Vorticity and Deformation in Fluid Mechanics

A contribution to their kinematical properties

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*Communicated by C. TRUESDELL**

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Much of the material in this paper was contained in my 1948 Paris thesis, “Recherches sur la théorie des tourbillons.” Besides adding new results, I have radically recast the whole, shifting emphasis to the purely kinematical aspects of fluid motion.

Since my studies of vorticity began many years ago in Bucharest, it is only proper here that I express my gratitude to Professors POMPEIU and VALCOVICI. The incomparable teaching and writings of Professor H. VILLAT, as well as his personal advice and assistance to me in Paris, will never be forgotten; may he find here the expression of deep gratitude of his devoted pupil.

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PRELIMINARIES

The classical theory of vorticity. The theory of vorticity is the creation of Cauchy, Stokes, Helmholtz, and Kelvin. In his great prize memoir of 1815 Cauchy derived his celebrated first integral of the hydrodynamical equations:¹

$$(C) \quad \frac{w^i}{\rho} = \frac{W^\alpha}{\rho_0} x^i{}_{,\alpha},$$

where w^i is the vorticity vector and ρ is the density, W^α and ρ_0 are their values at $t = 0$ in the motion $x^i = x^i(X^1, X^2, X^3, t)$, the X^α being material and the x^i spatial co-ordinates so selected that $x^1(X^1, X^2, X^3, 0) = X^1$, etc., and finally $x^i{}_{,\alpha} \equiv \partial x^i / \partial X^\alpha$.

Equations (C) furnish the general solution of the differential equations

$$(H) \quad \frac{D}{Dt} \left(\frac{w^i}{\rho} \right) = \frac{w^j}{\rho} u^i{}_{,j},$$

where u^i is the velocity vector, $u^i{}_{,j}$ is its covariant derivative with respect to x^j , and

$$(0.1) \quad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(\right)_{,i} u^i$$

is the material derivative operator. While equations (H) were derived in major special cases by d'Alembert in 1749 and in full generality by Euler in 1752–1755, their important kinematical consequences were first obtained by Helmholtz in 1858.² To Helmholtz are due the concepts of vortex-line, vortex-surface, vortex-tube, and vortex strength, as well as theorems establishing their permanence. An explanation for this delay is found in the fact that the vorticity components themselves,

$$(0.2) \quad w^i \equiv \frac{1}{2} \sqrt{g} e^{ijk} u_{j,k},$$

where e^{ijk} is the permutation symbol, $g \equiv \det g_{ij}$, and g_{ij} is the metric tensor of a real co-ordinate system in real Euclidean 3-space, were not recognized as measures of the instantaneous local rate of rotation until after the classical researches of Cauchy, MacCullagh, and Stokes in 1841–1845. Indeed, Cauchy obtained (C) by analytical insight alone, it being reserved for Kirchhoff many years later to show how Cauchy's equations can yield the Helmholtz theorems.

¹ *E.g.* pp. 204–205 of LAMB, *Hydrodynamics*, 6th ed., Cambridge (1932).

² *Ibid.*, p. 205.

Kelvin created the basic concept of circulation and discovered its relation to the vorticity field.

To these pioneer efforts many other savants added their researches, resulting in the elegant and unified theory presented in the classical books on fluid mechanics.

All this work presupposes the fluid inviscid and in barotropic flow subject to conservative extraneous force—if we prefer a kinematical description, circulation-preserving motion. The general case, when Kelvin's circulation theorem does *not* hold, has been attacked in two ways. Beginning with Beltrami in 1871, several authors have analysed the pure kinematics of rotational motion. These researches, as well as the classical theory, may be found in Truesdell's *Kinematics of Vorticity*.¹ The second approach, where the first major result was obtained by V. Bjerknes in 1896, studies the specific mechanisms of vorticity creation or change through baroclinic flow, viscosity, *etc.* My earlier work consists mainly in such analysis for the case of a viscous incompressible fluid.²

Deformation. The deformation (or better, rate of deformation) is defined by

$$(0.3) \quad e_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}).$$

We cannot properly speak of a theory of deformation in fluid mechanics. So far as I know, Appell in 1903 was the first to give differential equations for the deformation analogous to the equations (H) for the vorticity.³ Let $\dot{u}^i$ be the acceleration field:

$$(0.4) \quad \dot{u}^i \equiv \frac{Du^i}{Dt},$$

and put

$$(0.5) \quad e_{ij}^* \equiv \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}).$$

Then Appell's result, purely kinematical, is

$$(A) \quad \frac{De_{ij}}{Dt} = e_{ij}^* - e_i^r e_{rj} - w_i^k w_{kj},$$

where

$$(0.6) \quad w_{ij} \equiv \frac{1}{2}(u_{i,j} - u_{j,i})$$

is the vorticity tensor. Up to now there has been no attempt to find an integral for these equations analogous to the integral (C) for the equations (H). It is precisely such an integral that I will derive in Chapter II.

¹ Indiana Univ. Science Series No. 19 (1954).

² C. R. Acad. Sci. Paris **223** (1946), 1095–1096; **224** (1947), 534–535; **225** (1947), 664–666.

³ J. Math. Pures Appl. (5) **9** (1903), 11–12.

The plan of this work. Chapter I presents the theory of vorticity from a purely kinematical point of view. A generalization of Zorawski's criterion for material vortex-surfaces is given. An attempt is made to clear up the question of the existence of a potential for the acceleration. In Chapter II a finite equation for the deformation, analogous to Cauchy's formula (C) for the vorticity, is obtained. Some applications of this formula are given. A geometrical interpretation of Appell's condition for the acceleration in a two-dimensional motion is given, along with some extensions for accelerational volumes in a three-dimensional space. The case of a rigid motion is studied in some detail. Chapter III deals with material lines and surfaces. In Chapter IV are presented some special problems on vorticity with particular emphasis on a motion such that the vorticity or its magnitude is constant for each particle.

Notations. Many of the formulae are valid in spaces of any number of dimensions. However, it is understood here that all considerations refer to Euclidean 3-space. The usual notations of tensor analysis are employed. Most of the special notations for kinematical quantities have already been introduced. Instead of the axial vorticity vector (0.2) it will usually be more convenient to employ the skew tensor (0.6), so that

$$(0.7) \quad u_{i,j} = e_{ij} + w_{ij}.$$

Similarly,

$$(0.8) \quad \dot{u}_{i,j} = e_{ij}^* + w_{ij}^*,$$

with e_{ij}^* defined by (0.5). Capital letters will denote initial values; thus, just as the X^α are the initial values of the x^i , the U^α are the initial values of the u^i , etc. Finally, we shall often replace " D/Dt " by a dot.

As is customary in works on differential geometry, all functions occurring will be assumed to have continuous derivatives of as many orders as needed.

CHAPTER I. VORTICITY

1. Generalization of Cauchy's equations. We begin with the kinematical identity

$$(1.1) \quad u_i x^i{}_{,\alpha} - U_\alpha = \int_0^t \dot{u}_i x^i{}_{,\alpha} dt + \frac{1}{2} \left(\int_0^t u^2 dt \right)_{,\alpha},$$

u being the speed and the integration, as in all time integrals in this work, being taken along the path of a single particle. Differentiating with respect to X^β yields

$$(1.2) \quad \begin{aligned} u_{i,j} x^i{}_{,\alpha} x^j{}_{,\beta} + u_i x^i{}_{,\alpha\beta} - U_{\alpha\beta} \\ = \int_0^t (\dot{u}_{i,j} x^i{}_{,\alpha} x^j{}_{,\beta} + \dot{u}_i x^i{}_{,\alpha\beta}) dt + \frac{1}{2} \left(\int_0^t u^2 dt \right)_{,\alpha\beta}. \end{aligned}$$

This basic formula implies many consequences, as will shortly appear. We may say that it contains potentially the entire theory of vorticity and deformation.

If we take the skew part of (1.2), we get the *vorticity formula*:

$$(1.3) \quad w_{ij} x^i{}_{,\alpha} x^j{}_{,\beta} - W_{\alpha\beta} = \int_0^t w_{ij}^* x^i{}_{,\alpha} x^j{}_{,\beta} dt.$$

Multiplying by $X^\alpha{}_{,k} X^\beta{}_{,l}$ yields

$$(1.4) \quad w_{kl} = \left(W_{\alpha\beta} + \int_0^t w_{ij}^* x^i{}_{,\alpha} x^j{}_{,\beta} dt \right) X^\alpha{}_{,k} X^\beta{}_{,l}.$$

In the circulation-preserving case $w_{ij}^* = 0$; in view of the relation $x^i{}_{,\alpha} X^\alpha{}_{,j} = \delta_j^i$, our result then becomes the tensor form of Cauchy's equations (C).

It is useful to notice the following equivalent forms of (1.3), valid in rectangular Cartesian systems:

$$(1.5) \quad \begin{aligned} w_{ij} x^i{}_{,\alpha} x^j{}_{,\beta} - W_{\alpha\beta} &= \frac{1}{2} \int_0^t w_{ij}^* \frac{\partial(x^i, x^j)}{\partial(X^\alpha, X^\beta)} dt \\ &= \frac{1}{2} \int_0^t \sum_{i=1}^3 \frac{\partial(x^i, \dot{u}^i)}{\partial(X^\alpha, X^\beta)} dt. \end{aligned}$$

2. Flux of vorticity. As is well known, a material element of surface ds^{ij} is related to its initial value $dS^{\alpha\beta}$ by the equation

$$(2.1) \quad ds^{ij} = dS^{\alpha\beta} x^i{}_{,\alpha} x^j{}_{,\beta}.$$

Multiplying both sides of (1.3) by $dS^{\alpha\beta}$ yields

$$(2.2) \quad w_{ij} ds^{ij} = W_{\alpha\beta} dS^{\alpha\beta} + \int_0^t w_{ij}^* ds^{ij} dt.$$

This result is an integral form of the classical relations

$$(2.3) \quad \frac{D}{Dt} (w_{ij} ds^{ij}) = w_{ij}^* ds^{ij},$$

$$\frac{D}{Dt} \oint_C u_i dx^i = \oint_C \dot{u}_i dx^i, \quad \frac{D}{Dt} \int_S w_{ij} ds^{ij} = \int_S w_{ij}^* ds^{ij}.$$

Thus (1.4) might have been obtained by starting with (2.3)₁, integrating to obtain (2.2), and observing that three components of ds^{ij} may be selected arbitrarily.

From (2.2) or (2.3) we may observe that even in a motion which is not circulation-preserving there may be certain surfaces for which the flux of vorticity is preserved; for example, any material surface so oriented as to remain tangent to the vorticity of the acceleration. Whether or not any material surfaces such that $w_{ij}^* ds^{ij} = 0$ can exist over a period of time is an open question.

3. Beltrami's equation. Generalizing (H), Beltrami derived the kinematical identity

$$(3.1) \quad \dot{w}_{ij} + w_i^k e_{jk} + w_j^k e_{ik} = w_{ij}^*.$$

As was noticed by Appell,¹ it follows that

$$(3.2) \quad \frac{Dw^2}{Dt} = -2Iw^2 + 2w_i w_j e^{ij} + w_{ij}^* w^{ij},$$

where $I = e^k_k$ is the expansion or first deformation invariant. The identity used for the derivation is

$$(3.3) \quad w_i^j w_j^k e^i_k = -Iw^2 + w_i w_j e^{ij}.$$

The result (3.2) reveals the scalar whose value determines the rate of change of vorticity magnitude.

4. On the existence of an acceleration-potential. We shall finish this chapter with some remarks on the existence of a potential for the acceleration. It would be extremely interesting to know under what conditions such a potential is preserved if it exists at time $t = 0$. A mathematical formulation for this problem is as follows: for what class of flows does $W_{\alpha\beta}^* = 0$ imply $w_{ij}^* = 0$ for all t ?

¹ *Loc. cit. ante.*

Now

$$(4.1) \quad \frac{D}{Dt} (u_i x^i_{,\alpha}) = \ddot{u}_i x^i_{,\alpha} + \dot{u}_i u^i_{,\alpha}.$$

Differentiation with respect to X^β yields

$$(4.2) \quad \frac{D}{Dt} (\dot{u}_{i,j} x^i_{,\alpha} x^j_{,\beta}) = (\ddot{u}_{i,j} + \dot{u}_{i,k} u^k_{,j}) x^i_{,\alpha} x^j_{,\beta},$$

whose skew part is

$$(4.3) \quad \frac{D}{Dt} (w^*_{ij} x^i_{,\alpha} x^j_{,\beta}) = (w^{**}_{ij} + \tfrac{1}{2}[\dot{u}_{i,k} u^k_{,j} - \dot{u}_{j,k} u^k_{,i}]) x^i_{,\alpha} x^j_{,\beta},$$

where

$$(4.4) \quad w^{**}_{ij} \equiv \tfrac{1}{2}(\ddot{u}_{i,j} - \ddot{u}_{j,i}).$$

In order that the initial vanishing of w^*_{ij} shall imply its continual vanishing, for the same fluid particles, it is thus necessary and sufficient that

$$(4.5) \quad w^{**}_{ij} + \tfrac{1}{2}[\dot{u}_{i,k} u^k_{,j} - \dot{u}_{j,k} u^k_{,i}] = 0,$$

a result to be contrasted with the vorticity equation (H).

The class of motions satisfying (4.5) would be an interesting object of research.

CHAPTER II. DEFORMATION

5. Appell's equations. The quadric $f^* = e_{ij}^* dx^i dx^j$ can be analyzed and interpreted in a way analogous to that usual for the deformation quadric $f = e_{ij} dx^i dx^j$. The quadric f^* , which was introduced by Appell, we may call the quadric of deformation of acceleration. It is natural to ask if the coefficients of f^* can be determined from those of f . That such is not the case is shown by the equations (A) of Appell, which we shall now briefly derive.

By differentiating Euler's acceleration formula follows

$$(5.1) \quad \frac{D}{Dt} (u_{i,j}) + u_{i,k} u^k_{,j} = \dot{u}_{i,j}.$$

Taking the symmetric part yields

$$(5.2) \quad \dot{e}_{ij} + u_{i,k} u^k_{,j} + u_{j,k} u^k_{,i} = e_{ij}^*,$$

which is easily put into the form (A).

The trace of equations (A) is

$$(5.3) \quad \dot{I} + e^i_k e^k_i - 2w^2 = I^*,$$

an identity discovered in a slightly different form by Appell¹ and employed in various later studies.²

6. Finite formula for the deformation. The symmetric part of (1.2) is

$$(6.1) \quad e_{ij} x^i_{,\alpha} x^j_{,\beta} + u_i x^i_{,\alpha\beta} - E_{\alpha\beta} = \int_0^t (e_{ij}^* x^i_{,\alpha} x^j_{,\beta} + \dot{u}_i x^i_{,\alpha\beta}) dt + \frac{1}{2} \left(\int_0^t u^2 dt \right)_{,\alpha\beta}.$$

This result can be improved as follows. First,

$$(6.2) \quad \int_0^t \dot{u}_i x^i_{,\alpha\beta} dt = u_i x^i_{,\alpha\beta} - \int_0^t u_i u^i_{,\alpha\beta} dt.$$

¹ *Loc. cit.*

² LICHTENSTEIN, *Hydromechanik* (1929), pp. 409–414; D. POMPEIU, *Rend. Circ. Math. Palermo* **54** (1930), 119; I. CARSTOIU, *Mathematica* **20** (1944), 172–173; C. JACOB, *Bull. Math. Soc. Roumaine Sci.* **46** (1946), 81–90; BILIMOVITCH, *Acad. Serbe Sci. Publ. Inst. Math.* **2** (1948), 37–52; C. TRUESDELL, pp. 101–105 of *The Kinematics of Vorticity*.

On the other hand,

$$(6.3) \quad \frac{1}{2}u^2_{,\alpha\beta} = u^i_{,\alpha} u_{i,\beta} + u^i_{,\alpha\beta} u_i.$$

Hence (6.1) can be written

$$(6.4) \quad \begin{aligned} e_{ij} x^i_{,\alpha} x^j_{,\beta} - E_{\alpha\beta} &= \int_0^t (e^*_{ij} x^i_{,\alpha} x^j_{,\beta} + u^i_{,\alpha} u_{i,\beta}) dt \\ &= \int_0^t (e^*_{ij} + u^k_{,j} u_{k,i}) x^i_{,\alpha} x^j_{,\beta} dt. \end{aligned}$$

This relation, the *deformation formula*, is somewhat similar to the basic vorticity formula (1.3), the difference lying in the term $u^k_{,j} u_{k,i}$ in the integrand. Since this term depends on both vorticity and deformation, no very simple conservation law can result from (6.4). In fact, for the deformation to be preserved (tensorially) by the motion it is necessary and sufficient that

$$(6.5) \quad e^*_{ij} + u^k_{,j} u_{k,i} = 0.$$

Thus, for example, if $E_{\alpha\beta} = 0$ it cannot follow that $e_{ij} = 0$ unless the stringent condition (6.5) is satisfied. More generally, (6.4) shows how the deformation is changed in time for a given particle.

It is easy to see that (6.4) is the general solution of Appell's equation (A) or (5.2).

In a circulation-preserving motion, the formula (6.4) becomes¹

$$(6.6) \quad e_{ij} = [E_{\alpha\beta} - U_\gamma X^\gamma_{,k} x^k_{,\alpha\beta}] X^\alpha_{,i} X^\beta_{,j} + \chi_{,ij},$$

where we have used the customary notation

$$(6.7) \quad \chi = \int_0^t (Q + \frac{1}{2}u^2) dt,$$

Q being the acceleration-potential.

7. Two remarks on Bernoulli's equation. In the case when a velocity-potential exists, $u_i = \phi_{,i}$, (6.1) becomes

$$(7.1) \quad \begin{aligned} \phi_{,ij} x^i_{,\alpha} x^j_{,\beta} + \phi_{,i} x^i_{,\alpha\beta} &= \int_0^t \left\{ \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 \right)_{,ij} x^i_{,\alpha} x^j_{,\beta} \right. \\ &\quad \left. + \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 \right)_{,i} x^i_{,\alpha\beta} \right\} dt + \frac{1}{2} \left(\int_0^t u^2 dt \right)_{,\alpha\beta}, \end{aligned}$$

which reduces to

$$(7.2) \quad (\phi - \Phi - \chi)_{,\alpha\beta} = 0.$$

¹ I. CARSTOIU, C. R. Acad. Sci. Paris **236** (1953), 2209-2211.

Since this equation must hold for all values $\alpha, \beta = 1, 2, 3$, it is equivalent to

$$(7.4) \quad \phi = \Phi + \chi + F(t),$$

an integrated form of the Bernoulli equation. Thus (6.1) or (6.4) may be regarded as a generalization of Bernoulli's equation to motions of an arbitrary continuum.

Another such generalization, for the case of steady motion, may be derived from the identity

$$(7.5) \quad \int_0^t u^i u_i dt = \frac{1}{2} u^2 - \frac{1}{2} U^2.$$

Now

$$(7.6) \quad \begin{aligned} u^i u_i &= \left(\frac{\partial u^i}{\partial t} + u^i{}_{,j} u^j \right) u_i \\ &= \frac{1}{2} \frac{\partial u^2}{\partial t} + e_{ij} u^i u^j. \end{aligned}$$

Hence in a steady motion (7.6) becomes

$$(7.7) \quad \int_0^t e_{ij} u^i u^j dt = \frac{1}{2} u^2 - \frac{1}{2} U^2,$$

which is our result. While to evaluate the integral considerably more knowledge of the motion is required than the right hand side might lead us to expect, nevertheless special cases of this formula may be interesting. For example, if on a certain stream-line the velocity lies upon the asymptotic cone of the deformation quadric, the speed is constant. This result becomes obvious from (7.6), which shows that an equivalent condition is that the acceleration be normal to the velocity.

8. Some applications of the deformation formula. Equivalent to the basic deformation formula (6.4) is

$$(8.1) \quad e_{ij} = \left[E_{\alpha\beta} + \int_0^t (e_{kl}^* + u^m{}_{,k} u_{m,l}) x^k{}_{,\alpha} x^l{}_{,\beta} dt \right] X^\alpha{}_{,i} X^\beta{}_{,j}.$$

To obtain the dilatation, multiply by g^{ij} . But $g^{ij} X^\alpha{}_{,i} X^\beta{}_{,j} = (C^{-1})^{\alpha\beta}$, where $(C^{-1})^{\alpha\beta}$ is the tensor reciprocal to Green's finite deformation tensor.¹ Hence

$$(8.2) \quad I = (C^{-1})^{\alpha\beta} \left[E_{\alpha\beta} + \int_0^t (e_{kl}^* + u^m{}_{,k} u_{m,l}) x^k{}_{,\alpha} x^l{}_{,\beta} dt \right].$$

¹ *E.g.*, see §14 of C. TRUESDELL, *J. Rational Mech. Analysis* **1**, 125-300 (1952).

This formula may be regarded as an integral form of (5.3). If we have a motion which is initially isochoric, $I = 0$ when $t = 0$, and hence $(C^{-1})^{\alpha\beta} E_{\alpha\beta} = 0$. A necessary and sufficient condition that such a motion remain isochoric is then

$$(8.3) \quad 0 = (C^{-1})^{\alpha\beta} \int_0^t (e_{kl}^* + u_{,k}^m u_{m,l}) x_{,\alpha}^k x_{,\beta}^l dt.$$

To calculate the material derivative of the invariant $e^{ij} e_{ij}$, note that from (A) we have

$$(8.4) \quad e^{ij} \dot{e}_{ij} = e^{ij} e_{ij}^* - e^{ij} (e_i^r e_{rj} + w_i^k w_{kj}).$$

Now from the Cayley-Hamilton equation follows the identity

$$(8.5) \quad e^i_j e^j_k e^k_i = I e^i_j e^j_i + I II + 3III,$$

where I, II, III are the principal invariants of e^i_j . Substituting this result and (3.3) in (8.4) yields

$$(8.6) \quad \frac{1}{2} \frac{D}{Dt} (e^{ij} e_{ij}) = e^{ij} e_{ij}^* - I (e^i_j e^j_i + II - w^2) - 3III - w_i w_j e^{ij}.$$

9. Geometrical interpretation of (5.3). Let us now come back to the equation (5.3), which we shall study from an integral point of view. In this section we employ rectangular Cartesian co-ordinates.

For plane isochoric motion (5.3) assumes the form

$$(9.1) \quad \frac{\partial \dot{u}_1}{\partial x^1} + \frac{\partial \dot{u}_2^2}{\partial x^2} + 2 \frac{\partial (u_1, u_2)}{\partial (x^1, x^2)} = 0.$$

Integration over a domain D bounded by a closed curve C yields

$$(9.2) \quad \oint_C (\dot{u}_1 dx^2 - \dot{u}_2 dx^1) + 2 \iint_D \frac{\partial (u_1, u_2)}{\partial (x^1, x^2)} d\sigma = 0,$$

a result which has an elegant geometrical interpretation, as follows now.

Let $\mathcal{Q}_C$ be the area bounded by the curve C . Consider the velocity and acceleration vectors at the points upon C . The termini of these vectors trace out two curves, which bound the areas $\mathcal{Q}_V$ and $\mathcal{Q}_A$. We shall call these areas, respectively, the *velocital area* and *accelerational area*. If the origins of these vectors be put at the origin of co-ordinates, their termini trace out other curves which enclose certain areas $\mathcal{Q}_H$ and $\mathcal{Q}_{H'}$; we shall call these areas the *hodographic areas* for the velocity and acceleration. In terms of these areas, (9.2) asserts that

$$(9.3) \quad \mathcal{Q}_A = \mathcal{Q}_C + \mathcal{Q}_{H'} - 2\mathcal{Q}_H,$$

which is similar to the identity

$$(9.4) \quad \mathcal{Q}_V = \mathcal{Q}_C + \mathcal{Q}_H,$$

which was given by Pompeiu as a geometrical interpretation of the continuity equation for plane isochoric motion. Following Pompeiu, we shall give an elementary proof of (9.3). Consider, at the time t , three fluid particles P_1, P_2, P_3 . The area of the triangle formed with $P_1(x_1^1, x_1^2)$, $P_2(x_2^1, x_2^2)$ and $P_3(x_3^1, x_3^2)$ is preserved in time. Hence, the determinant

$$(9.5) \quad |1 \ x_i^1 \ x_i^2| = \text{const. in time.}$$

After two material differentiations, this gives

$$(9.6) \quad |1 \ \dot{u}_{1i} \ x_i^2| + 2 |1 \ \dot{u}_{1i} \ u_{2i}| + |1 \ x_i^1 \ u_{2i}| = 0.$$

Hence it is easy to show that the area of the triangle formed by the termini of the acceleration vectors of P_1, P_2, P_3 is

$$(9.7) \quad |1 \ x_i^1 + \dot{u}_{1i} \ x_i^2 + \ddot{u}_{2i}| = |1 \ x_i^1 \ x_i^2| + |1 \ \dot{u}_{1i} \ \dot{u}_{2i}| - 2 |1 \ u_{1i} \ u_{2i}|,$$

which for the special case of the triangle $P_1 P_2 P_3$, is precisely the formula (9.3) which we wish to prove. The generalization to any closed polygon is obvious, and hence by a limit process the result can be proved for any smooth curve C .

The formula (9.3), valid for any closed fluid line at a fixed time t , takes on a different aspect if one follows during the motion the same fluid line C . Let $\mathcal{Q}_A^0$ be the area $\mathcal{Q}_A$ at $t = 0$, etc. We shall have

$$(9.8) \quad \mathcal{Q}_A - \mathcal{Q}_A^0 = \mathcal{Q}_{H'} - \mathcal{Q}_H^0 - 2(\mathcal{Q}_H - \mathcal{Q}_H^0).$$

On the other hand, one has

$$(9.9) \quad \mathcal{Q}_A - \mathcal{Q}_V = \mathcal{Q}_{H'} - 3\mathcal{Q}_H.$$

10. The case of a plane rigid motion. Consider the case of a rigid plane motion, for instance the motion of a plate in its own plane. The formula (5.3) is then identically satisfied. We shall write $w_{ij} = \omega_{ij}$ and $w = \omega$ for a rigid motion. Under these circumstances, one has¹

$$(10.1) \quad \mathcal{Q}_H = \omega^2 \mathcal{Q}_C, \quad \mathcal{Q}_{H'} = (\dot{\omega}^2 + \omega^4) \mathcal{Q}_C.$$

Hence, the formulae (9.4) and (9.3) become respectively

$$(10.2) \quad \mathcal{Q}_V = (1 + \omega^2) \mathcal{Q}_C, \quad \mathcal{Q}_A = [(1 - \omega^2)^2 + \dot{\omega}^2] \mathcal{Q}_C.$$

11. Accelerational volumes.² Guided by the studies of Pompeiu, Valcovici, Bilimovitch, Jacob, and Truesdell for velocital volumes, we shall now present an

¹ I. CARSTOIU, Bull. Sc. École Poly. Timisoara **11** (1944), 172-173.

² I. CARSTOIU, C. R. Acad. Sci. Paris **227** (1948), 1337-1339.

analysis of the accelerational volumes. Let S be a closed fluid surface which encloses the volume $\mathfrak{V}_s$, and let $\mathfrak{V}_v$ be the velocital volume and $\mathfrak{V}_A$ the accelerational volume of this surface, defined as in §9. Let $\mathfrak{V}_H$ and $\mathfrak{V}_{H'}$ be the hodo-graphic volumes for the velocity and acceleration, and introduce the invariant

$$(11.1) \quad \phi^* = w_i^* w_j^* e^{*ij}.$$

One has

$$(11.2) \quad \mathfrak{V}_A = \iiint_{\mathfrak{V}_s} \left[1 + \dot{u}_{i,i} + \frac{\partial(\dot{u}_2, \dot{u}_3)}{\partial(x^2, x^3)} + \frac{\partial(\dot{u}_3, \dot{u}_1)}{\partial(x^3, x^1)} + \frac{\partial(\dot{u}_1, \dot{u}_2)}{\partial(x^1, x^2)} + \frac{\partial(\dot{u}_1, \dot{u}_2, \dot{u}_3)}{\partial(x^1, x^2, x^3)} \right] d\mathfrak{V}_s,$$

which can be written as

$$(11.3) \quad \mathfrak{V}_A = \mathfrak{V}_s + \mathfrak{V}_{H'} + \iiint_{\mathfrak{V}_s} (I^* - II^* + w^{*2}) d\mathfrak{V}_s,$$

where

$$(11.4) \quad \mathfrak{V}_{H'} = \iiint_{\mathfrak{V}_s} (III^* + \phi^*) d\mathfrak{V}_s,$$

I^* , II^* , III^* , are the principal invariants of e_{ij}^* , and w^* is the magnitude of the vector w_i^* .

These formulae are to be compared with those for the velocital volume given by Jacob, Bilimovitch, and Truesdell.

12. Rigid motion. These formulae simplify greatly in the case of a rigid motion. First, Jacob and Truesdell's formulae show that the velocital volume reduces to the elegant expression given by Valcovici:¹

$$(12.1) \quad \mathfrak{V}_v = (1 + \omega^2) \mathfrak{V}_s,$$

where ω is the magnitude of the angular velocity ω . Let us compute the invariants I^* , II^* , III^* for a rigid motion.² Appell's equations (A) give

$$(12.2) \quad \begin{aligned} e_{11}^* &= -(\omega_3^2 + \omega_2^2), & e_{22}^* &= -(\omega_1^2 + \omega_3^2), & e_{33}^* &= -(\omega_2^2 + \omega_1^2), \\ e_{23}^* &= \omega_2 \omega_3, & e_{31}^* &= \omega_3 \omega_1, & e_{12}^* &= \omega_1 \omega_2. \end{aligned}$$

Hence

$$(12.3) \quad I^* = -2\omega^2, \quad II^* = -\omega^4, \quad III^* = 0.$$

¹ Bull. Math. Phys. École Poly. Bucarest **3** (1932-1933), 179.

² I. CARSTORU, Bull. Sci. Acad. Roumaine **29** (1946), 207-209.

On the other hand, we have

$$(12.4) \quad w^{*2} = \dot{\omega}^2$$

and one easily finds that

$$(12.5) \quad \phi^* = -(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2.$$

The formulae (11.3) and (11.4) become

$$(12.6) \quad \begin{aligned} \mathcal{V}_A &= [(1 - \omega^2)^2 + \dot{\omega}^2 - (\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2] \mathcal{V}_s, \\ \mathcal{V}_{H'} &= -(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}})^2 \mathcal{V}_s. \end{aligned}$$

One can now express the invariants I^* , II^* and ϕ^* in terms of these different volumes. We have

$$(12.7) \quad I^* = 2 \frac{\mathcal{V}_s - \mathcal{V}_V}{\mathcal{V}_s}, \quad II^* = -\frac{1}{4} I^{*2}, \quad \phi^* = \frac{\mathcal{V}_{H'}}{\mathcal{V}_s},$$

which formulae give geometrical interpretations of these invariants for the case of rigid motion.

CHAPTER III. MATERIAL LINES AND SURFACES.

13. Preliminaries. In this chapter we shall study the infinitesimal vectors

$$(13.1) \quad v_i \equiv w_{ij} dx^j, \quad e_i \equiv e_{ij} dx^j$$

associated at time t with a given infinitesimal field dx^i which changes in time according to the rule $dx^i = x^i_{,\alpha} dX^\alpha$. Thus if the field dx^i is at one instant tangent to a congruence of material curves, it will remain so for all time. Setting $v_i = 0$ yields a differential system for the vortex-lines.

14. Change in v_i induced by the motion. Multiplying (1.3) by dX^β yields

$$(14.1) \quad v_i x^i_{,\alpha} - V_\alpha = \int_0^t v_i^* x^i_{,\alpha} dt,$$

where we have put $v_i^* = w_{ij}^* dx^j$. Consider a single material line which is initially a vortex-line. Then $V_\alpha = 0$. In order that the material line be again a vortex-line at time t it is necessary and sufficient that $v_i = 0$, or, equivalently, since the motion is continuous, $v_i x^i_{,\alpha} = 0$. Hence our necessary and sufficient condition is

$$(14.2) \quad \int_0^t v_i^* x^i_{,\alpha} dt = 0.$$

For this condition to hold for all time it is necessary and sufficient that $v_i^* = 0$: that is, that the vorticity of the acceleration be tangent to this particular line, and hence parallel to the vorticity. While this result is due to Zorawski, the foregoing proof seems to be simpler than the others in the literature.

Equivalent to (14.1) is

$$(14.3) \quad v_i = \left(V_\alpha + \int_0^t v_j^* x^j_{,\alpha} dt \right) X^\alpha_{,i}.$$

While this result is but (1.4) rewritten, it shows a plainer analogy to Cauchy's equations (C).

15. Differential equation for v_i . The material derivative of (14.1) is

$$(15.1) \quad \dot{v}_i x^i_{,\alpha} + v_i u^i_{,\alpha} = v_i^* x^i_{,\alpha}.$$

Hence

$$(15.2) \quad \dot{v}_i + v_j u^j_{,i} = v_i^*,$$

a result analogous to Beltrami's equation (3.1).

From (15.2) follows

$$(15.3) \quad \frac{1}{2} \frac{Dv^2}{Dt} = v_i^* v^i - e_{ij} v^i v^j.$$

In the circulation-preserving case, or, more generally, when v_i^* is perpendicular to v^i , we get by integrating (15.3)

$$(15.4) \quad v^2 - V^2 = -2 \int_0^t e_{ij} v^i v^j dt.$$

Hence for the magnitude of v^i to be preserved at all times it is necessary and sufficient that v^i lie upon the asymptotic cone of the deformation quadric f . Also, by (14.3),

$$(15.5) \quad v^2 = v_i v^i = V_\alpha V_\beta X^\alpha_{,i} X^\beta_{,i},$$

so that we get finally

$$(15.6) \quad V^2 = V_\alpha V_\beta X^\alpha_{,i} X^\beta_{,i} + 2 \int_0^t e_{ij} v^i v^j dt.$$

16. Material surfaces. A material surface $x^i = x^i(\lambda, \mu)$ is defined by any initial surface $X^\alpha = X^\alpha(\lambda, \mu)$ and by the equations $x^i = x^i(X^\alpha(\lambda, \mu), t)$. At all times a given material point on this surface corresponds to the parameters λ, μ .

Any surface normal to the field v^i is a vortex-surface. In a circulation preserving motion these surfaces are material, and hence a material surface normal to v_i at one time is carried by the motion into another such surface at all times. We now discover a related property of the vortex-surfaces in a circulation-preserving motion.

We assume V_α normal to the surface $X^\alpha = X^\alpha(\lambda, \mu)$, and we use rectangular Cartesian co-ordinates X^α and x^i . Then

$$(16.1) \quad \frac{V_1}{\frac{\partial(X^2, X^3)}{\partial(\lambda, \mu)}} = \frac{V_2}{\frac{\partial(X^3, X^1)}{\partial(\lambda, \mu)}} = \frac{V_3}{\frac{\partial(X^1, X^2)}{\partial(\lambda, \mu)}} = \frac{V}{\sqrt{EG - F^2}},$$

where E, F, G are the Gauss coefficients for this surface. Inverting the differentiations on the right-hand side of (14.3) in the circulation-preserving case yields three equations of the type

$$(16.2) \quad \frac{\rho_0}{\rho} v_1 = V_1 \frac{\partial(x^2, x^3)}{\partial(X^2, X^3)} + V_2 \frac{\partial(x^2, x^3)}{\partial(X^3, X^1)} + V_3 \frac{\partial(x^2, x^3)}{\partial(X^1, X^2)}.$$

Substituting in these equations the values of V_α given by (16.1), we get at once

$$(16.3) \quad \frac{\rho_0}{\rho} v_1 = \frac{V_1}{\sqrt{EG - F^2}} \frac{\partial(x^2, x^3)}{\partial(\lambda, \mu)}, \dots,$$

that is

$$(16.4) \quad \frac{\frac{v_1}{\partial(x^2, x^3)}}{\frac{\partial(\lambda, \mu)}{\partial(\lambda, \mu)}} = \frac{\frac{v_2}{\partial(x^3, x^1)}}{\frac{\partial(\lambda, \mu)}{\partial(\lambda, \mu)}} = \frac{\frac{v_3}{\partial(x^1, x^2)}}{\frac{\partial(\lambda, \mu)}{\partial(\lambda, \mu)}} = \frac{\rho}{\rho_0} \frac{V}{\sqrt{EG - F^2}} = \frac{v}{\sqrt{eg - f^2}}.$$

Hence follows the interesting property

$$(16.5) \quad \frac{\rho}{\rho_0} \frac{ds}{dS} = \frac{v}{V},$$

which may be compared with a somewhat analogous result of Appell concerning vortex-lines.¹ In particular, for an incompressible fluid, the preservation of the length of v^i is necessary and sufficient for the preservation of the respective areas of the successive configurations of a vortex-surface.

17. The case when v_i is an exact differential. Suppose

$$(17.1) \quad v_i = w_{ij} dx^j = d\Phi_i.$$

Then $w_{ij} = \Phi_{i,j}$, and hence

$$(17.2) \quad \Phi_{i,j} + \Phi_{j,i} = 0.$$

The general solution Φ_i of this equation is the general vector representing a rigid displacement. Hence w_{ij} is uniform in space. Thus for v_i to be an exact differential it is necessary and sufficient that the motion be one of uniform vorticity.

18. Change in e_i induced by the motion. From (6.4) we have

$$(18.1) \quad e_i x^i_{,\alpha} - E_\alpha = \int_0^t \{e_i^* x^i_{,\alpha} + (e_k^i + w_k^i)(e_i + v_i)x^k_{,\alpha}\} dt,$$

where $e_i^* = e_{ij}^* dx^j$.

Let f and f^* be the quadratic forms based on e_{ij} and on e_{ij}^* , respectively. Then, since $f = e_i dx^i$, $f^* = e_i^* dx^i$, multiplying (18.1) by dX^α yields

$$(18.2) \quad f - F = \int_0^t \{f^* + (e_i + v_i)(e^i + v^i)\} dt.$$

As a simple application, we can find at once the form of f^* for a rigid motion. For, in this case

¹ Cf. §97 of C. TRUESDELL, *The Kinematics of Vorticity*.

$$(18.3) \quad f = F = 0, \quad e_i = 0, \quad v_i = \omega_{ij} dx^j,$$

so that (18.2) yields

$$(18.4) \quad f^* = -\omega_{ij} \omega^i_k dx^j dx^k.$$

19. Differential equation for e_i . The material derivative of (18.1), solved for $\dot{e}_i$, is

$$(19.1) \quad \dot{e}_i = e_i^* + (e^k_i + w^k_i) v_k.$$

Hence follows

$$(19.2) \quad \frac{1}{2} \frac{De^2}{Dt} = e_i^* e^i + e_{ik} v^i e^k.$$

The results of this and the preceding section could have been deduced from those of §§14–15 by noting the evident identity

$$(19.3) \quad \dot{e}_i + \dot{v}_i = e_i^* + v_i^*.$$

20. The case when e_i is an exact differential. In order that $e_i = d\Psi_i$ it is necessary and sufficient that

$$(20.1) \quad e_{ij} = \Psi_{i,j}, \quad \text{where} \quad \Psi_{i,j} = \Psi_{j,i}.$$

Hence there must exist a function Θ such that

$$(20.2) \quad \Psi_i = \Theta_{,i},$$

and hence

$$(20.3) \quad e_{ij} = \Theta_{,ij}.$$

This condition is satisfied in an irrotational motion.

On a vortex-line we have $0 = v_i = (u_{i,j} - e_{ij}) dx^j = du_i - e_i$; hence for a portion of a vortex-line we have

$$(20.4) \quad \int_{P_1}^{P_2} e_i = u_i \Big|_{P_1}^{P_2}.$$

21. Consequences of the vanishing of e_i . The differential system $e_i = 0$, or

$$(21.1) \quad e_{ij} dx^j = 0,$$

is compatible if and only if

$$(21.2) \quad |e_{ij}| = 0;$$

that is, if and only if the quadric of deformation is degenerate.

If (21.2) is satisfied, and if none of the principal minors ε^i of e_{ij} is zero, (21.1) is equivalent to

$$(21.3) \quad \frac{dx^1}{\varepsilon^1} = \frac{dx^2}{\varepsilon^2} = \frac{dx^3}{\varepsilon^3}.$$

From (21.1) and (20.4) we get at once the theorem: *if at each point of a certain vortex-line the deformation quadric is degenerate, the velocity is constant along that line.* Such is the case, for example, in any plane motion.

CHAPTER IV. SOME SPECIAL PROBLEMS ON VORTICITY

22. Inversion of Beltrami's equation. In this chapter we investigate some special problems concerning the vorticity. The equation (3.1) can be written as follows:

$$(22.1) \quad -w_i^* = -\dot{w}_i + (e_i^j - I\delta_i^j) w_j.$$

Put

$$(22.2) \quad \Delta \equiv \det (e_i^j - I\delta_i^j) = I II + III,$$

and assume $\Delta \neq 0$. Then we can solve (22.1) for w_i . The result may be simplified by means of the quadratic forms

$$(22.3) \quad \Phi' \equiv e^{ij} \dot{w}_i \dot{w}_j, \quad \Phi^* \equiv e^{ij} w_i^* w_j^*$$

and their adjoints

$$(22.4) \quad \Psi' \equiv g^{ip} \delta_{jln}^{ikm} e_k^l e_m^n \dot{w}_i \dot{w}_p, \quad \Psi^* \equiv g^{ip} \delta_{jln}^{ikm} e_k^l e_m^n w_i^* w_p^*.$$

Then the solution of (22.1) is given by

$$(22.5) \quad \Delta w_i = \frac{1}{2} \left[\frac{\partial \Psi'}{\partial \dot{w}_i} - \frac{\partial \Psi^*}{\partial w_i^*} + I \left\{ \frac{\partial \Phi'}{\partial \dot{w}_i} - \frac{\partial \Phi^*}{\partial w_i^*} \right\} \right].$$

23. The case of an incompressible fluid. Suppose now the fluid incompressible. The formula (22.4) reduces to

$$(23.1) \quad w_i = \frac{1}{2III} \left[\frac{\partial \Psi'}{\partial \dot{w}_i} - \frac{\partial \Psi^*}{\partial w_i^*} \right],$$

whence we draw the following immediate consequences:

a) If the vector $\dot{w}_i$ lies on the supplementary cone of the quadric of deformation, that is $\Psi' = 0$, then

$$(23.2) \quad w_i = - \frac{1}{2III} \frac{\partial \Psi^*}{\partial w_i^*},$$

and

$$(23.3) \quad w^i w_i^* = - \frac{\Psi^*}{III}.$$

b) If, on the other hand, the vorticity of the acceleration lies on the same cone, then

$$(23.4) \quad w_i = \frac{1}{2III} \frac{\partial \Psi'}{\partial w_i},$$

and

$$(23.5) \quad \frac{1}{2} \frac{Dw^2}{Dt} = \frac{\Psi'}{III}.$$

If in this case, we have also

$$(23.6) \quad \Psi' = 0:$$

the magnitude of vorticity will be constant with respect to the time for each particle. In the case when a potential for the acceleration exists, the vorticity itself will lie on the asymptotic cone of the quadric of deformation, as can be seen immediately from the formula (3.2).

24. Vorticity constant along a path.¹ In this section we suppose that the fluid is incompressible and the vorticity of acceleration either vanishes, that is, a potential for the acceleration exists, or lies on the supplementary cone of the quadric of deformation. We have therefore

$$(24.1) \quad w_i = \frac{1}{2III} \frac{\partial \Psi'}{\partial w_i}.$$

Suppose now that $w_i = 0$: the vorticity is constant in time for each particle. The formula (24.1) then gives $w_i = 0$ except in the case when the third invariant vanishes. Thus a necessary condition for a motion with non-zero vorticity constant in time for each particle is that the quadric of deformation be degenerate.

This circumstance does occur in the case of a plane motion. The quadric of deformation reduces indeed in this case to a cylinder and the vorticity of each particle is constant in time. It would be interesting to know if other motions satisfying this condition exist.

Supposing $w_i^* = 0$, one has from (22.1), along a vortex-line $e_{ij} dx^j = 0$, that is the vector e_i does vanish in this case. The equations (21.3) represent therefore, in the case under consideration, the vortex-lines.

25. Vorticity magnitude constant along a path.² We now obtain a property of fluid motion in which the magnitude of vorticity is constant for each particle. As noted in §23, for an isochoric circulation-preserving motion the vorticity lies on the asymptotic cone of the quadric of deformation. There is no need to

¹ I. CARSTOIU, C. R. Acad. Sci. Paris, to be published.

² I. CARSTOIU, Bull. Sci. Acad. Roumaine **28** (1946), 503-504, 589-592.

assume that this cone is real, since the condition of incompressibility insures it. It can easily be shown that for a vortex-line in such a motion we have the relation¹

$$(25.1) \quad \frac{dl}{dL} = \frac{w}{W},$$

where dl and dL are the magnitudes of corresponding elements of arc on a vortex-line at the time t and $t = 0$. It follows at once that for $w = \text{const.}$ (with respect to the time)

$$(25.2) \quad dl = dL.$$

Therefore, in an isochoric circulation-preserving motion such that the vorticity magnitude is constant for each particle, the length of each portion of a vortex-line is preserved in time. Vice versa, if (25.2) holds, the vorticity magnitude for each particle will be preserved during its motion.

¹ C. TRUESDELL, *loc. cit.* p. 180.