A Theorem on Rearrangements and its Application to Certain Delay Differential Equations

F. H. BROWNELL

University of Washington, Seattle, Washington, and Institute for Advanced Study, Princeton, New Jersey

&

W. K. ERGEN

Oak Ridge National Laboratory, Oak Ridge, Tennessee

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1. Introduction. Let $f(x)$ be real valued and possess a continuous derivative $f'(x)$ for all real x such that everywhere

$$(1.1) \quad f(x) > 0, \quad f'(x) > 0, \quad \text{and} \quad f(0) = 1.$$

Under the conditions (1.1) we consider over $t > 0$ the necessarily nonlinear delay differential equation

$$(1.2) \quad x'(t) = -\frac{c}{\theta} \int_0^\theta (\theta - h) \{f(x(t-h)) - 1\} dh$$

for given constants $c > 0$ and $\theta > 0$. We remark (theorem 7) that a unique solution of (1.2) over $t > 0$ is determined by every real valued, continuous function over $-\theta \leq t \leq 0$. Define over $t > 0$

$$(1.3) \quad [R(y)](t) = F(y(t)) - \left[\frac{1}{\theta} \int_0^\theta f(y(t-h)) dh \right] y(t) + \frac{1}{2c} [y'(t)]^2,$$

where $F(y) = 1 + \int_0^y f(z) dz$. Then we shall find (lemma 3) that every solution $x(t)$ of (1.2) must also satisfy for $t_2 > t_1 > \theta$

$$(1.4) \quad \int_{t_1}^{t_2} x(t) \{f(x(t)) - f(x(t-\theta))\} dt = -\theta \{[R(x)](t_2) - [R(x)](t_1)\}.$$

By applying a simple modification (theorem 2) of a well-known rearrangement theorem of Hardy, Littlewood, & Polya to (1.4), we are able to show (theorem 9) that every solution $x(t)$ of (1.2) approaches in a certain sense as $t \rightarrow +\infty$ the set of periodic solutions of (1.2). Also (theorems 4 and 5) each nonzero periodic solution $x(t)$ of (1.2) has $x(t) = x(t - \theta)$, so θ is a multiple of the period, and $x(t)$ satisfies everywhere

$$(1.5) \quad \gamma + 1 = F(x(t)) - x(t) + \frac{1}{2c} [x'(t)]^2$$

for some constant $\gamma > 0$. From (1.5) the periodic solutions of (1.2) can be explicitly computed. In particular if $f(x)$ is analytic on the real axis, then (theorem 11) there are only a finite number of essentially distinct periodic solutions of (1.2), and each solution of (1.2) must approach just one of these as $t \rightarrow +\infty$.

In view of the present virtually complete lack of information about the behavior of nonlinear delay differential equation solutions of large amplitude, so that the equation is not "almost linear," the use here of rearrangement theorems to get such information appears particularly significant. Presumably such techniques should give some information about much more general equations than the rather special (1.2).

2. Rearrangements. We first want to note a simple rearrangement theorem of Hardy, Littlewood, & Polya ([2], p. 278). Let S be an interval of the real line, $\mathfrak{F}$ the σ -algebra of Lebesgue measurable subsets of S , and μ ordinary Lebesgue measure on the real line. By a point automorphism of the measure space $(S, \mathfrak{F}, \mu)$ we mean a one-to-one point function φ from S onto S such that $\varphi(A) \in \mathfrak{F}$, $\varphi^{-1}(A) \in \mathfrak{F}$, and $\mu(\varphi(A)) = \mu(A)$ for every set $A \in \mathfrak{F}$. Two real valued, ($\mathfrak{F}$) measurable functions $f(s)$ and $g(s)$ over $s \in S$ are said to be similarly ordered if $g(s') \geq g(s)$ whenever $f(s') > f(s)$, the relation clearly being symmetric.

Theorem 1. *For the above measure space $(S, \mathfrak{F}, \mu)$ let f and g be two nonnegative real valued, ($\mathfrak{F}$) measurable functions which are similarly ordered, and let φ be a point automorphism of $(S, \mathfrak{F}, \mu)$; then the following Lebesgue integrals exist as extended real numbers and satisfy the following inequality:*

$$(2.1) \quad \int_S g(s)f(s) \, d\mu(s) \geq \int_S g(s)f(\varphi(s)) \, d\mu(s) \geq 0.$$

This statement merely rephrases a well-known theorem ([2], p. 278, theorem 378), and the proof will not be reproduced here. We remark that the theorem remains true for a perfectly arbitrary measure space $(S, \mathfrak{F}, \mu)$. Also the condition of similar ordering can be considerably weakened. All we really need to assume in its place is that for all real α and β

$$(2.2) \quad \mu(F_\alpha \cap G_\beta) = [\min \mu(F_\alpha), \mu(G_\beta)],$$

where $F_\alpha = \{s \in S \mid f(s) \geq \alpha\}$ and $G_\beta = \{s \in S \mid g(s) \geq \beta\}$. The proof of theorem 1 under condition (2.2) depends upon the persistence of this condition for the

dyadic-fraction valued, simple approximations to f and g . For these approximations (2.1) reduces to the easy finite case ([2], p. 263–264), from which (2.1) for f and g follows by the monotone convergence theorem.

Theorem 2. Let φ be a point automorphism of the above measure space $(S, \mathfrak{F}, \mu)$, and let $f \in L_1(S, \mathfrak{F}, \mu) \cap L_2(S, \mathfrak{F}, \mu)$ and $g \in L_2(S, \mathfrak{F}, \mu)$ be such that for some real constants b and $r > 0$ both $f(s) \geq 0$ and $g(s) - rf(s) + b \geq 0$ over $s \in S$ and also

$$(2.3) \quad g(s') - g(s) \geq r\{f(s') - f(s)\}$$

wherever $f(s') > f(s)$. Then it follows that

$$(2.4) \quad \begin{aligned} \int_S g(s) \{f(s) - f(\varphi(s))\} d\mu(s) &\geq r \int_S f(s) \{f(s) - f(\varphi(s))\} d\mu(s) \\ &= \frac{r}{2} \int_S [f(s) - f(\varphi(s))]^2 d\mu(s). \end{aligned}$$

Proof. First notice that (2.3) says that $h(s) = g(s) - rf(s) + b$ is similarly ordered with $f(s)$. Since $h(s) \geq 0$ and $f(s) \geq 0$, $\int_S h(s)f(s) d\mu(s) \geq \int_S h(s)f(\varphi(s)) d\mu(s)$ by theorem 1. Also $f \in L_1$ makes $b\int_S f(s) d\mu(s) = b\int_S f(\varphi(s)) d\mu(s)$ finite, so by subtracting this and using $f \in L_2$ and $g \in L_2$ we get the first part of (2.4) with all integrals existing and finite.

Also

$$\begin{aligned} \int_S \{f(s) - f(\varphi(s))\}^2 d\mu(s) &= \int_S [f(s)]^2 d\mu(s) - 2 \int_S f(s)f(\varphi(s)) d\mu(s) + \int_S [f(\varphi(s))]^2 d\mu(s) \\ &= 2 \int_S f(s) \{f(s) - f(\varphi(s))\} d\mu(s) \end{aligned}$$

gives the rest of (2.4). Q.E.D.

3. Periodic Solutions. Returning to (1.2),

$$x'(t) = -\frac{c}{\theta} \int_0^\theta (\theta - h) \{f(x(t-h)) - 1\} dh,$$

to be precise we define a real valued function $x(t)$ to be a solution of (1.2) over $t > 0$ if $x(t)$ is defined continuous over all real $t \geq -\theta$ and possesses a continuous derivative $x'(t)$ satisfying (1.2) over all $t > 0$. Define $[R(y)](t)$ by (1.3),

$$[R(y)](t) = F(y(t)) - \left[\frac{1}{\theta} \int_0^\theta f(y(t-h)) dh \right] y(t) + \frac{1}{2c} [y'(t)]^2,$$

for use in the following lemma.

Lemma 3. *If $x(t)$ is a solution of (1.2) over $t > 0$ under conditions (1.1), then for every pair of real t_1 and t_2 with $t_2 > t_1 \geq \theta$ the equation (1.4) is satisfied,*

$$\int_{t_1}^{t_2} x(t) \{f(x(t)) - f(x(t - \theta))\} dt = -\theta \{[R(x)](t_2) - [R(x)](t_1)\}.$$

Proof. Since $x(t)$ is a solution of (1.2) over $t > 0$, for $t > \theta$ exists

$$\begin{aligned} \frac{d}{dt} [R(x)](t) &= x'(t)f(x(t)) - \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} x'(t) \\ &\quad - x(t) \frac{d}{dt} \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} \\ &\quad - x'(t) \frac{d}{dt} \left\{ \frac{1}{\theta} \int_0^\theta (\theta-h)[f(x(t-h)) - 1] dh \right\} \\ (3.1) \quad &= x'(t)f(x(t)) - \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} x'(t) \\ &\quad - x(t) \frac{1}{\theta} [f(x(t)) - f(x(t-\theta))] \\ &\quad - x'(t) \left\{ f(x(t)) - \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} \\ &= -\frac{1}{\theta} x(t) \{f(x(t)) - f(x(t-\theta))\}, \end{aligned}$$

since

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} &= -\frac{1}{\theta} \int_0^\theta \frac{\partial}{\partial h} \{f(x(t-h))\} dh \\ &= \frac{1}{\theta} \{f(x(t)) - f(x(t-\theta))\} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{1}{\theta} \int_0^\theta (\theta-h)[f(x(t-h)) - 1] dh \right\} &= -\frac{1}{\theta} \int_0^\theta (\theta-h) \frac{\partial}{\partial h} \{f(x(t-h))\} dh \\ &= \left[-\frac{1}{\theta} (\theta-h)f(x(t-h)) \right]_{h=0}^{h=\theta} \\ &\quad + \frac{1}{\theta} \int_0^\theta (-1)f(x(t-h)) dh \\ &= f(x(t)) - \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \end{aligned}$$

for $t > \theta$ so that $x'(t - h)$ exists and is continuous for $0 \leq h \leq \theta$. But (3.1) clearly implies (1.4). Q.E.D.

By a periodic solution of (1.2) we mean a function $x(t)$ defined and periodic over all real t which is also a solution of (1.2) over $t > 0$, and thus actually over $-\infty < t < +\infty$ by the periodicity. Since $x(t) \equiv 0$ is clearly the only constant solution of (1.2) under conditions (1.1), a periodic solution $x(t) \not\equiv 0$ of (1.2) always has a well-defined period.

To study (1.4), we note that $\frac{d}{dx} \{F(x) - x\} = f(x) - 1$, which is positive for $x > 0$ and negative for $x < 0$ by conditions (1.1). Thus for real $v \geq 0$ the equation $F(x) - x = v^2 + 1$ has exactly two roots $x = g_1(v)$ and $x = g_2(v)$ with $g_1(v') < g_1(v) < 0 < g_2(v) < g_2(v')$ for $0 < v < v'$, and $g_1(0) = 0 = g_2(0)$.

With these definitions we have the following theorem, where we mean by a nowhere constant function one for which there exists no interval of positive length on which the function is constant.

Theorem 4. Under conditions (1.1) and for $\gamma > 0$

$$\begin{aligned} P(\gamma) &= \sqrt{\frac{2}{c}} \int_{g_1(\sqrt{\gamma})}^{g_2(\sqrt{\gamma})} \frac{dx}{\sqrt{\gamma + 1 + x - F(x)}} \\ (3.2) \quad &= 2 \sqrt{\frac{2}{c}} \int_0^{\frac{1}{2}\pi} \left\{ \frac{\sqrt{\gamma} \cos s}{1 - f(g_1(\sqrt{\gamma} \cos s))} + \frac{\sqrt{\gamma} \cos s}{f(g_2(\sqrt{\gamma} \cos s)) - 1} \right\} ds \end{aligned}$$

is finite and has

$$(3.3) \quad \lim_{\gamma \rightarrow 0^+} P(\gamma) = \frac{2\pi}{\sqrt{cf'(0)}},$$

$$(3.4) \quad \left(2 \sqrt{\frac{2}{c}} \right) \sqrt{\gamma} \leq P(\gamma).$$

Now if $x(t)$ is a periodic solution of (1.2) under conditions (1.1), and if $x(t) = x(t - \theta)$ for all real t , then for some real constant $\gamma \geq 0$ the following hold over all real t :

$$(1.5) \quad \gamma + 1 = F(x(t)) - x(t) + \frac{1}{2c} [x'(t)]^2,$$

$$(3.5) \quad x''(t) = -c\{f(x(t)) - 1\},$$

$$(3.6) \quad 1 = \frac{1}{\theta} \int_0^\theta f(x(t - h)) dh.$$

If also $x(t) \not\equiv 0$, then $\gamma > 0$, $x(t)$ is nowhere constant, the period T of $x(t)$ is well defined, $T = P(\gamma)$, and $\theta = kP(\gamma)$ for some integer $k \geq 1$.

Conversely, under conditions (1.1) and for every $\gamma > 0$, equation (1.5) has a nowhere constant solution $x_\gamma(t)$ over $-\infty < t < +\infty$, which is unique except for

translations along the t axis and which satisfies (3.5) everywhere. Moreover, this $x_\gamma(t)$ is periodic with well-defined period $T = P(\gamma)$; for every $\theta = kP(\gamma)$, k being an integer ≥ 1 , this $x_\gamma(t)$ is a periodic solution of (1.2); $x_\gamma(t) = x_\gamma(t - \theta)$; and (3.6) holds over $-\infty < t < +\infty$. For $\gamma = 0$, $x_\gamma(t) \equiv 0$ is the unique solution of (1.5), so $f(x_\gamma(t)) = 1$ and (3.6) and (1.2) hold everywhere for all $\theta > 0$.

Proof. The first integral in (3.2) clearly exists and is finite for $\gamma > 0$, the zero of the denominator at the endpoints causing no trouble since $\frac{d}{dx} \{F(x) - x\} = f(x) - 1 \neq 0$ here for $\gamma > 0$. We show that the second integral exists and is finite and equal to the first by making the change of variable $x = g_1(\sqrt{\gamma} \cos s)$ for $x < 0$, $x = g_2(\sqrt{\gamma} \cos s)$ for $x > 0$,

$$\begin{aligned} & [\gamma + 1 + x - F(x)]^{-\frac{1}{2}} \frac{dx}{ds} \\ &= -[\gamma + 1 - (1 + \gamma \cos^2 s)]^{-\frac{1}{2}} \frac{2\sqrt{\gamma} \cos s}{f(g_i(\sqrt{\gamma} \cos s)) - 1} (\sqrt{\gamma} \sin s) \\ &= \frac{\pm 2\sqrt{\gamma} \cos s}{f(g_i(\sqrt{\gamma} \cos s)) - 1}, \end{aligned}$$

since

$$g'_i(v) = \frac{2v}{f(g_i(v)) - 1}$$

exists for $v > 0$ by the standard implicit function theorem. In order to get (3.3) from (3.2), notice that $F(x) = 1 + x + \frac{1}{2}[f'(0) + r_0(x)]x^2$ with $\lim_{x \rightarrow 0} r_0(x) = 0$ makes $g_i(v) = (-1)^i \sqrt{2}[f'(0) + r_0(g_i(v))]^{-\frac{1}{2}}v$, and hence

$$g_i(v) = (-1)^i \sqrt{\frac{2}{f'(0)}} (1 + r_i(v))v$$

with $\lim_{v \rightarrow 0} r_i(v) = 0$, $i = 1$ or 2 . Putting this into (3.2) and using the mean value theorem with condition (1.1) gives

$$\begin{aligned} & (-1)^i [f(g_i(\sqrt{\gamma} \cos s)) - 1]^{-1} \sqrt{\gamma} \cos s \\ &= \left[f'(\alpha_i(\gamma, s)g_i(\sqrt{\gamma} \cos s)) \sqrt{\frac{2}{f'(0)}} (1 + r_i(\sqrt{\gamma} \cos s)) \right]^{-1} \rightarrow \sqrt{2f'(0)} \end{aligned}$$

as $\gamma \rightarrow 0+$ uniformly over $0 \leq s \leq \frac{1}{2}\pi$, where $0 < \alpha_i < 1$. Thus (3.3) follows.

Passing over (3.4) for a moment, assume $x(t)$ to be a periodic solution of (1.2) with $x(t) = x(t - \theta)$ everywhere. As in (3.1),

$$\frac{d}{dt} \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} = \frac{1}{\theta} \{f(x(t)) - f(x(t-\theta))\} = 0$$

now, so

$$\frac{1}{\theta} \int_0^\theta f(x(t-h)) dh = \beta,$$

constant over $-\infty < t < +\infty$. To evaluate β integrate (1.2) from t_1 to $t_1 + \theta$, getting

$$\begin{aligned} 0 &= x(t_1 + \theta) - x(t_1) = \int_{t_1}^{t_1+\theta} x'(t) dt \\ &= -\frac{c}{\theta} \int_0^\theta (\theta - h) \left\{ \int_{t_1}^{t_1+\theta} f(x(t-h)) dt - \theta \right\} dh = -c \int_0^\theta (\theta - h)(\beta - 1) dh \\ &= -\frac{1}{2} c \theta^2 (\beta - 1) \end{aligned}$$

since

$$\int_{t_1}^{t_1+\theta} f(x(t-h)) dt = \int_0^\theta f(x(t_1 + \theta - h - s)) ds = \beta \theta.$$

Thus $0 = -\frac{1}{2} c \theta^2 (\beta - 1)$ makes $\beta = 1$ for (3.6). This in turn makes $[R(x)](t) = F(x(t)) - x(t) + [x'(t)]^2/(2c)$; and, since $x(t) = x(t - \theta)$ and (3.1) make $[R(x)](t)$ constant over $-\infty < t < +\infty$, we have (1.5) with $\gamma \geq 0$ from $F(x) - x \geq 1$ by conditions (1.1). Now (3.5) follows by differentiating (1.2) and using (3.6), $x''(t) = -cf(x(t)) + c \left\{ \frac{1}{\theta} \int_0^\theta f(x(t-h)) dh \right\} = -cf(x(t)) + c$. Finally if $x(t) \equiv 0$, then $\gamma > 0$ obviously from (1.5), $x(t)$ is nowhere constant from (1.5) and (3.5), T is well defined, $\theta = kT$ for some integer $k \geq 1$ from $x(t) = x(t - \theta)$, and

$$T = 2 \int_{g_1(\sqrt{\gamma})}^{g_2(\sqrt{\gamma})} \frac{dx}{|x'(t)|} = \sqrt{\frac{2}{c}} \int_{g_1(\sqrt{\gamma})}^{g_2(\sqrt{\gamma})} \frac{dx}{\sqrt{\gamma + 1 + x - F(x)}} = P(\gamma)$$

from (1.5).

The converse statement is completely obvious except for showing $x_\gamma(t)$, determined by (1.5), satisfies (1.2) and (3.6) everywhere for every $\theta = kP(\gamma)$, k a positive integer, when $\gamma > 0$. Here we get (3.6) first by noting with $T = P(\gamma)$ that

$$\begin{aligned} \int_0^T f(x_\gamma(t)) dt &= \sqrt{\frac{2}{c}} \int_{g_1(\sqrt{\gamma})}^{g_2(\sqrt{\gamma})} \frac{f(x) dx}{\sqrt{\gamma + 1 + x - F(x)}} \\ &= 2 \sqrt{\frac{2}{c}} \int_0^{\frac{1}{2}\pi} \left\{ \frac{f(g_1(\sqrt{\gamma} \cos s))}{1 - f(g_1(\sqrt{\gamma} \cos s))} + \frac{f(g_2(\sqrt{\gamma} \cos s))}{f(g_2(\sqrt{\gamma} \cos s)) - 1} \right\} \sqrt{\gamma} \cos s ds, \end{aligned}$$

so that

$$T - \int_0^T f(x_\gamma(t)) dt = 2 \sqrt{\frac{2}{c}} \int_0^{\frac{1}{2}\pi} \{+1 - 1\} \sqrt{\gamma} \cos s ds = 0.$$

Thus $T' = \int_0^T f(x_\gamma(t)) dt = \int_\tau^{\tau+T} f(x_\gamma(t)) dt$ for any τ since $x_\gamma(t)$ has period T , or $\theta = kT = \int_0^\theta f(x_\gamma(t-h)) dh$ for (3.6).

Finally to get (1.2) we note as in (3.1) that

$$\begin{aligned} \frac{d}{dt} \left\{ -\frac{c}{\theta} \int_0^\theta f(x_\gamma(t-h))(\theta-h) dh \right\} &= -cf(x_\gamma(t)) + c \left\{ \frac{1}{\theta} \int_0^\theta f(x_\gamma(t-h)) dh \right\} \\ &= -c\{f(x_\gamma(t)) - 1\}. \end{aligned}$$

Thus using (3.5), which follows from (1.5) by differentiation,

$$x''_\gamma(t) = \frac{d}{dt} \left\{ -\frac{c}{\theta} \int_0^\theta (\theta-h)f(x_\gamma(t-h)) dh \right\}$$

and thus

$$x'_\gamma(t) = -\frac{c}{\theta} \int_0^\theta (\theta-h) \left\{ f(x_\gamma(t-h)) - 1 \right\} dh + \alpha.$$

Integrating over the period T , $0 = \int_0^T x'_\gamma(t) dt = 0 + \alpha T$ by using (3.6) again, so $\alpha = 0$ and (1.2) follows.

Returning to prove (3.4) for $\gamma > 0$, since $x_\gamma(t)$ satisfies (1.2) with $\theta = P(\gamma)$, we have

$$\begin{aligned} x'_\gamma(t) &= \frac{c}{P(\gamma)} \int_0^{P(\gamma)} (P(\gamma)-h) \{1 - f(x_\gamma(t-h))\} dh \\ &\leq \frac{c}{P(\gamma)} \int_0^{P(\gamma)} (P(\gamma)-h) dh = \frac{1}{2}cP(\gamma) \end{aligned}$$

for $-\infty < t < +\infty$ since $f(x) > 0$ for all real x . From (1.5) we must have $x_\gamma(t) = 0$, $x'_\gamma(t) = \sqrt{2c\gamma}$ for some real t , so $\sqrt{2c\gamma} \leq \frac{1}{2}cP(\gamma)$ or $\left(2\sqrt{\frac{2}{c}}\right)\sqrt{\gamma} \leq P(\gamma)$ for (3.4). Q.E.D.

Theorem 5. If $x(t)$ is a periodic solution of (1.2) under conditions (1.1), then $x(t) = x(t-\theta)$ for all real t and the conclusions of theorem 4 follow.

Proof. This statement is trivial if $x(t) \equiv 0$, so assume $x(t) \not\equiv 0$ and hence has a well-defined period $T > 0$. Thus by lemma 3

$$\begin{aligned} (3.7) \quad \int_{t_1}^{t_1+nT} x(t) \{f(x(t)) - f(x(t-\theta))\} dt &= -\theta \{[R(x)](t_1+nT) - [R(x)](t_1)\} \\ &= 0 \end{aligned}$$

by the periodicity of $x(t)$ for $t_1 \geq \theta$ and integer $n \geq 1$, which we will take so large that $nT \geq 2\theta$. Now define $\varphi(t) = t - \theta$ for $t_1 + \theta < t \leq t_1 + nT$, $\varphi(t) = t - \theta + nT$ for $t_1 < t \leq t_1 + \theta$, $\varphi(t_1) = t_1$. Thus for the measure space $(S, \mathfrak{F}, \mu)$ with $\mathfrak{F}$ the Lebesgue measurable subsets of $S = [t_1, t_1 + nT]$ and μ Lebesgue measure

on $[t_1, t_1 + nT]$ we see that φ is a point automorphism. Also $x(\varphi(t)) = x(t - \theta)$ for $t \neq t_1$ by the periodicity of $x(t)$, so (3.7) and theorem 2 give

$$\begin{aligned} 0 &= \int_{\cdot} x(t) \{f(x(t)) - f(x(\varphi(t)))\} d\mu(t) \geq \frac{1}{2}r \int_{\cdot} [f(x(t)) - f(x(\varphi(t)))]^2 d\mu(t) \\ (3.8) \quad &= \frac{1}{2}r \int_{t_1}^{t_1+nT} [f(x(t)) - f(x(t - \theta))]^2 dt. \end{aligned}$$

Here $m = \min_{0 \leq t \leq r} x(t)$ and $M = \max_{0 \leq t \leq r} x(t)$ exist and are finite by the continuity of $x(t)$, $r = [\max_{m \leq x \leq M} f'(x)]^{-1} > 0$ has $x(t') - x(t) \geq r\{f(x(t')) - f(x(t))\}$ by the mean value theorem whenever $f(x(t')) > f(x(t))$, and $b = rf(M) - m$ has $x(t) - rf(x(t)) + b \geq 0$ everywhere, so that the conditions of theorem 2 are satisfied. But (3.8) implies $f(x(t)) = f(x(t - \theta))$ for almost (μ) all t , or $x(t) = x(t - \theta)$ for all real t since $x(t)$ is continuous and f monotone. Q.E.D.

4. Convergence to Periodic Solutions. We shall get convergence by a compactness argument, for which we need the following unpublished result on boundedness of solutions due to S. Tamor.

Lemma 6. *If $x(t)$ is a solution of (1.2) over $t > 0$ under conditions (1.1), then either*

- a) $x(t) > 0$ for all $t \geq 0$,
- b) $x(t) < 0$ for all $t \geq 0$, or
- c) $x(t) = 0$ for some $t \geq 0$, of which $t_1 = (\inf \text{ such } t)$ is the least.

Moreover, in case a) $x'(t) < 0$ for $t \geq \theta$ and $\lim_{t \rightarrow +\infty} x(t) = 0 = \lim_{t \rightarrow +\infty} x'(t)$; in case b) $x'(t) > 0$ for $t \geq \theta$ and $\lim_{t \rightarrow +\infty} x(t) = 0 = \lim_{t \rightarrow +\infty} x'(t)$; and in case c) for $t \geq t_1$

$$(4.1) \quad x'(t) \leq \frac{1}{2}\theta c \quad \text{and} \quad x(t) \leq \frac{1}{2}\theta^2 c,$$

while for $t \geq t_1 + \theta$

$$(4.2) \quad -\frac{1}{2}\theta c \{f(\frac{1}{2}\theta^2 c) - 1\} \leq x'(t) \quad \text{and} \quad -\frac{1}{2}\theta^2 c \{f(\frac{1}{2}\theta^2 c) - 1\} \leq x(t).$$

Proof. The first statement is obvious. In case a), $x(t) > 0$ on $t \geq 0$ makes $f(x(t - h)) - 1 > 0$ on $t \geq \theta$, $0 \leq h \leq \theta$, so $x'(t) < 0$ on $t \geq \theta$ from (1.2) and $x(t)$ is strictly monotone decreasing. Thus $\lim_{t \rightarrow +\infty} x(t) \geq 0$ exists, and if $\lim_{t \rightarrow +\infty} x(t) > 0$ then $x(t) \geq \rho$ for $t \geq 0$ would follow for some $\rho > 0$, so $x'(t) \leq -\frac{1}{2}c\theta\{f(\rho) - 1\} < 0$ on $t \geq \theta$ would require $x(t) = 0$ for some $t \leq \theta + \frac{2x(\theta)}{c\theta\{f(\rho) - 1\}}$. This contradicts $x(t) > 0$ on $t \geq 0$, so $\lim_{t \rightarrow +\infty} x(t) = 0$ must follow in case a), and thus $\lim_{t \rightarrow +\infty} x'(t) = 0$ also from (1.2). The conclusion in case b) follows similarly.

Now $f(x) > 0$ makes $x'(t) \leq \frac{1}{2}\theta c$ for $t \geq 0$ from (1.2). Also in case c) for $t \geq t_1$ and $x(t) > 0$, there exists t' with $t_1 \leq t' \leq t$, $x(t') = 0$, and $x(\tau) > 0$ for $t' < \tau \leq t$. Here $x'(\tau) \leq \frac{1}{2}\theta c$ makes $x(\tau) \leq \frac{1}{2}\theta^2 c$ for $t' \leq \tau \leq t' + \theta$, and thus for $\tau = t$ if

$t \leq t' + \theta$. If $t > t' + \theta$, then $x(\tau) > 0$ for $t' < \tau \leq t$ makes $x'(\tau) \leq 0$ for $t' + \theta \leq \tau \leq t$ from (1.2), so $x(t) \leq x(t' + \theta) \leq \frac{1}{2}\theta^2 c$ gives (4.1) for case c).

But from (4.1) in case c), $x'(t) \geq -\frac{1}{2}\theta c\{f(\frac{1}{2}\theta^2 c) - 1\}$ over $t \geq t_1 + \theta$ from (1.2). This in turn gives $x(t) \geq -\frac{1}{2}\theta^2 c\{f(\frac{1}{2}\theta^2 c) - 1\}$ over $t \geq t_1 + \theta$ for (4.2) by exactly the same argument as for (4.1). Q.E.D.

To be sure that the statements in this section are not rather vacuous, we remark that every real valued function continuous over a closed t interval of length θ determines a unique solution of (1.2) over all larger t . To see this, first note that (1.2) over $t > t_0$ is equivalent to the integral equation

$$(4.3) \quad x(t) = x(t_0) - \frac{c}{\theta} \int_{t_0}^t \int_0^\theta (\theta - h) \{f(x(s - h)) - 1\} dh ds.$$

Letting $M = \max x(t)$ and $m = \min x(t)$ over $t_0 - \theta \leq t \leq t_0$ and arguing as in lemma 6 makes it clear for every τ having $t_0 < \tau \leq t_0 + \theta$ that if $x(t)$ is a solution of (1.2) over $t_0 < t \leq \tau$ then

$$(4.4) \quad m - \frac{1}{2}\theta^2 c\{f(M + \frac{1}{2}\theta^2 c) - 1\} \leq x(t) \leq M + \frac{1}{2}\theta^2 c$$

over $t_0 - \theta \leq t \leq \tau$.

Now the Picard method of successive approximations applies equally well to the integral equation (4.3), with $f'(x)$ bounded over $m - 1 \leq x \leq M + 1$ by conditions (1.1), as to corresponding integral equations for ordinary differential equations. Thus the given $x(t)$, continuous over $t_0 - \theta \leq t \leq t_0$, has a unique continuous extension over $t_0 - \theta \leq t \leq t_0 + \delta$ satisfying (4.3) over $t_0 \leq t \leq t_0 + \delta$ for some $\delta > 0$ depending upon M and m . Replacing M and m by $M' = M + \frac{1}{2}\theta^2 c$ and $m' = m - \frac{1}{2}\theta^2 c\{f(M + \frac{1}{2}\theta^2 c) - 1\}$, we see from (4.4) that δ' , with $0 < \delta' < \theta$, can be chosen uniformly over the interval $[t_0, t_0 + \theta]$. Thus a finite number of such δ' extensions give a unique solution throughout $[t_0, t_0 + \theta]$, and a finite number of such θ intervals starting at zero reach any real $t > 0$.

Hence we see, leaving the details of proof to the reader, that we can state our result precisely as follows.

Theorem 7. *If $x_0(t)$ is a real valued, continuous function over $-\theta \leq t \leq 0$, there exists under conditions (1.1) a real valued function $x(t)$, continuous over all real $t \geq -\theta$, such that $x(t) = x_0(t)$ for $-\theta \leq t \leq 0$ and $x'(t)$ exists as a continuous function and satisfies (1.2) over $t > 0$; moreover, such an $x(t)$ is unique.*

For not necessarily periodic solutions we now get the following analogue of theorem 5.

Lemma 8. *If $x(t)$ is a solution of (1.2) over $t > 0$ under conditions (1.1), then $\int_0^{+\infty} [f(x(t)) - f(x(t - \theta))]^2 dt < +\infty$.*

Proof. As in theorem 5, consider the interval $[\theta, n\theta]$ for integer $n \geq 3$ and on it define φ by $\varphi(t) = t - \theta$ for $2\theta < t \leq n\theta$, $\varphi(t) = t - \theta + (n - 1)\theta$ for $\theta < t \leq 2\theta$, $\varphi(\theta) = \theta$. Thus φ is a point automorphism for the Lebesgue measure space over $[\theta, n\theta]$. Also from lemma 6, $M = (\sup_{t \geq 0} |x(t)|) < +\infty$, so that $r = [\max_{|x| \leq M} f'(x)]^{-1}$ exists with $0 < r < +\infty$ and $x(t') - x(t) \geq r\{f(x(t')) -$

$f(x(t))\}$ whenever $f(x(t')) > f(x(t))$; also $x(t) - rf(x(t)) + b \geq 0$ for $b = M + rf(M)$. Thus by theorem 2 we have

$$\begin{aligned}
 \frac{1}{2}r \int_{2\theta}^{n\theta} [f(x(t)) - f(x(t - \theta))]^2 dt &\leq \frac{1}{2}r \int_{\theta}^{n\theta} [f(x(t)) - f(x(\varphi(t)))]^2 dt \\
 &\leq \int_{\theta}^{n\theta} x(t) \{f(x(t)) - f(x(\varphi(t)))\} dt \\
 &= \int_{2\theta}^{n\theta} x(t) \{f(x(t)) - f(x(t - \theta))\} dt \\
 (4.5) \qquad &\qquad \qquad + \int_{\theta}^{2\theta} x(t) \{f(x(t)) - f(x(t + (n - 2)\theta))\} dt \\
 &= -\theta \{[R(x)](n\theta) - [R(x)](2\theta)\} + \int_{\theta}^{2\theta} x(t)f(x(t)) dt \\
 &\qquad \qquad \qquad - \int_{\theta}^{2\theta} x(t)f(x(t + (n - 2)\theta)) dt,
 \end{aligned}$$

by using lemma 3 also. But lemma 6 shows the right side of (4.5) to be bounded over integers $n \geq 3$, so thus $\int_{\theta}^{n\theta} [f(x(t)) - f(x(t - \theta))]^2 dt$ is bounded over such n as desired. Q.E.D.

Let Ω_{θ} be the set of all periodic solutions of (1.2), which we have characterized by theorems 4 and 5 for $\theta > 0$ given. Ω_{θ} can never be void, since $x(t) \equiv 0$ is always a member. In terms of Ω_{θ} we can state our general convergence theorem as follows.

Theorem 9. *If $x(t)$ is a solution of (1.2) over $t > 0$ under conditions (1.1), then*

$$(4.6) \qquad 0 = \lim_{n \rightarrow +\infty} \{ \inf_{z \in \Omega_{\theta}} [\max_{-\theta \leq t \leq 2\theta} |x(t + n\theta) - z(t)|] \}.$$

Moreover, there exists some $z(t)$ in Ω_{θ} and some sequence $\{n_p\}$, $n_{p+1} > n_p$, such that

$$0 = \lim_{p \rightarrow +\infty} \{ \max_{-\theta \leq t \leq 2\theta} |x(t + n_p \theta) - z(t)| \}.$$

Proof. Let X_{θ} be the set of all real valued, continuous functions $y(t)$ over $-\theta \leq t \leq 2\theta$ such that $-\frac{1}{2}\theta^2 c \{f(\frac{1}{2}\theta^2 c) - 1\} \leq y(t) \leq \frac{1}{2}\theta^2 c$ and $-\frac{1}{2}\theta c \{f(\frac{1}{2}\theta^2 c) - 1\} \leq \frac{y(t') - y(t)}{t' - t} \leq \frac{1}{2}\theta c$ for $t' \neq t$ both in $[-\theta, 2\theta]$. X_{θ} becomes a complete metric space under the uniform norm metric

$$||z - y|| = (\sup_{-\theta \leq t \leq 2\theta} |z(t) - y(t)|),$$

and it is well known that the uniform bounds on the difference quotients make X_{θ} compact under this metric. Also defining $[L_n(y)](t) = y(t + n\theta)$, we see that lemma 6 requires $[L_n(x)](t)$, contracted to $-\theta \leq t \leq 2\theta$, to be in X_{θ} for n suf-

ficiently large, $x(t)$ being our given solution of (1.2). Similarly every $z(t)$ in Ω_θ , when contracted to $[-\theta, 2\theta]$, is in X_θ .

Now if (4.6) failed, we would have a sequence $\{n_p\}$, $n_{p+1} > n_p$, with

$$(\inf_{z \in \Omega_\theta} \|L_{n_p}(x) - z\|) \geq \rho > 0,$$

and by the compactness of X_θ we can take $\lim_{p \rightarrow \infty} \|L_{n_p}(x) - y\| = 0$ for some $y \in X_\theta$. Thus if we can prove that any given sequence $\{n_p\}$, $n_{p+1} > n_p$, with $y \in X_\theta$ and $\lim_{p \rightarrow \infty} \|L_{n_p}(x) - y\| = 0$ requires $y(t)$ to be the contraction to $[-\theta, 2\theta]$ of a unique $z(t)$ in Ω_θ , we will have proved both (4.6) and the other statement of the theorem, again using the compactness of X_θ for the latter.

Now lemma 8 clearly requires

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} \int_{n\theta}^{(n+2)\theta} [f(x(t)) - f(x(t-\theta))]^2 dt \\ &= \lim_{n \rightarrow +\infty} \int_0^{2\theta} [f([L_n(x)](t)) - f([L_n(x)](t-\theta))]^2 dt, \end{aligned}$$

so thus $\lim_{p \rightarrow +\infty} \|L_{n_p}(x) - y\| = 0$ with $n_{p+1} > n_p$ and $y \in X_\theta$ requires $\int_\theta^{2\theta} [f(y(t)) - f(y(t-\theta))]^2 dt = 0$. $y(t)$ being continuous and $f(x)$ monotone, we have $y(t) = y(t-\theta)$ over $0 \leq t \leq 2\theta$. Thus $z(t) = y(t-m\theta)$ with integer m such that $-\theta \leq t-m\theta \leq 2\theta$ defines a unique periodic $z(t)$ over all real t .

Also by the equivalence of (1.2) with its corresponding integral equation (4.3), it is clear that $\lim_{p \rightarrow \infty} \|L_{n_p}(x) - y\| = 0$ makes $y(t)$ satisfy (4.3) for $t_0 = 0$ and $0 \leq t \leq 2\theta$, and hence $y(t)$ is a solution of (1.2) over $0 < t \leq 2\theta$. Thus also $y(t) = y(t-\theta)$ over $0 \leq t \leq 2\theta$ makes $z(t)$, the periodic extension of $y(t)$, satisfy (1.2) for all real t , and thus $z(t)$ is in Ω_θ . Q.E.D.

Extend $P(\gamma)$ of (3.2) to $\overline{P(\gamma)}$ by defining $\overline{P(\gamma)} = P(\gamma)$ for $\gamma > 0$ and $P(0) = 2\pi/\sqrt{cf'(0)}$, so $\overline{P(\gamma)}$ is continuous over $\gamma \geq 0$. Let Γ be the set in the plane which is the graph of $T = \overline{P(\gamma)}$ over $\gamma \geq 0$, and let $\Gamma_\theta = \bigcup_{k=1}^\infty (\Gamma \cap A_{k,\theta})$ for $A_{k,\theta} = \left\{ (\gamma, T) \mid T = \frac{\theta}{k} \right\}$, horizontal straight lines. We can amplify theorem 9 as follows.

Theorem 10. *If conditions (1.1) are satisfied and if Γ_θ has each of its points isolated in Γ_θ , then Γ_θ is a finite set $\{(\gamma_j, T_j)\}_{0 \leq j \leq m}$ with $T_j = \overline{P(\gamma_j)}$ and $0 < \gamma_1 < \gamma_2 < \dots < \gamma_m$, where $\gamma_0 = 0$ may or may not have $(\gamma_0, \overline{P(\gamma_0)}) \in \Gamma_\theta$, but $(\gamma_j, \overline{P(\gamma_j)}) \in \Gamma_\theta$ for $1 \leq j \leq m$. Moreover, for every γ_j with $0 \leq j \leq m$ there exists $z_j(t)$ in Ω such that $z_0(t) \equiv 0$ and that $z_j(t)$ is a nowhere constant solution of (1.5) for γ_j with period T_j if $1 \leq j \leq m$; every $z(t)$ in Ω_θ is a t axis translate of some $z_j(t)$; and every solution $x(t)$ of (1.2) over $t > 0$ determines a unique one of these $z_j(t)$ such that*

$$(4.7) \quad 0 = \lim_{n \rightarrow +\infty} \left\{ \max_{-\theta \leq t \leq 2\theta} |x(t+n\theta) - z_j(t+\beta_n)| \right\}$$

for some sequence of real β_n with $|\beta_n| \leq \frac{1}{2}\theta$.

Proof. First since $\overline{P(\gamma)}$ is continuous and positive over $\gamma \geq 0$ and $\rightarrow +\infty$ as $\gamma \rightarrow +\infty$ by (3.4), $\overline{P(\gamma)}$ has a minimum value $\lambda > 0$ over all $\gamma \geq 0$. Thus only the finite number p of integers k having $1 \leq k \leq \theta/\lambda$ can have $\Gamma \cap A_{k,\theta} \neq \emptyset$, so $\Gamma_\theta = \bigcup_{k=1}^p (\Gamma \cap A_{k,\theta})$. Γ being closed as the graph of a continuous function over the closed interval $[0, +\infty)$, we see that Γ_θ is closed in the plane. Since it is also bounded from $(2\sqrt{2/c})\sqrt{\gamma} \leq \overline{P(\gamma)} \leq \theta$ by (3.4), Γ_θ is compact. Since its points are all isolated, it is thus finite.

The remaining statements are obvious from theorems 4 and 5 except for (4.7). To get (4.7) note that since $\gamma_j \neq \gamma_{j'}$ for $j \neq j'$ and since $g_2(\sqrt{\gamma})$ is strictly monotone in $\gamma \geq 0$, there exists $\delta > 0$ such that $2\delta \leq |g_2(\sqrt{\gamma_j}) - g_2(\sqrt{\gamma_{j'}})|$ for integers $j \neq j'$ with $0 \leq j, j' \leq m$. Then clearly from (1.5)

$$\left(\max_{t \text{ real}} |z_j(t + \beta) - z_{j'}(t)| \right) \geq |g_2(\sqrt{\gamma_j}) - g_2(\sqrt{\gamma_{j'}})| \geq 2\delta$$

for any real β . Thus using the uniform norm metric introduced for X_θ in the proof of theorem 9 and remembering $z(t + \theta) = z(t)$ for $z(t)$ in Ω_θ , we see that every pair $z(t)$ and $\tilde{z}(t)$ in Ω_θ having $\|z - \tilde{z}\| < 2\delta$ must be translates of the same $z_j(t)$.

Now from (4.6) there exists finite N such that for every integer $n \geq N$ some ${}_nz(t)$ in Ω_θ has $\|L_n(x) - {}_nz\| < \delta$. Actually ${}_nz(t)$ is the translate, ${}_nz(t) = z_{j_n}(t + \beta_n)$, of a unique $z_j(t)$; for if ${}_nz(t)$ and ${}_nz(t)$ both have $\|L_n(x) - {}_nz\| < \delta$ and $\|L_n(x) - {}_n\tilde{z}\| < \delta$, then $\|{}_nz - {}_n\tilde{z}\| \leq \|{}_nz - L_n(x)\| + \|L_n(x) - {}_n\tilde{z}\| < 2\delta$ requires ${}_nz(t)$ and ${}_n\tilde{z}(t)$ to be translates of the same $z_j(t)$. Also $j_{n+1} = j_n$ for $n \geq N$, and hence $j_n = j_N$ there. For remembering that $z(t + \theta) = z(t)$ for $z(t)$ in Ω_θ , we see that

$$\begin{aligned} \|{}_{n+1}z - {}_nz\| &= \max_{0 \leq t \leq \theta} |{}_{n+1}z(t) - {}_nz(t)| \\ &\leq \left(\max_{0 \leq t \leq \theta} |{}_{n+1}z(t) - x(t + (n+1)\theta)| \right) + \left(\max_{0 \leq t \leq \theta} |x(t + (n+1)\theta) - {}_nz(t)| \right) \\ &= \left(\max_{0 \leq t \leq \theta} |{}_{n+1}z(t) - x(t + (n+1)\theta)| \right) + \left(\max_{\theta \leq t' \leq 2\theta} |x(t' + n\theta) - {}_nz(t')| \right) \\ &\leq \|{}_{n+1}z - L_{n+1}(x)\| + \|L_n(x) - {}_nz\| < 2\delta, \end{aligned}$$

requiring ${}_{n+1}z(t)$ and ${}_nz(t)$ to be translates of the same $z_j(t)$.

Thus we have shown for each $n \geq N$ that every $z(t)$ in Ω_θ with $\|L_n(x) - z\| < \delta$ must be a translate of a unique $z_j(t)$, and that this $z_j(t)$ is the same for all $n \geq N$. Thus (4.6) implies (4.7). Q.E.D.

Theorem 11. If $f(x)$ is analytic in the real axis as well as satisfying conditions (1.1), then $\overline{P(\gamma)}$ is analytic in $\sqrt{\gamma}$ (and hence in γ for $\gamma > 0$), Γ_θ contains only isolated points and so is finite, and the other conclusions of theorem 10 follow includ-

ing (4.7). The first few terms of the power series in $\sqrt{\gamma}$ for $\overline{P(\gamma)}$, with $\delta > \gamma \geq 0$, are

$$(4.8) \quad \overline{P(\gamma)} = \frac{2\pi}{\sqrt{cf'(0)}} \left\{ 1 + \left[\frac{5 \left(\frac{f''(0)}{f'(0)} \right)^2 - 3 \frac{f'''(0)}{f'(0)}}{24 f'(0)} \right] \gamma + a_3 \gamma^{\frac{3}{2}} + \dots \right\}.$$

Proof. To see that $\overline{P(\gamma)}$ is analytic in $\sqrt{\gamma}$, recall that for $v \geq 0$ the $g_\alpha(v)$, $\alpha = 1$ or 2, are defined to be the two roots x of

$$v^2 = F(x) - x - 1 = \sum_{p=2}^{\infty} \frac{f^{(p-1)}(0)}{p!} x^p$$

with $g_1(v) < 0 < g_2(v)$ for $v > 0$ and $g_1(0) = 0 = g_2(0)$. Since

$$\frac{d}{dx} \{F(x) - x - 1\} = f(x) - 1 \neq 0$$

for real $x \neq 0$ and since $F(x) = 1 + \int_0^x f(y) dy$ is analytic, the standard implicit function theorem makes $g_\alpha(v)$ analytic in $v > 0$. Near $v = 0$ the defining equations can be taken to be

$$g_\alpha(v) = (-1)^\alpha v \left[\frac{1}{2} f'(0) + \sum_{p=3}^{\infty} \frac{f^{(p-1)}(0)}{p!} (g_\alpha(v))^{p-2} \right]^{-\frac{1}{2}}$$

with $f'(0) > 0$, so directly from the method of successive approximations we see that the $g_\alpha(v)$ are analytic in $v = 0$ with $g_1(v) = g_2(-v)$. Thus

$$h_\alpha(v) = \frac{v}{(-1)^\alpha [f(g_\alpha(v)) - 1]}$$

is analytic in real $v > 0$, and actually in $v = 0$ with

$$h_\alpha(0) = \frac{1}{f'(0) [\frac{1}{2} f'(0)]^{-\frac{1}{2}}} = \frac{1}{\sqrt{2} f'(0)}$$

as was evaluated in getting (3.3). Thus using the Fubini theorem and the Cauchy integral formula on the second integral in (3.2), we see that $\overline{P(\gamma)}$ is analytic in $\sqrt{\gamma}$ at each $\gamma \geq 0$.

Thus $\overline{P(\gamma)} = \theta/k$ has isolated roots $\gamma \geq 0$, Γ_θ contains only isolated points, and the first statement is proved. A simple power series expansion of $h_\alpha(v)$ about $v = 0$ readily gives (4.8). Q.E.D.

5. Computational Remarks. The set Ω_θ of periodic solutions of (1.2), which we have just seen to contain all "limits" of any solution of (1.2), is easily computed in practice. Γ , the graph of $T = \overline{P(\gamma)}$, is found by numerical integration of equation (3.2); and its points of intersection with the horizontal straight lines $T = \theta/k$ form Γ_θ . Each $(\gamma, T) \in \Gamma_\theta$, together with $\gamma = 0$, deter-

mine each of the $z(t)$ in Ω_θ , except for t axis translations, according to theorem 4.

We have seen using (3.4) that $\overline{P(\gamma)}$ has a positive minimum over $\gamma \geq 0$; thus if we adjust the positive parameters c and θ so that

$$(5.1) \quad \theta \sqrt{c} < \left(\min_{\gamma \geq 0} \left\{ 2 \sqrt{2} \int_0^{\frac{1}{2}\pi} \left[\frac{\sqrt{\gamma} \cos s}{1 - f(g_1(\sqrt{\gamma} \cos s))} + \frac{\sqrt{\gamma} \cos s}{f(g_2(\sqrt{\gamma} \cos s)) - 1} \right] ds \right\} \right),$$

the positive right side depending only on $f(x)$ and not c or θ , we shall have Γ_θ to be void and $z(t) \equiv 0$ to be the only function in Ω_θ . Hence $\lim_{t \rightarrow +\infty} x(t) = 0$ for every solution $x(t)$ of (1.2) over $t > 0$ by theorem 9. Clearly condition (5.1) is important for engineering stability applications [3].

We also remark that for γ near zero we have the $z(t)$ in Ω_θ to be given approximately by

$$z(t) = \lambda \cos \left(\frac{2\pi}{T} t + \beta \right),$$

$$\lambda = \frac{1}{2}[g_2(\sqrt{\gamma}) - g_1(\sqrt{\gamma})] = \sqrt{\frac{2}{f'(0)}} \sqrt{\gamma} + O(\gamma).$$

Thus from (4.8) we have the approximate formulae

$$(5.2) \quad T - T_0 = T_0 \left(\frac{5 \left(\frac{f''(0)}{f'(0)} \right)^2 - 3 \frac{f'''(0)}{f'(0)}}{48} \right) \lambda^2,$$

$$z(t) = \lambda \cos \left(\frac{2\pi}{T} t + \beta \right), \quad T_0 = \frac{2\pi}{\sqrt{cf'(0)}}.$$

With $T = \theta/k = (\theta/\theta_0) T_0$ and θ considered to be the varying parameter, these approximate formulae can also be obtained by a perfectly general procedure, applicable to any delay differential equation, which is based on the Schmidt branch equations for integral equations [1].

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