*On a Problem of Type, with Application to Elliptic Partial Differential Equations**

ROBERT FINN

University of Southern California, Los Angeles, California

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1. Let $\varphi(x, y)$ be a function of class $C^{(n)}$, defined for all finite real values of x and y . The function $\varphi(x, y)$ then represents a surface S over the (x, y) plane which has a simply covered projection on this plane. It is known that for any such surface S , there will exist a complex valued function $f(x, y)$ which maps the whole of S in a 1-1 conformal manner onto a simply-connected open set T of the plane. Loewner has raised the question whether there exist surfaces S such that T will have a finite boundary point. No such surfaces have been discovered. Under certain circumstances, *e.g.* if S is a surface of revolution or if $(\varphi_x^2 + \varphi_y^2)^{\frac{1}{2}} < K \log (2 + x^2 + y^2)$ for some constant K , it is easy to show that T covers the plane. It has been shown by Blanc & Fiala [4]¹ that if the surface integral of the Gaussian curvature of S is bounded below, then T must cover the plane. It is easy to construct surfaces S for which the surface integral of the Gaussian curvature is not bounded below, but such that T covers the plane.

In the present paper we consider surfaces S which are defined by solutions of variational problems

$$(1) \quad \delta \iint F(p^2 + q^2) dx dy = 0, \quad p = \varphi_x, \quad q = \varphi_y.$$

In addition to the usual requirement of ellipticity,

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¹ Numbers in brackets refer to references at end of paper.

$$(2a) \quad F_{pp} F_{qq} - F_{pq}^2 > 0,$$

we assume that for some constant K , $1 \leq K < \infty$, the integrand $F(p^2 + q^2) = F(\Omega^2)$ satisfies, for any solution of (1), the condition

$$(2b) \quad 1/K \leq FM \leq K,$$

where $M = [F - 2\Omega^2 F']$, $\Omega^2 = p^2 + q^2$ and $F' = dF/d\Omega^2$. It should be noted that conditions (2a) and (2b) are satisfied in a class of variational problems (1) including as a special case that for minimal surfaces,

$$(3) \quad \delta \iint \sqrt{1 + p^2 + q^2} \, dx \, dy = 0.$$

We shall show that for any surface S represented by the solution of such a problem, the image set T will cover the plane.

Theorem 1: *Let $\varphi(x, y)$ be a solution, for all finite values of x and y , of the variational problem (1). Suppose that (2a) is satisfied for all p, q and that (2b) is satisfied for the given function $\varphi(x, y)$. Then the surface S represented by $\varphi(x, y)$, considered as a Riemann surface on which angles are defined by its intrinsic geometry, is of parabolic type; that is, the domain T has no finite boundary point.*

The proof of theorem 1 will be given in sections 5, 6, and 7.

2. For the case of a minimal surface, theorem 1 appears as an immediate consequence of S. Bernstein's theorem [1] that if $\varphi(x, y)$ is a solution, for all finite values of x and y , of the equation for minimal surfaces,

$$(3) \quad (1 + q^2)\varphi_{xx} - 2pq\varphi_{xy} + (1 + p^2)\varphi_{yy} = 0, \quad p = \varphi_x, \quad q = \varphi_y,$$

then $\varphi(x, y)$ is necessarily a linear function: $\varphi = Ax + By + C$ for some constants A, B, C .

Bernstein's proof is based on a geometrical theorem concerning the behavior of surfaces of non-positive Gaussian curvature. E. Hopf pointed out a topological difficulty in the proof²; Radó [15] gave an independent proof of the property of minimal surfaces³. The topological difficulty has been overcome by E. Hopf [11] and by E. J. Mickle [13]. Still other proofs of Bernstein's theorem on minimal surfaces have been given by L. Bers [2] and by E. Heinz [10].

Let us set $W = \sqrt{1 + p^2 + q^2}$, $\alpha = (1 + q^2)/W$, $\beta = -pq/W$, $\gamma = (1 + p^2)/W$. For any solution of the equation (3), α, β and γ may be considered as Beltrami coefficients of a Riemannian metric

$$(4) \quad ds^2 = \gamma \, dx^2 - 2\beta \, dx \, dy + \alpha \, dy^2$$

spread over the (x, y) plane. For linear elliptic partial differential equations, at

² In [8] this observation is erroneously attributed to Radó.

³ Section 5 of this paper is based on the method of Radó.

each point, the metric induced in a corresponding manner by the coefficients of the second order terms will be quasi-conformal⁴ in relation to the Euclidean metric of the plane, with eccentricity independent of the particular solution considered. In the present instance this is clearly not the case. However, we observe that the metric (4) coincides with the metric induced over the (x, y) plane by the intrinsic geometry of the surface represented by $\varphi(x, y)$. That is, *for any solution $\varphi(x, y)$ of (3), the Riemannian metric induced over the (x, y) plane by taking as Beltrami coefficients the coefficients $\gamma, -\beta, \alpha$ of (3), is conformally related to the intrinsic metric of the surface defined by $\varphi(x, y)$.*

This property of the coefficients can be used to characterize the equation for minimal surfaces. In particular, then, it characterizes the type of behavior in the large discovered by Bernstein. Suppose now that the word *conformal* in the preceding italicized statement is replaced by the word *quasi-conformal*. This leads to the characterization of a class of partial differential equations

$$(5) \quad \begin{aligned} a(p, q)\varphi_{xx} + 2b(p, q)\varphi_{xy} + c(p, q)\varphi_{yy} &= 0, \\ p &= \varphi_x, \quad q = \varphi_y, \quad ac - b^2 = 1, \end{aligned}$$

by the condition that for some constant ϵ , $1 \leq \epsilon < \infty$,

$$(6) \quad a\gamma + c\alpha - 2b\beta \leq 2\epsilon$$

for every solution of (5). There is some evidence to indicate that for such equations the conclusion of Bernstein remains true.

Theorem 2: *Let $\varphi(x, y)$ satisfy (5) and (6) for all finite values of x and y . Suppose further that the surface S represented by $\varphi(x, y)$, considered as a Riemann surface on which angles are defined by its intrinsic geometry, is of parabolic type. Then $\varphi(x, y)$ is a linear function: $\varphi = Ax + By + C$ for some constants A, B, C .*

The proof of theorem 2 follows directly from a lemma, proved in section 8. This lemma has also been used in another connection [7].

From theorems 1 and 2 we obtain theorem 3.

Theorem 3: *Let $\varphi(x, y)$ be a solution, for all finite values of x and y , of the variational problem (1). Suppose that (2a) is satisfied for all (p, q) , and that (2b) and (6) are satisfied for the given function $\varphi(x, y)$. (In (6) we set $a = F_{pp}/\Delta$, $b = F_{pq}/\Delta$, $c = F_{qq}/\Delta$, $\Delta = (F_{pp}F_{qq} - F_{pq}^2)^{\frac{1}{2}}$.) Then $\varphi(x, y)$ is a linear function, $\varphi = Ax + By + C$.*

Bers has proved [3] that the conclusion of theorem 3 (*i.e.* of Bernstein's theorem) is valid for certain classes of equations (5) which are characterized by conditions other than those considered in this paper. Very simple examples show that a criterion which in [8] is proved to imply removability of isolated singularities of solutions of (5) is not sufficient in the present case.

3. We insert here some definitions and remarks.

Let $w = u + iv$ be a complex valued function of class $C^{(\nu)}$ defined in a domain

⁴ A brief definition of quasi-conformality and of related concepts appears in paragraph 3.

D of the $z = x + iy$ plane. We shall say that w represents a quasi-conformal mapping of eccentricity not larger than ϵ , $1 \leq \epsilon \leq \infty$, provided that

- (i) the Jacobian $\begin{pmatrix} u & v \\ x & y \end{pmatrix}$ of the transformation vanishes only at isolated points.
- (ii) at all points where $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \neq 0$,

$$(u_x^2 + u_y^2 + v_x^2 + v_y^2) \begin{pmatrix} x & y \\ u & v \end{pmatrix} \leq 2\epsilon.$$

The geometrical significance of these conditions is that, except perhaps at isolated points, infinitesimal circles are mapped onto infinitesimal ellipses, for which the ratio of major to minor axis remains uniformly bounded in D .

Let R_1 and R_2 be Riemannian manifolds defined in a domain D of the (x, y) plane by the metrics

$$ds_1^2 = \lambda_1(x, y) [e_1 dx^2 + 2f_1 dx dy + g_1 dy^2]$$

$$ds_2^2 = \lambda_2(x, y) [e_2 dx^2 + 2f_2 dx dy + g_2 dy^2],$$

$$e_1 g_1 - f_1^2 = e_2 g_2 - f_2^2 = 1.$$

We shall say that R_1 and R_2 are in quasi-conformal relation in D provided that the ratio $\lambda_2 ds_1^2 / \lambda_1 ds_2^2$ is uniformly bounded and bounded away from zero in D . This is equivalent to the assertion that for some ϵ , $1 \leq \epsilon < \infty$,

$$e_1 g_2 + g_1 e_2 - 2f_1 f_2 \leq 2\epsilon$$

in D .

Let $\phi(x, y)$ be a solution of an equation of type (5). Then the points where $p^2 + q^2 = 0$, the points where $p_x^2 + p_y^2 = 0$, and the points where $q_x^2 + q_y^2 = 0$ are isolated points except when ϕ is a linear function. This is a consequence of a theorem of Carleman [5] and of the theory of transformation of second order elliptic equations to normal form.

Throughout this paper the symbol K will refer to a constant, $1 \leq K < \infty$, the value of which will in general change with the context. Unless otherwise specified, all functions considered are assumed of class $C^{(n)}$.

4. As already indicated, sections 5, 6, and 7 are devoted to a proof of theorem 1 and are accordingly based on the hypotheses of this theorem. Theorem 2 (and hence theorem 3) is a consequence of the material in sections 8 and 9, in which sections the additional hypothesis (6) is used. Since all three theorems are obviously correct for a linear solution of (1), we may assume throughout that we are dealing with a solution which is not a linear function, but which is defined and of class $C^{(n)}$ for all (x, y) . Theorem 1 is then established directly, and the others follow from an absurdity to which this hypothesis leads.

5. Using appropriate quasi-conformal mappings, in this section we shall generalize the solution of a type-problem given by Radó in the case when the mappings which occur are conformal [15].

We introduce three functions ω_1 , ω_2 and ψ by means of the relations

$$\begin{aligned} \text{a)} \quad & d\omega_1 = -pF_q dx - (F - pF_p) dy, \\ (7) \quad \text{b)} \quad & d\omega_2 = (F - qF_q) dx + qF_p dy, \\ \text{c)} \quad & d\psi = F_q dx - F_p dy. \end{aligned}$$

The Euler-Lagrange equation $(F_p)_x + (F_q)_y = 0$ is a common integrability condition for the three differentials (7). The existence of the functions ω_1 and ω_2 was first observed by Radó [16].

Let $\alpha = x$, $\beta = \omega_1(x, y)$. Condition (2b) implies that the Jacobian β_y of this transformation does not vanish, and it is easily seen that the mapping from (x, y) to (α, β) is in fact one to one in the large. We shall prove that *the mapping $(x, y) \leftrightarrow (\alpha, \beta)$ covers the entire (α, β) plane.*

A simple computation gives the equations

$$\begin{aligned} (8) \quad & y_\alpha = (\omega_2)_\beta, \\ & y_\beta = -\frac{1}{FM} (\omega_2)_\alpha. \end{aligned}$$

Condition (2b) thus implies that $y + i\omega_2$, considered as a function of $\gamma = \alpha + i\beta$, is a one to one quasi-conformal mapping, the eccentricity of which does not exceed a finite bound depending only on K .

Since for fixed $x = \alpha$, $\beta_y \neq 0$, the boundary of the γ -image of the (x, y) plane consists of semi-infinite and (at most two) infinite straight lines. If the set of such lines is void, then there is nothing left to prove. Otherwise we distinguish two cases, according as there does or does not exist a line with two accessible points. We shall prove only that the latter case does not occur, the modifications required for the former case being obvious.

In the latter case, there will exist a simply connected region D of the type indicated in figure 1. For any sequence of points in D which tends toward infinity or toward one of the boundary lines, we must have $y \rightarrow \infty$. For clearly $x + iy \rightarrow \infty$ and $x = \alpha$ is bounded in D .

Thus the function $f(\gamma) = y + i\omega_2$ maps D in a one to one quasi-conformal manner onto a (unbounded) simply connected domain D_1 in the f -plane. That portion of the boundary of D which is indicated by dotted lines in figure 1 corresponds in a one to one manner to the finite part of the boundary of D_1 . All other (finite or infinite) boundary points of D correspond to ∞ in the f -plane. Let $\zeta(f)$, $\zeta = \xi + i\eta$ be a conformal map which carries D_1 onto the unit disc C about the origin and ∞ into the point (-1) . Let $\tilde{D}$ be that simply connected domain (semi-strip) in the γ -plane whose boundary consists of the dotted lines

FIGURE 1

on the boundary of D and the two semi-infinite extensions of these lines, and let $h(\gamma)$ be the conformal map which takes $\tilde{D}$ onto the unit disc $\tilde{C}$ with $h(\infty) = -1$. In this mapping the portion of the boundary of D indicated by solid lines corresponds to a certain non-degenerate continuum in and on the boundary of $\tilde{C}$.

In the ζ -plane this corresponds to a single point, $\zeta = -1$. We may assume that the points $\pm i$ in the ζ -plane correspond to the points $\pm i$ in the h -plane. Now consider the function $h\{\gamma[f(\zeta)]\}$. This represents a one to one quasi-conformal mapping onto a region with finite area and hence, as is easily ascertained, has finite Dirichlet integral,

$$D = \iint \{ |h_\xi|^2 + |h_\eta|^2 \} d\xi d\eta < \infty.$$

Denote by Γ_r the circular arc of radius r interior to $|\zeta| \leq 1$ with $\zeta = -1$ as center; denote by L_r the length of the image of Γ_r , and let w_r be the distance be-

tween the end points of this image. Then clearly $L_r \geq w_r > \delta^2$ for some $\delta^2 > 0$, and w_r is non-decreasing in r . Then, integrating over the annulus indicated in figure 1,

$$\begin{aligned} D &> \int_{\rho}^1 \frac{dr}{r} \int_{\theta_1(r)}^{\theta_2(r)} |h_{\theta}|^2 d\theta \\ &\geq \frac{1}{\pi} \int_{\rho}^1 \frac{dr}{r} \left\{ \int_{\theta_1}^{\theta_2} |h_{\theta}| d\theta \right\}^2 = \frac{1}{\pi} \int_{\rho}^1 \frac{1}{r} L_r^2 dr \\ &\geq \frac{1}{\pi} \int_{\rho}^1 [w_r]^2 \frac{dr}{r} \\ &> \frac{[w_{\rho}]^2}{\pi} \log \frac{1}{\rho}. \end{aligned}$$

Thus

$$w_{\rho}^2 < \frac{\pi^2 D}{\log \frac{1}{\rho}},$$

which $\rightarrow 0$ as $\rho \rightarrow 0$. This contradiction proves that the $\gamma = \alpha + i\beta$ plane is covered.

6. We shall now prove that if $\psi(x, y)$ is defined by the relation (7c), then $\phi + i\psi$ is quasi-conformal in the metric of $\alpha + i\beta$.

An elementary computation yields the relations

$$\begin{aligned} \phi_{\alpha} &= \frac{M}{2F'} \psi_{\beta} \\ (9) \quad -\phi_{\beta} &= \frac{1}{2FF'} \psi_{\alpha}, \quad F' = \frac{dF}{d(\Omega^2)}, \quad \Omega^2 = p^2 + q^2. \end{aligned}$$

It is thus sufficient to demonstrate the existence of a finite constant $K \geq 1$ such that

$$(10) \quad 1/K \leq FF' \leq K, \quad 1/K \leq M/F' \leq K.$$

Let $\mu_0^2 = \min_{\Omega^2 \leq 1} [(\Omega F')^2]'$. Then

$$F'(\Omega^2) \geq \begin{cases} \mu_0, & \Omega^2 \leq 1 \\ \mu_0/\Omega, & \Omega^2 \geq 1, \end{cases}$$

from which, choosing an appropriate constant of integration, we get

$$F(\Omega^2) \geq \begin{cases} 1 + \mu_0 \Omega^2, & \Omega^2 \leq 1 \\ 2\mu_0 \Omega, & \Omega^2 \geq 1, \end{cases}$$

so that

$$FF' \geq \begin{cases} \mu_0, & \Omega^2 \leq 1 \\ 2\mu_0^2, & \Omega^2 \geq 1, \end{cases}$$

which proves the left side of the first inequality.

To prove the right side of this inequality, we use the assumption that

$$1/K < FM = F^2 - 2\Omega^2 FF' < K.$$

Set $R(\Omega^2) = 2FF' = (F^2)'$. Then

$$1/K < F^2(0) + \int_0^{\Omega^2} R(\xi) d\xi - \Omega^2 R(\Omega^2) < K.$$

Integrating by parts, we get

$$1/K < F^2(0) - \int_0^{\Omega^2} \xi R'(\xi) d\xi < K.$$

Now set

$$S(\Omega^2) = \int_0^{\Omega^2} \xi R'(\xi) d\xi.$$

Then

$$S < -\frac{1}{K} + F^2(0), \quad -S < K - F^2(0),$$

so in any event there is a constant $K > 0$ such that $|S| < K$ for all Ω^2 . We have

$$\begin{aligned} S'(\Omega^2) &= \Omega^2 R'(\Omega^2) \\ R(\Omega^2) &= R(1) + \int_1^{\Omega^2} \frac{1}{\xi} S'(\xi) d\xi \\ &= R(1) - S(1) + \frac{1}{\Omega^2} S(\Omega^2) + \int_1^{\Omega^2} \frac{1}{\xi^2} S(\xi) d\xi, \end{aligned}$$

which establishes an upper bound on FF' for $\Omega^2 \geq 1$. For $\Omega^2 \leq 1$ the existence of a bound is obvious.

There remains yet the second inequality (10). We need merely observe that from inequalities of the form

$$1/K < FM < K, \quad 1/K < FF' < K$$

follows immediately

$$1/K^2 < M/F' < K^2.$$

Thus $\phi + i\psi$ is quasi-conformal as a function of $\alpha + i\beta$.

7. The function $\phi + i\psi$ is quasi-conformal in the metric of the surface represented by $\phi(x, y)$.

From the relations $M/F' > 1/K$, $FF' < K$, we find $2\Omega^2 F' < F - F'/K$, from which $2(\Omega F')^2 < K$. The desired result then follows from the following lemma (cf. [9]).

Lemma: Let $\phi(x, y)$ be a solution of the Euler-Lagrange equation $(pF')_x + (qF')_y = 0$ corresponding to the variational problem $\delta \iint F(p^2 + q^2) dx dy = 0$ with $F_{pp} F_{qq} - F_{pq}^2 = [(\Omega F')^2]' > 0$ for all p and q . Define $\psi(x, y)$ by the relations $\psi_y = 2pF'$, $-\psi_x = 2qF'$. Then $\phi + i\psi$ will be quasi-conformal in the metric of the surface represented by $\phi(x, y)$ if and only if $(\Omega F')^2$ is bounded.

Proof. A simple computation shows that the eccentricity of the mapping is

$$\epsilon = \frac{1}{2} \Omega^2 \frac{4(\Omega F')^2 + 4F'^2 + 1}{2\Omega^2 F' \sqrt{1 + \Omega^2}}.$$

In section 6 we have proved the existence of a constant $\mu_0 > 0$ such that if $\Omega^2 \geq 1$ then $\Omega F' \geq \mu_0$. But if $(\Omega F')^2 < K$ and $\Omega^2 \geq 1$, then $F'^2 < K$. Thus for $\Omega^2 \geq 1$,

$$\epsilon < \frac{1}{2\mu_0} \frac{\Omega}{\sqrt{1 + \Omega^2}} (4K + 4K + 1),$$

and for $\Omega^2 \leq 1$ the result is obvious.

To prove the converse, observe that

$$\epsilon > \frac{1}{2} \frac{2\Omega^2 F'}{\sqrt{1 + \Omega^2}} = \frac{(\Omega F')^2}{\sqrt{(F')^2 + (\Omega F')^2}} > \frac{(\Omega F')^2}{\sqrt{2} (\Omega F')} \text{ for } \Omega \geq 1.$$

To prove theorem 1, we need now merely use the known property of simply-connected Riemann surfaces that the type is invariant under quasi-conformal mapping.

8. In this section we shall use the hypothesis (6) to construct a function $\chi(p, q)$ which is quasi-conformal in the metric of the surface S , and bounded.

We first consider (p, q) as independent variables and define the Riemannian manifold R_1 by means of the metric over the (p, q) plane,

$$ds_1^2 = \alpha dp^2 + 2\beta dp dq + \gamma dq^2,$$

where

$$\alpha = \frac{1 + q^2}{W}, \quad \beta = -\frac{pq}{W}, \quad \gamma = \frac{1 + p^2}{W}, \quad W = (1 + p^2 + q^2)^{\frac{1}{2}}.$$

It is easy to verify that the function⁵

$$\chi(p, q) = \frac{p + iq}{1 + \sqrt{1 + p^2 + q^2}}$$

is analytic in the metric of R_1 . Now define the manifold R_2 by means of the metric

$$ds_2^2 = adp^2 + 2bdpdq + cdq^2,$$

where

$$a = F_{pp}/\Delta, \quad b = F_{pq}/\Delta, \quad c = F_{qq}/\Delta, \quad \Delta = (F_{pp}F_{qq} - F_{pq}^2)^{\frac{1}{2}}.$$

The condition

$$(6) \quad a\gamma + c\alpha - 2b\beta \leq 2\epsilon, \quad 1 \leq \epsilon < \infty,$$

implies that the ratio ds_1^2/ds_2^2 is bounded and bounded away from zero. Thus the function $\chi(p, q)$ is quasi-conformal on the manifold R_2 .

Using the relations

$$p_y = q_x$$

$$ap_x + 2bp_y + cq_y = 0,$$

we compute easily that

$$adp^2 + 2bdpdq + cdq^2 = (p_y^2 - p_x q_y)[cdx^2 - 2bxdy + ady^2].$$

But

$$(p_y^2 - p_x q_y) = \frac{1}{a}(aq_x^2 + 2bq_x q_y + cq_y^2),$$

so that the relation $ac - b^2 = 1$ implies that the expression $p_y^2 - p_x q_y$ can vanish only at points where q_x and q_y vanish simultaneously. We have already observed that such points are isolated. From this we conclude immediately that $\chi(p, q)$ is quasi-conformal on the manifold R_3 with metric

$$ds_3^2 = cdx^2 - 2bxdy + ady^2.$$

Finally we consider the manifold R_4 defined over the (x, y) plane by the metric of the surface,

$$ds_4^2 = \gamma dx^2 - 2\beta xdy + \alpha dy^2.$$

The quasi-conformality of $\chi(p, q)$ on R_4 follows from another application of the condition (6).

⁵ For a geometrical interpretation of this function, see E. HOPF [12]. See also S. A. CHAPLYGIN [6], L. BERS [2], E. HEINZ [10], R. FINN [7].

9. If the surface S were of parabolic type, that is, if S could be mapped conformally onto a plane region without finite boundary point, then under the hypotheses of theorem 2 the function $\chi(p, q) = (p + iq)/(1 + \sqrt{1 + p^2 + q^2})$ would represent a quasi-conformal mapping of this region into the interior of the unit disc. The non-existence of such a mapping has been demonstrated by various authors (*e.g.* [14]). This contradiction proves theorem 2. Theorem 3 then follows immediately from theorem 1.

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