}.$$

Since u and u_d must be real, we see that a steady flow is possible only if

$$1 - \kappa a^2 d \geq 0.$$

If this condition is assumed to hold, from (21.31) follows

$$\begin{aligned} v' &< 0 & \text{for} & \quad 0 < x \leq d, \\ v' &= 0 & \text{for} & \quad x = 0, \\ v' &> 0 & \text{for} & \quad 0 > x \geq -d. \end{aligned}$$

Hence we have a velocity profile as indicated in Figure 12. The greatest velocity $v_0 = v(0)$ is given by

$$(21.35) \quad v_0 = \frac{4}{\kappa\eta a} \left\{ (2 - u_d) - \log \frac{2}{u_d} \right\}.$$

We shall not go into further detail, since the results for the more important case of Poiseuille flow will turn out to be completely analogous to those obtained here.

22. Poiseuille flow. a. Basic formulas. We investigate now a steady flow through a circular pipe (see Figure 13). Using cylindrical coordinates (see §18c), we assume the velocity field to be of the form

FIGURE 13

$$(22.1) \quad \{v^i\} = \{0, 0, v^z(r)\}.$$

The extra-stress is assumed to have the following form:

$$(22.2) \quad ||t^{ik}|| = \begin{vmatrix} q_r & 0 & s \\ 0 & r^{-2}q_\theta & 0 \\ 0 & 0 & q_z \end{vmatrix}, \quad q_i = q_i(r), \quad s = s(r).$$

For convenience in notation, we write

$$(22.3) \quad v^z(r) = v, \quad \partial_r v^z(r) = v', \quad \partial_r q_i(r) = q'_i, \quad \partial_r s(r) = s'.$$

From (18.24), (18.25) and (18.27), it follows that

$$(22.4) \quad \operatorname{div} \mathbf{v} = \nabla_i v^i = 0,$$

$$(22.5) \quad \dot{v}^i = 0, \quad \{\nabla_j t^{ij}\} = \{q'_r + r^{-1}(q_\theta - q_r), 0, s' + r^{-1}s\}.$$

By (22.4), the condition of incompressibility (18.21) is automatically satisfied.
By (22.5), the dynamical equation (18.22) yields

$$q'_i + r^{-1}(q_r - q_\theta) - \partial_r \varphi = 0,$$

$$r^{-2} \partial_\theta \varphi = 0,$$

$$s' + r^{-1} s - \partial_z \varphi = 0.$$

A simple analysis of these equations shows that they can be reduced to

$$(22.6) \quad s = -\frac{1}{2}ar + br^{-1},$$

$$(22.7) \quad \varphi = -za + \sigma(r),$$

$$(22.8) \quad q'_r + r^{-1}(q_r - q_\theta) = \sigma'(r),$$

where a and b are constants and $\sigma(r)$ is a function of r .

By (22.1), (22.2), (18.20) and (18.26), we obtain for the quantities occurring in the constitutive equation (18.23)

$$\|d_{ik}\| = \|d^{ik}\| = \|d^i_k\| = \|d_i^k\| = \frac{1}{2}v' \begin{vmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{vmatrix},$$

$$\|t^{ik}\| = -v' \begin{vmatrix} 0 & 0 & q_r \\ & 0 & 0 \\ & & 2s \end{vmatrix}, \quad \text{tr}(DT) = d^{ij}t_{ji} = v' s,$$

$$\|t^i_j d^{jk} + t^k_j d^{ij}\| = \frac{1}{2}v' \begin{vmatrix} 2s & 0 & q_r + q_z \\ & 0 & 0 \\ & & 2s \end{vmatrix}.$$

Hence, with the abbreviation

$$q = q_r + q_\theta + q_z = \text{tr } T,$$

by (18.23) we have for incompressible linear fluent bodies

$$\begin{aligned} -v' \begin{vmatrix} 0 & 0 & q_r \\ & 0 & 0 \\ & & 2s \end{vmatrix} &= \lambda_1 \begin{vmatrix} q_r & 0 & s \\ & r^{-2}q_\theta & 0 \\ & & q_z \end{vmatrix} + (\mu_1 + \alpha_2 q) \cdot \frac{1}{2}v' \begin{vmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{vmatrix} \\ &+ (\alpha_3 - 1) \cdot \frac{1}{2}v' \begin{vmatrix} 2s & 0 & q_r + q_z \\ & 0 & 0 \\ & & 2s \end{vmatrix} + (\lambda_2 q + \alpha_1 v' s) \begin{vmatrix} 1 & 0 & 0 \\ & r^{-2} & 0 \\ & & 0 \end{vmatrix}. \end{aligned}$$

This matrix equation corresponds to the following four scalar equations:

$$\begin{aligned}
 (22.9) \quad & 0 = \lambda_1 q_r + \lambda_2 q + (\alpha_3 + \alpha_1 - 1)v' s, \\
 & 0 = \lambda_1 q_z + \lambda_2 q + (\alpha_3 + \alpha_1 + 1)v' s, \\
 & 0 = \lambda_1 q_\theta + \lambda_2 q + \alpha_1 v' s, \\
 & 0 = \lambda_1 s + \{\mu_1 + \alpha_2 q + \alpha_3(q_r + q_z) + (q_r - q_z)\} \cdot \frac{1}{2}v'.
 \end{aligned}$$

The above system of equations has the same form as the system (21.10)–(21.13). Hence the basic formulas (21.17) and (21.18) are valid also for the rotationally symmetric flow (22.1), except that the meaning of v' and s is now given by (22.2) and (22.3) instead of by (21.2) and (21.3).

For flow through a circular pipe, we must require that $s(r)$ is continuous for $r = 0$. Hence we must put $b = 0$ in (22.6). Assuming $a > 0$ (flow in the positive z -direction), we get

$$(22.10) \quad s = -\frac{1}{2}ar, \quad a > 0.$$

We assume that we have a constant extraneous force g acting parallel to the pipe (*e.g.* gravity). By (18.9) and (22.7), it then follows that

$$(22.11) \quad p = \varphi - \rho\psi = -(a - \rho g)z + \sigma(r).$$

Let l be the distance between two cross-sections I and II of the pipe. Then the difference of the average pressures $\bar{p}_I$ and $\bar{p}_{II}$ at I and II, respectively, is

$$(22.12) \quad \bar{p}_I - \bar{p}_{II} = (a - \rho g)l,$$

since the contributions of $\sigma(r)$ cancel. For the normal stresses, we get by (22.11)

$$(22.13) \quad s_i = q_i - p = (a - \rho g)z + q_i(r) - \sigma(r), \quad i = r, \theta, z.$$

This expression shows that the difference

$$(22.14) \quad \Delta\bar{p}_z = \bar{p}_{zI} - \bar{p}_{zII}$$

of the average normal pressures in the flow direction,

$$\bar{p}_{zI} = -\bar{s}_{zI}, \quad \bar{p}_{zII} = -\bar{s}_{zII},$$

is equal to the pressure head $\bar{p}_I - \bar{p}_{II}$. Hence (22.12) yields

$$(22.15) \quad a = \rho g + \frac{\Delta\bar{p}_z}{l}.$$

With a replaced by $\frac{1}{2}a$ and x replaced by r , the expression (21.30) is formally the same as (22.10). Hence, with the condition of adherence $v(R) = 0$, we obtain by analogy to (21.34)

$$(22.16) \quad v = \frac{2}{\nu\eta a} \left\{ (u - u_R) - \log \frac{u}{u_R} \right\},$$

$$(22.17) \quad u = 1 + \sqrt{1 - \nu a^2 r^2}, \quad u_R = 1 + \sqrt{1 - \nu a^2 R^2}, \quad \nu = 4\alpha\tau_d^2\eta^{-2} = \frac{1}{4}\kappa.$$

In order for a steady flow to be possible, we have to assume

$$1 - \nu a^2 R^2 \geq 0, \quad a^2 \leq \nu^{-1} R^{-2}.$$

By (21.17), we then have

$$\begin{aligned} v' &= 0 & \text{for } r &= 0, \\ v' &< 0 & \text{for } 0 < r &\leq R. \end{aligned}$$

Hence the velocity profile has the form indicated in Figure 13, the greatest velocity $v_0 = v(0)$ being

$$(22.18) \quad v_0 = \frac{2}{\nu\eta a} \left\{ (2 - u_R) - \log \frac{2}{u_R} \right\}.$$

b. Rate of volume discharge, mean velocity. The volume discharge per unit of time is

$$(22.19) \quad Q = 2\pi \int_0^R v(r) r \, dr.$$

Substituting (22.16), (22.17) for $v(r)$ in (22.19) and integrating, we obtain after a rather elaborate computation

$$(22.20) \quad Q = \frac{\pi R^4 a}{4\eta} \cdot \frac{4}{3} \cdot \frac{2u_R - 1}{u_R^2}.$$

The mean velocity is

$$(22.21) \quad \bar{v} = \frac{Q}{\pi R^2} = \frac{R^2 a}{4\eta} \cdot \frac{4}{3} \cdot \frac{2u_R - 1}{u_R^2}, \quad u_R = 1 + \sqrt{1 - 4\alpha(\tau_d \eta^{-1} a R)^2}.$$

As in the case of simple shearing flow, we must distinguish between the cases $\alpha > 0$ and $\alpha < 0$ in order to analyse (22.21). In the case $\alpha = 0$ we have $u_R = 2$, *i.e.* (22.21) reduces to

$$(22.22) \quad \bar{v} = \frac{R^2 a}{4\eta},$$

which is the same formula as for classical viscous fluids¹.

For the sake of simplicity we assume that there are no extraneous forces. By (22.15)

$$a = \frac{\Delta \bar{p}_z}{l}$$

is then the pressure gradient in the flow direction.

¹ Many textbooks use the notation 2μ for the viscosity η .

α . The case when $\alpha > 0$. In terms of the shearing yield time $\tau_s = \sqrt{\alpha\tau_d}$ (see (21.26)) the formula (22.21) becomes

$$(22.23) \quad \begin{aligned} \bar{v} &= \frac{R^2}{4\eta} a \cdot \frac{4}{3} \cdot \frac{2(1 + \sqrt{1 - \xi^2}) - 1}{(1 + \sqrt{1 - \xi^2})^2}, \\ &= \frac{R^2}{4\eta} \cdot a \left\{ 1 + \frac{\xi^2}{6} \cdot \frac{2 - \frac{3}{4}\xi^2}{(1 - \xi^2)^{\frac{3}{2}} + (1 - \frac{3}{2}\xi^2 + \frac{3}{8}\xi^4)} \right\}, \quad \xi = \frac{2\tau_s}{\eta} aR. \end{aligned}$$

FIGURE 14

Plotting the mean velocity $\bar{v}$ against the pressure gradient, we get the graph in Figure 14. For

$$a \leq \frac{\eta}{6\tau_s R}, \quad \bar{v} \leq \frac{R}{24\tau_s},$$

the classical formula (22.22) is a good approximation to (22.23), the error for $\bar{v}$ being smaller than 3%. A second approximation to (22.23) is given by

$$\bar{v} \approx \frac{R^2}{4\eta} a \left(1 + \frac{\xi^2}{6} \right), \quad \xi = 2\tau_s \eta^{-1} aR.$$

For

$$\frac{\eta}{6\tau_s R} < a < \frac{\eta}{2\tau_s R}, \quad \frac{R}{24\tau_s} < \bar{v} < \frac{R}{6\tau_s},$$

the mean velocity $\bar{v}$ produced by a given pressure gradient a is appreciably larger than the one predicted by the classical formula (22.22). However, the deviation is never greater than 34 %.

The formula (22.23) yields real values for the velocity $\bar{v}$ only if $|\xi| \leq 1$, *i.e.* if

$$(22.24) \quad a \leq \frac{\eta}{2\tau_s R}.$$

We see that the largest possible value of the pressure gradient and the mean velocity are

$$(22.25) \quad a_{\text{cr}} = \frac{\eta}{2\tau_s R}, \quad \bar{v}_{\text{cr}} = \frac{R}{6\tau_s}.$$

If the pressure gradient a exceeds the critical value a_{cr} , the flow necessarily becomes unsteady. The critical velocity, $\bar{v}_{\text{cr}}$, which occurs when a reaches a_{cr} , is exactly equal to $\frac{4}{3}$ of the velocity predicted by the classical formula (22.22).

§. *The case when $\alpha < 0$.* Introducing the shearing limit time $\tau'_s = \sqrt{|\alpha|} \tau_a$ (see (21.28)), we put (22.21) into the form

$$(22.26) \quad \begin{aligned} \bar{v} &= \frac{R^2}{4\eta} \cdot a \cdot \frac{4}{3} \cdot \frac{2(1 + \sqrt{1 + \zeta^2} - 1)}{(1 + \sqrt{1 + \zeta^2})^2} \\ &= \frac{R^2}{4\eta} \cdot a \left[1 - \frac{\zeta^2}{6} \frac{2 + \frac{2}{3}\zeta^2}{(1 + \zeta^2)^{\frac{3}{2}} + (1 + \frac{2}{3}\zeta^2 + \frac{2}{3}\zeta^4)} \right], \quad \zeta = 2\tau'_s \eta^{-1} aR. \end{aligned}$$

This relation between the mean pressure gradient a and the velocity $\bar{v}$ is represented by the graph in Figure 15. For

$$a \leq \frac{\eta}{6\tau'_s R}, \quad \bar{v} \leq \frac{R}{24\tau_s}$$

the classical formula (22.22) is again a good approximation to (22.26) (error smaller than 3 %). A second approximation is now

$$\bar{v} \approx \frac{R^2 a}{4\eta} \left(1 - \frac{\zeta^2}{6} \right), \quad \zeta = \frac{2\tau'_s}{\eta} aR.$$

For

$$\frac{\eta}{6\tau'_s R} < a, \quad \frac{R}{24\tau'_s} < \bar{v},$$

the deviation from the classical law is no longer negligible, the value of $\bar{v}$ being smaller than the one predicted by the classical formula (22.22). If the pressure gradient a increases, the flow becomes more and more resistant. No matter how large a is, the mean velocity never exceeds the limit value

$$\bar{v}_{\text{lim}} = \frac{1}{3} \frac{R}{\tau'_s}.$$

c. *Normal stresses.* By the analogy between the equations (21.10)–(21.13) and the equations (22.9), the formulas for the normal stresses (21.10) become valid for Poiseuille flow if we replace q_x, q_y, q_z there by q_r, q_z, q_θ , respectively. In particular, we get

$$q_r - q_\theta = \tau_d(\alpha_3 - 1)v' s.$$

FIGURE 15

From this result and from (22.8), (22.10), we infer that

$$q'_r - \sigma' = \frac{1}{2} a \tau_d (\alpha_3 - 1) v';$$

hence

$$q_r - \sigma = \frac{1}{2} a \tau_d (\alpha_3 - 1) v + \text{const.}$$

Since the pressure p (and therefore, by (22.11), $\sigma(r)$) is determined only up to an arbitrary constant, we can choose this constant so that

$$\sigma = q_r - \frac{1}{2} a \tau_d (\alpha_3 - 1) v.$$

Substituting this expression into (22.13), we get the normal stresses

$$(22.27) \quad s_i = (a - \rho g)z + (q_i - q_r) + \frac{1}{2} a \tau_d (\alpha_3 - 1) v.$$

Observing (21.19), we get

$$(22.28) \quad q_z - q_r = 2\tau_d v' s, \quad q_\theta - q_r = -\tau_d(\alpha_3 - 1)v' s.$$

By (21.17) and (22.10) we obtain

$$(22.29) \quad v' s = \frac{a^2 r^2}{\eta u}, \quad u = 1 + \sqrt{1 - \nu a^2 r^2}, \quad \nu = 4\alpha \frac{\tau_d^2}{\eta^2}.$$

Combining (22.27), (22.28) and (22.29) yields the final result

$$(22.30) \quad \begin{aligned} s_r &= (a - \rho g)z + \frac{1}{2}a\tau_d(\alpha_3 - 1)v, \\ s_z &= (a - \rho g)z + \frac{1}{2}a\tau_d \left((\alpha_3 - 1)v + \frac{4ar^2}{\eta u} \right), \\ s_\theta &= (a - \rho g)z + \frac{1}{2}a\tau_d(\alpha_3 - 1) \left(v - \frac{2ar^2}{\eta u} \right), \end{aligned}$$

where v and u are given by (22.6) and (22.7).

At the center of the pipe ($r = 0$), we have

$$(22.31) \quad (s_r)_0 = (s_z)_0 = (s_\theta)_0 = (a - \rho g)z + \frac{1}{2}a\tau_d(\alpha_3 - 1)v_0,$$

where v_0 is given by (22.8). At the wall of the pipe ($r = R$, $v = 0$) we get

$$(22.32) \quad \begin{aligned} (s_r)_R &= (a - \rho g)z, \\ (s_z)_R &= (a - \rho g)z + \frac{2\tau_d a^2 R^2}{\eta u_R}, \\ (s_\theta)_R &= (a - \rho g)z - (\alpha_3 - 1) \frac{\tau_d a^2 R^2}{\eta u_R}, \\ u_R &= 1 + \sqrt{1 - \nu a^2 R^2}, \quad \nu = 4\alpha \tau_d^2 \eta^{-2}. \end{aligned}$$

The formulas (22.30) show that the normal stresses deviate from the value

$$(22.33) \quad s_r = s_z = s_\theta = -p = (a - \rho g)z,$$

which is valid for classical viscous fluids. The excess stresses depend on r , and they have different values at the center (see (22.31)) and at the periphery (see (22.32)) of the pipe. This dependence on r is rather complicated, and nothing can be said about it in general without specification of the material constants α , α_3 , τ_d , η . For slow flow, the excess stresses will be small, so that the deviation from the classical value (22.33) can be neglected.

23. Couette flow. A steady flow between two concentric cylinders with a velocity field of the form (cylindrical coordinates)

$$(23.1) \quad \{v^i\} = \{0, v^\theta(r), 0\}$$

is called a Couette flow (see Figure 16). The extra stress is assumed to have the following form:

$$(23.2) \quad || t^{ik} || = \begin{vmatrix} q_r & r^{-1} s & 0 \\ r^{-1} s & r^{-2} q_\theta & 0 \\ 0 & 0 & q_z \end{vmatrix}, \quad q_i = q_i(r), \quad s = s(r).$$

We proceed here in the same way as we did in §21 and §22. We omit the details of the calculations, which are straightforward but rather elaborate. For convenience in notation we again denote derivatives with respect to r by primes and write

FIGURE 16

$$(23.3) \quad v = v^\theta(r)$$

for the angular velocity at the distance r from the center.

If there are no extraneous forces, the dynamical equations (18.22) reduce to

$$q'_r - r^{-1}(q_r + q_\theta) - \partial_r p = -\rho r v^2$$

$$r s' + 2s - \partial_\theta p = 0$$

$$\partial_z p = 0.$$

A simple analysis shows that the pressure depends only on r and that the above equations are equivalent to

$$(23.4) \quad s = r^{-2} a, \quad a = \text{const.}$$

$$(23.5) \quad q'_r - p' - r^{-1}(q_r + q_\theta) = -\rho r v^2.$$

The torque per unit height (z -direction) exerted on the cylinder $r = \text{const.}$ by the fluid outside it is

$$(23.6) \quad r \cdot (2\pi r) \cdot s = 2\pi a$$

and is therefore independent of r . Without loss of generality we assume that $a > 0$.

By using the abbreviation

$$(23.7) \quad q = q_r + q_\theta + q_z = \text{tr } T$$

we put the constitutive equation (18.23) into the form

$$\begin{aligned} -v' \begin{vmatrix} 0 & q_r & 0 \\ & 2r^{-1}s & 0 \\ & & 0 \end{vmatrix} &= \lambda_1 \begin{vmatrix} q_r & r^{-1}s & 0 \\ & r^{-2}q_\theta & 0 \\ & & q_z \end{vmatrix} + (\mu_1 + \alpha_2 q) \cdot \frac{1}{2} v' \begin{vmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{vmatrix} \\ &+ (\alpha_3 - 1) \cdot \frac{1}{2} v' \begin{vmatrix} 2rs & q_\theta + q_r & 0 \\ & 2r^{-1}s & 0 \\ & & 0 \end{vmatrix} + (\lambda_2 q + \alpha_1 r v' s) \begin{vmatrix} 1 & 0 & 0 \\ & r^{-2} & 0 \\ & & 1 \end{vmatrix}; \end{aligned}$$

hence

$$\begin{aligned} 0 &= \lambda_1 q_r + \lambda_2 q + (\alpha_3 + \alpha_1 - 1) r v' s, \\ 0 &= \lambda_r q_\theta + \lambda_2 q + (\alpha_3 + \alpha_1 + 1) r v' s, \\ (23.8) \quad 0 &= \lambda_1 q_z + \lambda_2 q + \alpha_1 r v' s, \\ 0 &= \lambda_1 s + \{\mu_1 + \alpha_2 q + \alpha_3(q_r + q_\theta) + (q_r - q_\theta)\} \cdot \frac{1}{2} r v'. \end{aligned}$$

These equations can be obtained formally from (21.10)–(21.13) by replacing q_x, q_y, q_z, v' therein by $q_r, q_\theta, q_z, r v'$, respectively. Hence we infer from (21.17) and (23.4) that the following formula holds:

$$(23.9) \quad v' = \frac{4a}{\eta r^3} \cdot \frac{1}{1 \pm \sqrt{1 - \kappa a^2 r^{-2}}}, \quad \kappa = 16\alpha\tau_a^2 \eta^{-1}.$$

We shall consider further only the special case when the inner cylinder ($r = r_1$) is at rest and the outer cylinder ($r = r_2$) rotates with the angular velocity V (see Figure 16). Assuming that the material adheres to the walls, we have

$$(23.10) \quad v(r_1) = 0, \quad v(r_2) = V, \quad r_2 > r_1.$$

According to (23.6), $2\pi a$ is the torque per unit height which must be applied to the outer cylinder in order to produce the angular velocity V . As in the case of simple shearing flow and Poiseuille flow, we have to distinguish between the cases $\alpha > 0$ and $\alpha < 0$. For $\alpha = 0$, the integration of (23.9) together with the conditions (23.10) yields

$$(23.11) \quad V = \frac{a}{\eta} \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}.$$

Thus just as for shearing flow and Poiseuille flow so too in Couette flow the case $\alpha = 0$ yields the same formula as for classical viscous fluids.

α . The case when $\alpha > 0$. It follows from (23.9) and (23.10) that

$$(23.12) \quad \begin{aligned} V &= \frac{1}{2\tau_s} \left\{ 2 \operatorname{arc} \operatorname{tg} w_2 + \frac{1}{w_2} - 2 \operatorname{arc} \operatorname{tg} w_1 - \frac{1}{w_1} \right\} \\ &= \frac{1}{2\tau_s} \left\{ 2 \operatorname{arc} \operatorname{tg} \frac{w_2 - w_1}{1 + w_1 w_2} - \frac{w_2 - w_1}{w_1 w_2} \right\}, \end{aligned}$$

where

$$(23.13) \quad w_1 = \frac{r_1^2}{a\gamma} + \sqrt{\left(\frac{r_1^2}{a\gamma}\right)^2 - 1}, \quad w_2 = \frac{r_2^2}{a\gamma} + \sqrt{\left(\frac{r_2^2}{a\gamma}\right)^2 - 1}, \quad \gamma = 4\tau_s \eta^{-1},$$

τ_s being the shearing yield time (21.26). The equation (23.12) is a relation between the angular velocity V of the outer cylinder and a , which is, up to the factor 2π , the torque necessary to produce this velocity. To a given a correspond two different values of V , according to which sign we give the square roots in (23.13). We take both roots as positive; this choice is characterized by the condition

$$\lim_{a \rightarrow 0} V = 0,$$

and hence the corresponding flow can be obtained from the state of rest ($V = 0$, $a = 0$) by a continuous transition. As in the case of simple shearing flow, we cannot expect that the choice of the negative sign for the roots in (23.13) corresponds to a physical situation.

The equation (23.12) yields a real value for V only if

$$a \leq \gamma^{-1} r_1^2 = \frac{\eta}{4\tau_s} r_1^2.$$

For values of a larger than the *critical torque*

$$a_{\text{cr}} = \frac{\eta}{4\tau_s} r_1^2$$

no steady flow is possible. The corresponding *critical angular velocity* is

$$(23.14) \quad V_{\text{cr}} = \frac{1}{2\tau_s} \left\{ 2 \arctan \frac{w^* - 1}{w^* + 1} + \frac{1}{w^* - 1} \right\},$$

where

$$w^* = \left(\frac{r_2}{r_1} \right)^2 + \sqrt{\left(\frac{r_2}{r_1} \right)^4 - 1}.$$

An analysis of (23.14) shows that

$$V_{\text{cr}} = \frac{1}{\tau_s} \cdot \frac{d}{r_2} + o\left(\frac{d}{r_2}\right),$$

where

$$d = r_2 - r_1.$$

Comparison with the results of §21b α shows that, for $d \ll r_2$, we get approximately the same value as for simple shearing flow, the rate of shear being $V_{\text{cr}} r_2 d^{-1}$.

We obtain from (23.12) and (23.13)

$$(23.15) \quad \begin{aligned} \frac{dV}{da} &= \frac{1}{2\tau_s a} \left(\frac{1}{w_1} - \frac{1}{w_2} \right) > 0, \\ \frac{d^2 V}{da^2} &= \frac{1}{\tau_s a^2} \left(\frac{1}{w_1(w_1^2 - 1)} - \frac{1}{w_2(w_2^2 - 1)} \right) > 0, \\ \left(\frac{dV}{da} \right)_{a=a_{\text{cr}}} &= \frac{1}{2\tau_s a_{\text{cr}}} \left(1 - \frac{1}{w^*} \right), \quad \lim_{a \rightarrow a_{\text{cr}}} \frac{d^2 V}{da^2} = \infty, \\ \left(\frac{dV}{da} \right)_{a=0} &= \frac{1}{\eta} \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}, \quad \left(\frac{d^2 V}{da^2} \right)_{a=0} = 0. \end{aligned}$$

These relations show that the graph of $V = V(a)$ has the form shown in Figure 17. It is clear from (23.15) that the classical law (23.11) (represented by the dotted line in Figure 17) is a good approximation to (23.12) for small a , *i.e.* for small angular velocities V . For large values of a , V will be greater than the value predicted by the classical formula.

3. *The case when $\alpha < 0$.* In this case we get

$$(23.16) \quad V = \frac{1}{2\tau_s} \left\{ \log \frac{(w_2 - 1)(w_1 - 1)}{(w_2 + 1)(w_1 + 1)} - \frac{w_2 - w_1}{w_2 w_1} \right\},$$

where

$$(23.17) \quad w_1 = \frac{r_1^2}{a\gamma'} + \sqrt{\left(\frac{r_1^2}{a\gamma'} \right)^2 + 1}, \quad w_2 = \frac{r_2^2}{a\gamma'} + \sqrt{\left(\frac{r_2^2}{a\gamma'} \right)^2 + 1},$$

$$\gamma' = 4\tau_s' \eta^{-1},$$

τ'_s being the shearing limit time (21.28). We take the positive sign of the square roots in (23.17) for the same reasons as in the case $\alpha > 0$. The equation (23.16) yields a real value of V for all values of a . We have now

FIGURE 17

$$\begin{aligned}
 \frac{dV}{da} &= \frac{1}{2\tau'_s a} \left(\frac{1}{w_1} - \frac{1}{w_2} \right) > 0, \\
 (23.18) \quad \frac{d^2 V}{da^2} &= \frac{1}{\tau'_s a^2} \left(\frac{1}{w_2(w_2^2 + 1)} - \frac{1}{w_1(w_1^2 + 1)} \right) < 0, \\
 \left(\frac{dV}{da} \right)_{a=0} &= \frac{1}{\eta} \cdot \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}, \quad \left(\frac{d^2 V}{da^2} \right)_{a=0} = 0.
 \end{aligned}$$

We see that the graph of $V = V(a)$ now has the form shown in Figure 18. From (23.18) follows again that the classical law (23.11) (dotted line in Figure 18) is a good approximation to (23.16) for small angular velocities. For large values of a , V will be smaller than the value predicted by the classical formula; *i.e.* the flow stiffens. The angular velocity never exceeds the limit value

$$V_{\text{lim}} = \frac{1}{\tau_s'} \log \frac{r_2}{r_1}$$

no matter how large a torque is applied. With $d = r_2 - r_1$, we have

$$V_{\text{lim}} = \frac{1}{\tau_s'} \cdot \frac{d}{r_2} + o\left(\frac{d}{r_2}\right),$$

which shows that for

$$r_2 - r_1 = d \ll r_2$$

FIGURE 18

we get again approximately the same value as for simple shearing flow (see §21bβ).

Remarks. The normal stresses could be determined by (23.5) and (23.8); they will be complicated functions of r , depending on the values of the material constants η , τ_d , α , α_3 . The same remarks as made at the end of §22 apply here.

We could also have considered the case where the outer cylinder is at rest and the inner cylinder rotates. If we then denote the angular velocity of the inner cylinder by V , we can see easily that the results would be the same as those given above. The case where both cylinders rotate is more difficult to analyze.

REFERENCES

- [1] P. R. HALMOS, *Finite Dimensional Vector Spaces*, Princeton 1942.
- [2] A. LICHNÉROWICZ, *Eléments de Calcul Tensoriel*, Paris 1950.
- [3] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Cambridge 1927.
- [4] J. C. MAXWELL, "On the dynamical theory of gases," *Phil. Trans. R. Soc. London (A)*, v.**157**, pp. 49–88 (1867) = *Papers* **2**, pp. 26–78. See pp. 30–31.
- [5] H. RICHTER, "Zur Elastizitätstheorie endlicher Verformungen," *Math. Nachr.* v.**8**, pp. 65–73 (1952).
- [6] R. S. RIVLIN & J. L. ERICKSEN, "Stress-deformation relations for isotropic materials," to appear in the next issue.
- [7] C. TRUESDELL, "Hypo-elasticity," following in this issue. Also "The simplest rate theory of pure elasticity," *Proceedings of the Second U. S. Army Ordnance Symposium*, at Chicago April, 1954, to appear in *Comm. Pure Appl. Math. N. Y. U.* **8** (1955).
- [8] C. TRUESDELL, "The mechanical foundations of elasticity and fluid dynamics," *J. Rational Mech. Anal.* v.**1**, pp. 125–300 (1952); v.**2**, pp. 593–616 (1953).
- [9] S. ZAREMBA, "Sur une conception nouvelle des forces intérieures dans un fluide en mouvement," *Mém. Sci. Math. No. 82* (1937), and "Sur une forme perfectionnée de la théorie de la relaxation," *Bull. Int. Acad. Sci. Cracovie* 1903, pp. 594–614.