*Essential Singularities of Solutions of a Class of Linear Partial Differential Equations in Three Variables**

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I. The method of integral operators and its application for the study of pole-like singularities of the first kind and essential singularities. Various problems of importance in physical applications require development of a theory of multi-valued solutions and of singularities of functions satisfying linear partial differential equations. When developing this theory in the case of differential equations in two variables†

$$(1.1) \quad \begin{aligned} \frac{\partial^2 \tilde{\psi}}{\partial y^2} + \frac{\partial^2 \tilde{\psi}}{\partial z^2} + \tilde{F}\tilde{\psi} &= -\frac{\partial^2 \psi}{\partial Z \partial Z^*} + F\psi = 0, \\ Z &= \tfrac{1}{2}(z + iy), \quad Z^* = -\tfrac{1}{2}(z - iy), \\ \tilde{\psi} \equiv \tilde{\psi}(z, y) &= \psi(Z, Z^*) \equiv \psi, \quad \tilde{F} = 4F \end{aligned}$$

where $F = F(Z, Z^*)$ is an entire function, we associate with ψ the analytic function $\psi(Z, 0)$ of one complex variable Z . The integral operator which transforms the function $\psi(Z, 0)$ into that complex solution of (1.1) whose real part is

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† We use here a somewhat unusual complex notation Z and Z^* (instead of, say, $\zeta = y + iz$, $\bar{\zeta} = y - iz$) since the notation (x, Z, Z^*) has been used in the case of three variables in a number of previous papers and we wish to avoid modifications when comparing our results with various formulas used in the previous papers. In the case of two variables we employ usually the variables $Z = \tfrac{1}{2}(z + iy)$ and $\bar{Z} = -Z^* = \tfrac{1}{2}(z - iy)$, so that for z and y real, Z and $\bar{Z}$ are conjugate to each other and

$$\frac{\partial}{\partial Z} = \frac{1}{2} \left(\frac{\partial}{\partial z} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial Z^*} = -\frac{1}{2} \left(\frac{\partial}{\partial z} + i \frac{\partial}{\partial y} \right).$$

$\psi(Z, Z^*)$ preserves many properties of the operator "Re" (Real part of), so that by using this integral operator it is possible to develop the theory of solutions of the equation (1.1). See [2, 3].[‡] The theory of linear partial differential equations (1.1) developed along these lines preserves many characteristic aspects of the theory of harmonic functions of two real variables. In particular, it is interesting that many relations between the properties of the sequence $\{a_{m0}\}$ of coefficients of the series development $\psi(Z, Z^*) = \sum a_{mn} Z^m Z^{*n}$ of a solution ψ on one hand, and the location and character of the singularities of $\psi(Z, Z^*)$ on the other hand, are independent of the coefficient F of the equation (1.1). For instance, if $F \equiv 0$ and the sequence $\{a_{m0}\}$ satisfies the Hadamard conditions, insuring that $\sum_{m=0}^{\infty} a_{m0} Z^m$ is a meromorphic function which possesses poles at the points $\{y_m, z_m\}$, $m = 1, 2, 3, \dots$, the solution $\psi(Z, Z^*)$ will have as its only singularities the so-called pole-like singularities at the points $\{y_m, z_m\}$ (i.e., $\psi(Z, Z^*)$ will be infinite of an integral order at these points, but will have at these points branch points of infinite order). These facts show that one can use the theory of integral operators of the first kind to classify singularities and in particular, accumulation points of pole-like singularities of the first kind. See for details [3] pp. 135, 140 ff, [4] pp. 318, 319.

Remark 1.1. An integral operator $\mathbf{p}_2$,

$$\mathbf{p}_2(g) \equiv \left[g(Z) + \int_{\sigma=0}^Z T(Z, \bar{Z}; \sigma) g(\sigma) d\sigma \right],$$

transforming functions g of one complex variable into solutions of (1.1) and possessing the property that

$$T(Z, 0, \sigma) = T(0, \bar{Z}, \sigma) = 0,$$

is denoted as an integral operator of the first kind. We note further that in the case of the differential equation

$$\frac{1}{4}\Delta_2\tilde{\psi} + \frac{1}{2}A\tilde{\psi}_z + \frac{1}{2}B\tilde{\psi}_y + C\tilde{\psi} \equiv \psi_{\bar{z}z^*} + a\psi_{\bar{z}} + a^*\psi_{\bar{z}^*} + F\psi = 0,$$

$$\tilde{\psi} \equiv \tilde{\psi}(z, y), \quad A \equiv A(z, y), \quad B \equiv B(z, y),$$

where

$$a = \frac{1}{2}A[(\zeta + \zeta^*)/2, (\zeta - \zeta^*)/2i] + iB[(\zeta + \zeta^*)/2, (\zeta - \zeta^*)/2i^*],$$

$$a^* = \frac{1}{2}(A - iB), \quad F = C[(\zeta + \zeta^*)/2, (\zeta - \zeta^*)/2i],$$

the real* solutions (analogous to harmonic functions)

$$\Psi(\zeta, \zeta^*) = \frac{1}{2}[\mathbf{p}_2(g(\zeta)) + \mathbf{p}_2^*(\bar{g}(\zeta^*))]$$

[‡] Numbers in brackets refer to the bibliography at the end.

* I.e., solutions which are real for real values of z and y (or when $\zeta^* = \bar{\zeta}$ is the conjugate of ζ).

are connected with the associate function $g(\zeta)$ by the relation

$$\Psi(\zeta, 0) = \frac{1}{2}[g(\zeta) + R^*(0, \zeta)g^*(0)],$$

where

$$R(\zeta, \zeta^*) = \exp \left[- \int_0^{\zeta^*} ad\zeta^* \right].$$

It is of considerable interest to generalize this technique to the case of differential equations in three variables. In the present paper we consider the differential equation

$$(1.2) \quad \Delta_3 \tilde{\phi} + F(r^2)\tilde{\phi} \equiv \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} + \frac{\partial^2 \tilde{\phi}}{\partial z^2} + F(r^2)\tilde{\phi} = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial Z \partial Z^*} + F\phi = 0,$$

$$\tilde{\phi}(x, y, z) = \phi(X), \quad X = (x, Z, Z^*), \quad Z = \frac{1}{2}(z + iy), \quad Z^* = -\frac{1}{2}(z - iy),$$

where $F(r^2)$ is an entire function of $r^2 = x^2 + y^2 + z^2 = x^2 - 4ZZ^*$, and we show that the methods and results obtained in the theory of harmonic functions of three variables can be utilized in the study of solutions of (1.2). However, the situation is now more complicated than in the case of equation (1.1), since it is necessary first to develop the theory of singularities of harmonic functions of *three* variables.

In the case of two variables the theory of harmonic functions is connected in a particularly simple way with the algebra of functions of one complex variable. As has been shown in [1, 5, 6, 8] in the case of three variables, instead of the operator "Re" we can use an integral operator which transforms functions of two complex variables into harmonic functions of three variables. This operator $B_3(f, \mathfrak{L}, X_0)$ transforms analytic functions $f(u, \zeta)$ of $u = (x + Z\zeta + Z^*\zeta^{-1})$ and ζ into harmonic functions $H(X)$ of three variables,

$$(1.3) \quad H(X) = B_3(f, \mathfrak{L}, X_0), \quad B_3(f, \mathfrak{L}, X_0) = \frac{1}{2\pi i} \int_{\mathfrak{L}} f(u, \zeta) \frac{d\zeta}{\zeta},$$

$$|X - X_0| < \epsilon, \quad X = (x, Z, Z^*), \quad X_0 = (x_0, Z_0, Z_0^*),$$

$\mathfrak{L}$ is a closed curve in the ζ plane, $\epsilon > 0$ and sufficiently small*.

Remark 1.2. $B_3(f, \mathfrak{L}, X_0)$ is closely connected with the Legendre formula for harmonic polynomials and is a generalization of the Whittaker representation for harmonic functions. The operator $B_3(f, \mathfrak{L}, X_0)$ transforms analytic functions of two variables u and ζ into harmonic functions of three variables, *i.e.*, it defines a mapping of the algebra of the functions of two complex variables into

* For simplicity's sake we shall assume in this section that X_0 is the origin, $x = Z = Z^* = 0$.

the linear space of harmonic functions. However, the variables u and ζ play a different role. It is therefore useful to make a further step in representing $f(u, \zeta)$ in terms of a function

$$(1.4) \quad \chi(Z, Z^*) = (ZZ^*)^{\frac{1}{2}} \chi_1(Z, Z^*) + \chi_2(Z, Z^*)$$

of two complex variables Z, Z^* , where χ_1 and χ_2 are analytic functions of Z, Z^* which are regular in the neighborhood of $Z_0 = Z_0^* = 0$. In this way we obtain a new representation for harmonic functions of three variables

$$(1.5) \quad H(X) = C_3(\chi) \equiv \frac{1}{\pi i} \int_{|\zeta|=1} \int_{T=0}^1 u^{\frac{1}{2}} \frac{d[u^{\frac{1}{2}} \chi(u\zeta^{-1} T^2, u\zeta(1-T)^2)]}{du} dT \frac{d\zeta}{\zeta}.$$

See [1] and [7] pp. 467ff.* The formula (1.5) can be used for a representation of solutions of (1.2) in terms of functions (1.4). For every differential equation (1.2) there exists an entire function $M(r^2, \sigma^2)$, $r^2 = x^2 - 4ZZ^*$, $M(0, \sigma^2) = 0$ such that

$$(1.6) \quad \phi(X) = p_3(H(X)) \equiv H(X) + \int_{\sigma=0}^1 M(r^2, \sigma^2) H(X\sigma^2) d\sigma$$

are solutions of the differential equation (1.2), and every solution of (1.2) regular at the origin can be represented in the form (1.6), where H is conveniently chosen harmonic function. See [7] p. 500. In the case of the integral operator of the first kind (*i.e.* in the case of a conveniently chosen function M) we obtain for the inverse of the operator

$$(1.7) \quad \phi(X) = p_3[C_3(\chi(Z, Z^*))], \quad X = (x, Z, Z^*)$$

the formula

$$(1.8) \quad \chi(Z, Z^*) = \phi[2(ZZ^*)^{\frac{1}{2}}, Z, Z^*].$$

Remark 1.3. For many purposes the fact that (in analogy to the situation in two variables) the operation transforming $\phi(x, Z, Z^*)$ into $\chi(Z, Z^*)$, (see (1.8)), is *independent* of the coefficient F of equation (1.2) is of great importance. However, when using the method of operators in the case of equations of three variables we encounter an important complication which we shall discuss now.

An operator transforming an analytic function (of one or two variables) into a solution of a differential equation (of two or three variables) is at first defined in the small, say in the neighborhood of the origin O . When the associate function f and the solution produced $T(f, O)$ are continued in the large, then in the case of harmonic functions of two variables the analytic continuation to a

* We note that in [7] p. 468, in lines 6 and 8 from the bottom, there is a misprint: $[T(1-T)^{-1}]^k$ should be replaced by $[T^{-1}(1-T)]^k$.

point P coincides as a rule with $T(f, P)$. This is *not* (in general) the case for equations in three variables, since the operator transforming analytic functions of two complex variables into solutions of the differential equation (1.2) involves integration. Fortunately in many instances the laws connecting the analytic continuation of $\mathbf{B}_s(f, \mathfrak{L}, X_0)$ to a point X_1 with $\mathbf{B}_s(f, \mathfrak{L}, X_1)$ are comparatively simple. (For details see §2 and §3.)

If one wishes to develop a theory of harmonic functions of three variables based on the mapping of these functions onto the algebra of functions $f(u, \zeta)$ or $\chi(Z, Z^*)$, then, following the patterns in the case of harmonic functions of two variables, it is useful at first to study those harmonic functions whose associates f are rational, *i.e.* functions

$$(1.9) \quad f(u, \zeta) = \frac{p(u, \zeta)}{q(u, \zeta)}$$

where p and q are polynomials in u and ζ . The harmonic functions obtained in this way possess (in general) line singularities along algebraic curves. See [1]. These algebraic curves are given by the equations

$$(1.10) \quad \frac{\partial Q}{\partial \zeta} = 0, \quad Q = 0, \quad Q = q[(x + Z\zeta + Z^*\zeta^{-1}), \zeta].$$

They may (in the real x, y, z space) degenerate to a point. For instance, if $f = (u + a_1 - a_2 i)^{-1}$ where the a_k are real and a_1 is positive, we obtain

$$H(\mathbf{X}) = \mathbf{B}_s(f, [|\zeta| = 1], O) = \frac{1}{\sqrt{(x + a_1 - a_2 i)^2 + y^2 + z^2}}.$$

If $a_2 > 0$ then $H(\mathbf{X})$ will have $[x = -a_1, y^2 + z^2 = a_2^2]$ as a branch line. If $a_2 = 0$ this branch line shrinks to a point.

II. An essential singularity of special type at infinity. The periodic and doubly-periodic functions are among the simplest examples of analytic functions possessing an essential singularity at infinity. These functions possess various useful properties. In particular they are connected in a simple way with other well known functions, and they satisfy an ordinary differential equation. In considering solutions of partial differential equations it is of interest to introduce solutions with periodically located pole-like singularities discussed in Section I.

We shall obtain functions of this type by applying the integral operators to functions $f(u, \zeta)$ with periodically located poles. The singularities of the function $f(u, \zeta)$ produce discontinuities of the resulting solution of the differential equation, so that if $f(u, \zeta)$ has a singularity at a point $u^{(0)}$ and $x^{(0)} = \operatorname{Re} u^{(0)}$, then (in general) $\mathbf{B}_s(f, \mathfrak{L}, X_1)$ differs from $\mathbf{B}_s(f, \mathfrak{L}, X_2)$, $X_n = (x_n, Z_n, Z_n^*)$, $n = 1, 2$, if $x_1 < x^{(0)} < x_2$. This is the reason that, instead of applying integral operators to functions $f(u, \zeta)$ which are periodic in u , it is more convenient to apply them to functions which have periodically located singularities in the half plane $[\operatorname{Re} u < 0]$ but are regular for $[\operatorname{Re} u > 0]$.

Let

$$(2.1) \quad \psi(u) = \frac{\Gamma'(u)}{\Gamma(u)} = -C + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+u} \right),$$

where Γ is the gamma function, and let us define the harmonic functions*

$$(2.2a) \quad \Psi_c^{(m)}(X, a) = \frac{1}{2\pi i} \int_{|\xi|=1} \psi(u - ia) \left(\frac{\xi^m + \xi^{-m}}{2} \right) \frac{d\xi}{\xi},$$

$$x > 0, \quad m = 0, 1, 2, \dots$$

$$(2.2b) \quad \Psi_s^{(m)}(X; a) = \frac{1}{2\pi i} \int_{|\xi|=1} \psi(u - ia) \frac{\xi^m - \xi^{-m}}{2i} \frac{d\xi}{\xi}, \quad x > 0.$$

For $a > 0$ the functions (2.2a), (2.2b) will have pole-like singularities on the circles $[y^2 + z^2 = a^2, x = -n], n = 0, 1, 2, \dots$ and along the negative x -axis $[x \leq 0, y = z = 0]$. Various relations are known for $\psi(u)$. Each of these relations leads to formulas for the functions $\Psi_c^{(m)}$ and $\Psi_s^{(m)}$, $m = 0, 1, 2, \dots$.

Examples.

1. From

$$(2.3) \quad \psi(u) = \log u - \frac{1}{2u} - \frac{1}{2} \sum_{k=3}^{\infty} (k-2)k^{-1} \sum_{r=1}^{\infty} (u+r)^{-k}, \quad \operatorname{Re} u > 0,$$

(see [15] p. 261, example 18) follows for $x > 0$,

$$(2.4) \quad \Psi_c^{(m)}(X; a) = L_m(X; a) - \frac{1}{2} F_{c,m}^{(1)}(X; a) - \frac{1}{2} \sum_{k=3}^{\infty} (k-2)k^{-1} \sum_{r=1}^{\infty} F_{c,m}^{(k)}(X + rI; a),$$

where

$$(2.4a) \quad L_m(X; a) = \frac{1}{4\pi i} \int_{|\xi|=1} \log(u - ia) (\xi^m + \xi^{-m}) \xi^{-1} d\xi,$$

$$(2.4b) \quad F_{c,m}^{(k)}(X + rI; a) = \frac{1}{4\pi i} \int_{|\xi|=1} (u + r - ia)^{-k} (\xi^m + \xi^{-m}) \xi^{-1} d\xi,$$

$$m = 0, 1, 2, \quad I = (1, 0, 0).$$

Remark 2.1. The functions $F_{c,m}^{(k)}(X; a)$ are studied more in detail in [1], [6] p. 446. For $|m| \leq k$ and $a = 0$ they are negative harmonics and their derivatives.

For $|m| \geq 1$ and $a = 0$, we have $F_{c,m}^{(1)}(X; 0) = C \cos m\phi \tan^m(\vartheta/2) R^{-1}$,

* The functions considered in this section are often singular at the origin, and therefore here (in contrast to §1) we choose X_0 as a point different from the origin.

$C = \text{const.}$, where (R, ϑ, ϕ) are polar coordinates. One obtains explicit expressions for $F_{C,m}^{(n)}(\mathbf{X}; 0)$, $n > 1$, by computing the derivatives of $F_{C,m}^{(1)}(\mathbf{X}; 0)$.

2. The formula

$$(2.5) \quad \psi(ku) = \frac{1}{k} \sum_{\nu=0}^{k-1} \psi\left(u + \frac{\nu}{k}\right) + \log k, \quad k = 2, 3, 4, \dots$$

(see [11] p. 3) leads us to the relation

$$(2.6) \quad \Psi_C^{(m)}(k\mathbf{X}; ka) = \frac{1}{k} \sum_{\nu=0}^{k-1} \Psi_C^{(m)}\left(\mathbf{X} + \frac{\nu}{k} \mathbf{I}\right) + \delta_{m1} \log k, \quad \begin{array}{l} \delta_{m1} = 1 \text{ for } m = 1, \\ \delta_{m1} = 0 \text{ for } m > 1. \end{array}$$

In a similar way as above we can form harmonic functions with pole-like singularities located at the points $\{x = -m, y = \pm n\}$, $m = 0, 1, 2, \dots$, $n = 0, 1, 2, \dots$, or at $\{x = -m, y = \pm n, z = \pm q\}$, $m = 0, 1, 2, \dots$, $n = 0, 1, 2, \dots$, $q = 0, 1, 2, \dots$. Using the operator* $\mathbf{p}_s(f)$, we obtain solutions of the differential equation $\Delta_s \phi + F\phi = 0$. The functions

$$(2.7) \quad \mathbf{p}_s[\Psi_C^{(m)}] = \Psi_C^{(m)}(\mathbf{X}; a) + \int_0^1 M(r^2, \sigma^2) \Psi_C^{(m)}(\sigma^2 \mathbf{X}; a) d\sigma,$$

$$M(r^2, \sigma^2) = \sum_{n=1}^{\infty} B^{(n)}(r^2) \sigma^2 (1 - \sigma^2)^{n-1}$$

(see [7], (4.4) and [6] p. 426), are singular along the same curves as $\Psi_C^{(m)}(\mathbf{X}; a)$ and possess an essential singularity at infinity.

Obviously, any relation for analytic functions involving additions, subtractions, differentiations and integrations can be (by applying the integral operator $\mathbf{B}_3$) reformulated as a relation between corresponding harmonic functions in three variables. In order to be able to formulate relations for harmonic functions (and in general for solutions ϕ) corresponding to relations involving multiplication of associate functions (as well as for some other purposes which will not be specified here), it is useful to introduce composition rules which correspond to the multiplication of the associate functions. These composition rules are defined with respect to a fixed point which we choose, as a rule, as the origin. Since we associate with every harmonic function $H(\mathbf{X})$ associates of two different kinds, namely, $f(u, \zeta)$ and $H(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*)$, we define two composition rules [denoted by $*$ and $\odot$] which correspond to the multiplication of the associates $f(u, \zeta)$ and $H(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*)$, respectively.

Suppose that $H_k(\mathbf{X})$, $k = 1, 2$, are two harmonic functions which are regular at the origin. Let us denote by

$$(2.8) \quad \phi^{(k)}(u, \zeta) = \sum_{n=1}^{\infty} \sum_{m=-n}^n a_{nm} u^n \zeta^m$$

* For more details about $\mathbf{p}_s(f)$, see [6] p. 427 and [7] p. 500 ff.

the associates of $H_k(\mathbf{X})$ (with respect to the origin $\mathbf{X} = O$) so that $H_k(\mathbf{X})$ admits in the neighborhood of the origin the representation

$$(2.9) \quad H_k(\mathbf{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} \phi^{(k)}(u, \zeta) \frac{d\zeta}{\zeta}, \quad k = 1, 2.$$

Then we define

$$(2.10a) \quad H_1(\mathbf{X}) * H_2(\mathbf{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} \phi^{(1)}(u, \zeta) \phi^{(2)}(u, \zeta) \frac{d\zeta}{\zeta}, \quad \mathbf{X} \in N(O).$$

Remark 2.2. We note that the associates $\phi^{(k)}(u, \zeta)$ have to be normalized in (2.8) since $\int_{|\zeta|=1} u^n \zeta^{m-1} d\zeta = 0$ for $|m| > n \geq 0$ and since on the other hand $\int_{|\zeta|=1} u^{n_1+n} \zeta^{m_1+m-1} d\zeta$ does not necessarily vanish even if $|m| > n$.

Similarly, if $\chi^{(k)}(Z, Z^*) = H_k(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*)$ are the associates of $H_k(\mathbf{X})$, we define

$$(2.10) \quad H_1(\mathbf{X}) \odot H_2(\mathbf{X}) = \frac{1}{\pi i} \int_{|\zeta|=1} \int_{T=0}^1 u^{\frac{1}{2}} \frac{d[u^{\frac{1}{2}} \chi^{(1)} \chi^{(2)}]}{du} dT \frac{d\zeta}{\zeta},$$

$$\chi^{(k)} = \chi^{(k)}(u\zeta^{-1} T^2, u\zeta(1-T)^2), \quad k = 1, 2.$$

Thus from every equation connecting the associates we obtain a relation between the corresponding harmonic functions, whereby the multiplication has to be replaced by the compositions introduced above. We shall indicate an example of the formulas which one obtains by using the first composition rule (*). We can proceed also in a similar way by using the second composition rule.

For instance, from

$$\log \Gamma(\tfrac{1}{2} + u) + \log \Gamma(\tfrac{1}{2} - u) = \log \pi - \log \cos \pi u,$$

(see [10], [15] p. 239) we have

$$(2.11) \quad \psi'(\tfrac{1}{2} + u) + \psi'(\tfrac{1}{2} - u) - \pi^2 - [\psi(\tfrac{1}{2} + u) - \psi(\tfrac{1}{2} - u)]^2 = 0.$$

Therefore for the functions $\Psi_c^{(m)}$ introduced in (2.2a) and for $[-\frac{1}{2} < x < \frac{1}{2}]$ follows

$$(2.11a) \quad \frac{\partial \Psi_c^{(m)}(\tfrac{1}{2} + x, Z, Z^*; a)}{\partial x} + \frac{\partial \Psi_c^{(m)}(\tfrac{1}{2} - x, Z, Z^*; a)}{\partial x} - \pi^2$$

$$- [\Psi_c^{(m)}(\tfrac{1}{2} + x, Z, Z^*; a) - \Psi_c^{(m)}(\tfrac{1}{2} - x, Z, Z^*; a)]^2$$

$$[\Psi_c^{(m)}(\tfrac{1}{2} + x, Z, Z^*; a) - \Psi_c^{(m)}(\tfrac{1}{2} - x, Z, Z^*; a)] = 0.$$

Remark 2.3. Using the fact that for $m = 1, 2, \dots$

$$i \frac{\partial \Psi_c^{(m)}}{\partial x} = \frac{i}{2} \left(\frac{\partial \Psi_c^{(m-1)}}{\partial Z} + \frac{\partial \Psi_c^{(m-1)}}{\partial Z^*} \right) - \frac{1}{2} \left(\frac{\partial \Psi_s^{(m-1)}}{\partial Z} - \frac{\partial \Psi_s^{(m-1)}}{\partial Z^*} \right)$$

and that similar formulas exist for $\frac{\partial \Psi_s^{(m)}}{\partial x}$, we can replace in the left hand side of (2.11a) the derivatives of $\Psi_c^{(m)}$ by the derivatives of $\Psi_c^{(m-1)}$ and $\Psi_s^{(m-1)}$.

III. A representation for solutions of equation (1.2) in terms of their associates.

As we mentioned in §1 (see also [2], [3] pp. 170 ff., [4]) in the case of functions satisfying the linear partial differential equation (1.1) where F is an entire function, there exist various simple relations between the properties of the coefficients of the power series development of $\psi(Z, 0)$ on one hand and the location and character of the singularities of $\psi(Z, Z^*)$ on the other hand. Many of these relations are completely independent of the coefficient F of the equation (1.1). In deriving these relations the starting points are various classical theorems referring to the singularities of an analytic function of a complex variable given by its series development at the origin. Applying then the integral operator of the first kind we obtain the desired relations.

One of the aims of the present paper is to show that this procedure can be generalized to the differential equations (1.2) in *three* variables, where $F = F(r^2)$ is an entire function of $r^2 = x^2 + y^2 + z^2 = x^2 - 4ZZ^*$. Due to the fact that we operate now with functions of three variables there arise certain additional complications (see §1). As we mentioned in §1, we modify our procedure somewhat; namely, we substitute into a solution $\phi(x, Z, Z^*)$ of (1.2) $x = 2(ZZ^*)^{\frac{1}{2}}$ and obtain the associate function

$$(3.1) \quad \phi(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*) = \chi(Z, Z^*) = (ZZ^*)^{\frac{1}{2}} \chi_1(Z, Z^*) + \chi_2(Z, Z^*),$$

where χ_1 and χ_2 are analytic functions of two complex variables which are regular at the origin. Again one can show that between the coefficients of the power series developments of χ on one hand and the character and the location of singularities of $\phi(X)$ on the other hand there exist relations which are independent of F . In the case of three variables, there arises an additional difficulty mentioned in §1. Because of this difficulty we consider at first examples of applications of our method in a special case where the study of functions of two complex variables can be reduced to problems in the theory of one complex variable.

For our purposes it is useful to modify somewhat a formula derived previously (see e.g. [7]) which expresses the solution ϕ of the differential equation (1.2) in terms of its associate $f(u, \xi)$.

Lemma. *Let $\phi(X)$ be a (real) solution of the differential equation (1.2) which is regular at the origin, and let the series development of $\chi(Z, Z^*) = \phi(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*)$*

in the neighborhood of the origin be

$$(3.2) \quad \phi(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*) = \sum_{m=0}^{\infty} \sum_{k=-m}^m M_{m+k, m} Z^{\frac{1}{2}(m-k)} Z^{*\frac{1}{2}(m+k)},$$

$$M_{m-k, m} = \bar{M}_{m+k, m}.$$

If for $|k| \geq k_0$, k_0 sufficiently large

$$(3.3) \quad |M_{m+k, m}| \leq \left(\frac{ek_1}{m}\right)^{m/k_1} \left(\frac{ek_2}{m+k}\right)^{(m+|k|)/k_2},$$

where k_1 and k_2 are conveniently chosen positive constants and $e = 2.71 \dots$, then the corresponding solution $\phi(X)$ of (1.2) can be represented in the form

$$(3.4) \quad \phi(X) = \mathbf{p}_3[H(X)]$$

(see (1.6)), where

$$(3.4a) \quad H(X) = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \sum_{k=-m}^m \int_{|\zeta|=1} M_{m+k, m} u^m \zeta^k \frac{(m-k)!(m+k)!}{(2m)!} \frac{d\zeta}{\zeta}.$$

Proof. Let us consider at first the case of a harmonic function, i.e., the case of (1.2) when $F = 0$. The series development of a real harmonic function $H(X)$ which is regular at the origin can be written in the form

$$(3.5) \quad H(X) = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \sum_{k=-m}^m \int_{|\zeta|=1} S_{m+k, m} u^m \zeta^k \frac{d\zeta}{\zeta}, \quad S_{m+k, m} = \bar{S}_{m-k, m}.$$

If we substitute $x = 2(ZZ^*)^{\frac{1}{2}}$ we obtain

$$(3.6) \quad \begin{aligned} H(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*) &= \frac{1}{2\pi i} \sum_{m=0}^{\infty} \sum_{k=-m}^m S_{m+k, m} \int_{|\zeta|=1} (Z^{\frac{1}{2}} \zeta^{\frac{1}{2}} + Z^{*\frac{1}{2}} \zeta^{-\frac{1}{2}})^{2m} \zeta^{k-1} d\zeta \\ &= \sum_{m=0}^{\infty} \sum_{k=-m}^m \binom{2m}{m+k} S_{m+k, m} Z^{\frac{1}{2}(m-k)} Z^{*\frac{1}{2}(m+k)}. \end{aligned}$$

If we compare the last line of (3.6) with the right hand side of (3.4a) we obtain

$$(3.7) \quad S_{m+k, m} = M_{m+k, m} / \binom{2m}{m+k},$$

which yields the representation (3.4a) for $H(X)$; i.e., we have proved our assertion in the case $F = 0$. According to (4.12), page 502 of [7] the corresponding solution (i.e. the solution possessing the same associate χ as the harmonic function H) of (3.2) can be obtained by using the formula (1.6).

IV. Some Relations between properties of subsequences of coefficients of the

series development of $\phi(X)$ and the character and location of singularities of $\phi(X)$. In Section 3 solutions of the differential equation (1.2) are represented as integrals over series in u and ζ (see (3.4) and (3.4a)). In this section we shall show that in certain cases these series can be summed, thus yielding new representations for solutions possessing singularities of certain types. This procedure yields results concerning relations between properties of subsequences of coefficients of the series developments of solutions $\phi(X)$ on one hand and the location and character of singularities of $\phi(X)$ on the other hand.

In Section 3 (see (3.4), (3.4a) and (3.7)), we considered subsequences $\{S_{m+k,m}\}$, $m = 0, 1, 2, \dots$, of the coefficients of the development (3.4) of the solution $\phi(X)$ as a series of particular solutions $\mathbf{p}_3[\int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta]$ and $\mathbf{p}_3[\int_{|\zeta|=1} u^m \zeta^{-k-1} d\zeta]$. In analogy to the procedure in the case of one complex variable (see [9]) we associate with the subsequence $\{S_{m+k,m}\}$ the numbers

$$(4.1) \quad l_p^{(k)} = \overline{\lim_{m \rightarrow \infty}} |D_{m,p}^{(k)}|^{1/m}, \quad p = 0, 1, 2, \dots,$$

where

$$(4.2) \quad D_{m,p}^{(k)} = \begin{vmatrix} S_{m+k,m} & S_{m+1+k,m+1} & \cdots & S_{m+p+k,m+p} \\ S_{m+k+1,m+1} & S_{m+2+k,m+2} & \cdots & S_{m+p+1+k,m+1+p} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m+p+k,m+p} & S_{m+p+1+k,m+p+1} & \cdots & S_{m+2p+k,m+2p} \end{vmatrix}.$$

Theorem 4.1. *Let a solution $\phi(X)$ of (1.2) admit in a sufficiently small sphere with center at the origin the development*

$$(4.3) \quad \phi(X) = \sum_{k=0}^{\infty} \{\mathbf{p}_3[Q_k(X)] + \mathbf{p}_3[R_k(X)]\},$$

where

$$(4.3a) \quad \mathbf{p}_3[Q_k(X)] = \sum_{m=0}^{\infty} S_{m+k,m} \mathbf{p}_3 \left(\frac{1}{2\pi i} \int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta \right),$$

$$(4.3b) \quad \mathbf{p}_3[R_k(X)] = \sum_{m=0}^{\infty} S_{m-k,m} \mathbf{p}_3 \left(\frac{1}{2\pi i} \int_{|\zeta|=1} u^m \zeta^{-k-1} d\zeta \right).$$

Let us assume further that for $|n_1 - n_2| \geq \alpha_0$, $\alpha_0 < \infty$, sufficiently large,

$$(4.4) \quad \left| \frac{(n_1 + n_2)!}{n_1! n_2!} S_{n_1, n_2} \right| \leq \left(\frac{ek_1}{n_1} \right)^{n_1/k_1} \left(\frac{ek_2}{n_2} \right)^{n_2/k_2},$$

where $k, \nu = 1, 2$, are conveniently chosen constants and $e = 2.71 \dots$.

Then for $k > k_0$ the $Q_k(X)$ and $R_k(X)$ are entire functions of x, Z, Z^* . If further we associate with $Q_k(X)$, ($k = 0, 1, \dots, k_0$) the sequence of numbers $l_p^{(k)}$, $P =$

0, 1, 2, $\dots$ (see (4.1)), then in four cases (see below) there exist relations between the properties of the sequence $l_P^{(k)}$, $P = 0, 1, 2, \dots$ on one hand and the location and the character of singularities of $Q_k(\mathbf{X})$ on the other hand. (Analogous results hold for the series $R_k(\mathbf{X})$, $k = 0, 1, \dots, k_0$.)

I. From some P on, say from $P = P_k$, the $l_P^{(k)}$ vanish. Then we can determine certain polynomials, $P_k(U)$, (of degree $n \equiv n_k$) and if

$$(4.5) \quad P_k(U) = 0, \quad \frac{dP_k(U)}{dU} \neq 0$$

(i.e. if $P_k(U)$ has no multiple roots*) then $\mathbf{p}_3[Q_k(\mathbf{X})]$ can be represented for $[x^2 + y^2 + z^2 < \infty]$ in the form

$$(4.6) \quad \mathbf{p}_3[Q_k(\mathbf{X})] = \sum_{m=0}^{\infty} a_m^{(k)} \mathbf{p}_3[L_m(\mathbf{X}, \mathfrak{s}_k^1)]$$

where the $L_m(\mathbf{X}, \mathfrak{s}_k^1)$ are algebraic functions of x, Z, Z^* (given by (4.19a) and (4.19b)). The functions $\zeta_q = \zeta_q(x, Z, Z^*)$ appearing in (4.18), (4.19a) and (4.19b) are solutions of the equation $\zeta^n P_k(x + Z\zeta + Z^*\zeta^{-1}) = 0$. The function $\mathbf{p}_3[Q_k(\mathbf{X})]$ has only finitely many pole-like singularities along the curves

$$(4.7) \quad \mathfrak{s}_k^1 = \left[\zeta^n P_k = 0, \frac{\partial(\zeta^n P_k)}{\partial \zeta} = 0 \right], \quad P_k \equiv P_k(x + Z\zeta + Z^*\zeta^{-1}).$$

(See remark 1.3.)

II. For $P \geq P_k$, $(l_{P-1}^{(k)}/l_P^{(k)})$ become constant, say $\rho_k = \rho_k^{(0)}$. Then we have the same situation as in case **I** with the single difference that the representations (4.19a) and (4.19b) for $\mathbf{p}_3[Q_k(\mathbf{X})]$ do not hold in the whole space but hold only in the sphere

$$(4.8) \quad x^2 + y^2 + z^2 < (\rho_k^{(0)})^2.$$

On the boundary of this sphere $\mathbf{p}_3[Q_k(\mathbf{X})]$ has at least one point belonging to a singularity of different type than those considered in the present paper.

III. The sequence $[l_P^{(k)}/l_{P-1}^{(k)}]$ tends to zero for $P \rightarrow \infty$ and (4.5) is satisfied. If

$$(4.9) \quad \sum_{P=P_0}^{\infty} \left(\frac{l_P^{(k)}}{l_{P-1}^{(k)}} \right)^{\gamma}$$

converges for $\gamma = 1$ then $Q_k(\mathbf{X})$ can be represented in the form (4.16), where the $L_m(\mathbf{X}, \mathfrak{s}_k^1)$ have the form (4.29), and the $\zeta_q = \zeta_q(x, Z, Z^*)$ are solutions of the transcendental equation

$$(4.10) \quad P_k(x + \zeta Z + \zeta^{-1} Z^*) = 0$$

* If the left hand side of (4.5) vanishes the corresponding formulas are more complicated (see for details [1], pp. 646–657) and for brevity's sake we do not discuss this case here.

Let us add further that using results of MANDELBROJT [12, 13] we can determine also the arguments of the zeros of the $P_k(U)$ and so (4.5) can be expressed in terms of relations between the coefficients $S_{m+k,m}$ of the sequence $\{S_{m+k,m}\}$.

which for sufficiently small values of $(x^2 + y^2 + z^2)$ lie in the unit circle $[|\zeta| \leq 1]$. The ζ_a are (in general) infinitely many valued functions.

If the smallest integer γ for which (4.9) converges is greater than 1 we have the same situation as above with the single difference that the $L_m(\mathbf{X}, \mathfrak{s}_k^1)$ have the form (4.30).*

IV. The sequence $[l_p^{(k)}/l_{p-1}^{(k)}]$ tends to a positive constant $1/R$. In this case we have a situation similar to that in case **III**, but the representation obtained is valid only in a sphere of finite radius R .

Proof. According to our hypotheses, for $|n_1 - n_2| \geq \alpha_0$ the S_{n_1, n_2} satisfy the inequality (4.4). Therefore if we write

$$(4.11) \quad \phi(\mathbf{X}) = \sum_{k=0}^{k_0} \mathbf{p}_3 [Q_k(\mathbf{X}) + R_k(\mathbf{X})] + \sum_{k_0+1}^{\infty} \mathbf{p}_3 [Q_k(\mathbf{X}) + R_k(\mathbf{X})],$$

the second sum will represent an entire function of x, Z, Z^* . Consequently it suffices to investigate the expressions $\mathbf{p}_3[Q_k(\mathbf{X})]$ and $\mathbf{p}_3[R_k(\mathbf{X})]$, for $k = 0, 1, \dots, k_0$. In a sufficiently small neighborhood of the origin we can write

$$(4.12) \quad Q_k(\mathbf{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} q_k(u) \zeta^{k-1} d\zeta,$$

where

$$(4.13) \quad q_k(u) = \sum_{m=k}^{\infty} S_{m+k, m} u^m.$$

As indicated in the statement of the theorem, we consider four cases.

I. Since from some P onward the $l_p^{(k)}$ vanish, according to [9, p. 131] $q_k(u)$ is an entire function in u divided by a polynomial in u , say $P_k(u)$, i.e.

$$(4.14) \quad q_k(u) = \sum_{m=0}^{\infty} a_m^{(k)} \frac{u^m}{P_k(u)}.$$

Let $n = n_k$ be the degree of $P_k(u)$.

For every value x, Z, Z^* for which $\operatorname{Re}[P_k(x + Z\zeta + Z^*\zeta^{-1})]$ and $\operatorname{Im}[P_k(x + Z\zeta + Z^*\zeta^{-1})]$ do not vanish simultaneously for $|\zeta| = 1$, the series (4.14) converges uniformly and absolutely, since the $a_m^{(k)}$ are coefficients of an entire function. Let

$$P_k(U) = b_0 + b_1 U + \dots + b_n U^n, \quad n = n_k, \quad b_\nu = b_\nu^{(k)},$$

where b_ν are constants, and let $U_\nu = U_\nu^{(k)}$, $\nu = 0, 1, \dots, n_k$, be the zeros of $P_k(U) = 0$. The equation

$$(4.15) \quad x + \frac{1}{2}iy(\zeta + \zeta^{-1}) + \frac{1}{2}z(\zeta - \zeta^{-1}) = U_\nu, \quad \nu = 0, 1, 2, \dots, n_k,$$

* Other possibilities for which the theorem slightly changes its form are omitted here but could be discussed in a similar manner.

represents for every fixed ζ a straight line in the (real) x, y, z -space. If therefore ζ varies along $|\zeta| = 1$ the envelope of these curves will form the circle

$$\mathfrak{C}'_\nu = [x = \operatorname{Re} U_\nu, y^2 + z^2 = (\operatorname{Im} U_\nu)^2]$$

with the center on the x -axis. Since according to (4.5) $P_k(U)$ has no multiple roots, $\zeta = \zeta_\nu(\mathbf{X})$ will be a branch point of the first order and $Q_k(\mathbf{X})$ will have along $\mathfrak{C}'_\nu$ a branch line of the first order. (See [1].)

Remark 4.1. In the case when (4.5) does not hold, i.e. $P_k(U) = (U - U_\nu)^{m_\nu} T_k(U)$, $T_k(U_\nu) \neq 0$ and $m_\nu > 1$, an analogous consideration shows that the circle $\mathfrak{C}'_\nu$ is a branch line of the order $2m_\nu - 1$. If we carry out the integration in (4.12) for a fixed value of $\mathbf{X}$, say $\mathbf{X}_0 = (x_0, y_0, z_0)$, it can happen that $\zeta^n P_k[x + \frac{1}{2}iy(\zeta + \zeta^{-1}) + \frac{1}{2}z(\zeta - \zeta^{-1})]$ vanishes for some value of ζ , $|\zeta| = 1$. If the point $\mathbf{X}_0$ does not lie on (4.7), we can change the curve of integration a little so that

$$(4.15a) \quad \zeta^n P(x + Z\zeta + Z^*\zeta^{-1})$$

does not vanish when ζ varies along the new curve of integration, and therefore the series on the right hand side of (4.14) will converge uniformly when $\mathbf{X}$ varies in the neighborhood of the point $\mathbf{X}_0 = (x_0, y_0, z_0)$.

If on the other hand two (or more) $\zeta_N(\mathbf{X}_0)$, $|\zeta_N(\mathbf{X}_0)| = 1$ for which (4.15a) vanishes are equal, say $\zeta_N(\mathbf{X}_0) = \zeta_{N+1}(\mathbf{X}_0)$, then the discriminant of (4.7) will vanish, and this case as mentioned above was excluded. The corresponding point $\mathbf{X}_0$ belongs to the set of singularities of the function.

Let us now proceed to the evaluation of the right hand side of (4.12) for points $\mathbf{X}$ different from those satisfying (4.7). According to (4.10) and (4.12)

$$(4.16) \quad Q_k(\mathbf{X}) = \frac{1}{2\pi i} \int_{|\zeta|=1} q_k(u) \zeta^{k-1} d\zeta = \sum_{m=0}^{\infty} a_m^{(k)} L_m(\mathbf{X}, \mathfrak{s}_k^1),$$

where

$$(4.17) \quad \begin{aligned} L_m(\mathbf{X}, \mathfrak{s}_k^1) &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(x + Z\zeta + Z^*\zeta^{-1})^m \zeta^{k-1}}{P_k(x + Z\zeta + Z^*\zeta^{-1})} d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(Z^* + x\zeta + Z\zeta^2)^m}{\zeta^{m-n-k+1} P_k^*(Z^* + x\zeta + Z\zeta^2)} d\zeta, \end{aligned}$$

and

$$(4.18) \quad P_k^*(Z^* + x\zeta + Z\zeta^2) = \zeta^n P_k(x + Z\zeta + Z^*\zeta^{-1}) = \zeta_0 \prod_{q=1}^{2n} (\zeta - \zeta_q),$$

$$\zeta_0 = Z^n, \quad \zeta_q = \zeta_q^{(k)}(\mathbf{X}), \quad q = 1, 2, \dots, 2n.$$

By a lemma which will be formulated and proved subsequently (see p. 554), (4.5) implies that the $\zeta_q^{(k)}(\mathbf{X})$, $q = 1, 2, \dots, 2n$ for $(x^2 + y^2 + z^2)$ sufficiently

small are different from each other. According to [1], p. 644 and $\mathcal{J}$, if $\tau \equiv m + 1 - n - k > 0$, then

$$(4.19a) \quad L_m(\mathbf{X}, \mathfrak{s}_k^1) = \sum_q \frac{(Z^* + x\zeta_q + Z\zeta_q^2)^m}{\left\{ \frac{\partial}{\partial \zeta} [\zeta^n P_k^*(Z^* + x\zeta + Z\zeta^2)] \right\}_{\zeta=\zeta_q}} + R_m(\mathbf{X}),$$

$$R_m(\mathbf{X}) = \frac{\begin{vmatrix} P_k^*(Z^*) & 0 & \cdot & Z^* \\ \frac{\partial P_k^*}{\partial \zeta} & P_k^*(Z^*) & \cdot & mxZ^{*m-1} \\ \frac{\partial^2 P_k^*}{\partial \zeta^2} & 1! \binom{2}{1} \frac{\partial P_k^*}{\partial \zeta} & \cdot & mZ^{*2}[ZZ^* + (m-1)x^2] \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^{\tau-1} P_k^*}{\partial \zeta^{\tau-1}} & 1! \binom{\tau-1}{1} \frac{\partial^{\tau-2} P_k^*}{\partial \zeta^{\tau-2}} & \cdot & \frac{\partial^{\tau-1}(Z^* + x\zeta + Z\zeta^2)^m}{\partial \zeta^{\tau-1} \zeta} \end{vmatrix}}{1 \cdot 1! 2! \cdots (\tau-1)! [P_k^*(Z^*)]^\tau},$$

where $P_k^* = P_k^*(Z^* + x\zeta + Z\zeta^2)$ and after the differentiation with respect to ζ is performed ζ_q should be inserted instead of ζ . If $\nu \equiv (n+k) - (m+1) - \tau \geq 0$, then

$$(4.19b) \quad L_m(\mathbf{X}, \mathfrak{s}_k^1) = \sum_q \frac{(Z^* + x\zeta_q + Z\zeta_q^2)^m \zeta_q^\nu}{\frac{\partial}{\partial \zeta_q} [P_k^*(Z^* + x\zeta_q + Z\zeta_q^2)]}.$$

$\sum_q$ means the summation over those $\zeta_q(\mathbf{X})$ which for sufficiently small $|\mathbf{X}|$ satisfy the inequality $|\zeta_q(\mathbf{X})| < 1$. Expressions (4.19a) and (4.19b) are algebraic functions of x, Z, Z^* , and one obtains the analytic continuation of our functions to other space sheets, replacing $\zeta_q(\mathbf{X})$, indicated above, by conveniently chosen other branches of these functions. (Harmonic functions of this kind have been studied in more detail in [1]). They are finitely many valued functions which (in general) are singular along algebraic curves.* In the case of the differential equations (1.2) when $F \equiv 0$, $L_m(\mathbf{X}, \mathfrak{s}_k^1)$ has to be replaced by

$$(4.20) \quad \mathbf{p}_3(L_m(\mathbf{X}, \mathfrak{s}_k^1)) = L_m(\mathbf{X}, \mathfrak{s}_k^1) + \int_0^1 M(r^2, \sigma^2) L_m(\mathbf{X}\sigma^2, \mathfrak{s}_k^1) d\sigma$$

(M has been defined in (1.6)). As has been shown in [7], the only singularities of the functions (4.20) lie on the same algebraic curves $\mathfrak{s}_k^1$ as in the case of the harmonic equation.

Before proceeding further we shall prove the lemma used before. In order to be able to use it in the following we shall assume only that $P_k(U)$ is an analytic function of U which is regular in a (sufficiently large) neighborhood of the origin.

* It can happen that in the (real) x, y, z -space these curves shrink to points.

Lemma. Let $P_k(U)$ be an analytic function of U which is regular in $[|U| < M]$, M sufficiently large. Then

$$(4.21) \quad P_k(U) = 0, \quad \frac{dP_k(U)}{dU} = 0$$

implies

$$(4.22) \quad P_k(x + Z\zeta + Z^*\zeta^{-1}) = 0, \quad \frac{dP_k(x + Z\zeta + Z^*\zeta^{-1})}{d\zeta} = 0,$$

and vice versa.

Proof. It is clear that if two roots of $P_k(U) = 0$ are equal, say if $P_k(U) = (U - a_{k1})(U - a_{k2})R_k(U)$, where $a_{k1} = a_{k2}$ and $R_k(U)$ is a function, regular in $[|U| < M]$, then the corresponding pair of roots $\zeta_{k\nu}^{(1)}$ and $\zeta_{k\nu}^{(2)}$, $\nu = 1, 2$, of the equation $(x - a_{k\nu}) + Z\zeta + Z^*\zeta^{-1} = 0$, $\nu = 1, 2$, are also equal, and therefore (4.22) holds. On the other hand if (4.22) holds, this means that at least one root of the equation

$$(4.23) \quad x - \frac{1}{2}iy(\zeta + \zeta^{-1}) + \frac{1}{2}z(\zeta - \zeta^{-1}) = a_{k1}$$

is (identically) equal to a root of the equation (4.23) with a_{k1} replaced by a_{k2} . Consequently

$$(4.24) \quad [-(x - a_{k1}) + ((x - a_{k1})^2 - 4Z^*Z)^{\frac{1}{2}}]Z^{-1} \\ = [(x - a_{k2}) + ((x - a_{k2})^2 - 4Z^*Z)^{\frac{1}{2}}]Z^{-1},$$

and we have to show that then the conjugate roots are also equal. From (4.24) follows that

$$(4.25) \quad (a_{k1} - a_{k2}) = ((x - a_{k2})^2 - 4ZZ^*)^{\frac{1}{2}} - ((x - a_{k1})^2 - 4ZZ^*)^{\frac{1}{2}}$$

and

$$(4.26) \quad (a_{k1} - a_{k2})^2 - (x - a_{k2})^2 + (x - a_{k1})^2 \\ = 2[((x - a_{k2})^2 - 4ZZ^*)((x - a_{k1})^2 - 4ZZ^*)]^{\frac{1}{2}}.$$

But (4.26) can be satisfied only if

$$(4.27) \quad a_{k1} = a_{k2},$$

which implies (4.21) (unless (4.23) has a double root). If (for some special values of x, y, z) (4.23) has a double root, then the corresponding point lies on a branch line which shrinks to a singular point if $\text{Im } a_{k1} = 0$. See [1], pp. 644 ff.

II. In this case, it follows from [9] (p. 133 see 3⁰) that we have the same situation as in case I, but only for $|X| < \rho_k^{(0)}$, $|X| = x^2 + y^2 + z^2$.

III. As we indicated, we consider first the case when (4.9) converges for $\gamma = 1$.

Then there exist constants, say C_N , $|C_N| \leq |C_{N+1}|$ such that

$$(4.28) \quad q_k(u) = \sum_{m=0}^{\infty} \frac{a_m^{(k)} u^m}{\prod_{\mu=1}^{\infty} (1 - u C_{\mu}^{-1})}$$

exists. If* $|C_{N_0}| < 1$, $|C_{N_0+1}| > 1$ and $\nu = (N_0 + k) - (m + 1) \geq 0$, then

$$(4.29) \quad L_m(\mathbf{X}, \mathfrak{s}_k^1) = \sum_{q=1}^{N_0} \frac{(Z^* + x\zeta_q + Z\zeta_q^2)^m \zeta_q^{\nu}}{\left\{ \zeta_q \frac{\partial}{\partial \zeta_q} \left[\prod_{\mu=1}^{N_0} (\zeta_q - C_{\mu}^{-1}(Z^* + x\zeta_q + Z\zeta_q^2)) \right] \right\} \left\{ \prod_{\mu=N_0+1}^{\infty} (1 - C_{\mu}^{-1}(x + Z\zeta_q + Z^*\zeta_q^{-1})) \right\}},$$

where ζ_q are solutions of $P_k(x + \zeta Z + \zeta^{-1} Z^*) = 0$, $P_k(u) = \prod_{\mu=1}^{\infty} (1 - u C_{\mu}^{-1})$, which for values of x, y, z belonging to a sufficiently small neighborhood of the origin lie in the unit circle. In the case where $(m + 1) - (N_0 + k) \equiv \tau > 0$, instead of (4.19a) we obtain

$$(4.30) \quad L_m(\mathbf{X}, \mathfrak{s}_k^1) = \sum_{q=1}^{N_0} \frac{(Z^* + x\zeta_q + Z\zeta_q^2)^m}{\left\{ \frac{\partial}{\partial \zeta_q} [\zeta_q^{\tau} P_k(Z^* + x\zeta_q + Z\zeta_q^2)] \right\}} H_2(\mathbf{X}, \zeta_q) + R_m^*(\mathbf{X}),$$

where

$$H_2(\mathbf{X}, \zeta_q) = \prod_{\mu=N_0+1}^{\infty} [1 - (x + Z\zeta_q + Z^*\zeta_q^{-1}) C_{\mu}^{-1}]$$

and $R_m^*(\mathbf{X})$ is an expression constructed similarly to $R_m(\mathbf{X})$, see p. 553.

Remark 4.2. The ζ_q are solutions of $P_k(x + Z\zeta + Z^*\zeta^{-1}) = 0$ and are in general infinitely many valued functions of x, y, z . Replacing the initially chosen branches $\zeta_q(x, Z, Z^*)$ by conveniently chosen other branches, we obtain the analytic continuation of the function under consideration to other space sheets of the function.

In the case where γ (the smallest integer for which (4.9) exists) satisfies $1 < \gamma < \infty$, we write instead of (4.28)

$$(4.31) \quad q_k(u) = \sum_{m=0}^{\infty} a_m^{(k)} \frac{u^m}{\sum_{N=1}^{\infty} \left\{ (1 - u C_N^{-1}) \exp \left[- \sum_{H=1}^{\gamma-1} H^{-1} (u C_N^{-1})^H \right] \right\}}.$$

If† $(m + 1) - (N_0 + k) > 0$, we obtain a formula similar to (4.30), but in

* If there are some C_N for which $|C_N| = 1$ we change the integration curve $[|\zeta| = 1]$ a little so that the corresponding C_N lie inside the new curve.

† N_0 has the same meaning as before.

this case

$$H_2(X, \zeta_q) = \left\{ \sum_{N=1}^{N_0} \exp \left[- \sum_{H=1}^{\gamma-1} H^{-1} ((x + Z\zeta_p + Z^* \zeta_q^{-1}) C_N^{-1})^H \right] \right\} \\ \times \left\{ \prod_{N=N_0+1}^{\infty} \left[1 - ((x + Z\zeta_q + Z^* \zeta_q^{-1}) C_N^{-1})^H \right] \right\} \\ \times \exp \left[- \sum_{H=1}^{\gamma-1} H^{-1} ((x + Z\zeta_q + Z^* \zeta_q^{-1}) C_N^{-1})^H \right] \right\}.$$

Finally, if we have to choose for γ not a constant but a set of numbers, say $\gamma = \gamma_P$, $P = 1, 2, \dots$, which increase with P (so that $P_k(u)$ is an entire function of infinite order) we obtain similar formulas for $L_m(X, \mathfrak{s}_k^1)$, where the $H_2(X, \zeta_q)$ has to be modified in an obvious way.

IV. In case IV we proceed exactly in the same way as in case III and obtain the results mentioned in the theorem.

V. The case of circle-singularities whose centers lie on an arbitrary straight line. In §4 we derived necessary and sufficient conditions for the coefficients $S_{m+k, m}$ of the developement

$$\sum_{m=0}^{\infty} S_{m+k, m} \int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta, \quad -m \leq k \leq m,$$

in order that it represent a function element of a harmonic function with singularities along the circles whose centers lie on the x -axis (*i.e.* on $y = z = 0$.)

These results can be generalized to the case where the axis is an arbitrary straight line α^1 passing through the origin. Indeed, if we have the development

$$(5.1) \quad H(X) = \sum_{m=0}^{\infty} \sum_{K=-m}^m S_{m+K, m} \int_{|\zeta|=1} u^m \zeta^{K-1} d\zeta, \quad \bar{S}_{m-K, m} = S_{m+K, m},$$

and if the centers of the circle-singularities of the harmonic function lie on α^1 , then by a convenient rotation, say by the transformation

$$(5.2) \quad \begin{aligned} Z &= P_1 \equiv \beta_{11} Z_1 + \beta_{12} x_1 + \beta_{13} Z_1^* \\ x &= P_0 \equiv \beta_{21} Z_1 + \beta_{22} x_1 + \beta_{23} Z_1^* \\ Z^* &= P_{-1} \equiv \beta_{31} Z_1 + \beta_{32} x_1 + \beta_{33} Z_1^*, \end{aligned}$$

α^1 is transformed into the x_1 -axis.* The function $H_1(X_1)$ which we obtain by substituting the new variables Z_1, x_1, Z_1^* has singularities of the type considered in §4.

* As is well known (see *e.g.* [16] p. 8ff.), the β_{mn} , $m = 1, 2, 3, n = 1, 2, 3$ can be represented by three independent (real) parameters.

The function element of $H_1(X_1)$ can be represented in the form

$$(5.3) \quad H_1(X_1) = \sum_{m=0}^{\infty} \sum_{\kappa=-m}^m T_{m+\kappa, m} \int_{|\zeta|=1} u_1^m \zeta^{\kappa-1} d\zeta,$$

$$(5.3a) \quad u_1 = Z_1 \zeta + x_1 + Z_1^* \zeta^{-1},$$

$$(5.3b) \quad T_{m+\kappa, m} = \sum_{K=-m}^m \alpha_{mK}^{(\kappa)} S_{m+K, m},$$

where the $\alpha_{mK}^{(\kappa)}$ are constants which depend only upon the transformation (5.2).

If a function $H(X) \equiv H_1(X_1)$ has singularities introduced in §4 along the new x_1 -axis, we shall say that $H(X)$ belongs to the class $\mathbf{C}(m, k; \alpha^1)$.

Since $H_1(X_1)$ is of the class $\mathbf{C}(m, k; [y_1 = z_1 = 0])$, the subsequences $\{T_{m+k, m}\}$, $m = 0, 1, 2, \dots$, satisfy the conditions indicated in §4, while $\{T_{m+\kappa, m}\}$, $m = 0, 1, 2, \dots$, $\kappa \neq k$, satisfies the inequality

$$(5.4) \quad |T_{m+\kappa, m}| \leq (ep/m)^{m/p}, \quad \kappa \neq k,$$

where p is a conveniently chosen constant.

The sequence $\{S_{m+K, m}\}$, $-m \leq K \leq m$, and the sequence $\{T_{m+\kappa, m}\}$, $-m \leq \kappa \leq m$, are connected with each other by relation (5.3b). The constants depend only upon the rotation (5.2). We proceed now to the derivation of the relations permitting us to express the $T_{m+\kappa, m}$ in terms of the β_{mn} , $m = 1, 2, 3$, $n = 1, 2, 3$, [see (5.2)] and the coefficients of the series development $S_{m+K, m}$, $-m \leq K \leq m$, of $H(X)$. Under the transformation (5.2) a homogeneous harmonic polynomial goes over into a polynomial of the same degree; therefore

$$(5.5) \quad \sum_{K=-m}^m S_{m+K, m} \int_{|\zeta|=1} u^m \zeta^{K-1} d\zeta = \sum_{\kappa=-m}^m T_{m+\kappa, m} \int_{|\zeta|=1} u_1^m \zeta^{\kappa-1} d\zeta.$$

Here $T_{m+\kappa, m}$ and $S_{m+K, m}$ are connected by the relation (5.3b).

If now one writes

$$(5.6) \quad u = Z\zeta + x + Z^* \zeta^{-1} \equiv P_1 \zeta + P_0 + P_{-1} \zeta^{-1}, \quad P_k \equiv P_k(Z_1, x_1, Z_1^*),$$

where P_1, P_0, P_{-1} are the right hand sides of (5.2), a formal computation yields

$$(5.7) \quad (P_1 \zeta + P_0 + P_{-1} \zeta^{-1})^m = \sum_{\sigma=-m}^m L_{m, \sigma} \zeta^{\sigma}.$$

If m is even

$$(5.7a) \quad L_{m, -(2\sigma+1)} = \sum_{s=0}^{\frac{1}{2}m-\sigma-1} \binom{m}{m-1-2s} \cdot \binom{m-1-2s}{\frac{1}{2}m+\sigma-s} P_0^{1+2s} P_1^{\frac{1}{2}m-\sigma-1-s} P_{-1}^{\frac{1}{2}m+\sigma-s},$$

$$(5.7b) \quad L_{m,-2\sigma} = \sum_{s=0}^{\frac{1}{2}m-\sigma} \binom{m}{m-2s} \binom{m-2s}{\frac{1}{2}m+\sigma-s} P_0^{2s} P_1^{\frac{1}{2}m-\sigma-s} P_{-1}^{\frac{1}{2}m+\sigma-s}.$$

For $L_{m,2\sigma+1}$ and $L_{m,2\sigma}$, $\sigma > 0$, we obtain analogous formulas with P_1 and P_{-1} interchanged. Differentiating $(P_1 \zeta + P_0 + P_{-1} \zeta^{-1})^m$ with respect to P_0 and dividing by m , we obtain coefficients $L_{m-1,\sigma}$ for an odd $(m-1)$.

It follows from (5.5), (5.6) and (5.7) that

$$(5.8) \quad \sum_{K=-m}^m S_{m+K,m} L_{m,-K} = \frac{1}{2\pi i} \sum_{\kappa=-m}^m T_{m+\kappa,m} \int_{|\zeta|=1} (Z_1 \zeta + x_1 + Z_1^* \zeta^{-1})^m \zeta^{\kappa-1} d\zeta.$$

Notation. In accordance with [7] we denote the function $\chi(Z, Z^*) = H(2(ZZ^*)^{\frac{1}{2}}, Z, Z^*)$ as the $\mathbf{C}_3$ -associate of $H(X)$.

Lemma. The $\mathbf{C}_3$ -associate $\chi(Z, Z^*)$ of the spherical harmonic

$$(2\pi i)^{-1} \int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta$$

is given by the relation

$$(5.9) \quad \chi(Z, Z^*) = \mathbf{C}_3 \left(\frac{1}{2\pi i} \int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta \right) = \binom{2m}{m-k} Z^{\frac{1}{2}(m-k)} Z^{*\frac{1}{2}(m+k)}.$$

Proof. Substituting $x = 2(ZZ^*)^{\frac{1}{2}}$ into

$$\frac{1}{2\pi i} \int_{|\zeta|=1} u^m \zeta^{k-1} d\zeta,$$

we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{|\zeta|=1} [(Z\zeta)^{\frac{1}{2}} + (Z^* \zeta^{-1})^{\frac{1}{2}}]^{2m} \zeta^{k-1} d\zeta &= \frac{1}{2\pi i} \int_{|\zeta|=1} \sum_{M=0}^{2m} \binom{2m}{M} Z^{\frac{1}{2}M} Z^{*\frac{1}{2}(2m-M)} \zeta^{M-m+k-1} d\zeta \\ &= \binom{2m}{m-k} Z^{\frac{1}{2}(m-k)} Z^{*\frac{1}{2}(m+k)}. \end{aligned}$$

If we insert on the right and left hand sides of (5.8) $x_1 = 2(Z_1 Z_1^*)^{\frac{1}{2}}$ then we obtain

$$(5.10) \quad \sum_{K=-m}^m S_{m+K,m} \{ \tilde{L}_{m,-K} \}_{Z_1^{\frac{1}{2}(m-\kappa)} Z_1^{*\frac{1}{2}(m+\kappa)}} = \binom{2m}{m-\kappa} T_{m+\kappa,m}$$

where $\tilde{L}_{m,K}$ are the expressions $L_{m,K}(P_1, P_0, P_{-1})$ [see (5.7a), (5.7b) etc.] in which P_n is replaced by

$$\tilde{P}_n \equiv \beta_{m1} Z_1 + 2\beta_{m2} (Z_1 Z_1^*)^{\frac{1}{2}} + \beta_{m3} Z_1^*, \quad m = 2 - n, \quad n = 1, 2, 3,$$

and where $\{ \cdots \}_{Z_1^{\frac{1}{2}(m-\kappa)} Z_1^{*\frac{1}{2}(m+\kappa)}}$ denotes the coefficient of the power

$$Z_1^{\frac{1}{2}(m-\kappa)} Z_1^{*\frac{1}{2}(m+\kappa)}, \quad \kappa = 0, 1, \cdots, 2m,$$

of the development in the bracket $\{\cdots\}$. The coefficients of $S_{m+\kappa, m}$ in (5.10) divided by $\binom{2m}{m-\kappa}$ are the constants $\alpha_{m\kappa}^{(\kappa)}$ [see (5.3b)] which we wanted to determine.

A similar procedure can be applied if instead of an axis passing through the origin, the axis of the singularity circles is an arbitrary straight line. Let us assume that the new x -axis is $\alpha^1 = [y = -\alpha, z = -\beta]$, so that the new coordinate system x_2, Z_2, Z_2^* is connected with the system x, Z, Z^* by the relations $x_2 = x, Z_2 = Z - A, Z_2^* = Z^* - B$, where $A = \frac{1}{2}(i\alpha + \beta)$, $B = \frac{1}{2}(i\alpha - \beta)$. In the case under consideration we have

$$\begin{aligned} (5.11) \quad \frac{1}{2\pi i} \int_{|\zeta|=1} [x + Z\zeta + Z^*\zeta^{-1}]^M \zeta^{k-1} d\zeta \\ = \frac{1}{2\pi i} \int_{|\zeta|=1} [x_2 + (Z_2 + A)\zeta + (Z_2^* + B)\zeta^{-1}]^M \zeta^{k-1} d\zeta. \end{aligned}$$

Substituting $x_2 = 2(Z_2 Z_2^*)^{\frac{1}{2}}$ into the right hand side of (5.11) we obtain

$$\begin{aligned} (5.12) \quad \frac{1}{2\pi i} \int_{|\zeta|=1} [(Z_2 + A)\zeta + 2(Z_2 Z_2^*)^{\frac{1}{2}} + (Z_2^* + B)\zeta^{-1}]^M \zeta^{k-1} d\zeta \\ = \frac{1}{2\pi i} \int_{|\zeta|=1} \sum_{\sigma} \sum_{\rho} \left\{ \begin{matrix} M \\ \sigma, \rho \end{matrix} \right\} (Z_2 + A)^{\sigma} (2(Z_2 Z_2^*)^{\frac{1}{2}})^{\rho} (Z_2^* + B)^{M-\sigma-\rho} \zeta^{2\sigma-M+\rho+k-1} d\zeta, \end{aligned}$$

where $\left\{ \begin{matrix} M \\ \sigma, \rho \end{matrix} \right\}$ are trinomial coefficients, *i.e.*, coefficients of the development

$$(5.13) \quad (a + b + c)^M = \sum_{\sigma} \sum_{\rho} \left\{ \begin{matrix} M \\ \sigma, \rho \end{matrix} \right\} a^{\sigma} b^{\rho} c^{M-\sigma-\rho}.$$

Integrating the right hand side of (5.12) with respect to ζ , we see that all terms for which $\rho \neq M - 2\sigma - k$ vanish, so that the left hand side of (5.12) equals

$$(5.14) \quad \sum_{\sigma} \left\{ \begin{matrix} M \\ \sigma, M - 2\sigma - k \end{matrix} \right\} (Z_2 + A)^{\sigma} (4Z_2 Z_2^*)^{\frac{1}{2}(M-k)-\sigma} (Z_2^* + B)^{\sigma-k}.$$

Developing this expression in a series of

$$Z_2^{\frac{1}{2}(\mu-\kappa)} Z_2^{*\frac{1}{2}(\mu+\kappa)}, \quad |\kappa| \leq \mu, \mu = 0, 1, \cdots, m,$$

and replacing

$$\binom{2\mu}{\mu - \kappa} Z_2^{\frac{1}{2}(\mu - \kappa)} Z_2^{*\frac{1}{2}(\mu + \kappa)} \quad \text{by} \quad \frac{1}{2\pi i} \int_{|\zeta|=1} u_2^\mu \zeta^{\kappa-1} d\zeta, \quad u_2 \equiv x_2 + Z_2 \zeta + Z_2^* \zeta^{-1},$$

we obtain the desired representation of

$$\frac{1}{2\pi i} \int_{|\zeta|=1} u^\mu \zeta^{\kappa-1} d\zeta$$

as a development in

$$\frac{1}{2\pi i} \int_{|\zeta|=1} u_2^\mu \zeta^{\kappa-1} d\zeta.$$

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