*Non-linear Elliptic Equations Without Non-linear Entire Solutions*¹

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§1. Introduction. A) The equation of minimal surfaces

$$(1 + \varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_x^2)\varphi_{yy} = 0$$

occupies a special position in the theory of quasi-linear elliptic equations since, just like its linear analogue, the Laplace equation, it can be investigated by methods of complex function theory. In particular, the following properties of solutions of this equation are known.

a. *Every entire solution is a linear function.* (By an entire solution we mean here and hereafter one which is defined for all values of x and y .)

b. *A single-valued solution defined in a neighborhood of the point at infinity is always such that its gradient attains a finite limit at infinity. The same is true for a multiple-valued solution, provided that its gradient is finitely many-valued.*

c. *At a finite point a single-valued solution can have at most a removable isolated singularity.*

B) Theorem **a** is due to S. Bernstein [2],² who obtained it as a corollary of his celebrated geometric theorem. The latter theorem implies that every elliptic equation of the form

$$A\varphi_{xx} + 2B_{xy} + C\varphi_{yy} = 0$$

possesses only constant entire bounded solutions. Other proofs of Bernstein's theorem on minimal surfaces were given by T. Rado [19], myself [3], and E. Heinz [13]. I proved Theorems **b** and **c** by the same method as Bernstein's

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² Numbers in brackets refer to the bibliography. Cf. also HOPF [14], MICKLE [18].

theorem. The other methods do not yield these results. A different proof of Theorem **c** was given later by Finn [11].

I conjectured [4] that Theorems **a** and **c** remain true for elliptic equations of the form

$$(\rho\varphi_x)_x + (\rho\varphi_y)_y = 0, \quad \rho = \rho(\sqrt{\varphi_x^2 + \varphi_y^2})$$

where $\rho(q)$ is a given function such that $q\rho(q)$ is increasing and bounded. For Theorem **c** this was verified by Finn [11], who, in fact, proved a stronger result on removable irregularities. As far as **a** is concerned, the conjecture is almost certainly wrong. Simple examples show that neither of the Theorems **a** and **c** implies the other. On the other hand, it is easy to see that Theorem **b** implies Theorem **a**, at least for quasi-linear elliptic equations of the form

$$A(\varphi_x, \varphi_y)\varphi_{xx} + 2B(\varphi_x, \varphi_y)\varphi_{xy} + C(\varphi_x, \varphi_y)\varphi_{yy} = 0.$$

C) The main purpose of this paper is to indicate two classes of equations of the form stated above for which Theorem **a** holds. (Another such class was discovered, independently, by Finn [12].)

One of these classes (called “equations of finite conformal radius and slowly growing characteristic”) contains the equation of minimal surfaces. The proof in this case is very similar to my proof of Bernstein’s theorem, except that pseudo-analytic functions and quasi-conformal mappings are used instead of analytic functions. In particular, a recent result of Agmon on quasi-conformal mappings is used at an essential point. I believe that Theorem **b** remains true in this case, but I was unable to prove it.

The other class of equations (“equations of finite conformal radius and slowly growing eccentricity”) is of interest primarily because it contains the potential equation of a subsonic adiabatic gas flow. For this class of equations Theorems **a** and **b** can be proved by a relatively simple quasi-conformality argument. The same argument also yields a stronger form of Theorem **c**: an isolated singularity of a solution is removable if not the solution itself but merely its first derivatives are assumed to be single-valued. For minimal surfaces such a statement is false, as is shown by the example $\varphi = \arctan(y/x)$.

D) In §2 we collect information on quasi-conformal mappings which are used in the proofs. The definitions of the concepts: conformal radius, eccentricity, characteristic, are given in §§3, 4. In §5 we give a version of the familiar hodograph linearization of quasi-linear equations which is convenient for our purposes. The proofs of the results described above will be found in §§5, 6. The last section contains some examples.

§2. Lemmas on quasi-conformal mappings. A) A homeomorphism

$$(2.1) \quad \zeta = \zeta(z) = \xi(x, y) + i\eta(x, y)$$

of a domain D in the z -plane ($z = x + iy$) will be called *smooth* if the derivatives

$\xi_x, \xi_y, \eta_x, \eta_y$ exist and are continuous, and $\xi_x \eta_y - \xi_y \eta_x > 0$. The *eccentricity* of a smooth homeomorphism at a point of D is defined as the value of

$$(2.2) \quad E = \frac{\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2}{2(\xi_x \eta_y - \xi_y \eta_x)} = \frac{x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2}{2(x_\xi y_\eta - x_\eta y_\xi)}$$

at this point. It is also the eccentricity of the homeomorphism inverse to (2.1) (of course, at the corresponding point in the ζ -plane), and it is also the eccentricity of any homeomorphism obtained from (2.1) by preceding or following it by a conformal mapping.

If some restriction is imposed on the value of E , the mapping is called quasi-conformal. It is called of *bounded eccentricity* if the ratio (2.2) is uniformly bounded.³

B) A smooth homeomorphism (2.1) is said to be *conformal with respect to a Riemann metric*

$$(2.3) \quad g_{11}(x, y)dx^2 + 2g_{12}(x, y)dx dy + g_{22}(x, y)dy^2$$

(where $g_{11}g_{22} - g_{12}^2, g_{11} > 0$) if

$$d\xi^2 + d\eta^2 = \lambda(g_{11}dx^2 + 2g_{12}dx dy + g_{22}dy^2),$$

$\lambda = \lambda(x, y)$ being a positive function. The eccentricity of such a mapping is easily seen to equal

$$E = \frac{g_{11} + g_{22}}{2(g_{11}g_{22} - g_{12}^2)^{1/2}}.$$

In order that the mapping (2.1) be conformal with respect to (2.3) it is necessary and sufficient that $\xi(x, y)$ and $\eta(x, y)$ satisfy the system of Beltrami equations determined by the metric. Korn [15] and Lichtenstein [16, 17] proved that the Beltrami equations are solvable in the small if the coefficients of the metric (2.3) are Hölder continuously differentiable or even only Hölder continuous. In this case the solutions are twice (once) Hölder continuously differentiable. (Here and hereafter we call a function Hölder continuous on a set if it satisfies a uniform Hölder condition on every compact subset.)

Combining this result with the general uniformization theorem one obtains the following well-known statement. If the coefficients of the metric (2.3) are defined and Hölder continuous in a plane domain D (or, more generally, on a Riemann surface $\tilde{D}$ of genus zero), then there exists a smooth homeomorphism $\zeta = \zeta(z)$ of D (of $\tilde{D}$) onto a plane domain Δ which is conformal with respect to (2.3). The domain Δ , is, of course, determined but for a conformal transformation.

C) It is well known that the domain

$$(2.4) \quad D: \quad 0 < r_0 < |z| < +\infty$$

³ We need not consider here quasi-conformal mappings which do not have continuous derivatives.

cannot be mapped topologically onto the domain

$$(2.5) \quad \Delta: \quad 0 < \rho_0 < |\zeta| < \rho_1 < +\infty$$

by a mapping of bounded eccentricity. We state below two (certainly known) lemmas which extend this result. For the convenience of the reader we insert the very simple proofs.

Lemma 1. *Let (2.1) be a smooth homeomorphism of (2.4) onto (2.5). Then*

$$(2.6) \quad \iint_{\Delta} E \, d\xi \, d\eta = +\infty.$$

Lemma 2. *Under the hypotheses of Lemma 1 an estimate of the form*

$$(2.7) \quad E = o(\log |z|), \quad z \rightarrow \infty$$

cannot hold in D .

Before proving these lemmas we state two immediate consequences.

Corollary 1. *If (2.1) is a smooth homeomorphism of the whole z -plane onto the disc $|\zeta| < \rho < +\infty$, then*

$$(2.8) \quad \iint_{|\zeta| < \rho} E \, d\xi \, d\eta = +\infty.$$

Corollary 2. *If (2.1) is a smooth homeomorphism of the whole z -plane onto the disc $|\zeta| < \rho < +\infty$, then an estimate of the form (2.7) cannot hold.*

The derivation of the corollaries from the lemmas may be left to the reader.

D) We shall prove both lemmas simultaneously. Using polar coordinates ($z = re^{i\theta}$, $\zeta = \rho e^{i\omega}$) we note that the image of any circle $|z| = r$ in D has at least length $2\pi\rho_0$. Hence, using the definition of E , we have that

$$(2.9) \quad 2\pi\rho_0 \leq \int_0^{2\pi} |\zeta_\theta(re^{i\theta})| \, d\theta \leq \int_0^{2\pi} r E^{\frac{1}{2}} \left(\frac{\partial(\xi, \eta)}{\partial(x, y)} \right)^{\frac{1}{2}} d\theta.$$

Dividing both sides of this inequality by $r^{\frac{1}{2}}$ and using Schwarz' inequality we see that

$$\frac{2\pi\rho_0}{r^{\frac{1}{2}}} \leq \left\{ 2\pi \int_0^{2\pi} E \frac{\partial(\xi, \eta)}{\partial(x, y)} r \, d\theta \right\}^{\frac{1}{2}}.$$

Squaring and integrating with respect to dr from r_0 to $r > r_0$ yields the relation

$$4\pi^2 \rho_0^2 \log \frac{r}{r_0} \leq 2\pi \int_{r_0}^r \int_0^{2\pi} E \frac{\partial(\xi, \eta)}{\partial(x, y)} r \, dr \, d\theta = 2\pi \iint_{\Delta_r} E \, d\xi \, d\eta,$$

where Δ_r is the image of the annulus $r_0 < |z| < r$ under the mapping (2.1). Letting r approach $+\infty$ we obtain (2.6).

On the other hand, dividing both sides of (2.9) by r , integrating with respect

to dr , and then applying Schwarz' inequality, we have that

$$\begin{aligned} 2\pi\rho_0 \log \frac{r}{r_0} &\leq \int_{r_0}^r \int_0^{2\pi} E^{\frac{1}{2}} \left(\frac{\partial(\xi, \eta)}{\partial(x, y)} \right)^{\frac{1}{2}} dr d\theta \\ &\leq \left\{ \int_{r_0}^r \int_0^{2\pi} \frac{\partial(\xi, \eta)}{\partial(x, y)} r dr d\theta \cdot \int_{r_0}^r \int_0^{2\pi} \frac{E}{r} dr d\theta \right\}^{\frac{1}{2}} \\ &= \left\{ \iint_{\Delta_r} d\xi d\eta \cdot \int_{r_0}^r \int_0^{2\pi} \frac{E}{r} dr d\theta \right\}^{\frac{1}{2}} \end{aligned}$$

so that

$$(2.10) \quad \frac{4\pi\rho_0^2}{\rho_1^2 - \rho_0^2} \left(\log \frac{r}{r_0} \right)^2 \leq \int_{r_0}^r \int_0^{2\pi} \frac{E}{r} dr d\theta.$$

But if (2.7) holds, then

$$\int_{r_0}^r \int_0^{2\pi} \frac{E}{r} dr d\theta = o(1) \left(\log \frac{r}{r_0} \right)^2, \quad r \rightarrow +\infty,$$

which contradicts (2.10).

E) **Lemma 3.** *Let (2.1) be a smooth homeomorphism of the disc $|z| < r$. If the eccentricity of (2.1) is bounded, then the finite limit*

$$\zeta(re^{i\omega}) = \lim_{t \rightarrow r} \zeta(te^{i\omega})$$

exists for almost all ω , and the function $\zeta(re^{i\omega})$ is not constant on any set of positive (linear) measure.

This result was proved by Agmon [1].

§3. Distorted gradient, conformal radius, eccentricity. A) From now on we consider a fixed equation of the form

$$(3.1) \quad A_{11}(\varphi_x, \varphi_y)\varphi_{xx} + 2A_{12}(\varphi_x, \varphi_y)\varphi_{xy} + A_{22}(\varphi_x, \varphi_y)\varphi_{yy} = 0$$

under the following hypotheses. The coefficients $A_{ik}(u, v)$ are defined in a simply connected domain (ellipticity domain) Ω of the w -plane ($w = u - iv$) containing the origin. In this domain the ellipticity condition

$$(3.2) \quad A_{11}A_{22} - A_{12}^2 > 0, \quad A_{11} > 0$$

is satisfied. Also, the $A_{ik}(u, v)$ possess partial derivatives of the first order, and on every compact subdomain of Ω these derivatives satisfy a uniform Hölder condition. Ω may be the whole plane; if it is not, the interesting case is that in which the coefficients become singular or cease to satisfy the ellipticity condition at each finite boundary point of Ω .

B) In Ω

$$(3.3) \quad A_{11}(u, v)du^2 + 2A_{12}(u, v)du dv + A_{22}(u, v)dv^2$$

is a Riemannian metric. Hence (cf. §2, B)) there exists a uniquely determined complex-valued function $w^* = w^*(w) = u^*(u, v) - iv^*(u, v)$ having the following properties. (i) The mapping $w \rightarrow w^*$ is a smooth homeomorphism of Ω onto the domain $|w^*| < R \leq +\infty$. (ii) This homeomorphism is conformal with respect to the metric (3.3). (iii) The conditions

$$u^* = v^* = 0, \quad u_u^* = 1, \quad u_v^* = 0 \quad \text{at } w = 0$$

are satisfied. In view of our hypotheses on the coefficients we also have that (iv) the functions $u^*(u, v)$, $v^*(u, v)$ have continuous second derivatives.

The function $w^*(w)$ will be called the *distorted gradient* belonging to equation (3.1), and the number R the *conformal radius* of this equation.

C) At a point (u, v) of Ω the eccentricity of the mapping is

$$(3.4) \quad E(u, v) = \frac{A_{11} + A_{22}}{2(A_{11}A_{22} - A_{12}^2)^{\frac{1}{2}}}.$$

Clearly this quantity may be considered as a function of w^* . We set

$$e(r) = \max_{|w^*| \leq r} E(u, v), \quad 0 \leq r < R$$

and call the function $e(r)$ the eccentricity of equation (3.1). The eccentricity will be said to *grow slowly* if either $R = +\infty$ and

$$(3.5) \quad e(r) = o(\log r), \quad r \rightarrow \infty,$$

or $R < +\infty$ and

$$(3.6) \quad e(r) = O(|R - r|^{\epsilon-1}), \quad r \rightarrow R,$$

where $0 < \epsilon < 1$.

D) *The conformal radius of an equation of slowly growing eccentricity is finite if and only if the ellipticity domain Ω is not the whole plane.*

In fact, assume that $R = +\infty$, (3.5) holds and Ω is not the whole plane. Then there exists an analytic function $Z = Z(w)$ mapping Ω onto $|Z| < 1$. The mapping of the w^* -plane onto $|Z| < 1$ defined by setting $Z = Z(w)$, $w^* = w^*(w)$, w in Ω , is a smooth homeomorphism, and its eccentricity at a point w^* is given by (3.4) and thus bounded by $e(|w^*|)$. Noting (3.5) we observe a contradiction with Corollary 2 of §2, C).

In the same way we can show that if Ω is the whole plane and $R < +\infty$, assumption (3.6) leads to a contradiction to Corollary 1.

§4. Characteristic. A) Equation (3.1) can be written as a system of two first

order equations for the unknown functions $u = \varphi_x, v = \varphi_y$

$$(4.1) \quad \begin{aligned} A_{11} u_x + 2A_{12} u_y + A_{22} v_y &= 0 \\ u_y - v_x &= 0. \end{aligned}$$

Interchanging, at this stage purely formally, dependent and independent variables, we obtain the system

$$(4.2) \quad \begin{aligned} A_{11} y_v - 2A_{12} x_v + A_{22} x_u &= 0 \\ x_v - y_u &= 0. \end{aligned}$$

Now introduce as new independent variables the real and imaginary part of the distorted gradient. System (4.2) becomes

$$(4.3) \quad \begin{aligned} x_{u^*} &= T_1 y_{u^*} - S_1 y_{v^*} \\ x_{v^*} &= S_1 y_{u^*} + T_1 y_{v^*} \end{aligned}$$

where

$$(4.4) \quad \begin{aligned} T_1(u^*, v^*) &= \frac{A_{12}(u, v)}{A_{22}(u, v)}, \\ S_1(u^*, v^*) &= + \frac{[A_{11}(u, v)A_{22}(u, v) - A_{12}(u, v)^2]^{\frac{1}{2}}}{A_{22}(u, v)}. \end{aligned}$$

The functions T_1, S_1 have Hölder continuous derivatives. Set

$$(4.5) \quad S_0 = -\frac{S_1}{S_1^2 + T_1^2}, \quad T_0 = \frac{T_1^2}{S_1^2 + T_1^2}.$$

Eliminating from (4.3) one of the dependent variables by differentiation we are led to the second order equations

$$(4.6) \quad x_{u^*u^*} + x_{v^*v^*} + \alpha_0 x_{u^*} + \beta_0 x_{v^*} = 0$$

$$(4.7) \quad y_{u^*u^*} + y_{v^*v^*} + \alpha_1 y_{u^*} + \beta_1 y_{v^*} = 0$$

where

$$(4.8) \quad \alpha_j(u^*, v^*) = \frac{1}{S_j} \left(\frac{\partial S_j}{\partial u^*} - \frac{\partial T_j}{\partial v^*} \right), \quad \beta_j(u^*, v^*) = \frac{1}{S_j} \left(\frac{\partial S_j}{\partial v^*} + \frac{\partial T_j}{\partial u^*} \right),$$

$j = 0, 1.$

A direct computation shows that

$$(4.9) \quad \alpha_0^2 + \beta_0^2 = \alpha_1^2 + \beta_1^2.$$

We set

$$(4.10) \quad \chi(r) = \max_{|w^*| \leq r} |\alpha_j(u^*, v^*)^2 + \beta_j(u^*, v^*)^2|, \quad 0 \leq r < R,$$

and call the function $\chi(r)$ the *characteristic* of equation (3.1).

B) *Slow growth* of the characteristic is defined in the same way as slow growth of the eccentricity. We shall be concerned only with the case $R < +\infty$. In this case we call the characteristic slowly growing if

$$\chi(r) = O(|R - r|^{\epsilon-1}), \quad r \rightarrow R$$

for some ϵ , $0 < \epsilon < 1$.

We recall now a result from the theory of pseudo-analytic functions.⁴ If $\alpha(x, y)$, $\beta(x, y)$ are real-valued Hölder continuous functions defined for

$$x^2 + y^2 < R^2$$

and satisfying an inequality of the form

$$\alpha^2 + \beta^2 \leq \text{const.} (R - (x^2 + y^2)^{\frac{1}{2}})^{\epsilon-1}, \quad 0 < \epsilon < 1$$

then the equation

$$\Psi_{xx} + \Psi_{yy} + \alpha\Psi_x + \beta\Psi_y = 0$$

can be considered as obtained by elimination from the elliptic system

$$\varphi_x = \tau\psi_x + \sigma\psi_y$$

$$\varphi_y = -\sigma\psi_x + \tau\psi_y$$

with Hölder continuously differentiable bounded coefficients $\tau(x, y)$, $\sigma(x, y)$ and σ positive and bounded away from zero. In other words, there exist functions τ , σ defined for $x^2 + y^2 < R^2$, having the properties stated above and such that

$$\alpha = \frac{\sigma_x + \tau_y}{\sigma}, \quad \beta = \frac{\sigma_y - \tau_x}{\sigma}.$$

We conclude that if equation (3.1) has a finite conformal radius and a slowly growing characteristic, then there exist four real-valued functions

$$\sigma_j(u^*, v^*), \quad \tau_j(u^*, v^*), \quad j = 0, 1$$

defined for $|w^*| < R$ such that

$$(4.11) \quad \sigma_j > 0, \quad \frac{1 + \sigma_j^2 + \tau_j^2}{\sigma_j} \leq \text{const.}$$

⁴ Concerning this theory, see BERS [5, 6, 8, 9].

$$(4.12) \quad \alpha_j = \frac{1}{\sigma_j} \left(\frac{\partial \sigma_j}{\partial u^*} - \frac{\partial \tau_j}{\partial v^*} \right), \quad \beta_j = \frac{1}{\sigma_j} \left(\frac{\partial \sigma_j}{\partial v^*} + \frac{\partial \tau_j}{\partial u^*} \right).$$

The proof of the result quoted above will be found in Bers [8].

§5. Parametric representation of solutions. A) Let $\varphi(x, y)$ be a (single-valued) solution of (3.1) defined in a domain D of the z -plane, $z = x + iy$. By calling φ a solution we imply that the second derivatives of φ exist and are continuous, and that the function $w = \varphi_x - i\varphi_y$ takes on only values in Ω .

Keeping φ fixed we may consider $A_{ik}(\varphi_x, \varphi_y)$ as given functions of (x, y) . In D

$$(5.1) \quad A_{22}(\varphi_x, \varphi_y)dx^2 - 2A_{12}(\varphi_x, \varphi_y)dx dy + A_{11}(\varphi_x, \varphi_y)dy^2$$

is a Riemannian metric. There exists a smooth homeomorphism

$$(5.2) \quad \zeta = \zeta(z) = \xi(x, y) + i\eta(x, y)$$

of D onto a plane domain Δ which is conformal with respect to the metric (5.1). We claim that the second derivatives of the functions ξ, η exist and are Hölder continuous. In order to verify this we must know that the coefficients $A_{ik}(\varphi_x, \varphi_y)$ of the metric are Hölder continuously differentiable with respect to (x, y) , that is that the second derivatives of φ are Hölder continuous. But this follows from a classical result of Schauder [20] on linear elliptic equations, since φ may be considered as a solution of a linear elliptic equation with Hölder continuous coefficients.

B) If φ is not a linear function, then the expression

$$j = \varphi_{xy}^2 - \varphi_{xx}\varphi_{yy} = -(u_x v_y - u_y v_x),$$

is positive, except at isolated points.

In fact, $u = \varphi_x$ and $v = \varphi_y$ satisfy system (4.1) which we may now consider as a linear elliptic system. This system implies at once that $j \geq 0$. On the other hand, the mapping (5.2) takes system (4.1) into the linear system of the form

$$(5.3) \quad \begin{aligned} u_\xi &= -tv_\xi - sv_\eta \\ u_\eta &= sv_\xi - tv_\eta \end{aligned}$$

where

$$\begin{aligned} t(\xi, \eta) &= -\frac{A_{12}(\varphi_x, \varphi_y)}{A_{22}(\varphi_x, \varphi_y)}, \\ s(\xi, \eta) &= +\frac{[A_{11}(\varphi_x, \varphi_y)A_{22}(\varphi_x, \varphi_y) - A_{12}(\varphi_x, \varphi_y)^2]^{\frac{1}{2}}}{A_{22}(\varphi_x, \varphi_y)} \end{aligned}$$

so that $s > 0$. From a known theorem on pseudo-analytic functions⁵ we con-

⁵ See BERS [9] where the corresponding statement is proved for Hölder-continuous s, t . For continuously differentiable s, t the result follows also from a theorem of CARLEMAN [10].

clude that $u_{\xi} v_{\eta} - u_{\eta} v_{\xi} = 0$ at most at isolated points. Since j differs from $u_{\xi} v_{\eta} - u_{\eta} v_{\xi}$ only by the positive factor $\xi_x \eta_y - \xi_y \eta_x$, the desired conclusion follows.

We call points at which $j = 0$ *exceptional points*.

C) *The distorted gradient*

$$w^* = w^*(\varphi_x - i\varphi_y)$$

of $\varphi(x, y)$, considered as a function of $\zeta = \zeta(z)$ is analytic.

In order to prove this we must show that the metrics (5.1) and (3.3) are conformally equivalent. For every solution of (3.1) the following identities hold

$$\begin{aligned} A_{11} \varphi_{xx}^2 + 2A_{12} \varphi_{xx} \varphi_{xy} + A_{22} \varphi_{xy}^2 &= (\varphi_{xx}^2 - \varphi_{xx} \varphi_{yy}) A_{22} \\ A_{11} \varphi_{xx} \varphi_{xy} + A_{12} (\varphi_{xx} \varphi_{yy} + \varphi_{xy}^2) + A_{22} \varphi_{xy} \varphi_{yy} &= -(\varphi_{xy} - \varphi_{xx} \varphi_{yy}) A_{12} \\ A_{11} \varphi_{xy}^2 + 2A_{12} \varphi_{xy} \varphi_{yy} + A_{22} \varphi_{yy}^2 &= (\varphi_{xy}^2 - \varphi_{xx} \varphi_{yy}) A_{11}. \end{aligned}$$

These identities imply that

$$\begin{aligned} A_{11} (d\varphi_x)^2 + 2A_{12} d\varphi_x d\varphi_y + A_{22} (d\varphi_y)^2 \\ = (\varphi_{xy}^2 - \varphi_{xx} \varphi_{yy}) (A_{22} dx^2 - 2A_{12} dx dy + A_{11} dy^2). \end{aligned}$$

If φ is not a linear function the factor $\varphi_{xy}^2 - \varphi_{xx} \varphi_{yy}$ is zero at isolated points only, so that the desired conclusion follows. If φ is linear, the statement is trivial.

D) *The variables x and y considered as functions of (ξ, η) satisfy the equations*

$$\begin{aligned} (5.4) \quad x_{\xi} &= \hat{T}_1 y_{\xi} + \hat{S}_1 y_{\eta} \\ x_{\eta} &= -\hat{S}_1 y_{\xi} + \hat{T}_1 y_{\eta}, \end{aligned}$$

where

$$\begin{aligned} (5.5) \quad \hat{T}_1(\xi, \eta) &= T_1(u^*, v^*) \\ \hat{S}_1(\xi, \eta) &= S_1(u^*, v^*). \end{aligned}$$

In order to prove this we note first that since $\zeta = \zeta(z)$ was assumed to be a smooth homeomorphism the functions $x(\xi, \eta)$, $y(\xi, \eta)$ have continuous derivatives. It will, therefore, suffice to show that equations (5.4) hold in the neighborhood of every point ζ_0 which is the image of a non-exceptional point z_0 . Now, in the neighborhood of z_0 the mapping $z \rightarrow w = \varphi_x - i\varphi_y$ is one-to-one. In the neighborhood of this point, therefore, we may introduce as new independent variables $u = \varphi_x$, $v = \varphi_y$, and also $u^* = \operatorname{Re} w^*$, $v^* = -\operatorname{Im} w^*$, and the transformations carried out in §4, A) are legitimate. Thus x, y considered as functions of u^* and v^* satisfy equations (4.3). On the other hand, $w^* = u^* - iv^*$ is an analytic function of $\zeta = \xi + i\eta$, and at ζ_0 the derivative $dw^*/d\zeta$ does not vanish, since the mapping $z \rightarrow w$ is one-to-one in the neighborhood of z_0 , so that

the same is true of the mapping $\zeta \rightarrow w^*$ in the neighborhood of the point ζ_0 . Thus we may consider w^* as an analytic function of ζ . But it is known, and immediately verified by a direct computation, that equations of the form (4.3) are preserved under a conformal mapping. Hence equations (5.4) hold in the neighborhood of the point ζ_0 .

E) We now reverse our procedure and assume that an analytic function $u^* - iv^* = w^*(\zeta)$ is given in a domain Δ , and that

$$|w^*(\zeta)| < R$$

where R is the conformal radius of equation (3.1). Then we can define in Δ the function (5.5) and may consider in Δ the linear system (5.4). Assume that $z(\zeta) = x(\xi, \eta) + iy(\xi, \eta)$ is a solution of this linear system which is univalent, that is a homeomorphism of Δ onto a domain D in the z -plane. Then w^* considered as a function of (x, y) defined in D is the distorted gradient of a (perhaps multiple-valued) solution φ of (3.1).

The proof of this statement may be left to the reader, since it consists merely in reversing the arguments used above.

F) The construction carried out in A) – C) can be extended to solutions of (3.1) having a multiple-valued gradient. More precisely, let $\tilde{D}$ be an unramified covering surface of a domain D in the z -plane. By this we mean that $\tilde{D}$ is a separable two-dimensional manifold on which there is defined a continuous mapping (projection) π into D , which is a homeomorphism in a properly chosen neighborhood of every point of $\tilde{D}$. In every such neighborhood we may introduce local coordinates (x, y) , by assigning to a point p of $\tilde{D}$ as its coordinates the coordinates of the point $\pi(p)$ in D . Hence if φ is a real-valued function defined in $\tilde{D}$ we may define its gradient $w = \varphi_x - i\varphi_y$ (which is independent of the local coordinates used) and we may decide when φ is a solution of equation (3.1).

Let φ be a solution of (3.1) defined in $\tilde{D}$; it need not be single-valued, but we assume that its gradient is. Then the metric (5.1) can be defined on $\tilde{D}$. If furthermore, $\tilde{D}$ is of genus zero, that is if $\tilde{D}$ is separated by every simple closed Jordan curve, then $\tilde{D}$ can be mapped onto a domain Δ in the ζ -plane by a transformation which is conformal with respect to the metric (5.1). As before we have that the distorted gradient w^* is an analytic function of ζ in Δ , and that x and y considered as functions of (ξ, η) satisfy equations (5.4).

G) The parametric representation of solutions of (3.1) described in A) – D) may be considered as a generalization of the Monge-Weierstrass representation of minimal surfaces. In order to make this analogy apparent it is advisable to consider together with (3.1) a certain “conjugate” equation. Since we shall not need it, however, we merely refer to our papers [7, 8] for further details.

§6. Equations with finite conformal radius and slowly growing characteristic.

A) We shall show that for such equations Bernstein's theorem is valid.

Theorem I. *Let (3.1) be an equation of finite conformal radius and slowly growing characteristic. Then every entire solution of (3.1) is a linear function.*

The proof given below is patterned after my proof of Bernstein's theorem on minimal surfaces.

B) Let $\varphi(x, y)$ be an entire solution of (3.1). We apply the results of 5, A) – D) taking for D the whole finite z -plane. The domain Δ is determined but for a conformal mapping and is simply connected. We may assume that it is the domain

$$(6.1) \quad \Delta: \quad |\zeta| < \rho \leq +\infty.$$

In Δ the distorted gradient $w^* = w^*(\varphi_x - i\varphi_y)$ is analytic and

$$(6.2) \quad |w^*| < R$$

where R is the conformal radius.

Consider x, y as functions of (ξ, η) . We claim that the following line-integrals are path independent:

$$(6.3) \quad \int (\hat{\tau}_0 x_\xi + \hat{\sigma}_0 x_\eta) d\xi + (-\hat{\sigma}_0 x_\xi + \hat{\tau}_0 x_\eta) d\eta$$

$$(6.4) \quad \int (\hat{\tau}_1 y_\xi + \hat{\sigma}_1 y_\eta) d\xi + (-\hat{\sigma}_1 y_\xi + \hat{\tau}_1 y_\eta) d\eta$$

where

$$\hat{\tau}_j(\xi, \eta) = \tau_j(u^*, v^*), \quad \hat{\sigma}_j(\xi, \eta) = \sigma_j(u^*, v^*), \quad u^* - iv^* = w^*, \quad j = 0, 1,$$

the functions τ_j, σ_j being defined in §4, B). It will suffice to consider the first of these integrals.

The path independence of (6.3) would follow if we could show that every point ζ_0 of Δ has a neighborhood $N(\zeta_0)$ such that

$$(6.5) \quad \oint_{\Gamma} (\hat{\tau}_0 x_\xi + \hat{\sigma}_0 x_\eta) d\xi + (-\hat{\sigma}_0 x_\xi + \hat{\tau}_0 x_\eta) d\eta = 0$$

for every simple closed rectifiable curve Γ situated in $N(\zeta_0)$, for in this case we could repeat verbatim Goursat's proof of Cauchy's integral theorem. As a matter of fact, since the derivatives x_ξ, x_η , are continuous, it is enough to prove that (6.5) holds for all points ζ_0 of Δ , except perhaps for a discrete set. We assume that φ is not a linear function and show that (6.5) holds in a sufficiently small neighborhood $N(\zeta_0)$ of a point ζ_0 in Δ which is the image of a non-exceptional point z_0 (cf. §5, B)). Now, reasoning as in §5, D) we can find a small disc $N(\zeta_0) \subset \Delta$ with center at ζ_0 such that in $N(\zeta_0)$ the mapping $\zeta \rightarrow w^*$ is one to one. For Γ in $N(\zeta_0)$ we have then that

$$(6.6) \quad \begin{aligned} \oint_{\Gamma} (\hat{\tau}_0 x_\xi + \hat{\sigma}_0 x_\eta) d\xi + (-\hat{\sigma}_0 x_\xi + \hat{\tau}_0 x_\eta) d\eta \\ = \oint_{\Gamma^*} (\tau_0 x_{u^*} - \sigma_0 x_{v^*}) du^* + (\sigma_0 x_{u^*} + \tau_0 x_{v^*}) dv^* \end{aligned}$$

where Γ^* is the image of Γ under the mapping $\zeta \rightarrow w^*$. This is verified by noting that u^* and $(-v^*)$ satisfy the Cauchy-Riemann equations with respect to (ξ, η) . Let z_0 be the point in the z -plane for which $\zeta(z_0) = \zeta_0$. We denote the inverse image of $N(\zeta_0)$ under the mapping $\zeta = \zeta(z)$ by $N(z_0)$. In $N(z_0)$ we may introduce as new independent variables the functions (u, v) and (u^*, v^*) . Then x and y considered as functions of (u, v) satisfy equations (4.2) and considered as functions of (u^*, v^*) equations (4.3). Hence x considered as a function of (u^*, v^*) satisfies equation (4.6), and noting (4.12) we conclude that

$$(6.7) \quad (\sigma_0 x_{u^*} + \tau_0 x_{v^*})_{v^*} - (\tau_0 x_{u^*} - \sigma_0 x_{v^*})_{u^*} = 0$$

in the domain interior to the curve Γ^* . From (6.6) and (6.7), (6.5) follows.

C) Having established the path-independence of the integral (6.3) we define the single-valued real-valued function $\tilde{x}(\xi, \eta)$ by the equation

$$(6.8) \quad \tilde{x}(\xi, \eta) = \int_{(0,0)}^{(\xi,\eta)} (\hat{\tau}_0 x_\xi + \hat{\sigma}_0 x_\eta) d\xi + (-\hat{\sigma}_0 x_\xi + \hat{\tau}_0 x_\eta) d\eta.$$

We claim that the mapping $\zeta \rightarrow \tilde{x} + ix$ has a non-vanishing Jacobian. In fact, if $\tilde{x}_\xi x_\eta - \tilde{x}_\eta x_\xi = 0$ at some point, then also

$$(6.9) \quad x_\xi^2 + x_\eta^2 = 0$$

since it follows from (6.8) that

$$(6.10) \quad \begin{aligned} \tilde{x}_\xi &= \hat{\tau}_1 x_\xi + \hat{\sigma}_1 x_\eta \\ \tilde{x}_\eta &= -\hat{\sigma}_1 x_\xi + \hat{\tau}_1 x_\eta. \end{aligned}$$

But (6.9) is impossible, since we have by construction that $\xi_x \eta_y - \xi_y \eta_x > 0$ and therefore $x_\xi y_\eta - x_\eta y_\xi > 0$.

We prove next that the mapping $\zeta \rightarrow \tilde{x} + ix$ is one-to-one. In fact, let a be any real number. The locus $x = a$ in Δ is a simple continuously differentiable open arc Λ , for it is the image of a straight line under a smooth homeomorphism. Along Λ the tangential derivative x_t of x vanishes. Hence the normal derivative, x_n , is different from zero and has constant sign. Otherwise the Jacobian $x_\xi y_\eta - x_\eta y_\xi$ would vanish somewhere on Λ . For the function $\tilde{x}$ we have by (6.10) that (upon proper orientation)

$$\tilde{x}_t = -\hat{\sigma}_1 x_n + \hat{\tau}_1 x_t.$$

But $\hat{\sigma}_1 > 0$. Hence $\tilde{x}_t$ has constant sign on Λ , so that $\tilde{x}$ is monotone on this curve. This proves the assertion.

D) We show now that $\rho = +\infty$. In fact, assume that $\rho < +\infty$. Then the mapping $\zeta \rightarrow z$ is a homeomorphism of $|\zeta| < \rho < +\infty$ onto the whole plane, so that

$$(6.11) \quad \lim_{|\zeta| \rightarrow \rho} (x^2 + y^2) = +\infty.$$

On the other hand, we have just seen that the mapping $\zeta \rightarrow \tilde{x} + i\tilde{y}$ is a smooth homeomorphism of $|\zeta| < \rho$. In view of (6.10) we have that

$$\frac{\tilde{x}_\xi^2 + \tilde{x}_\eta^2 + x_\xi^2 + x_\eta^2}{2(\tilde{x}_\xi x_\eta - \tilde{x}_\eta x_\xi)} = \frac{1 + \hat{\sigma}_1^2 + \hat{\tau}_1^2}{2\hat{\sigma}_1}$$

so that, by (4.11), the mapping $\zeta \rightarrow \tilde{x} + i\tilde{y}$ is of bounded eccentricity. Applying Agmon's result (see §2, E)) we conclude that

$$(6.12) \quad \overline{\lim}_{\lambda \rightarrow \rho} x(\lambda \cos \omega, \lambda \sin \omega) < +\infty \quad \text{for almost all } \omega.$$

In exactly the same way we can show that

$$(6.13) \quad \overline{\lim}_{\lambda \rightarrow \rho} y(\lambda \cos \omega, \lambda \sin \omega) < +\infty \quad \text{for almost all } \omega.$$

But (6.12) and (6.13) contradict (6.11).

E) We are ready to conclude. Since $\rho = +\infty$, w^* considered as a function of ζ is an entire analytic function. But it is bounded (see (6.2)), so that it is constant (Liouville's theorem). This implies that $w = \varphi_x - i\varphi_y$ is constant, so that $\varphi = ax + by + c$.

§7. Equations with finite conformal radius and slowly growing eccentricity.

A) For such equations, too, Bernstein's theorem is valid. In fact, we shall prove a somewhat stronger statement.

Theorem II. *Assume that equation (3.1) has a slowly growing eccentricity. Then it has non-linear entire solutions if and only if its conformal radius is infinite (or, according to §3, D), if and only if the ellipticity domain Ω is the whole plane).*

We assume first that the conformal radius is finite and show that an entire solution $\varphi(x, y)$ must be linear. Proceeding as in §6, B) we map the whole finite z -plane onto the domain (6.1) by a transformation conformal with respect to the metric (5.1). If $\rho = +\infty$, the linearity of φ follows as in §6, D). We show that the assumption $\rho < +\infty$ leads to a contradiction.

If $\rho < +\infty$ then the condition $|w^*| < R < +\infty$ implies, by a well-known consequence of Schwarz' lemma, that the analytic function of ζ, w^* , satisfies an inequality of the form

$$(7.1) \quad R - |w^*| \geq \text{const.} (\rho - |\zeta|).$$

In view of (3.6) we have that the quantity $E(u, v)$ defined in §3, C) considered as a function of ζ satisfies the inequality

$$(7.2) \quad E \leq \frac{\text{const.}}{(1 - |\zeta|)^{1-\epsilon}}, \quad 0 < \epsilon < 1.$$

But E is exactly the eccentricity of the mapping $\zeta \rightarrow z$ of $|\zeta| < \rho < +\infty$ onto the whole z -plane. Hence (7.2) contradicts Corollary 1 of §2, C).

B) We assume next that the conformal radius of equation (3.1) is infinite and choose a non-constant entire analytic function $w^*(\zeta) = u^* - iv^*$. With the aid of this function we define the functions (5.5) and consider the linear equations (5.4). I have shown elsewhere [7] that an elliptic system of the form (5.4) always possesses a univalent solution (x, y) . We consider a fixed such solution and denote the domain in the z -plane onto which it maps the ζ -plane by D . According to §5, E) the function w^* considered as a function of (x, y) defined in D is the distorted gradient of a (non-linear) solution φ of (3.1). In order that this solution be an entire solution D must be the whole plane.

The latter will certainly be the case if the quantity E , defined by (3.4), which may be considered as a function of ζ (u and v being uniquely determined by w^*), satisfies a relation of the form

$$(7.3) \quad E \leq o(1) \log |\zeta|, \quad |\zeta| \rightarrow \infty.$$

Indeed, under this condition the assumption that D is not the whole plane leads to a contradiction with Corollary 2 of §2, C), since if D is not the whole plane it can be mapped conformally onto the unit disc, and the resulting combined mapping of the plane onto the unit disc still has the eccentricity E .

On the other hand, in view of the definition of the function e , we have that (7.3) is a consequence of the relation

$$(7.4) \quad e[M(\rho)] = o(\log \rho), \quad \rho \rightarrow +\infty$$

where $M(\rho)$ is the maximum of the modulus of the analytic function $w^*(\zeta)$ on $|\zeta| = \rho$. Thus in order to know which entire analytic functions $w^*(\zeta)$ give rise to entire solutions of (3.1) one would have to know in which way the eccentricity function $e(r)$ grows. If $e = O(1)$, for instance, then any entire analytic function is acceptable. On the other hand, under the hypothesis (3.5) of slow growth the entire functions $w^* = a\zeta + b$, though perhaps no others, satisfy (7.4). Thus Theorem II is completely proved.

C) We prove next a local version of Bernstein's theorem.

Theorem III. *Assume that the conformal radius of equation (3.1) is finite and that the eccentricity grows slowly. Let $\varphi(x, y)$ be a solution of (3.1) defined for $a^2 < x^2 + y^2 < +\infty$ such that the gradient $w = \varphi_x - i\varphi_y$ of φ is m -valued. Then the finite limit*

$$(7.5) \quad w_\infty = \lim_{x^2+y^2 \rightarrow \infty} (\varphi_x - i\varphi_y)$$

exists and belongs to Ω .

Proof. Let $\tilde{D}$ denote the m -sheeted unramified unbounded covering surface of the domain $a < |z| < +\infty$. ($\tilde{D}$ may be thought of as the part of the Riemann surface of the function $z^{1/m}$ which covers the domain $a < |z| < +\infty$.) By the hypotheses of the theorem the function φ may be considered as defined on $\tilde{D}$, and w is single-valued on $\tilde{D}$. On the other hand, the surface $\tilde{D}$ is of genus

zero. Hence we may map it onto a necessarily doubly connected domain Δ in the ζ -plane by a transformation which is conformal with respect to the metric (5.1) (cf. §5, F)). Without loss of generality we assume that the boundary continuum of Δ corresponding to $|z| = a$ is non-degenerate, since we can, if necessary, consider instead of $\bar{D}$ the covering surface of the domain $a^2 + 1 < |z| < +\infty$. Since Δ is determined but for a conformal mapping, we may assume that it is the domain

$$(7.6) \quad \Delta: \quad 1 < |\zeta| < \rho \leq +\infty$$

with $|\zeta| \rightarrow \rho$ for $|z| \rightarrow \infty$.

We note that the distorted gradient $w^* = w^*(\varphi_x - i\varphi_y)$ is analytic in Δ and its modulus is bounded by the conformal radius $R < +\infty$. If we knew that $\rho = +\infty$, then we would have, by the theorem on removable singularities, that the limit

$$\lim_{\zeta \rightarrow \infty} w^* = w_\infty^*$$

exists, and, by the maximum modulus theorem, that $|w_\infty^*| < R$. This would imply the assertion of the theorem.

It remains to disprove the possibility of the inequality $\rho < +\infty$. But this can be accomplished exactly as in B) by noting that the quantity E defined by (3.4) and bounded by $e(|w^*|)$ is the eccentricity of the mapping $\zeta \rightarrow Z = \zeta^{1/m}$ of Δ onto the domain $a^{1/m} < |Z| < +\infty$. If $R < +\infty$, then (7.1) holds, and this contradicts Lemma 1 of §2, C).

D) **Theorem IV.** *Under the same hypotheses on the equation (3.1) as in Theorem III, let $\varphi(x, y)$ be a solution of this equation defined in $0 < x^2 + y^2 < a^2$. Assume that the gradient $w = \varphi_x - i\varphi_y$ of φ is m -valued. Then the finite limit*

$$(7.7) \quad w_0 = \lim_{x^2+y^2 \rightarrow 0} (\varphi_x - i\varphi_y)$$

exists and belongs to Ω .

The proof is so similar to that of Theorem III that it may be left to the reader. In the special case $m = 1$ this proof, together with the statement made in §5, E) implies the following

Corollary to Theorem IV. *Under the same conditions on equation (3.1) as in Theorem III assume that $\varphi(x, y)$ is a solution defined for $0 < x^2 + y^2 < a^2$ and having a single-valued gradient. Then this solution itself is single-valued and the singularity at 0 is removable.^{5a}*

Reasoning as in B) we can show that equations with an infinite conformal radius and slowly growing eccentricity always have single-valued solutions with non-removable isolated singularities at finite points.

^{5a} For the gas dynamical equation this theorem is also proved by FINN [11, p. 402].

§8. Examples. A) The distorted gradient (and hence also the conformal radius, the eccentricity and the characteristic) can be determined effectively for an elliptic equation of the form

$$(8.1) \quad A_1(\varphi_x)\varphi_{xx} + A_2(\varphi_y)\varphi_{yy} = 0$$

where $A_1(u)$, $A_2(v)$ are sufficiently smooth positive functions defined for

$$(8.2) \quad -l_{11} < u < l_{12}, \quad -l_{21} < v < l_{22},$$

with $0 < l_{ij} \leq +\infty$.

In fact, set

$$(8.3) \quad W(u - iv) = A_1(0)^{-\frac{1}{2}} \left[\int_0^u A_1(t)^{\frac{1}{2}} dt - i \int_0^v A_2(t)^{\frac{1}{2}} dt \right].$$

If all four quantities

$$(8.4) \quad \int_0^{-l_{11}} A_1^{\frac{1}{2}} du, \quad \int_0^{l_{12}} A_1^{\frac{1}{2}} du, \quad \int_0^{-l_{21}} A_2^{\frac{1}{2}} dv, \quad \int_0^{l_{22}} A_2^{\frac{1}{2}} dv$$

are infinite, the conformal radius of (8.1) is infinite, and $w^* = W$ is the distorted gradient. If one of the quantities (8.4) is finite, then (8.3) maps the ellipticity domain (8.2) conformally with respect to the metric $A_1 du^2 + A_2 dv^2$ onto a domain which is either a half-plane, or an infinite strip, or a semi-infinite strip, or a rectangle. In all cases the conformal radius is finite and the distorted gradient equals $w^* = F(W)$, where F is an analytic function which can be written down explicitly.

B) An equation of the form (3.1) of finite conformal radius can have non-linear entire solutions. In fact, $\varphi = xy$ is an entire solution of (8.1) provided that the ellipticity domain is the whole plane, *i.e.* provided that $l_{ij} = +\infty$, $i, j = 1, 2$.

C) An equation of the form (3.1) can have a non-linear entire solution with a uniformly bounded gradient.

To see this, let $A(u)$ be a sufficiently smooth positive function defined for $-1 < u < 1$ and such that

$$(8.5) \quad \int_{-1}^0 A(u) du = \int_0^1 A(u) du = +\infty.$$

Set

$$(8.6) \quad x(u) = \int_0^u A(u') du', \quad y(v) = -\int_0^v A(v') dv',$$

and denote the functions inverse to (8.6) by $u(x)$, $v(y)$ respectively. The function

$$(8.7) \quad \varphi(x, y) = \int_0^x u(x') dx' + \int_0^y v(y') dy'$$

is an entire solution of the equation

$$(8.8) \quad A(\varphi_x)\varphi_{xx} + A(\varphi_y)\varphi_{yy} = 0,$$

and $\varphi_x^2 + \varphi_y^2 < 1$. Note that A can be chosen so that

$$(8.9) \quad \int_{-1}^1 A(u)^{\frac{1}{2}} du < +\infty.$$

In this case the conformal radius of (8.8) is finite.

C) An equation of the form (3.1) may possess non-linear entire solutions without having single-valued solutions with non-removable isolated singularities.

In fact, let $A(u)$ be a sufficiently smooth positive function defined for $-\infty < u < +\infty$ and such that

$$(8.10) \quad K = \int_{-\infty}^{+\infty} A(u) du < +\infty.$$

Equation (8.8) can be written in the form of a conservation law

$$(8.11) \quad \frac{\partial \Lambda(\varphi_x)}{\partial x} + \frac{\partial \Lambda(\varphi_y)}{\partial y} = 0$$

where

$$(8.12) \quad \Lambda(u) = \int_0^u A(u') du'.$$

Since $|\Lambda| < K$ a theorem by Finn [11]⁶ implies that all (finite) isolated singularities of single-valued solutions of (8.11) are removable. But $\varphi = xy$ is an entire solution. Note that condition (8.10) does not imply the finiteness of the conformal radius.

D) Conversely, an equation of the form (3.1) may have single-valued solutions with non-removable isolated singularities without having non-linear entire solutions.

In fact, to every solution $\varphi(x, y)$ of the equation of minimal surfaces

$$(8.13) \quad (1 + \varphi_x^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_y^2)\varphi_{yy} = 0$$

one can associate a function $\psi(x, y)$ such that

$$(8.14) \quad \varphi_x = (1 + \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}}\psi_y, \quad \varphi_y = -(1 - \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}}\psi_x.$$

This function satisfies the equation

$$(8.15) \quad (1 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (1 - \psi_x^2)\psi_{yy} = 0$$

⁶ Cf. also BERS [8].

with the ellipticity domain $\psi_x^2 + \psi_y^2 < 1$. By Bernstein's theorem on minimal surfaces all entire solutions of (8.15) are linear. On the other hand the function

$$(8.16) \quad \psi(x, y) = \log [r + (1 + r^2)^{\frac{1}{2}}], \quad r^2 = x^2 + y^2,$$

is a solution of (8.15). The first derivatives of ψ are discontinuous at $x = y = 0$.

E) Let $\rho = \rho(q)$ be a positive sufficiently smooth function defined for $0 \leq q < \hat{q} \leq +\infty$, and assume that

$$(8.17) \quad \frac{1}{\rho(q)} \frac{d}{dq} [q\rho(q)] = \mu(q)^2 > 0.$$

The equation

$$(8.18) \quad (\rho\varphi_x)_x + (\rho\varphi_y)_y = 0, \quad \rho = \rho(\varphi_x^2 + \varphi_y^2)^{\frac{1}{2}}$$

can be written in the form (3.1) with

$$(8.19) \quad \begin{aligned} A_{11} &= \sin^2 \theta + \mu(q)^2 \cos^2 \theta \\ A_{12} &= [\mu(q)^2 - 1] \sin \theta \cos \theta \\ A_{22} &= \cos^2 \theta + \mu(q)^2 \sin^2 \theta \end{aligned}$$

where

$$(8.20) \quad \varphi_x - i\varphi_y = qe^{-i\theta}.$$

The ellipticity domain of this equation is the disc $u^2 + v^2 < \hat{q}^2$.

For equation (8.18) the distorted gradient can be written down explicitly. In fact, set

$$(8.21) \quad q^*(q) = q \exp \left[- \int_0^q [1 + \mu(q')] \frac{dq'}{q} \right].$$

Then

$$(8.22) \quad w^* = q^* e^{-i\theta}$$

as can be verified by a simple computation. Thus the conformal radius of (8.18) is infinite or finite according to whether the integral

$$(8.23) \quad \int_{\epsilon}^{\hat{q}} \frac{\mu(q)}{q} dq$$

does or does not diverge. It is possible to write down explicit conditions for slow growth of the eccentricity and of the characteristic of equation (8.18), but since these relations are rather complicated we do not do it here.

F) For

$$(8.24) \quad \rho^2 = \frac{1}{1+q^2}$$

equation (8.18) becomes the equation (8.13) of minimal surfaces. In this case

$$(8.25) \quad \hat{q} = +\infty, \quad q^* = \frac{(1+2^{\frac{1}{2}})q}{1+(1+q^2)^{\frac{1}{2}}}$$

and the conformal radius is $1+2^{\frac{1}{2}}$. Since the ellipticity domain is the whole plane, the eccentricity does not grow slowly (cf. §3, D)). A direct computation shows that the characteristic of (8.13) *vanishes identically*.

G) For

$$(8.26) \quad \rho = \left(1 - \frac{\gamma-1}{2} q^2\right)^{1/(\gamma-1)}, \quad \gamma > 1$$

equation (8.18) becomes the potential equation of a subsonic adiabatic gas flow. In this case

$$\hat{q} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}.$$

The equation of a subsonic adiabatic gas flow is (i) of finite conformal radius, (ii) of slowly growing eccentricity, (iii) not of slowly growing characteristic. We leave to the reader the verification of these statements. It is easy to see that (i) and (ii) hold for every equation of the form (8.18) provided that $\hat{q} < +\infty$ and that the function (8.17) is continuously differentiable in the closed interval $0 \leq q \leq \hat{q}$.

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