On a Three Dimensional Transmission Problem of Electromagnetic Theory

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§1. Introduction. The principal purpose of this paper is to establish an existence and uniqueness theorem for the following diffraction problem:

Let S be a closed surface separating two homogeneous media. Assume that the interior medium $\mathcal{V}$ has a permeability constant μ_0 , a dielectric constant ϵ_1 , and a conductivity constant σ_1 , and that the exterior medium has corresponding constants μ_0 , ϵ_0 , and $\sigma_0 = 0$. Let an electromagnetic field $(\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}})$ be incident from the exterior, so that the Maxwell equations

$$\text{curl } \mathbf{E}_{\text{inc}} = i\omega\mu_0 \mathbf{H}_{\text{inc}}$$

$$\text{curl } \mathbf{H}_{\text{inc}} = -\omega\epsilon_0 \mathbf{E}_{\text{inc}}$$

hold throughout Euclidean three-dimensional space when the stationary time dependence $\exp(-i\omega t)$ is assumed. The problem is to find a field $(\mathbf{E}, \mathbf{H})$ such that

(a) $(\mathbf{E} - \mathbf{E}_{\text{inc}}, \mathbf{H} - \mathbf{H}_{\text{inc}})$ satisfy the radiation condition

$$\lim_{r \rightarrow \infty} r \left[\frac{\mathbf{r}}{|\mathbf{r}|} \times \text{curl } (\mathbf{E} - \mathbf{E}_{\text{inc}}) + ik_0(\mathbf{E} - \mathbf{E}_{\text{inc}}) \right] = 0, \quad k_0 = \omega\sqrt{\epsilon_0\mu_0}.$$

(b) $(\mathbf{E}, \mathbf{H})$ satisfy Maxwell's equations in $\mathcal{V}$, with physical constants μ_0 , ϵ_1 and σ_1 .

(c) $(\mathbf{E}, \mathbf{H})$ satisfy Maxwell's equations outside of $\mathcal{V}$ with physical constants μ_0 , ϵ_0 and $\sigma_0 = 0$.

(d) The tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous across S .

Our method, which is that of integral equations, can be considered a generalization of the procedure first used by Rothe [15] and Sternberg [21, 22] for similar problems involving cylindrical symmetry. (Cf. Freudenthal [6], Barrar [3], and Dolph [5].) In particular, we replace the role of the usual Green's theorem for the scalar wave equation by the representation obtained by Stratton & Chu [23, p. 464] and use the latter in a manner entirely similar to that of Rothe or Sternberg referred to above.

Specifically, we establish the existence and uniqueness of the solution to this problem when the surface S is of class C^3 (see definition, (2.3)). Previously Müller [11] obtained similar results when the surface S is analytic by another method. However, Müller allowed both μ and ϵ to change across the surface while we have allowed ϵ and σ to change. Our method, however, is applicable in principle if μ also changes, although in this case our Fredholm integral equations would have to be replaced by different "belastete" integral equations. (Cf. Sternberg [21], Kneser [8] and Barrar [3].) Finally we believe that our present method is capable of extension to a medium with a variable dielectric constant (Cf. equation (5.5) and the discussion preceding it.)

We note in passing that Müller [13, 14] and Weyl [25, 26] have treated the case where S is a perfect conductor. Here the major difficulty is to study the eigensolutions of the resulting integral equation. In our case there are no eigensolutions of the integral equation, and it would be an interesting problem to show that as σ approaches infinity our solution approaches that of Weyl and Müller. (Cf. Rothe [15].) Historically, the conclusion that there are no eigenfunctions in problems of the type we are considering was first drawn by Sommerfeld [20] on physical grounds in the scalar case although he did not give a mathematical demonstration. Freudenthal [6], appears to have been the first to give a general rigorous proof. (Cf. the discussion in [6].) In the electromagnetic case, the fact that no eigenfunctions exist follows from a recent theorem of Saunders [16] if $\sigma_1 = 0$ in $\mathcal{V}$ and can be established by extending Freudenthal's original reasoning if $\sigma_1 \neq 0$ in $\mathcal{V}$. (See Theorem (4.2).)

The plan of the paper is as follows: In §2, we collect the necessary definitions and known lemmas from the potential theory of the scalar wave equation; in §3, we discuss the relationship between the representation of Stratton & Chu referred to above with that given by Levine & Schwinger [10]. This section has been included because we think the present proof, which starts with the Stratton & Chu representation, will be of interest to mathematicians. The original proof given by Levine & Schwinger employs a delta-function approach, and while their procedure can be justified by the Schwartz theory of distributions, the method of §3 obviates the necessity of considering delta-functions directly, in that the well-known singularity arguments of potential theory are used in their stead. The treatment of this section is an improvement over that given in Chapter VII of [5] which also avoided the delta-function. §4 is devoted to auxiliary existence and uniqueness theorems and includes a discussion of the

results of Rellich [14], Saunders [16], Freudenthal [6], and their generalizations to the electromagnetic case. In §5, the basic integral equations are formulated on the basis of the Stratton & Chu representation and their equivalence with the problem stated in this introduction is demonstrated. §6 is devoted to establishing the existence and uniqueness of the system of integral equation derived in §5.

§2. Notation, definitions and known lemmas. The following symbols will be used in the sequel:

- $\mathbf{r}$ = the vector from the point p to the point q ,
- $r = |\mathbf{r}| = r_{pq}$ = the Euclidean distance from the point p to the point q ,
- $k_0 = \omega \sqrt{\epsilon_0 \mu_0}$, real positive constant,
- $k_1^2 = \omega^2 \epsilon_1 \mu_0 + i\omega \mu \sigma_1$ where $\sigma_1 = 0$ is admitted, and
- $\phi = \exp(i k_0 r)/r$.

Where necessary, subscripts will be attached to differential operators to indicate which point is varying; e.g., $\text{grad}_p \phi$ indicates differentiation with respect to the point p , the point q being fixed.

Definition (2.1). If $f(p)$ is a function defined in a region $\mathcal{V}$ and if there exists a constant C such that

$$(2.1) \quad |f(p) - f(q)| \leq Cr^\alpha, \quad 0 < \alpha < 1,$$

for any two points p, q in $\mathcal{V}$, then $f(p)$ is said to be α -Hölder continuous in $\mathcal{V}$.

Further, if all the n^{th} order derivatives of $f(p)$ are α -Hölder continuous in $\mathcal{V}$, then $f(p)$ is said to be α - n -Hölder continuous in $\mathcal{V}$.

Definition (2.2). The norm $\|f\|_\alpha^\mathcal{V}$ is defined to be $[D + \max f(p)]$ for p in $\mathcal{V}$, where $D = \lim$ inferior of all constants C such that (2.1) holds.

The norm $\|f\|_{\alpha,n}^\mathcal{V}$ is defined to be equal to

$$\|f\|_{\alpha,n}^\mathcal{V} \equiv \sum_{i_1+i_2+i_3=n} \left\| \frac{\partial^{i_1, i_2, i_3}}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}} \right\|_\alpha^\mathcal{V}.$$

Definition (2.3). A surface S is said to be of the class C^3 if there exists a finite number m of images, $x^\nu = x^\nu(u, v)$, $y^\nu = y^\nu(u, v)$, $z^\nu = z^\nu(u, v)$, $\nu = 1, 2, \dots, m$ of the disk $u^2 + v^2 \leq 1$ that cover the surface S and such that the third derivatives of x^ν , y^ν , and z^ν with respect to u and v exist and are continuous. (We shall restrict attention to one such representation of S throughout.)

Definition (2.4). A function $f(p)$ defined on S is said to be α - n -Hölder continuous on S if $f(x^\nu, y^\nu, z^\nu) = F^\nu(u, v)$ is α - n -Hölder continuous in $u^2 + v^2 \leq 1$ for $\nu = 1, \dots, m$, where by definition

$$\|f\|_{\alpha,n}^S = \sum_{\nu=1}^m \|F^\nu\|_{\alpha,n}^{u^2+v^2 \leq 1}.$$

Definition (2.5). A completely continuous operator in a normed space carries bounded sets into sets which are sequentially compact.

Since our proof leans heavily on the properties of the kernel ϕ and since, as is well known, the properties of this kernel are deducible from the corresponding properties of the kernel $1/r$ of potential theory, we summarize the pertinent lemmas and reference the corresponding potential theory proof.

Lemma (2.1). *Let*

$$(2.2) \quad U = \int_{\mathfrak{V}} \phi f \, dv \quad \text{where } \|f\|_{\alpha}^{\mathfrak{V}} \text{ exists.}$$

Then

$$\nabla^2 U(p) + k_0^2 U(p) = -4\pi\lambda f(p),$$

where

$$\lambda = 1 \text{ for the point } p \text{ in the interior of } \mathfrak{V},$$

$$\lambda = 0 \text{ for the point } p \text{ not in the closure of } \mathfrak{V}.$$

(Cf. Kellogg [7, Theorem III, p. 156].)

Lemma (2.2). *Under the assumptions of Lemma (2.1), U is of the class C^1 throughout all space. (Cf. Courant & Hilbert [4, p. 228].)*

Lemma (2.3). *In equation (2.2) let $\|f\|_{\alpha,1}^{\mathfrak{V}}$ exist and let $\mathfrak{S}$ be of class C^3 , then $\|U\|_{\alpha,3}^{\mathfrak{V}}$ exists and there is a constant M such that*

$$\|U\|_{\alpha,3}^{\mathfrak{V}} \leq M \|f\|_{\alpha,1}^{\mathfrak{V}}.$$

Sullivan [24, Theorem XII, p. 163] proved the existence of $\|U\|_{\alpha,3}^{\mathfrak{V}}$. To establish the existence of M , one observes that from equation (2.2) it is easy to show that there exists an N such that

$$\max_{p \in \mathfrak{V}} |U| < N \max_{p \in \mathfrak{V}} |f|.$$

This however implies that the mapping (2.2) is closed in that if one defines

$$U_n = \int_{\mathfrak{V}} \phi f_n \, dv$$

then the existence of the limits

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\alpha,1}^{\mathfrak{V}} = 0$$

and

$$\lim_{n \rightarrow \infty} \|U_n - U\|_{\alpha,3}^{\mathfrak{V}} = 0$$

imply that the limit function U is given by (2.2). The existence of M now follows from Banach [2, Theorem 7, p. 41], which states that a linear closed map is bounded.

Lemma (2.4). *Let $W = \int_S \phi g \, ds$, and let $\|g\|_{\alpha,1}^S$ exist; then $\|W\|_{\alpha,2}^V$ exists, and there is a constant M such that*

$$\|W\|_{\alpha,2}^V \leq M \|g\|_{\alpha,1}^S.$$

Again Sullivan [24, Theorem VIII, p. 153] established the existence of $\|W\|_{\alpha,2}^V$. The existence of M can be deduced as in the preceding lemma.

Lemma (2.5). *Under the conditions of Lemma (2.4), the tangential derivatives of W are continuous across S while the normal derivative experiences a jump:*

$$\frac{\partial W}{\partial n_-} + \frac{\partial W}{\partial n_+} = -4\pi g.$$

(Cf. Courant & Hilbert [4, p. 237, remark 3].)

Lemma (2.6). *Let*

$$Y = \int \frac{\partial \phi}{\partial n} g \, ds,$$

where both Y and g are defined on S , and let $\|g\|_{\alpha,1}^S$ exist. Then for any ϵ such that $0 < \alpha < \alpha + \epsilon < 1$, the norm $\|Y\|_{\alpha+1,\epsilon}^S$ exists, and there is a constant M such that

$$\|Y\|_{\alpha+\epsilon,1}^S \leq M \|g\|_{\alpha,1}^S.$$

(Cf. Schauder [17, Lemma XI, p. 618].)

Lemma (2.7). *If W is defined as in Lemma (2.4) and g is bounded and continuous in V , then*

$$\frac{\partial W}{\partial x_j} = \int g \frac{\partial}{\partial x_j} \phi \, dv,$$

and there exists an M such that

$$\|W\|_{\alpha,1}^V < M \max_{p \in V} |g|, \quad 0 < \alpha < 1.$$

Lemma (2.8). *Let*

$$(2.3) \quad G(p) = \text{grad} \int \phi g \, ds,$$

where g is α -1-Hölder continuous on S , and let $G(p)$ be defined on S by its limit values; then there exists an M such that

$$\|G(p)\|_{\alpha,1}^V < M \|g\|_{\alpha,1}^S.$$

(For a proof of Lemmas (2.7) and (2.8) see Sullivan [24].)¹

¹ Sometimes results like those of Lemma (2.8) are stated as: the integral (2.3) takes $\|g\|_{\alpha,1}^S$ into $\|G\|_{\alpha,1}^V$. To see that this is a linear bounded transformation, one can map $\|G\|_{\alpha,1}^V$ into $\|G\|_{\mathcal{C}}^V$, where $\|G\|_{\mathcal{C}}^V$ is the norm of the space of continuous functions (=

Lemma (2.9). *Let*

$$J = \frac{\partial}{\partial n} \int \phi g \, ds,$$

where J and g are defined on S and where g is α -1-Hölder continuous on S . Then there exists an $M(\epsilon)$ such that

$$\|J\|_{\alpha+\epsilon,1}^S \leq M(\epsilon) \|g\|_{\alpha,1}^S$$

with

$$0 < \alpha < \alpha + \epsilon < 1.$$

(Cf. Korn [9].)²

3. Representation theorem. Let $\mathbf{W}$ be any twice continuously differentiable vector defined in the region $\mathcal{V}$. Then Stratton & Chu [23, p. 464] have shown that the following formula is true;

$$(3.1) \quad \begin{aligned} \lambda \mathbf{W}(p) = & \frac{1}{4\pi} \int_{\mathcal{V}} \phi [\text{curl curl } \mathbf{W} - k_0^2 \mathbf{W}] \, dv - \frac{\text{grad}}{4\pi} \int_{\mathcal{V}} \phi \text{div } \mathbf{W} \, dv \\ & - \frac{\text{grad}}{4\pi} \int_S \phi (\mathbf{W} \cdot \mathbf{n}) \, ds + \frac{1}{4\pi} \int_S \phi (\mathbf{n} \times \text{curl } \mathbf{W}) \, ds + \frac{\text{curl}}{4\pi} \int_S (\mathbf{n} \times \mathbf{W}) \phi \, ds, \end{aligned}$$

where

$\lambda = 1$ if P belongs to the interior of $\mathcal{V}$,

$\lambda = 0$ if P does not belong to the closure of $\mathcal{V}$.

Moreover, if one assumes that outside of $\mathcal{V}$, $\text{curl curl } \mathbf{W} - k_0^2 \mathbf{W} = 0$ and that $\mathbf{W}$ satisfies the radiation condition:

(a) $r\mathbf{W}(r)$ is uniformly bounded, with $\mathbf{r}$ the vector directed from the origin to P and,

(b) $\lim_{r \rightarrow \infty} r \left[\frac{\mathbf{r}}{r} \times \text{curl } \mathbf{W} + ik_0 \mathbf{W} \right] = 0$.

$\max_{p \in \mathcal{V}} \|G\|$ by the identity map. Since the transformation taking the norm $\|g\|_{\alpha,1}^S$ into the norm $\|G\|_{\mathcal{C}}^{\mathcal{V}}$ is continuous, it follows that the original transformation is also continuous by the following lemma (BANACH, [2], Lemma 2, p. 42). Let $y = U(x)$ be an additive transformation and let $z = V(y)$ be a bounded, one-to-one, linear transformation. If $\mathcal{V}[U(x)]$ is continuous, so is $U(x)$.

² SCHAUDER [18] showed that functions bounded in $\alpha + \epsilon$ norm are compact in the space of α normed functions. That is, any linear operator taking α -Hölder continuous functions into $\alpha + \epsilon$ -Hölder continuous functions $0 < \alpha < \alpha + \epsilon < 1$ is completely continuous.

Then Silver [19, p. 84] has shown that

$$(3.2) \quad \begin{aligned} \lambda' \mathbf{W}(p) = & \frac{\text{grad}}{4\pi} \int_s \phi(\mathbf{W} \cdot \mathbf{n}) \, ds - \frac{1}{4\pi} \int_s \phi(\mathbf{n} \times \text{curl } \mathbf{W}) \, ds \\ & - \frac{1}{4\pi} \text{curl} \int (\mathbf{n} \times \mathbf{W}) \phi \, ds, \end{aligned}$$

where

$\lambda' = 1$ if P does not belong to the closure of $\mathfrak{V}$,

$\lambda' = 0$ if P belongs to the interior of $\mathfrak{V}$.

On the other hand the corresponding Schwinger & Levine formulas may be written as

$$(3.3) \quad \begin{aligned} \lambda \mathbf{W}(p) = & \frac{1}{4\pi} \int \phi[\text{curl curl } \mathbf{W} - k_0^2 \mathbf{W}] \, dv \\ & + \frac{1}{4\pi k_0^2} \text{grad div} \int_{\mathfrak{V}} \phi[\text{curl curl } \mathbf{W} - k_0^2 \mathbf{W}] \, dv \\ & + \frac{1}{4\pi} \int \phi(\mathbf{n} \times \text{curl } \mathbf{W}) \, ds + \frac{1}{4\pi k_0^2} \text{grad div} \int \phi(\mathbf{n} \times \text{curl } \mathbf{W}) \, ds \\ & + \frac{1}{4\pi} \text{curl} \int_{\mathfrak{s}} \phi(\mathbf{n} \times \mathbf{W}) \, ds, \end{aligned}$$

or as

$$(3.4) \quad \begin{aligned} \lambda' \mathbf{W}(p) = & \frac{1}{4\pi} \int \phi(\mathbf{n} \times \text{curl } \mathbf{W}) \, ds + \frac{1}{4\pi k_0^2} \text{grad div} \int \phi(\mathbf{n} \times \text{curl } \mathbf{W}) \, ds \\ & + \frac{1}{4\pi} \text{curl} \int_{\mathfrak{s}} \phi(\mathbf{n} \times \mathbf{W}) \, ds, \end{aligned}$$

respectively, where the symbols have the same meaning as before. Now the terms involving the $1/k_0^2$ gradient divergence operator in the Schwinger & Levine formulas can clearly be obtained from (3.1) or (3.2) by applying this operator to both sides of these equations and adding the result to (3.1) or (3.2) respectively. If this is done one obtains from (3.1), for example, in addition to the terms of the Schwinger & Levine formula, the quantity

$$\lambda \frac{\text{grad div } \mathbf{W}}{k_0^2}$$

on the left and the sum

$$\begin{aligned} & -\frac{\text{grad}}{4\pi} \int \phi \operatorname{div} \mathbf{W} \, dv - \frac{\text{grad div grad}}{4\pi k_0^2} \int \phi \operatorname{div} \mathbf{W} \, dv \\ & -\frac{\text{grad}}{4\pi} \int \phi (\mathbf{W} \cdot \mathbf{n}) \, ds - \frac{\text{grad div grad}}{4\pi k_0^2} \int \phi (\mathbf{W} \cdot \mathbf{n}) \, ds \end{aligned}$$

on the right.

We shall now show that these two sets of remaining terms on the right are equal. In particular, if we let

$$\mathbf{F} = -\frac{\text{grad}}{4\pi} \int \phi \operatorname{div} \mathbf{W} \, dv,$$

we see that the sum of the first two of the remaining terms on the right can be written (since $\operatorname{curl} \operatorname{curl} \mathbf{F} = \operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F}$) as

$$\begin{aligned} \mathbf{F} + \frac{\text{grad div} \mathbf{F}}{k_0^2} &= \frac{\operatorname{curl} \operatorname{curl} \mathbf{F}}{k_0^2} + \frac{\nabla^2 \mathbf{F} + k_0^2 \mathbf{F}}{k_0^2} = \frac{\nabla}{4\pi} \left\{ \frac{\nabla^2 + k_0^2}{k_0^2} \int \phi \operatorname{div} \mathbf{W} \, dv \right\} \\ &= \frac{\lambda \nabla \operatorname{div} \mathbf{W}}{k_0^2}. \end{aligned}$$

Moreover, the remaining terms on the right are seen to cancel by setting

$$G = -\frac{1}{4\pi} \int \phi (\mathbf{W} \cdot \mathbf{n}) \, ds$$

and observing that

$$\nabla \left\{ G + \frac{\nabla^2 G}{k_0^2} \right\} = -\frac{\nabla}{4\pi} \left\{ \frac{k_0^2 + \nabla^2}{k_0^2} \right\} \int \phi (\mathbf{W} \cdot \mathbf{n}) \, ds = 0.$$

A similar argument allows one to obtain (3.4) from (3.2). Finally to write (3.3) or (3.4) in the form given by Schwinger & Levine, it is merely necessary to observe that

$$\frac{\text{grad div}}{4\pi k_0^2} \int \phi [\operatorname{curl} \operatorname{curl} \mathbf{W} - k_0^2 \mathbf{W}] \, dv = -\frac{\text{grad}_p}{4\pi k_0^2} \int \text{grad}_q \phi [\operatorname{curl} \operatorname{curl} \mathbf{W} - k_0^2 \mathbf{W}] \, dv$$

and that

$$\frac{\text{grad div}}{4\pi k_0^2} \int \phi (\mathbf{n} \times \operatorname{curl} \mathbf{W}) \, ds = -\frac{\text{grad}_p}{4\pi k_0^2} \int \text{grad}_q \phi (\mathbf{n} \times \operatorname{curl} \mathbf{W}) \, ds.$$

If this transformation is made and the tensor

$$f_{ij} = \left(\delta_{ij} - \frac{1}{k_0^2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) \frac{\phi}{4\pi} \quad i, j = 1, 2, \dots, 3$$

introduced, then (3.3) and (3.4) are precisely as given by Schwinger & Levine. This formulation has the advantage over those of Stratton & Chu that it does not involve either the normal derivative of $\mathbf{W}$ in the surface integral or the divergence of $\mathbf{W}$ in the volume integral. Moreover, when written in dyadic form, it is more analogous to familiar representations for Laplace's or the scalar wave equation.

§4. Auxiliary existence and uniqueness theorems.

Theorem (4.1). *Let*

$$(4.1) \quad \mathbf{W} = \mathbf{Q} + \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} \mathbf{W} \phi \, dv,$$

where $\|\mathbf{Q}\|_{\alpha,1}^{\mathcal{V}}$ exists. Then the integral equation (4.1) has one and only one solution, and moreover there exists a constant M such that

$$(4.2) \quad \|\mathbf{W}\|_{\alpha,1}^{\mathcal{V}} \leq M \|\mathbf{Q}\|_{\alpha,1}^{\mathcal{V}}.$$

Proof. Lemma (2.3) implies that the integral operator in (4.1) is completely continuous². Hence by the Fredholm alternative (Banach [2, p. 153]), the existence of a solution to (4.1) will follow if the homogeneous equation

$$(4.3) \quad \mathbf{W} = \frac{k_1^2 - k_0^2}{4\pi} \int \mathbf{W} \phi \, dv$$

can be shown to have only the trivial solution $\mathbf{W} = 0$. Once this has been established, (4.2) will follow since a one-to-one onto linear map has a bounded inverse [2, p. 146].

To show that (4.3) has only the trivial solution we observe that any rectangular component W of any solution of (4.3) satisfies the following boundary value problem:

$$(4.4) \quad \nabla^2 W + k_1^2 W = 0 \text{ in } \mathcal{V}$$

$$(4.5) \quad \nabla^2 W + k_0^2 W = 0 \text{ outside } \mathcal{V}, \quad k_0^2 = \text{a real positive constant.}$$

$$(4.6) \quad \lim_{r \rightarrow \infty} r \left[\frac{\partial W}{\partial r} - ik_0 W \right] = 0,$$

and where W and $\partial W / \partial n$ are continuous across $\mathcal{S}$. (Cf. Sternberg [21].) In order to see that $W = 0$ is the only solution to this boundary value problem, it is convenient to consider separately the cases where $\sigma_1 = 0$ and where $\sigma_1 \neq 0$.

If $\sigma_1 = 0$ in $\mathcal{V}$ as well as outside of $\mathcal{V}$, then let W^* be the conjugate of W . By Green's theorem applied to $\mathcal{V}$, one finds

$$(4.7) \quad \int_{\mathcal{S}} \left(W \frac{\partial W^*}{\partial n} - W^* \frac{\partial W}{\partial n} \right) ds = \int (W \nabla^2 W^* - W^* \nabla^2 W) \, dv = 0.$$

Rellich [14] proved that if W satisfies (4.7), (4.5), and (4.6) then W is identically zero outside of $\mathfrak{V}$. To see that $W = 0$ in $\mathfrak{V}$, one uses the representation

$$(4.7a) \quad \lambda W(p) = \int_s \left[W \frac{\partial}{\partial n} \phi - \frac{\partial W}{\partial n} \phi \right] ds$$

valid for any solution of (4.4). (Cf. Baker & Copson [1, p. 26].)

If $\sigma_1 \neq 0$ in $\mathfrak{V}$, then Freudenthal [6] proved that $W = 0$. Since his paper does not appear to be as widely known as that of Rellich, we shall sketch a slight modification of his proof for three dimensions. It follows from (4.3) that any solution of (4.5) satisfies

$$\begin{aligned} \frac{\partial W}{\partial r} - ik_0 W &= O(1/r^2) \\ \frac{\partial W^*}{\partial r} + ik_0 W^* &= O(1/r^2). \end{aligned}$$

Further, (4.3) implies asymptotically that $W = O(1/r)$. Thus if $\mathfrak{K}$ denotes any sphere with center at the singular point of ϕ , of radius r , whose interior $\mathcal{J}$ properly includes $\mathfrak{V}$, then (4.7) may be applied to $\mathfrak{K}$ and $\mathcal{J}$. Furthermore, we may write

$$\begin{aligned} \iiint_{\mathcal{J}} (W^* \nabla^2 W - W \nabla^2 W^*) dv &= \iiint_{\mathfrak{V}} (W^* \nabla^2 W - W \nabla^2 W^*) dv \\ &+ \iiint_{\mathcal{J}-\mathfrak{V}} (W^* \nabla^2 W - W \nabla^2 W^*) dv \\ &= (k_1^{*2} - k_1^2) \iiint_{\mathfrak{V}} |W|^2 dv = -2i \operatorname{Im} (k_1^2) \iiint_{\mathfrak{V}} |W|^2 dv \end{aligned}$$

in view of (4.4). Therefore

$$\frac{1}{2i} \iiint_{\mathcal{J}} (W^* \nabla^2 W - W \nabla^2 W^*) dv < 0, \text{ since } \operatorname{Im} (k_1^2) > 0.$$

The left-hand side of (4.7) can then be replaced by

$$\begin{aligned} \frac{1}{2i} \iint_{\mathfrak{K}} \left(W^* \frac{\partial W}{\partial r} - W \frac{\partial W^*}{\partial r} \right) ds &= \frac{1}{2i} \iint_{\mathfrak{K}} [W^* i \{k_0 W + O(1/r^2)\} \\ &- W \{-ik_0 W^* + O(1/r^2)\}] ds = k_0 \iint_{\mathfrak{K}} |W|^2 ds - \iint |W| O(1/r^2) ds < 0, \end{aligned}$$

since W can be written in the form $W = |W| e^{i\psi}$. From this it follows that $|W| = O(1/r^2)$. However by Rellich's theorem [14], this implies that $W \equiv 0$

outside $\mathcal{V}$. Again the representation theorem (4.7a) can be used to conclude that $W = 0$ in $\mathcal{V}$.

Theorem (4.2). *Let*

$$(4.8) \quad \mathbf{E} = \frac{k_1^2 - k_0^2}{4\pi} \int \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \text{grad} \int \phi (\mathbf{E} \cdot \mathbf{n}) \, ds,$$

$$(4.9) \quad \mathbf{H} = \frac{k_1^2 - k_0^2}{4\pi} \text{curl} \int \mathbf{E} \phi \, dv.$$

Then the system (4.8) and (4.9) has the unique solution $\mathbf{E} = \mathbf{H} = 0$.

By the representation theory of §3, we see that the $\mathbf{E}$ and $\mathbf{H}$ as defined by the above system have the following properties:

(a) In $\mathcal{V}$, $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations with

$$\mu = \mu_0, \quad \epsilon = \epsilon_1, \quad \sigma = \sigma_1.$$

(b) Outside $\mathcal{V}$, $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations with

$$\mu = \mu_0, \quad \epsilon = \epsilon_0, \quad \sigma = 0.$$

(c) The tangential components of $\mathbf{E}$, $\mathbf{H}$ are continuous across $\mathcal{S}$.

(d) $\mathbf{E}$ satisfies the radiation condition.

From these properties it follows that $\mathbf{E}$ satisfies

$$(4.10) \quad \text{curl curl } \mathbf{E} - k_0^2 \mathbf{E} = 0 \text{ outside of } \mathcal{V}$$

$$(4.11) \quad = (k_1^2 - k_0^2) \mathbf{E} \text{ in } \mathcal{V}.$$

Again it is convenient to separate the case $\sigma = 0$ in $\mathcal{V}$ from that of $\sigma \neq 0$ in $\mathcal{V}$. For $\sigma = 0$ in $\mathcal{V}$, Green's vector identity applied to $\mathbf{E}$ and its conjugate $\mathbf{E}^*$ over the region $\mathcal{V}$ implies that

$$(4.12) \quad i\omega\mu_0 \int_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}^*) + \mathbf{E}^* \times \mathbf{H} \, ds \\ = \int_{\mathcal{V}} (\mathbf{E} \cdot \text{curl curl } \mathbf{E}^* - \mathbf{E}^* \cdot \text{curl curl } \mathbf{E}) \, dv = 0.$$

Saunders [16] proved that if $\mathbf{E}$ and $\mathbf{H}$ satisfy (b), (4.12), and (d), then $\mathbf{E}$ and $\mathbf{H}$ are identically zero outside $\mathcal{V}$. To show that $\mathbf{E}$ and $\mathbf{H}$ are identically zero inside $\mathcal{V}$, one uses the Stratton & Chu representation.

If $\sigma_1 \neq 0$ in $\mathcal{V}$, we may argue in a manner parallel to that of the preceding theorem.

We shall first show that equation (4.8) implies that

$$(4.13) \quad \mathbf{r}_1 \times \text{curl } \mathbf{E} + ik_0 \mathbf{E} = \mathbf{r}_1 \times i\omega\mu_0 \mathbf{H} + ik_0 \mathbf{E} = O(1/r^2)$$

and that

$$(4.14) \quad \mathbf{r}_1 \times i\omega\mu_0 \mathbf{H}^* + ik_0 \mathbf{E}^* = O(1/r^2)$$

where

$$\mathbf{r}_1 = \frac{\mathbf{r}}{r}.$$

For this purpose we rewrite (4.8) using the fact that $\operatorname{div} \mathbf{E} = 0$ in $\mathfrak{U}$, as will be shown in §5.

$$\begin{aligned} \mathbf{E} &= \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathfrak{U}} \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \operatorname{grad} \int_{\mathfrak{U}} \operatorname{div} (\mathbf{E} \phi) \, dv \\ &= \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathfrak{U}} \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \operatorname{grad} \int_{\mathfrak{U}} (\mathbf{E} \cdot \operatorname{grad} \phi) \, dv. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{r}_1 \times \operatorname{curl}_p \mathbf{E} + ik_0 \mathbf{E} &= \frac{k_1^2 - k_0^2}{4\pi} \left\{ \int_{\mathfrak{U}} [\mathbf{r}_1 \times (\operatorname{grad}_p \phi \times \mathbf{E})] \, dv \right. \\ &\quad \left. + ik_0 \int_{\mathfrak{U}} \left(\phi \mathbf{E} \, dv - \frac{1}{k_0^2} \operatorname{grad}_p \right) \int_{\mathfrak{U}} (\operatorname{grad}_a \phi \cdot \mathbf{E}) \, dv \right\} \\ &= \frac{k_1^2 - k_0^2}{4\pi} \left\{ \int_{\mathfrak{U}} [(\mathbf{E} \cdot \mathbf{r}_1) \operatorname{grad}_p \phi - (\mathbf{r}_1 \cdot \operatorname{grad}_p \phi) \mathbf{E}] \, dv \right. \\ &\quad \left. + ik_0 \int_{\mathfrak{U}} \phi \mathbf{E} \, dv - \frac{1}{k_0^2} \operatorname{grad}_p \int_{\mathfrak{U}} (\operatorname{grad}_a \phi \cdot \mathbf{E}) \, dv \right\} \\ &= \frac{k_1^2 - k_0^2}{4\pi} \left\{ \operatorname{grad}_p \left[\int_{\mathfrak{U}} [(\mathbf{E} \cdot \mathbf{r}_1) \phi + \frac{i}{k_0} (\operatorname{grad}_p \phi \cdot \mathbf{E})] \, dv \right] \right. \\ &\quad \left. + \int_{\mathfrak{U}} [ik_0 \phi \mathbf{E} - (\mathbf{r}_1 \cdot \operatorname{grad}_p \phi) \cdot \mathbf{E}] \, dv \right\}. \end{aligned}$$

Now asymptotically this last expression is:

$$\cong \frac{k_1^2 - k_0^2}{4\pi} \left\{ \operatorname{grad}_p \int_{\mathfrak{U}} (\mathbf{E} \cdot \mathbf{r}_1) \left[\phi + \frac{i}{k_0} \frac{\partial \phi}{\partial r} \right] dv + \int_{\mathfrak{U}} \left(ik_0 \phi - \frac{\partial \phi}{\partial r} \right) \mathbf{E} \, dv \right\},$$

from which (4.13) follows at once and (4.14) follows by taking the appropriate conjugates.

If we now apply Green's vector identity to the sphere $\mathcal{K}$ whose center is at the singular point of ϕ , whose radius is r , and whose interior $\mathfrak{g}$ properly includes $\mathfrak{U}$, we find that the right hand side yields:

$$\begin{aligned}
& \int_{\mathcal{G}} (\mathbf{E} \cdot \text{curl curl } \mathbf{E}^* - \mathbf{E}^* \cdot \text{curl curl } \mathbf{E}) dv \\
&= \int_{\mathcal{V}} (\mathbf{E} \cdot \text{curl curl } \mathbf{E}^* - \mathbf{E}^* \cdot \text{curl curl } \mathbf{E}) dv \\
(4.15) \quad &+ \int_{\mathcal{G}-\mathcal{V}} (\mathbf{E} \cdot \text{curl curl } \mathbf{E}^* - \mathbf{E}^* \cdot \text{curl curl } \mathbf{E}) dv \\
&= (k_1^{*2} - k_1^2) \int |\mathbf{E}|^2 dv = -2i \operatorname{Im}(k_1^2) \int |\mathbf{E}|^2 dv \quad \text{since } \operatorname{Im}(k_1^2) > 0,
\end{aligned}$$

or

$$\frac{1}{2i} \int_{\mathcal{G}} (\mathbf{E} \cdot \text{curl curl } \mathbf{E}^* - \mathbf{E}^* \cdot \text{curl curl } \mathbf{E}) dv < 0.$$

On the other hand, the left-hand side may be rearranged and used in connection with (4.13) and (4.14) to give

$$\begin{aligned}
& \int_{\mathcal{K}} [(\mathbf{E}^* \times i\omega\mu_0 \mathbf{H}) + \mathbf{E} \times i\omega\mu_0 \mathbf{H}^*] \cdot \mathbf{r}_1 ds \\
&= - \int_{\mathcal{K}} [\mathbf{E}^* \cdot (\mathbf{r}_1 \times i\omega\mu_0 \mathbf{H}) + \mathbf{E} \cdot (\mathbf{r}_1 \times i\omega\mu_0 \mathbf{H}^*)] ds \\
(4.16) \quad &= - \int_{\mathcal{K}} [\mathbf{E}^* \{ -ik_0 \mathbf{E} \times O(1/r^2) \} + \mathbf{E} \cdot \{ -ik_0 \mathbf{E}^* + O(1/r^2) \} \cdot ds \\
&= 2ik_0 \int |\mathbf{E}|^2 ds - \int |\mathbf{E}| O(1/r^2) ds.
\end{aligned}$$

Now (4.15) and (4.16) imply that

$$(4.17) \quad \int |\mathbf{E}|^2 dv - \int |\mathbf{E}| O(1/r^2) dv < 0.$$

Thus $|\mathbf{E}| = O(1/r^2)$. However by (4.10) $\operatorname{div} \mathbf{E} = 0$ outside $\mathcal{V}$. Thus each rectangular component of E_i of $\mathbf{E}$ satisfies $(\nabla^2 + k_0^2)E_i = 0$, $i = 1, 2, 3$. Now (4.17) implies that $|E_i| = O(1/r^2)$. Thus it follows as before from Rellich's theorem [14] that $E_i = 0$, i.e., that $\mathbf{E} \equiv 0$ outside $\mathcal{V}$. Equation (4.9) now implies that $\mathbf{H} \equiv 0$ outside of $\mathcal{V}$, and once again the representation theorem and the continuity properties (c) imply that $\mathbf{E} \equiv \mathbf{H} \equiv 0$ in $\mathcal{V}$ as well.

§5. The integral equation. Using the formulas (3.1) and (3.2) we shall now derive our basic integral equation. We set $\mathbf{W} = \mathbf{E} - \mathbf{E}_{\text{inc}}$, then

$$\begin{aligned} \operatorname{curl} \operatorname{curl} \mathbf{W} - k_0^2 \mathbf{W} &= \operatorname{curl} \operatorname{curl} \mathbf{E} - k_0^2 \mathbf{E} \begin{cases} = (k_1^2 - k_0^2) \mathbf{E} \text{ in } \mathcal{V}, \\ = 0 \text{ outside of } \mathcal{V}, \end{cases} \\ \operatorname{div} \mathbf{W} &= 0. \end{aligned}$$

We note further that since the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous across $\mathcal{S}$, $(\operatorname{curl} \mathbf{W} \times \mathbf{n})$ and $(\mathbf{W} \times \mathbf{n})$ are both continuous across $\mathcal{S}$. Thus combining (3.1) and (3.2) one finds that

$$\begin{aligned} \mathbf{E}(p) - \mathbf{E}_{\text{inc}}(p) &= \lambda \{\mathbf{E}(p) - \mathbf{E}_{\text{inc}}(p)\} + \lambda' \{\mathbf{E}(p) - \mathbf{E}_{\text{inc}}(p)\} \\ &= \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} \phi \mathbf{E} \, dv - \operatorname{grad} \int_{\mathcal{S}} \phi (\mathbf{E}^- - \mathbf{E}^+) \cdot \mathbf{n} \, ds \end{aligned}$$

where $\mathbf{E}^-$ is the limit that $\mathbf{E}$ takes on $\mathcal{S}$ from the interior of $\mathcal{V}$, and where $\mathbf{E}^+$ is the limit that $\mathbf{E}$ takes on $\mathcal{S}$ from the exterior of $\mathcal{V}$. Since, as is shown in Stratton [23, p. 483],

$$k_1^2 (\mathbf{E}^+ \cdot \mathbf{n}) = k_0^2 (\mathbf{E}^- \cdot \mathbf{n})$$

independently of whether σ_1 is zero or not, one finally obtains the formula

$$(5.1) \quad \mathbf{E}(p) = \mathbf{E}_{\text{inc}}(p) + \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \operatorname{grad} \int_{\mathcal{S}} \phi (\mathbf{E} \cdot \mathbf{n}) \, ds,$$

where we have abbreviated $(\mathbf{E}^- \cdot \mathbf{n})$ by $(\mathbf{E} \cdot \mathbf{n})$.

Equation (5.1) is our basic formula. We now show that any solution of (5.1) where $\|E\|_{\alpha,1}^{\mathcal{V}}$ exists satisfies all the conditions of §1, (a), (b), (c), and (d). First of all we find that

$$(5.2) \quad \mathbf{H}(p) = \frac{\operatorname{curl} \mathbf{E}}{i\omega\mu_0} = \mathbf{H}_{\text{inc}} + \frac{k_1^2 - k_0^2}{4\pi i\omega\mu_0} \operatorname{curl} \int_{\mathcal{V}} \phi \mathbf{E} \, dv.$$

Now by Lemma (2.2), $\mathbf{H}(p)$ is continuous across $\mathcal{S}$, and by Lemmas (2.2) and (2.5), the tangential components of $\mathbf{E}(p)$ are continuous across $\mathcal{S}$. The radiation condition may be established by first observing that it is easy to show

$$(5.3) \quad \frac{\partial \mathbf{E}}{\partial r} - ik\mathbf{E} = O(1/r)$$

and that

$$(5.4) \quad r\mathbf{E} \text{ is bounded as } r \rightarrow \infty,$$

and then following step by step the procedure of Saunders [16].

Lemma (2.1) implies that

$$(\nabla^2 + k_0^2)\mathbf{E} = \lambda(k_0^2 - k_1^2)\mathbf{E},$$

so that any solution of (5.1) will be a solution of §1, (a), (b), (c), and (d) if $\operatorname{div} \mathbf{E} = 0$. To establish this relation, we apply the divergence operator to (5.1) to obtain

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \frac{k_1^2 - k_0^2}{4\pi} \operatorname{div} \int \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \nabla^2 \int \phi (\mathbf{E} \cdot \mathbf{n}) \, ds \\ &= \frac{k_1^2 - k_0^2}{4\pi} \int \operatorname{grad}_p \phi \mathbf{E} \, dv + \frac{k_1^2 - k_0^2}{4\pi} \int \phi (\mathbf{E} \cdot \mathbf{n}) \, ds \\ &= - \frac{(k_1^2 - k_0^2)}{4\pi} \int \operatorname{grad}_q \phi \mathbf{E} \, dv + \frac{k_1^2 - k_0^2}{4\pi} \int \phi (\mathbf{E} \cdot \mathbf{n}) \, ds \\ &= - \frac{(k_1^2 - k_0^2)}{4\pi} \int \phi \operatorname{div} \mathbf{E} \, dv. \end{aligned}$$

The last equality results from the usual argument involving the consideration of a sphere $\mathcal{K}_\delta$ of radius δ around the singularity of ϕ , an application of the divergence theorem to $\operatorname{div} (\phi \mathbf{E})$ in the region $\mathcal{V} - \mathcal{K}_\delta$, and then letting δ tend to zero. Finally then, we set $\mathbf{W} = \operatorname{div} \mathbf{E}$ in Theorem (4.1) and we see that all the conditions of that lemma are fulfilled and hence that $\mathbf{W} = \operatorname{div} \mathbf{E} = 0$ is the only solution to (5.1). Thus any solution of (5.1) satisfies all the conditions of the transmission problem.

If one assumes that the dielectric constant ϵ varies continuously in the medium $\mathcal{V}$ and that σ is zero in $\mathcal{V}$ so that $k_1^2 = \omega^2 \epsilon \mu$ is a continuously varying function in $\mathcal{V}$, it is possible by the methods of this section to establish the fact that the electric field $\mathbf{E}$ satisfies the integral equation

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\text{inc}} + \frac{1}{4\pi} \int \phi (k_1^2 - k_0^2) \mathbf{E} \, dv - \frac{\operatorname{grad}}{4\pi k_0^2} \int \phi (k_1^2 - k_0^2) (\mathbf{E} \cdot \mathbf{n}) \, ds \\ (5.5) \quad &+ \frac{\operatorname{grad}}{4\pi} \int \frac{(\operatorname{grad} k_1^2)}{k_1^2} \cdot \mathbf{E} \phi \, dv. \end{aligned}$$

Conversely, if $\mathbf{E}$ satisfies (5.5) then $\mathbf{E}$ will be the unique solution of Maxwell's equations corresponding to the diffraction of the incident field $\mathbf{E}_{\text{inc}}$ by the obstacle $\mathcal{V}$. However, we have not as yet succeeded in establishing the existence of a solution to (5.5).

§6. The main existence theorem. *Let $\mathbf{E}_{\text{inc}}$ and $\mathbf{H}_{\text{inc}}$ represent a solution of Maxwell's equations everywhere in free space. Then the equations (5.1) and (5.2) have one and only one solution.*

In order to prove this theorem we define $\mathbf{E}_0$ and $\mathbf{E}_1$ by the equations

$$(6.1) \quad \mathbf{E}_0 = \mathbf{E}_{\text{inc}} + \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} \mathbf{E}_0 \phi \, dv$$

$$(6.2) \quad \mathbf{E}_1 = -\frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \text{grad} \int_{\mathcal{S}} F_n \phi \, ds + \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} E_1 \phi \, dv,$$

where F_n is an arbitrary function of class α on $\mathcal{S}$. By Theorem (4.1) and Lemma (2.8), equations (6.1) and (6.2) have one and only one solution. Moreover, for these solutions, we have

$$(6.3) \quad \|\mathbf{E}_0\|_{\alpha,1}^{\mathcal{V}} < M_1 \|\mathbf{E}_{\text{inc}}\|_{\alpha,1}^{\mathcal{V}}$$

$$(6.4) \quad \|\mathbf{E}_1\|_{\alpha,1}^{\mathcal{V}} < M_2 \|F_n\|_{\alpha,1}^{\mathcal{S}}.$$

Adding (6.2) to (6.1) and setting $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$, we obtain

$$(6.5) \quad \mathbf{E} = \mathbf{E}_{\text{inc}} + \frac{k_1^2 - k_0^2}{4\pi} \int \phi \mathbf{E} \, dv - \frac{(k_1^2 - k_0^2)}{4\pi k_0^2} \text{grad} \int_{\mathcal{S}} F_n \phi \, ds.$$

Now equation (6.5) will be equal to the basic integral equation (5.1) and $\mathbf{E}$ will be a solution of (5.1) if and only if $F_n = (\mathbf{E} \cdot \mathbf{n})$ on the surface $\mathcal{S}$; that is, if and only if

$$(6.6) \quad F_n = (\mathbf{E}_1 \cdot \mathbf{n}) + (\mathbf{E}_0 \cdot \mathbf{n})$$

on $\mathcal{S}$, where $\mathbf{E}_0 \cdot \mathbf{n}$ is uniquely determined by (6.1) and the normal $\mathbf{n}$ of $\mathcal{S}$.

In order to satisfy (6.2) and (6.6) simultaneously, we shall determine a functional equation for F_n . From (6.1), (6.2) and Lemma (2.5) it follows that

$$(6.7) \quad (\mathbf{E}_0 \cdot \mathbf{n})|_{\mathcal{S}} = \mathbf{E}_{\text{inc}} \cdot \mathbf{n} + \frac{k_1^2 - k_0^2}{4\pi} \mathbf{n} \cdot \int_{\mathcal{V}} \mathbf{E}_0 \phi \, dv$$

$$(6.8) \quad (\mathbf{E}_1 \cdot \mathbf{n})|_{\mathcal{S}} = -\frac{(k_1^2 - k_0^2)}{2k_0^2} F_n + \frac{k_1^2 - k_0^2}{4\pi k_0^2} \frac{\partial}{\partial n} \int_{\mathcal{S}} F_n \phi \, ds$$

$$+ \frac{k_1^2 - k_0^3}{4\pi} \mathbf{n} \cdot \int_{\mathcal{V}} \mathbf{E}_1 \phi \, dv.$$

In view of (6.6) and (6.1), this last relationship may be written as

$$(6.9) \quad F_n - (\mathbf{E}_{\text{inc}} \cdot \mathbf{n}) - \frac{k_1^2 - k_0^2}{4\pi} \left(\mathbf{n} \cdot \int_{\mathcal{V}} \mathbf{E}_0 \phi \, dv \right)$$

$$= -\frac{(k_1^2 - k_0^2)}{2k_0^2} F_n + \frac{k_1^2 - k_0^2}{4\pi k_0^2} \frac{\partial}{\partial n} \int_{\mathcal{S}} F_n \phi \, ds + \frac{k_1^2 - k_0^2}{4\pi} \left(\mathbf{n} \cdot \int_{\mathcal{V}} \mathbf{E}_1 \phi \, dv \right)$$

which is of the form

$$(6.10) \quad F_n + L(F_n) = Y$$

where

$$L(F_n) = \frac{2k_0^2}{k_0^2 + k_1^2} \left\{ -\left(\frac{k_1^2 - k_0^2}{4k_0^2} \right) \frac{\partial}{\partial n} \int_{\mathcal{S}} F_n \phi \, ds + \frac{k_1^2 - k_0^2}{4\pi} \left(\mathbf{n} \cdot \int_{\mathcal{V}} \mathbf{E}_1 \phi \, dv \right) \right\}$$

and

$$Y = \frac{2k_0^2}{k_0^2 + k_1^2} \left\{ \mathbf{E}_{\text{inc}} \cdot \mathbf{n} - \frac{k_1^2 - k_0^2}{4\pi} \left(\mathbf{n} \cdot \int \mathbf{E}_0 \phi \, dv \right) \right\}.$$

Consequently we may consider (6.9) or (6.10) as an equation for F_n in the space of α -continuous functions. However, (6.3) and Lemma (2.7) imply that Y is also in this space. Moreover, (6.4) and Lemmas (2.7) and (2.9) imply that $L(F_n)$ is a completely continuous operator so that by the Fredholm alternative F_n will exist if it is unique. To see that F_n is unique, assume that there were two solutions F'_n and F''_n of (6.10). Then F'_n and F''_n would generate two fields $\mathbf{E}'_1$ and $\mathbf{E}''_1$ respectively by (6.2) and it would follow from (6.2) and (6.6) that the difference field $\mathbf{E}_2 = \mathbf{E}'_1 - \mathbf{E}''_1$ would satisfy

$$F'_n - F''_n = (\mathbf{E}'_1 - \mathbf{E}''_1) \cdot \mathbf{n} = (\mathbf{E}_2 \cdot \mathbf{n})$$

and the homogeneous integral equation

$$(6.11) \quad \mathbf{E}_2 = \frac{k_1^2 - k_0^2}{4\pi} \int_{\mathcal{V}} \mathbf{E}_2 \phi \, dv + \frac{k_1^2 - k_0^2}{4\pi k_0^2} \text{grad} \int (\mathbf{E}_2 \cdot \mathbf{n}) \phi \, ds.$$

If we now define $\mathbf{H}_2$ by the relation

$$(6.12) \quad \mathbf{H}_2 = \frac{k_1^2 - k_0^2}{4\pi i \omega \mu_0} \text{curl} \int_{\mathcal{V}} \mathbf{E}_2 \cdot \phi \, dv$$

we see that $\mathbf{E}_2$, $\mathbf{H}_2$ as defined by (6.11) and (6.12) satisfy the hypotheses of Theorem (4.2), and thus it follows that $\mathbf{E}_2 = \mathbf{H}_2 = 0$. This in turn implies that $F'_n = F''_n$ or that the F_n satisfying (6.9) is unique. Using this F_n in equation (6.2) as the generator of $\mathbf{E}_1$, we find that $\mathbf{E}_0 + \mathbf{E}_1$ is the unique solution to (5.1).

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