On the Singularities of Zonal Harmonic Expansions

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1. INTRODUCTION

1.1. The purpose of this note is to prove the following

Theorem: *Let*

$$(1) \quad a_0 + a_1 r P_1(\cos \theta) + a_2 r^2 P_2(\cos \theta) + \cdots + a_n r^n P_n(\cos \theta) + \cdots$$

be an expansion in terms of zonal harmonics with real coefficients a_n and

$$(2) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1.$$

We consider the power series with the same coefficients:

$$(3) \quad a_0 + a_1 \zeta + a_2 \zeta^2 + \cdots + a_n \zeta^n + \cdots .$$

Let $u(r, \theta)$ be the zonal harmonic function defined by (1), and $f(\zeta) = f(re^{i\theta})$ the analytic function defined by (3). Both functions are regular for $r = |\zeta| < 1$; moreover, the boundary point $(1, \theta)$ is a regular (singular) point of $u(r, \theta)$ if and only if the boundary point $\zeta = e^{i\theta}$ is a regular (singular) point of $f(\zeta)$.

As usual, the symbol P_n stands for the n^{th} Legendre polynomial, θ being the spherical distance from the pole. That the coefficients be real is not essential.

1.2. In spite of its simplicity, this theorem does not seem to be found in the literature. Its significance is obvious: by its aid, practically all theorems on power series concerning the regular or singular character of the points on the circle of convergence lead to analogous assertions for zonal harmonic expansions. As an interesting example we cite the well known theorem of Fabry on gap series, which yields the following result.

Concerning the series (1) we make the same assumptions as above. Moreover let

us denote by $N = N(n)$ the number of those among the coefficients $a_0, a_1, \dots, a_n$ which are different from zero, and suppose that

$$\lim_{n \rightarrow \infty} \frac{N(n)}{n} = 0.$$

Then all points on the sphere $r = 1$ are singular points of the zonal harmonic function $u(r, \theta)$ defined by (1).

1.3. The theorem in question is closely related to the investigation of Professor S. Bergman presented in the foregoing paper. I am indebted to him for many interesting conversations on similar subjects. The result to be established below has been obtained independently of the work of Professor S. Bergman.

1.4. It is rather natural to restrict this discussion to zonal harmonic expansions. Indeed, for expansions involving *arbitrary* solid harmonics, the domain of convergence is in general not a sphere. Let us consider for instance a two-dimensional harmonic expansion, which can be interpreted of course as a three-dimensional one; in this interpretation its domain of convergence is a cylinder.* By a rotation of the axis this cylinder can be brought into an arbitrary position; by a proper combination (addition) of two such expansions the domain of convergence will be the common part of two different cylinders. By repeating this procedure, solid harmonic expansions with more complicated domains of convergence might be constructed.

2. PREPARATION

We write

$$(1) \quad v(r, \theta) = \sum_{n=0}^{\infty} a_n r^{-n-1} P_n(\cos \theta),$$

$$(2) \quad f_n(r, \theta) = r^n P_n(\cos \theta),$$

and prove the following useful

Lemma. Assuming that the harmonic function $u(r, \theta)$ defined by the expansion 1 (1) is regular in a certain neighborhood of the origin bounded by a closed surface S , we have for the coefficients a_n the following representation:

$$(3) \quad a_n = \frac{1}{4\pi} \iint_{S'} \left(f_n \frac{\partial v}{\partial \nu} - v \frac{\partial f_n}{\partial \nu} \right) d\sigma.$$

Here S' is the surface arising from S by inversion with respect to the unit sphere, ν denotes the direction of the interior normal of S' , and $d\sigma$ the element of area on S' .

The function $v(r, \theta)$ defined by (1) is regular outside of S' . By Green's formula it is sufficient to prove (3) for a sphere S' of radius r where r is sufficiently large.

* Such a series might be convergent on an "unessential" set outside of the cylinder of convergence.

By employing the orthogonality relation of the zonal harmonics, we obtain for the integral on the right of (3):

$$\int_0^\pi \int_{-\pi}^\pi \{r^n P_n(\cos \theta) \cdot (n+1)a_n r^{-n-2} P_n(\cos \theta) - a_n r^{-n-1} P_n(\cos \theta) \cdot (-nr^{n-1})P_n(\cos \theta)\} r^2 \sin \theta d\theta d\varphi = (2n+1)a_n \cdot \frac{4\pi}{2n+1} = 4\pi a_n,$$

and this yields the assertion.

3. PROOF

3.1. Let z be the axis of symmetry, ρ the distance from the z -axis. In the meridian plane through the z -axis we introduce the complex quantity $\zeta = z + i\rho = re^{i\theta}$. In this plane, let M be any point set symmetrical with respect to the z -axis; we denote by $\bar{M}$ the set of those points in space which arise by rotating M about the z -axis.

3.2. We assume that the function $u(r, \theta)$ is regular at the point $r = 1, \theta$. We denote the set of the two points $e^{\pm i\theta}$ in the ζ -plane by P so that in fact $u(r, \theta)$ is regular on the line $\bar{P}$. For the sake of brevity we say that u is regular at $e^{i\theta}$. Let u be regular at all points on $r = 1$ satisfying the condition $\alpha \leq \theta \leq \beta$ where $0 < \alpha < \beta < \pi$ is assumed. Let $\rho < 1$; we denote by $K = K(\rho; \alpha, \beta)$ the boundary curve of the domain arising from the unit circle $r \leq 1$ by removing the common part of the strip $\cos \beta \leq \operatorname{Im} \zeta \leq \cos \alpha$ and of the ring $\rho \leq r \leq 1$. If ρ is sufficiently near to 1, the function v will be regular in the exterior of $\bar{K}$.

An important property of the domain bounded by K is its "convexity in the ρ -direction"; i.e., if any point $z + i\rho$ belongs to this domain, the point $z + ih\rho$ will also belong to the domain provided that $-1 \leq h \leq 1$.

3.3. For the coefficients a_n the representation 2 (1) holds, provided the surface S' be chosen in the exterior of $\bar{K}$ but otherwise arbitrarily close to $\bar{K}$. We may select a curve s in the ζ -plane symmetrical with respect to the z -axis and in the exterior of K and choose $S' = \bar{s}$. If $|\zeta|$ is sufficiently small,

$$(1) \quad f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \frac{1}{4\pi} \iint_{S'} \left(\frac{\partial v}{\partial \nu} \sum_{n=0}^{\infty} f_n \zeta^n - v \frac{\partial}{\partial \nu} \sum_{n=0}^{\infty} f_n \zeta^n \right) d\sigma.$$

It is convenient to use the Laplace representation

$$(2) \quad f_n(r, \theta) = \pi^{-1} \int_0^\pi (r \cos \theta + ir \sin \theta \cos t)^n dt,$$

so that

$$(3) \quad \sum_{n=0}^{\infty} f_n(r, \theta) \zeta^n = \pi^{-1} \int_0^\pi \{1 - \zeta(r \cos \theta + ir \sin \theta \cos t)\}^{-1} dt.$$

3.4. Combining the formulas (1), (3), and the formula arising from (3) by differentiating in the ν -direction, we conclude that $f(\zeta)$ will be regular in any simply connected domain of the ζ -plane containing the origin provided that the quantity

$$1 - \zeta(r \cos \theta + ir \sin \theta \cos t) = 1 - \zeta(z + i\rho \cos t)$$

remains different from zero; here (r, θ) is on S' (or $z + i\rho$ is on s) and $0 \leq t \leq \pi$.

Now the curve s is in the exterior of K but arbitrarily close to K ; in view of the convexity property of K the quantity $z + i\rho \cos t$ will be in a domain whose boundary is arbitrarily close to K . Hence $f(\zeta)$ is regular provided ζ^{-1} is in the exterior of K .

This establishes the regularity of $f(\zeta)$ on the arc $\zeta = e^{i\theta}$, $\alpha < \theta < \beta$.

3.5. Conversely let us assume that $f(\zeta) = f(re^{i\theta})$ is regular on the arc $\zeta = e^{i\theta}$, $\alpha \leq \theta \leq \beta$, $0 < \alpha < \beta < \pi$. Let $P > 1$; we denote by $L = L(P; \alpha, \beta)$ the boundary curve of the domain arising from the unit circle $r \leq 1$ by adding to it the common part of the strip $\cos \beta \leq \operatorname{Im} \zeta \leq \cos \alpha$ and of the ring $1 \leq r \leq P$. If $P > 1$ is sufficiently near to 1, the function $f(\zeta)$ will be regular in the interior of L . The domain bounded by L is again convex in the ρ -direction.

We have

$$(4) \quad a_n = \frac{1}{2\pi i} \int_{\gamma} \zeta^{-n-1} f(\zeta) d\zeta,$$

where the curve γ of the integration must be in the interior of L (including $\zeta = 0$) but can be chosen arbitrarily close to L . If r is small we have

$$(5) \quad u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{n=0}^{\infty} (\zeta^{-1} r)^n P_n(\cos \theta) \cdot \zeta^{-1} d\zeta$$

and

$$(6) \quad \sum_{n=0}^{\infty} (\zeta^{-1} r)^n P_n(\cos \theta) = \pi^{-1} \int_0^{\pi} \{1 - \zeta^{-1}(r \cos \theta + ir \sin \theta \cos t)\}^{-1} dt.$$

From these formulas we conclude the regularity of the harmonic function $u(r, \theta)$ provided that the quantity

$$1 - \zeta^{-1}(r \cos \theta + ir \sin \theta \cos t) = 1 - \zeta^{-1}(z + i\rho \cos t)$$

remains different from zero. Let $re^{i\theta} = z + i\rho$ be in the interior of L ; the same will be true for the segment $z + i\rho \cos t$, $0 \leq t \leq \pi$. We choose the curve γ so that it contains this segment, and on the other hand it is contained in L . Then the condition mentioned will be fulfilled.

This establishes the regularity of $u(r, \theta)$ for $r = 1$, $\alpha < \theta < \beta$.