A Property of Quasi-conformal Mappings

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It is known that a univalent analytic function $f(z)$, $|z| < 1$, has a finite radial limit $f(e^{i\omega})$ at almost every point $e^{i\omega}$, and that the function $f(e^{i\omega})$ is not constant on a set of positive measure.² The proof of this statement involves the classical theorems of Fatou and of F. and M. Riesz on boundary values of bounded (not necessarily univalent) analytic functions. The latter theorems are, however, not available in the case of bounded functions of bounded excentricity, so that in proving the corresponding statement for univalent functions of bounded excentricity we must use a different, more elementary, approach.

In this note we establish the following

Theorem A. *Let $\zeta = \zeta(z) = \xi + i\eta$ be a topological mapping of the unit disc $|z| < 1$. Assume that the derivatives $\xi_x, \xi_y, \eta_x, \eta_y$ exist and are continuous,*

$$\xi_x \eta_y - \xi_y \eta_x > 0,$$

and

$$(1) \quad \frac{\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2}{2(\xi_x \eta_y - \xi_y \eta_x)} \leq K < +\infty.$$

Then (i) the limit

$$(2) \quad \zeta(e^{i\omega}) = \lim_{r \rightarrow 1} \zeta(re^{i\omega})$$

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² See W. SEIDEL, Über die Ränderzuordnung bei konformen Abbildungen, *Math. Annalen*, vol. 104 (1931), pp. 182-243.

exists and is finite for almost all ω , and (ii) the function $\zeta(e^{i\omega})$ is not constant on any set of positive (linear) measure.

We remark that the smoothness conditions on $\zeta(z)$ can be weakened, and so can the condition of univalence. The present formulation, however, is sufficient for the applications to a problem in partial differential equations which was the occasion for the present note.³

We first prove Theorem A under the additional assumption that $\zeta(z)$ is bounded, $|\zeta(z)| < M$. Denote the length of the image of the segment $z = te^{i\omega}$, $\frac{1}{2} < t < r$ under the mapping $\zeta(z)$ by $l(r, \omega)$. We have, using polar coordinates ($z = te^{i\omega}$), that for $\frac{1}{2} < r < 1$

$$l(r, \omega) = \int_{\frac{1}{2}}^r \left| \frac{\partial \zeta(te^{i\omega})}{\partial t} \right| dt \leq 2^{\frac{1}{2}} K^{\frac{1}{2}} \int_{\frac{1}{2}}^r (\xi_x \eta_y - \xi_y \eta_x)^{\frac{1}{2}} t dt.$$

Integrating this with respect to ω from 0 to 2π we see by Schwarz' inequality that

$$\left\{ \int_0^{2\pi} l(r, \omega) d\omega \right\}^2 \leq 8K \iint_{\frac{1}{2} < |z| < r} (\xi_x \eta_y - \xi_y \eta_x) dx dy \cdot \pi(r^2 - \frac{1}{4}) \leq 8K \cdot \pi M^2 \cdot \frac{3}{4} \pi.$$

Thus

$$\int_0^{2\pi} l(1, \omega) d\omega \leq (6K)^{\frac{1}{2}} \pi M,$$

so that $l(1, \omega)$ is finite for almost all ω . This implies that the finite limit (2) exists for almost all ω .

Assume now that there exists a set Λ in the interval $0 \leq \omega < 2\pi$ which has a positive measure and is such that $\zeta(e^{i\omega}) = a$ for ω in Λ . Without loss of generality we may assume that $a = 0$. By the previous result there exists an ω_0 in Λ such that the image Γ of the segment $z = te^{i\omega_0}$, $0 \leq t < 1$ under the mapping $\zeta(z)$ is a Jordan arc of finite length which leads from the point $\zeta(0)$ to the point $\zeta = 0$. Let Δ be the bounded domain onto which $\zeta(z)$ maps the unit disc. $\zeta = 0$ is a boundary point of Δ . Let $\zeta_1 = \zeta^{\frac{1}{2}}$ denote a single-valued (in Δ) branch of the function. It maps Δ onto a bounded domain Δ_1 which contains a Jordan arc Γ_1 leading to $\zeta_1 = 0$. The arc Γ'_1 obtained from Γ_1 by the transformation $\zeta'_1 = -\zeta_1$ lies outside Δ_1 except for one end-point. Let $\zeta_2 = g(\zeta_1)$ be an analytic function which maps the domain exterior to Γ'_1 onto the sector $|\arg \zeta_2| < \frac{1}{4}\pi$, taking $\zeta_1 = 0$ into $\zeta_2 = 0$. This function is continuous on the closure of Δ_1 .

Next, let $F(\zeta_2)$ be a bounded univalent analytic function defined for $\operatorname{Re} \zeta_2 > 0$ and such that the limit

$$\lim_{\xi_2 \rightarrow 0+} F(\xi_2), \quad \xi_2 \text{ real}$$

³ See L. BERS, Non-linear elliptic equations without non-linear entire solutions, *J. Rational Mech. Anal.*, following in this issue.

does not exist. It is not difficult to construct such a function. By a well-known property of bounded analytic functions⁴ we have that the limit

$$\lim_{\zeta_2 \rightarrow 0} F(\zeta_2)$$

does not exist if ζ_2 approaches 0 along a Jordan curve within the sector $|\arg \zeta_2| < \frac{1}{4}\pi$. Set $\zeta_3(z) = F\{g[\zeta(z)^{\frac{1}{4}}]\}$. If z approaches a point $e^{i\omega}$, ω in Λ , along a radius, the point $\zeta_2 = g[\zeta(z)^{\frac{1}{4}}]$ approaches 0 within the sector considered. Hence the limit

$$\lim_{r \rightarrow 1} \zeta_3(re^{i\omega})$$

fails to exist for ω in Λ . Since, however, the function $\zeta_3(z)$ is bounded and satisfies the hypotheses of Theorem A, we arrived at a contradiction with the result already established. Thus Theorem A is proved for bounded functions.

If $\zeta(z)$ is unbounded, it maps $|z| < 1$ onto a simply connected domain Δ which is not the whole plane. (It is well known that the disc can not be mapped onto the whole plane by a mapping of bounded excentricity). There exists an elementary analytic function $\tilde{\zeta} = f(\zeta)$, involving only linear mappings and square roots, which maps Δ onto a bounded domain. The conclusion follows by applying the result already established to the function $\tilde{\zeta}(z) = f[\zeta(z)]$.

It is an open question whether the conclusions of Theorem A remain valid if $\zeta(z)$ is assumed to be bounded but not univalent.

⁴ See R. NEVANLINNA, *Eindeutige analytische Funktionen*, Berlin, Julius Springer (1936), p. 65.