Finite Plane Strain for Orthotropic Bodies

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1. Introduction. Some aspects of large elastic deformations of homogeneous anisotropic materials have been considered by Ericksen & Rivlin [1]. In particular they solve completely a number of problems for an incompressible material which is transversely isotropic with respect to a given direction. In this paper we consider the problem of finite plane strain for orthotropic and transversely isotropic materials. The flexure of a cuboid is then partly solved in terms of a general strain energy function for compressible orthotropic materials; the problem is solved completely for incompressible orthotropic materials. The finite extension and inflation of a tube, which is transversely isotropic with respect to its axis, is partly solved in terms of a general strain energy function for compressible materials; this problem was solved completely for incompressible materials by Ericksen & Rivlin [1].

2. Notation. Various equivalent forms of the natural state theory of elasticity have been discussed in a paper by Truesdell [2]. Here we summarize the notation we shall use in this paper. The points of an unstrained and unstressed body at time $t = 0$ are defined by a system of rectangular Cartesian coordinates x_i or by a general curvilinear system of coordinates θ_i . The curvilinear coordinates θ_i move with the body as it is deformed and form a curvilinear system in the strained body at time t . The covariant and contravariant metric tensors for the coordinate system θ_i in the unstrained body are denoted by g_{ij} and g^{ij} respectively, and for the coordinate system θ_i in the strained body, at time t , the corresponding metric tensors are G_{ij} and G^{ij} respectively. We write

$$g = |g_{ij}|, \quad G = |G_{ij}|, \quad (2.1)$$

Latin indices taking the values 1, 2, 3. The covariant strain tensor is γ_{ij} where

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}). \quad (2.2)$$

For elastic bodies which are initially homogeneous the contravariant stress tensor τ^{ij} , per unit area of the deformed body, is given by

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}} \left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right), \quad (2.3)$$

where W is a function of all the components of the strain tensor γ_{ij} and where

$$I_3 = G/g. \quad (2.4)$$

The function W represents the strain energy of the deformed body measured per unit volume of the undeformed body.

When the body is incompressible $I_3 = 1$ and equation (2.3) is replaced by

$$\tau^{ij} = \frac{1}{2} \left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right) + p G^{ij}, \quad (2.5)$$

where p is a scalar function of the coordinates θ_i .

When body forces are absent and the deformed body is in equilibrium then

$$\tau^{ij} ||_i = 0, \quad (2.6)$$

where the double line denotes covariant differentiation with respect to the deformed body, *i.e.* with respect to θ_i and the metric tensors G_{ij} , G^{ij} .

When the coordinates θ_i are taken to coincide with the rectangular Cartesian coordinates x_i in the undeformed body we denote the values of γ_{ij} , τ^{ij} by e_{ij} , t^{ij} respectively, so that

$$e_{rs} = \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} \gamma_{ij}, \quad (2.7)$$

and

$$\tau^{ij} = \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} t^{rs}. \quad (2.8)$$

Hence, for a compressible body,

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}} \left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}} \right) \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s}. \quad (2.9)$$

The corresponding result for an incompressible body is

$$\tau^{ij} = \frac{1}{2} \left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}} \right) \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} + p G^{ij}. \quad (2.10)$$

3. Orthotropy. As indicated by Ericksen & Rivlin [1], symmetries which may exist in a material impose invariance conditions on W . The discussion adopted

here is similar to that used by Green & Zerna [3] for infinitesimal elasticity. In general

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{31}). \quad (3.1)$$

Consider a change of rectangular Cartesian axes

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = -x_3. \quad (3.2)$$

If we denote the strain tensor corresponding to axes $\bar{x}_i$ by $\bar{e}_{rs}$ then

$$\bar{e}_{\alpha\beta} = e_{\alpha\beta}, \quad \bar{e}_{33} = e_{33}, \quad \bar{e}_{\alpha 3} = -e_{\alpha 3} \quad (\alpha, \beta = 1, 2).$$

The strain energy W , which is an invariant, can be expressed in terms of $\bar{e}_{rs}$. If the expression for W in terms of $\bar{e}_{rs}$ can be found from (3.1) simply by replacing e_{rs} by $\bar{e}_{rs}$, then the body is said to be elastically symmetrical with respect to the plane $x_3 = 0$. In this case W must be restricted to the form

$$W = W(e_{11}, e_{22}, e_{33}, e_{31}^2, e_{23}^2, e_{31}e_{23}, e_{12}). \quad (3.3)$$

If, in addition, the body is elastically symmetrical with respect to the plane $x_1 = 0$ it follows that W must be restricted to the form

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}^2, e_{23}^2, e_{31}^2, e_{12}e_{23}e_{31}). \quad (3.4)$$

From (3.4) it is seen that elastic symmetry with respect to the planes $x_1 = 0$, $x_3 = 0$ implies elastic symmetry with respect to the plane $x_2 = 0$. Also, since the undeformed body is homogeneous, there are three orthogonal planes of symmetry at each point, and the body is said to be *orthotropic*.

The stress strain relations for an orthotropic compressible material follow from (2.9) and (3.4). Thus

$$\begin{aligned} \sqrt{I_3} \tau^{ij} = & \frac{\partial W}{\partial e_{11}} \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^1} + \frac{\partial W}{\partial e_{22}} \frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^2} + \frac{\partial W}{\partial e_{33}} \frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^3} \\ & + \left[e_{12} \frac{\partial W}{\partial e_{12}^2} + \frac{e_{23} e_{31}}{2} \frac{\partial W}{\partial (e_{12} e_{23} e_{31})} \right] \left[\frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^2} + \frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^1} \right] \\ & + \left[e_{23} \frac{\partial W}{\partial e_{23}^2} + \frac{e_{31} e_{12}}{2} \frac{\partial W}{\partial (e_{12} e_{23} e_{31})} \right] \left[\frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^3} + \frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^2} \right] \\ & + \left[e_{31} \frac{\partial W}{\partial e_{31}^2} + \frac{e_{12} e_{23}}{2} \frac{\partial W}{\partial (e_{12} e_{23} e_{31})} \right] \left[\frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^1} + \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^3} \right]. \end{aligned} \quad (3.5)$$

When the body is incompressible $e_{12} e_{23} e_{31}$ may be expressed in terms of $e_{11}, e_{22}, e_{33}, e_{12}^2, e_{23}^2, e_{31}^2$ by means of the condition $I_3 = 1$, so that (3.4) becomes

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}^2, e_{23}^2, e_{31}^2). \quad (3.6)$$

Hence, from (2.10) and (3.6),

$$\begin{aligned}
 \tau^{ij} = & \frac{\partial W}{\partial e_{11}} \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^1} + \frac{\partial W}{\partial e_{22}} \frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^2} + \frac{\partial W}{\partial e_{33}} \frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^3} \\
 & + e_{12} \frac{\partial W}{\partial e_{12}^2} \left(\frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^2} + \frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^1} \right) \\
 & + e_{23} \frac{\partial W}{\partial e_{23}^2} \left(\frac{\partial \theta^i}{\partial x^2} \frac{\partial \theta^j}{\partial x^3} + \frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^2} \right) \\
 & + e_{31} \frac{\partial W}{\partial e_{31}^2} \left(\frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^1} + \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^3} \right) \\
 & + p G^{ij}.
 \end{aligned} \tag{3.7}$$

4. Plane strain superposed on uniform extension. The strained body may be defined by a set of rectangular Cartesian coordinates y_i , and we take the y_3 -axis to be parallel to the x_3 -axis, and the planes $y_3 = 0$, $x_3 = 0$ to coincide. We now suppose that the body is deformed by a uniform finite extension parallel to the x_3 -axis, with constant extension ratio λ , and that subsequently the body receives a finite plane strain parallel to the (x_1, x_2) plane. If we choose the curvilinear coordinate θ_3 so that $\theta_3 = y_3$ then

$$x_3 = y_3/\lambda = \theta_3/\lambda, \tag{4.1}$$

and

$$x_\alpha = x_\alpha(\theta_1, \theta_2), \quad y_\alpha = y_\alpha(\theta_1, \theta_2, t), \tag{4.2}$$

Greek indices taking the values 1, 2. It follows that

$$g_{ij} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} a^{11} & a^{12} & 0 \\ a^{12} & a^{22} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \quad g = \frac{a}{\lambda^2}, \tag{4.3}$$

$$G_{ij} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G^{ij} = \begin{bmatrix} A^{11} & A^{12} & 0 \\ A^{12} & A^{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = A, \tag{4.4}$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear coordinates θ_α in a plane $x_3 = 0$ of the undeformed body; $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear coordinates θ_α in a plane $y_3 = 0$ of the deformed body, and

$$a = |a_{\alpha\beta}|, \quad A = |A_{\alpha\beta}|, \quad I_3 = \lambda^2 A/a. \quad (4.5)$$

From (4.1) to (4.4),

$$\gamma_{\alpha 3} = 0, \quad \gamma_{33} = \frac{1}{2}(1 - 1/\lambda^2), \quad \gamma_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}). \quad (4.6)$$

Also, from (2.7), we have

$$e_{\alpha 3} = 0, \quad e_{33} = \lambda^2 \gamma_{33} = \frac{1}{2}(\lambda^2 - 1), \quad e_{\alpha\beta} = \frac{\partial \theta^\lambda}{\partial x^\alpha} \frac{\partial \theta^\mu}{\partial x^\beta} \gamma_{\lambda\mu}. \quad (4.7)$$

For a compressible body, from (4.7) and (3.5), we obtain

$$t^{\alpha 3} = \tau^{\alpha 3} = 0, \quad (4.8)$$

$$\sqrt{I_3} \tau^{\alpha\beta} = \frac{\partial W}{\partial e_{11}} \frac{\partial \theta^\alpha}{\partial x^1} \frac{\partial \theta^\beta}{\partial x^1} + \frac{\partial W}{\partial e_{22}} \frac{\partial \theta^\alpha}{\partial x^2} \frac{\partial \theta^\beta}{\partial x^2} + e_{12} \frac{\partial W}{\partial e_{12}^2} \left(\frac{\partial \theta^\alpha}{\partial x^1} \frac{\partial \theta^\beta}{\partial x^2} + \frac{\partial \theta^\alpha}{\partial x^2} \frac{\partial \theta^\beta}{\partial x^1} \right), \quad (4.9)$$

$$\tau^{33} = \lambda^2 t^{33} = \frac{\lambda^2}{\sqrt{I_3}} \frac{\partial W}{\partial e_{33}}, \quad (4.10)$$

where

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}^2). \quad (4.11)$$

When the material is incompressible we use (3.7) and (4.7) and find that (4.8) still holds, but now

$$\tau^{\alpha\beta} = \frac{\partial W}{\partial e_{11}} \frac{\partial \theta^\alpha}{\partial x^1} \frac{\partial \theta^\beta}{\partial x^1} + \frac{\partial W}{\partial e_{22}} \frac{\partial \theta^\alpha}{\partial x^2} \frac{\partial \theta^\beta}{\partial x^2} + e_{12} \frac{\partial W}{\partial e_{12}^2} \left(\frac{\partial \theta^\alpha}{\partial x^1} \frac{\partial \theta^\beta}{\partial x^2} + \frac{\partial \theta^\alpha}{\partial x^2} \frac{\partial \theta^\beta}{\partial x^1} \right) + p A^{\alpha\beta}, \quad (4.12)$$

$$\tau^{33} = \lambda^2 t^{33} = \lambda^2 \left(\frac{\partial W}{\partial e_{33}} + p \right). \quad (4.13)$$

The expressions for the stresses in (4.12) and (4.13) are evaluated from W expressed in the form (4.11). In addition we also have the condition

$$\lambda^2 A = a. \quad (4.14)$$

The equations of equilibrium (2.6) may be satisfied by introducing Airy's stress function ϕ . Thus

$$\tau^{\alpha\beta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\rho} \phi ||_{\gamma\rho}, \quad (4.15)$$

or

$$\phi ||_{\alpha\beta} = \epsilon_{\alpha\gamma} \epsilon_{\beta\rho} \tau^{\gamma\rho}, \quad (4.16)$$

where

$$\begin{aligned}\epsilon^{\alpha\gamma}\sqrt{A} &= \epsilon_{\alpha\gamma}/\sqrt{A} = 1 \text{ if } \alpha = 1, \gamma = 2 \\ &= -1 \text{ if } \alpha = 2, \gamma = 1 \\ &= 0 \text{ otherwise.}\end{aligned}\quad (4.17)$$

For a compressible body equations (4.9) and (4.16) yield

$$\phi_{||\alpha\beta} = \frac{\sqrt{I_3}}{\lambda^2} \left[\frac{\partial W}{\partial e_{11}} \frac{\partial x^2}{\partial \theta^\alpha} \frac{\partial x^2}{\partial \theta^\beta} + \frac{\partial W}{\partial e_{22}} \frac{\partial x^1}{\partial \theta^\alpha} \frac{\partial x^1}{\partial \theta^\beta} - e_{12} \frac{\partial W}{\partial e_{12}^2} \left(\frac{\partial x^1}{\partial \theta^\alpha} \frac{\partial x^2}{\partial \theta^\beta} + \frac{\partial x^2}{\partial \theta^\alpha} \frac{\partial x^1}{\partial \theta^\beta} \right) \right]. \quad (4.18)$$

In deriving (4.18) we have used the expression (4.5) for I_3 and relations of the form

$$\epsilon_{\alpha\gamma} \frac{\partial \theta^\gamma}{\partial x^1} = - \sqrt{\frac{A}{a}} \frac{\partial x^2}{\partial \theta^\alpha}, \quad \epsilon_{\alpha\gamma} \frac{\partial \theta^\gamma}{\partial x^2} = \sqrt{\frac{A}{a}} \frac{\partial x^1}{\partial \theta^\alpha}. \quad (4.19)$$

Similarly, for an incompressible body, we have, from (4.12) and (4.16),

$$\begin{aligned}\lambda^2 \phi_{||\alpha\beta} &= \frac{\partial W}{\partial e_{11}} \frac{\partial x^2}{\partial \theta^\alpha} \frac{\partial x^2}{\partial \theta^\beta} + \frac{\partial W}{\partial e_{22}} \frac{\partial x^1}{\partial \theta^\alpha} \frac{\partial x^1}{\partial \theta^\beta} \\ &\quad - e_{12} \frac{\partial W}{\partial e_{12}^2} \left(\frac{\partial x^1}{\partial \theta^\alpha} \frac{\partial x^2}{\partial \theta^\beta} + \frac{\partial x^2}{\partial \theta^\alpha} \frac{\partial x^1}{\partial \theta^\beta} \right) + \lambda^2 p A_{\alpha\beta}.\end{aligned}\quad (4.20)$$

5. Flexure of cuboid. This problem has been solved by Rivlin [4] for incompressible isotropic bodies and by Ericksen & Rivlin [1] for a cuboid with transverse isotropy in the plane of flexure. The isotropic problem was re-examined by Truesdell [2]. The isotropic problem was also considered again by Adkins, Green & Shield [5] as an example of plane strain, with the help of Airy's stress function. Here we use the theory developed in §4 to examine the problem of flexure of a compressible orthotropic cuboid. Consider an orthotropic cuboid bounded by the planes $x_1 = a_1$, $x_1 = a_2$; $x_2 = \pm b$; $x_3 = \pm c$, where the x_i -planes are parallel to the planes of elastic symmetry. The cuboid is deformed symmetrically with respect to the x_1 -axis so that

- (i) each plane initially normal to the x_1 -axis becomes, in the deformed state, a portion of a curved surface of a cylinder whose axis is the x_3 -axis;
- (ii) planes initially normal to the x_2 -axis become in the deformed state planes containing the x_3 -axis;
- (iii) there is a uniform extension λ in the direction of the x_3 -axis.

Using the notation of §4 we take cylindrical polar coordinates (r, θ, y_3) to define the strained body and we identify the curvilinear system θ_i with these coordinates so that

$$\begin{aligned}\theta_1 &= r, & \theta_2 &= \theta, & \theta_3 &= y_3, \\ y_1 &= r \cos \theta, & y_2 &= r \sin \theta.\end{aligned}\quad (5.1)$$

Also, if the x_i - and y_i -axes coincide,

$$x_1 = f(r), \quad x_2 = \lambda \theta / D, \quad (5.2)$$

where D is a constant. Hence

$$\begin{aligned}A_{\alpha\beta} &= \begin{bmatrix} 1, & 0 \\ 0, & r^2 \end{bmatrix}, & A^{\alpha\beta} &= \begin{bmatrix} 1, & 0 \\ 0, & \frac{1}{r^2} \end{bmatrix}, & A &= r^2 \\ a_{\alpha\beta} &= \begin{bmatrix} f'^2, & 0 \\ 0, & \frac{\lambda^2}{D^2} \end{bmatrix}, & a &= \frac{\lambda^2 f'^2}{D^2}, & I_3 &= \frac{D^2 r^2}{f'^2},\end{aligned}\quad (5.3)$$

a dash denoting differentiation with respect to r . From (5.2) and (4.7) it follows that

$$\begin{aligned}\gamma_{12} &= e_{12} = 0, & \gamma_{11} &= \frac{1}{2}(1 - f'^2), & \gamma_{22} &= \frac{1}{2}(r^2 - \lambda^2/D^2), \\ e_{11} &= \frac{1}{2}\left(\frac{1}{f'^2} - 1\right), & e_{22} &= \frac{1}{2}\left(\frac{D^2 r^2}{\lambda^2} - 1\right).\end{aligned}\quad (5.4)$$

If we now interpret (4.18) in polar coordinates, and assume that ϕ is independent of θ , then

$$\frac{d^2 \phi}{dr^2} = \frac{D r f'}{\lambda^2} \frac{\partial W}{\partial e_{22}}, \quad r \frac{d\phi}{dr} = \frac{r}{D f'} \frac{\partial W}{\partial e_{11}}. \quad (5.5)$$

Since

$$\frac{de_{11}}{dr} = -\frac{f''}{f'^3}, \quad \frac{de_{22}}{dr} = \frac{D^2 r}{\lambda^2},$$

it follows from (5.5) that

$$\frac{D}{f'} \frac{d^2 \phi}{dr^2} - \frac{D f''}{f'^2} \frac{d\phi}{dr} = \frac{D^2 r}{\lambda^2} \frac{\partial W}{\partial e_{22}} - \frac{f''}{f'^3} \frac{\partial W}{\partial e_{11}} = \frac{dW}{dr},$$

or

$$D \frac{d}{dr} \left(\frac{1}{f'} \frac{d\phi}{dr} \right) = \frac{dW}{dr},$$

so that

$$\frac{D}{f'} \frac{d\phi}{dr} = D \frac{d\phi}{dx_1} = W(x_1) + C, \quad (5.6)$$

where C is a constant. When the applied stresses over the curved surfaces of the deformed cylinder are zero, $d\phi/dr$ vanishes on these surfaces. Alternatively

$$\frac{d\phi}{dx_1} = 0 \quad (x_1 = a_1, \quad x_1 = a_2), \quad (5.7)$$

and therefore

$$-C = W(a_1) = W(a_2) = W_0 \text{ (say)}. \quad (5.8)$$

If condition (5.7) is satisfied it has been shown by Adkins, Green & Shield [5] that the couple, per unit length of the deformed cylinder, acting on each of the surfaces which are given by $x_2 = \pm b$ in the deformed state, is $M = \phi(a_2) - \phi(a_1)$, so that

$$M = \frac{1}{D} \int_{a_1}^{a_2} \{W(x_1) - W_0\} dx_1 = \frac{(a_1 - a_2)W_0}{D} + \frac{1}{D} \int_{a_1}^{a_2} W(x_1) dx_1. \quad (5.9)$$

This general result is also valid when the body is incompressible.

From (5.5) and (5.6)

$$W - W_0 = \frac{1}{f'^2} \frac{\partial W}{\partial e_{11}}, \quad (5.10)$$

which is an equation for f' when the form of W is known. When f' is found the problem for a compressible body is completely solved. The stresses over the ends $x_3 = \pm c$ of the cuboid are given by (4.10).

When the body is incompressible

$$f' = Dr, \quad f = \frac{1}{2}Dr^2 + B, \quad (5.11)$$

where B is a constant. If r_1, r_2 ($r_1 > r_2$) are the radii of the curved surfaces of the deformed body then

$$D = \frac{2(a_1 - a_2)}{r_1^2 - r_2^2}, \quad B = \frac{a_2 r_1^2 - a_1 r_2^2}{r_1^2 - r_2^2},$$

and (5.9) becomes

$$M = \frac{1}{2}(r_1^2 - r_2^2)W_0 - \int_{r_2}^{r_1} rW(r) dr. \quad (5.12)$$

This result, which is valid for an orthotropic incompressible cuboid, is the same as that found previously by Rivlin [4] for an incompressible isotropic cuboid and by Ericksen & Rivlin [1] for an incompressible transversely isotropic cuboid.

To obtain a detailed examination of the stress distribution for an incompressible orthotropic cuboid we replace (5.5) by

$$\frac{d^2 \phi}{dr^2} = \frac{D^2 r^2}{\lambda^2} \frac{\partial W}{\partial e_{22}} + p, \quad r \frac{d\phi}{dr} = \frac{1}{D^2} \frac{\partial W}{\partial e_{11}} + r^2 p, \quad (5.13)$$

where, from (5.4) and (5.11),

$$e_{11} = \frac{1}{2} \left(\frac{1}{D^2 r^2} - 1 \right), \quad e_{22} = \frac{1}{2} \left(\frac{D^2 r^2}{\lambda^2} - 1 \right). \quad (5.14)$$

The elimination of p from (5.13) leads to

$$\frac{1}{r} \frac{d\phi}{dr} = W(r) + C, \quad (5.15)$$

and thence to (5.12) if conditions (5.7) are satisfied. The value of p follows from (5.13) and (5.15).

6. Transverse isotropy. As already mentioned, problems have been solved completely by Ericksen & Rivlin [1] for incompressible bodies which are transversely isotropic with respect to a fixed direction. Here we consider one problem for a compressible body. If the body is transversely isotropic with respect to the x_3 -axis the equations of plane strain superposed on a uniform finite extension for a compressible body are given by (4.18), (4.9) and (4.10), where the strain energy function W in (4.11) is now restricted to the form

$$W = W(e_{11} + e_{22}, e_{11} e_{22} - e_{12}^2, e_{33}). \quad (6.1)$$

Alternatively,

$$W = W(I, J, e_{33}), \quad (6.2)$$

where

$$\begin{aligned} I &= a^{\alpha\beta} A_{\alpha\beta} = 2(e_{11} + e_{22} + 1), \\ J &= I_3/\lambda^2 = A/a = (2e_{11} + 1)(2e_{22} + 1) - 4e_{12}^2. \end{aligned} \quad (6.3)$$

Hence (4.18) becomes

$$\phi_{||\alpha\beta} = \frac{2\sqrt{J}}{\lambda} \left(a_{\alpha\beta} \frac{\partial W}{\partial I} + A_{\alpha\beta} \frac{\partial W}{\partial J} \right). \quad (6.4)$$

Apart from a small change in notation for J these are the same equations as those found by Adkins, Green & Shield [5] for isotropic compressible materials. Here, however, from (4.10),

$$\tau^{33} = \frac{\lambda}{\sqrt{J}} \frac{\partial W}{\partial e_{33}}. \quad (6.5)$$

7. Inflation and extension of a compressible hollow cylinder. Consider a circular cylinder of length l whose generators are parallel to the x_3 -axis and which has external and internal radii a_1, a_2 respectively in the unstrained state. The cylinder receives a uniform simple extension, of extension ratio λ parallel to its axis, and a uniform inflation so that the external and internal radii become r_1, r_2 respectively. We take the same system of coordinates in the deformed body as defined by (5.1) and we then have

$$x_1 = f(r) \cos \theta, \quad x_2 = f(r) \sin \theta,$$

where $f(r)$ is a function of r only (not the function used in §5). A simple calculation gives

$$a_{\alpha\beta} = \begin{bmatrix} f'^2, & 0 \\ 0, & f^2 \end{bmatrix}, \quad a^{\alpha\beta} = \begin{bmatrix} \frac{1}{f'^2}, & 0 \\ 0, & \frac{1}{f^2} \end{bmatrix}, \quad a = f^2 f'^2, \quad (7.1)$$

the metric tensors $A_{\alpha\beta}, A^{\alpha\beta}$ still being given by (5.3). Hence, from (6.3),

$$I = \frac{1}{f'^2} + \frac{r^2}{f^2}, \quad J = \frac{r^2}{f^2 f'^2}, \quad (7.2)$$

and, if ϕ is independent of θ , equations (6.4) yield

$$\frac{d^2 \phi}{dr^2} = \frac{2r}{\lambda f f'} \left(f'^2 \frac{\partial W}{\partial I} + \frac{\partial W}{\partial J} \right), \quad r \frac{d\phi}{dr} = \frac{2r}{\lambda f f'} \left(f^2 \frac{\partial W}{\partial I} + r^2 \frac{\partial W}{\partial J} \right). \quad (7.3)$$

From equations (7.3) we have

$$\lambda \left(\frac{f}{f'} - r \right) \frac{d^2 \phi}{dr^2} - \frac{\lambda f f''}{f'^2} \frac{d\phi}{dr} = 2 \left(r - \frac{r^2 f'}{f} - \frac{f^2 f''}{f'^3} \right) \frac{\partial W}{\partial I} + 2 \left(\frac{r}{f'^2} - \frac{r^2}{f f'} - \frac{r^2 f''}{f'^3} \right) \frac{\partial W}{\partial J}$$

or, using (7.2),

$$\begin{aligned} \lambda \frac{d}{dr} \left\{ \left(\frac{f}{f'} - r \right) \frac{d\phi}{dr} \right\} &= \left(\frac{\partial W}{\partial I} \frac{dI}{dr} + \frac{\partial W}{\partial J} \frac{dJ}{dr} \right) f^2 \\ &= f^2 \frac{dW}{dr}. \end{aligned}$$

This equation may be integrated to give

$$\begin{aligned} \lambda \left(\frac{f}{f'} - r \right) \frac{d\phi}{dr} &= \int_{r_1}^r f^2 \frac{dW}{dr} dr + K \\ &= \int_{a_1}^f f^2 \frac{dW}{df} df + K, \end{aligned} \quad (7.4)$$

where K is a constant. If there is no traction on the surface $r = r_1$ of the deformed cylinder then $d\phi/dr = 0$ there and K is zero, provided $f'/f \neq 1/r$. This latter condition is satisfied if the cylinder is inflated and not just deformed under simple extension.

In the case when a cylinder of compressible material is turned inside out and is in equilibrium under no forces on its curved surfaces then $d\phi/dr$ also vanishes on $r = r_2$ so that

$$\int_{r_1}^{r_2} f^2 \frac{dW}{dr} dr = \int_{a_1}^{a_2} f^2 \frac{dW}{df} df = 0, \quad (7.5)$$

which is the same form as for an isotropic material. Equation (7.5) is still valid for an incompressible cylinder.

To complete the problem for a compressible cylinder we must find f when W is known, either by eliminating $d\phi/dr$ from (7.4) and the second equation in (7.3), or by eliminating ϕ from (7.3).

The problem can be solved in general form for an incompressible body but this has already been discussed by Ericksen & Rivlin [1].

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