*The Multipole Expansion of Radiation Fields**

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Introduction. This paper grew out of an attempt to represent radiation fields as a superposition of fields of multipoles. The existence and uniqueness of a radiation field satisfying the ordinary boundary conditions (prescribed tangential component of the electric field at the boundary) was demonstrated recently by Weyl [6], Müller [3] and Saunders [5] but we avoid using their results partly because our method provides a different approach to the problem and partly because we are interested in relaxing the conditions to be imposed upon the bounding surface as well as those required of the tangential component of the electric field there. This degree of generality is necessary to establish the approximation and expansion theorems that motivated this paper and which are of intrinsic interest. For instance, the expansion theorem provides a means for computing the radiation field by a process involving only integrations over the bounding surface.

To facilitate the reading of the paper the results in each paragraph are presented first. Thus the main line of reasoning can be followed without going into the details of the proofs.

1. The bounding surface. We shall consider a connected domain $\mathfrak{D}$ whose complement $\mathfrak{R}$ is the union of a finite number of bounded regions. We shall assume that $\mathfrak{D}$ and $\mathfrak{R}$ have a common boundary Σ consisting of a finite number of disjoint smooth surfaces. More precisely, we shall assume that at each point of these surfaces there exist two spheres of fixed radius passing through it but otherwise contained in $\mathfrak{R}$ and $\mathfrak{D}$ respectively. If these conditions are satisfied, Σ has

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a continuously turning tangent plane. Moreover the unit vector normal to Σ satisfies a uniform Lipschitz condition; that is,

$$|\mathbf{n}_x - \mathbf{n}_y| \leq A |x - y|,$$

where $\mathbf{n}_x$ and $\mathbf{n}_y$ are the unit vectors normal to Σ at the points x and y and exterior to $\mathfrak{R}$, $|x - y|$ is the distance between these points and A is a fixed constant.

Another consequence of the assumptions is the following. For sufficiently small λ , as x runs over Σ the point $x + \lambda\mathbf{n}_x$ describes a collection of surfaces which we shall denote by Σ_λ . These surfaces have the same properties as those of Σ , and the unit vector normal to one of them at the point $x + \lambda\mathbf{n}_x$ coincides with $\mathbf{n}_x$.

Now we proceed to prove these assertions. Let ϵ be the radius of the spheres referred to above. Let x and y be two points on one of the components of Σ such that $|x - y| < \epsilon$ and z a third point on the same component such that $|x - y| = |x - z| = |y - z|$. Let π_1 denote the plane through the midpoint of the segment xy and perpendicular to it, and π'_1 its reflection through the point y . Then the point $y + \epsilon\mathbf{n}_y$ is between π_1 and π'_1 , for otherwise x would be at a distance less than ϵ from either $y + \epsilon\mathbf{n}_y$ or $y - \epsilon\mathbf{n}_y$, and this contradicts the assumption that the spheres through y with centers at $y + \epsilon\mathbf{n}_y$ and $y - \epsilon\mathbf{n}_y$ respectively have no other point in common with Σ than y itself. Similarly $y + \epsilon\mathbf{n}_y$ is between the plane π_2 through the midpoint of the segment yz and perpendicular to it and its reflection through y . Hence if $\mathbf{n}$ denotes the unit vector normal to the plane xyz and such that $\mathbf{n}_y \cdot \mathbf{n} > 0$, we conclude that $|\epsilon\mathbf{n} - \epsilon\mathbf{n}_y| \leq A |x - y|$, where A is a constant. Similarly we could show that $|\epsilon\mathbf{n} - \epsilon\mathbf{n}_x| \leq A |x - y|$. From this and the preceding inequality it follows that $\mathbf{n}_x$ satisfies a uniform Lipschitz condition.

Now let us consider the surface Σ_λ described by the point $x + \lambda\mathbf{n}_x$ as x runs over Σ . Let us assume that $\epsilon > \lambda > 0$ and denote by $\mathfrak{R}_\lambda$ the set of points at a distance less than λ from $\mathfrak{R}$. First we want to prove that Σ_λ coincides with the boundary of $\mathfrak{R}_\lambda$. Since $x + \lambda\mathbf{n}_x$ is the center of a sphere containing x but without other points in common with $\mathfrak{R}$, the distance between $x + \lambda\mathbf{n}_x$ and $\mathfrak{R}$ is λ , and therefore it belongs to the boundary of $\mathfrak{R}_\lambda$. Conversely if y is a point of the boundary of $\mathfrak{R}_\lambda$ the sphere with center at y and radius λ contains a point x of Σ but has no points of $\mathfrak{R}$ in its interior. This clearly implies that $y = x + \lambda\mathbf{n}_x$. Next we observe that the sphere with center at x and radius λ has its interior contained in $\mathfrak{R}_\lambda$, so that any sphere contained in this one and passing through $x + \lambda\mathbf{n}_x$ is otherwise contained in $\mathfrak{R}_\lambda$. Finally let us consider the sphere through x with center at $x + \epsilon\mathbf{n}_x$. The only point common to $\mathfrak{R}$ and this sphere is x , so that the sphere with center at $x + \epsilon\mathbf{n}_x$ and radius $\epsilon - \lambda$ contains $x + \lambda\mathbf{n}_x$ and is exterior to $\mathfrak{R}_\lambda$. The case when $\lambda < 0$ could be treated in a similar way.

2. The radiation condition. The problem we wish to solve is that of finding a vector field $\mathbf{E}(p)$ satisfying the equation

$$\nabla \times (\nabla \times \mathbf{E}) - c^2 \mathbf{E} = 0, \quad c = \text{real constant}, \quad (2.1)$$

in $\mathfrak{D}$ and whose tangential component on Σ takes prescribed values. The precise sense in which the last sentence is to be understood we shall discuss later. In addition we shall impose on $\mathbf{E}$ what is called the *radiation condition*. Let $\mathfrak{S}$ be a sphere of radius r with center at p , and denote by $\mathbf{n}$ the unit vector normal to the spherical surface. Then we shall require that $|r\mathbf{E}|$ be bounded and that

$$r[ci(\mathbf{n} \times \mathbf{E}) - \nabla \times \mathbf{E}]$$

tend uniformly to zero as $r \rightarrow \infty$. The fulfilment of this condition has several consequences. One is that the field can be calculated from the values of $\mathbf{E}$ and $\nabla \times \mathbf{E}$ on any surface enclosing Σ . Another is that the vanishing of the radiated energy

$$\text{Im} \left[\int_{\mathfrak{S}} \mathbf{E} \times (\nabla \times \bar{\mathbf{E}}) \cdot \mathbf{n} d\sigma \right] = 0,$$

the integral being extended over any surface $\mathfrak{S}$ enclosing Σ , implies the identical vanishing of $\mathbf{E}$. For a proof of these results we refer to [2] and [4].

3. The field of a magnetic current distribution on the bounding surface. In this paragraph we shall discuss in detail the field of magnetic current distributions on Σ . We shall consider general tangential current distributions on Σ as well as those which are absolutely continuous, that is, which are integrals of tangential vector-valued functions with respect to the surface area.

Let us denote by $\gamma(p, q)$ the function $(2\pi)^{-1} |p - q|^{-1} \exp(ci|p - q|)$ and by $\Gamma(p, q)$ its gradient with respect to q . Then the field of the magnetic current distribution $\mathbf{K}$ is given by

$$\mathbf{E}(p) = \int_{\Sigma} \Gamma(p, x) \times d\mathbf{K},$$

where the integral is to be interpreted as follows:

$$\mathbf{a} \cdot \int_{\Sigma} \Gamma(p, x) \times d\mathbf{K} = \int_{\Sigma} [\mathbf{a} \times \Gamma(p, x)] \cdot d\mathbf{K}$$

for every constant vector $\mathbf{a}$. It is readily seen that such a field satisfies the radiation condition.

If $\mathbf{K}$ is absolutely continuous and its derivative with respect to the area of Σ is $\mathbf{k}(x)$, then

$$\mathbf{E}(p) = \int_{\Sigma} \Gamma(p, x) \times \mathbf{k}(x) dx,$$

where dx denotes the element of area of Σ .

If p approaches Σ along the normal to it at the point y , then $\mathbf{E}(p) \times \mathbf{n}_y$ tends to

$$\mathbf{e}_s(y) = \mathbf{k}(y) + \int_{\Sigma} [\Gamma(y; x) \times \mathbf{k}(x)] \times \mathbf{n}_y dx$$

¹ $\bar{\mathbf{E}}$ stands for the complex conjugate of $\mathbf{E}$.

for p approaching y from $\mathfrak{D}$ or to

$$\mathbf{e}_e(y) = -\mathbf{k}(y) + \int_{\Sigma} [\Gamma(y, x) \times \mathbf{k}(x)] \times \mathbf{n}_y dx$$

for p approaching y from $\mathfrak{R}$. The limit is here an ordinary limit if $\mathbf{k}$ is continuous or a limit in the mean of order α if $\mathbf{k}$ is in the class $\mathfrak{L}^\alpha$, $1 \leq \alpha < \infty$. More precisely, if $\lambda > 0$ and $p = y + \lambda \mathbf{n}_y$, then

$$\int_{\Sigma} |\mathbf{E}(p) \times \mathbf{n}_y - \mathbf{e}_e(y)|^\alpha dy \rightarrow 0 \quad (3.1)$$

as $\lambda \rightarrow 0$, and similarly for p approaching y from $\mathfrak{R}$.

If $\mathbf{k}(x)$ satisfies a uniform Lipschitz condition of order β , $0 < \beta < 1$, and again $p = y + \lambda \mathbf{n}_y$, then

$$\begin{aligned} |\mathbf{E}(p) \times \mathbf{n}_y - \mathbf{e}_e(y)| &= O(|\lambda|^\beta) \\ |\nabla \times \mathbf{E}(p)| &= O(|\lambda|^{\beta-1}) \end{aligned}$$

uniformly in y , and

$$|[\nabla \times \mathbf{E}(y + \lambda \mathbf{n}_y) - \nabla \times \mathbf{E}(y - \lambda \mathbf{n}_y)] \times \mathbf{n}_y| \rightarrow 0$$

as $\lambda \rightarrow 0$, also uniformly in y .

We shall also discuss the integral operator

$$I(\mathbf{k}) = \int_{\Sigma} [\Gamma(y, x) \times \mathbf{k}(x)] \times \mathbf{n}_y dy \quad (3.2)$$

and shall show that it is completely continuous¹ in all spaces $\mathfrak{L}^\alpha$, $1 \leq \alpha \leq \infty$. Every solution of the equation $\mathbf{k} + I(\mathbf{k}) = 0$ is continuous and satisfies a uniform Lipschitz condition of any order β less than 1. Moreover if $I_t(\mathbf{h})$ denotes the transposed integral operator defined by

$$\int_{\Sigma} I_t(\mathbf{h}) \cdot \mathbf{k}(x) dx = \int_{\Sigma} \mathbf{h}(y) \cdot I(\mathbf{k}) dy,$$

and $\mathbf{K}$ is a magnetic current distribution such that $\int_{\Sigma} [\mathbf{h} - I_t(\mathbf{h})] \cdot d\mathbf{K} = 0$ for every continuous tangential $\mathbf{h}$, then $\mathbf{K}$ is absolutely continuous and its density $\mathbf{k}(x)$ satisfies also a uniform Lipschitz condition of any order β less than 1.

Finally we have the following results that are basic to our considerations: if the field of the magnetic current distribution $\mathbf{K}$ on Σ vanishes identically in $\mathfrak{R}$, then $\mathbf{K} = 0$. If for a magnetic current distribution $\mathbf{k}$ we have $\mathbf{e}_e(y) = 0$, then the field of $\mathbf{k}$ vanishes identically in $\mathfrak{D}$.

¹ For all results concerning completely continuous operators and Banach spaces we refer to [1].

We start by proving that $\mathbf{E}(y + \lambda \mathbf{n}_y) \times \mathbf{n}_y$ tends to $\mathbf{e}_e(y)$ or $\mathbf{e}_i(y)$ as asserted above. First we observe that if $\mathbf{k}(x)$ is continuous and we replace $\Gamma(p, q)$ by

$$\Gamma' = (2\pi)^{-1}(p - q) |p - q|^{-3}$$

in the integrals above, we obtain a new vector field whose difference with $\mathbf{E}(p)$ is everywhere continuously differentiable. Hence to prove continuity properties of $\mathbf{E}$ or its derivatives it is sufficient to establish them for the new field, which we shall denote by $\mathbf{E}'(p)$. Now to $\mathbf{E}'$ we may apply the standard arguments of potential theory, so that we shall restrict ourselves to a mere sketch of the line of reasoning wherever we feel it is well known. That $\mathbf{E}(p) \times \mathbf{n}_y$ tends to $\mathbf{e}_e(y)$ or $\mathbf{e}_i(y)$ for continuous $\mathbf{k}(x)$ is clear. When $\mathbf{k}(x)$ belongs to $\mathfrak{L}^\alpha$, we observe that

$$|\mathbf{E}(p) \times \mathbf{n}_y| \leq \int_{\Sigma} G(p, x) |\mathbf{k}(x)| dx,$$

where $G(p, x)$ is a positive function such that $G(y + \lambda \mathbf{n}_y, x)$ is integrable with respect to y and with respect to x uniformly in λ . By Hölder's inequality we have

$$|\mathbf{E}(p) \times \mathbf{n}_y|^\alpha \leq \left[\int_{\Sigma} G(p, x) dx \right]^{\alpha/\beta} \left[\int_{\Sigma} G(p, x) |\mathbf{k}(x)|^\alpha dx \right],$$

where $\beta = \alpha/(\alpha - 1)$. Replacing the first factor on the right by a constant, replacing p by $y + \lambda \mathbf{n}_y$, and integrating with respect to y , we obtain

$$\int_{\Sigma} |\mathbf{E}(y + \lambda \mathbf{n}_y)|^\alpha dy \leq A_\alpha \int_{\Sigma} |\mathbf{k}(x)|^\alpha dx,$$

where A_α is a constant depending on α . Then by approximating $\mathbf{k}(x)$ in $\mathfrak{L}^\alpha$ by a continuous $\mathbf{k}_1(x)$ and applying to the difference the preceding inequality we derive (3.1).

Next we consider the field

$$\mathbf{E}''(p) = \int_{\Sigma} \Gamma'(p, x) \times (\mathbf{a} \times \mathbf{n}_x) dx,$$

where $\mathbf{a}$ is a constant vector, and rewrite it as

$$\mathbf{E}''(p) = \mathbf{a} \int_{\Sigma} \Gamma'(p, x) \cdot \mathbf{n}_x dx - \int_{\Sigma} [\Gamma'(p, x) \cdot \mathbf{a}] \mathbf{n}_x dx.$$

The first integral is zero for p in $\mathfrak{D}$ and equal to $-2\mathbf{a}$ if p is in $\mathfrak{R}$. In the second term we substitute $y + \lambda \mathbf{n}_y$ for p , multiply on the right vectorially by $\mathbf{n}_y$, and subtract

$$\int_{\Sigma} [\Gamma'(y, x) \cdot \mathbf{a}] (\mathbf{n}_x \times \mathbf{n}_y) dx;$$

we obtain

$$\int_{\Sigma} \{[\Gamma'(y + \lambda \mathbf{n}_y, x) - \Gamma'(y, x)] \cdot \mathbf{a}\} (\mathbf{n}_x \times \mathbf{n}_y) dx.$$

Since $|\mathbf{n}_x \times \mathbf{n}_y| \leq A|x - y|$, the integrand is of order $|x - y|^{-1}$ for $|x - y| < \lambda$ and of order $\lambda|x - y|^{-2}$ for $|x - y| \geq \lambda$. Hence by integrating it follows that this integral is of order λ^β for any $\beta < 1$. Thus we have shown that $\mathbf{E}''(y + \lambda \mathbf{n}_y) \times \mathbf{n}_y$ minus its limit as $\lambda \rightarrow 0$ is of order λ^β for any $\beta < 1$.

On the other hand

$$\mathbf{E}''(p) = \nabla \times \int_{\Sigma} (2\pi|p - x|)^{-1} (\mathbf{a} \times \mathbf{n}_x) dx$$

and therefore

$$\begin{aligned} \nabla \times \mathbf{E}''(p) &= (\nabla \cdot \nabla) \int_{\Sigma} (2\pi|p - x|)^{-1} (\mathbf{a} \times \mathbf{n}_x) dx \\ &\quad - \nabla \left[\nabla \cdot \int_{\Sigma} (2\pi|p - x|)^{-1} (\mathbf{a} \times \mathbf{n}_x) dx \right]. \end{aligned}$$

Since the integral is a harmonic function of p , the first term on the right vanishes. For the second we have that

$$\begin{aligned} \nabla \cdot \int_{\Sigma} (2\pi|p - x|)^{-1} (\mathbf{a} \times \mathbf{n}_x) dx &= \int_{\Sigma} \Gamma'(p, x) \cdot (\mathbf{a} \times \mathbf{n}_x) dx \\ &= -\mathbf{a} \cdot \int_{\Sigma} \Gamma'(p, x) \times \mathbf{n}_x dx \\ &= \mathbf{a} \cdot \int \nabla_q \times \Gamma'(p, q) dq \\ &= 0, \end{aligned}$$

where the last integral is extended over the volume $\mathfrak{R}$. Hence $\nabla \times \mathbf{E}''(p) = 0$.

Now we shall investigate the behavior of $\mathbf{E}(p)$ in the vicinity of Σ when $\mathbf{k}$ satisfies a Lipschitz condition of order β , $\beta < 1$. Let y be a point on Σ . We want to show that $|\mathbf{E}(y + \lambda \mathbf{n}_y) \times \mathbf{n}_y - \mathbf{e}_s(y)| = O(\lambda^\beta)$. In order to do this we replace $\Gamma(p, x)$ by $\Gamma'(p, x)$ in the integral representing $\mathbf{E}(p) \times \mathbf{n}_y$ and subtract from $\mathbf{k}(x)$ the function $\mathbf{a} \times \mathbf{n}_x$, where $\mathbf{a}$ is such that $\mathbf{k}(y) = \mathbf{a} \times \mathbf{n}_y$. Thus we obtain

$$\int_{\Sigma} [\Gamma'(y + \lambda \mathbf{n}_y, x) \times (\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x)] \times \mathbf{n}_y dx,$$

and the argument reduces to showing that this integral minus its limit as $\lambda \rightarrow 0$ is of order λ^β . Now $\Gamma'(y + \lambda \mathbf{n}_y, x) - \Gamma'(y, x)$ is of order $|y - x|^{-2}$ for $|y - x| <$

λ and of order $\lambda |y - x|^{-3}$ for $|y - x| \geq \lambda$. On the other hand $\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x$ is of order $|y - x|^\beta$. Thus substituting and integrating we obtain the desired result.

Next we consider $\nabla \times \mathbf{E}(p)$. Here again we only have to show that

$$\nabla \times \int_{\Sigma} \mathbf{\Gamma}'(p, x) \times [\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x] dx$$

is of order $\lambda^{\beta-1}$. If we differentiate under the integral sign and replace p by $y + \lambda \mathbf{n}_y$ the integrand will be of order $[\lambda^2 + |y - x|^2]^{-\frac{3}{2}} |y - x|^\beta$, and by substituting and integrating the assertion follows.

Finally let us show that

$$|[\nabla \times \mathbf{E}(y + \lambda \mathbf{n}_y) - \nabla \times \mathbf{E}(y - \lambda \mathbf{n}_y)] \times \mathbf{n}_y| \rightarrow 0.$$

For this it will be enough to prove that

$$\left\{ \nabla \times \int_{\Sigma} \mathbf{\Gamma}'(y + \lambda \mathbf{n}_y, x) \times [\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x] dx \right. \\ \left. - \nabla \times \int_{\Sigma} \mathbf{\Gamma}'(y - \lambda \mathbf{n}_y, x) \times [\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x] dx \right\} \times \mathbf{n}_y$$

tends to zero as $\lambda \rightarrow 0$. By differentiating under the integral sign and subtracting we obtain the integral of a function which tends uniformly to zero in $|y - x| \geq \epsilon$ for every $\epsilon > 0$. For $|y - x| \leq \epsilon$, routine computation shows that on account of the fact that $|\mathbf{k}(x) - \mathbf{a} \times \mathbf{n}_x|$ is of order $|y - x|^\beta$, the function is dominated by a constant multiple of

$$|y - x|^{\beta-2} + \lambda |y - x|^{2+\beta} [\lambda^2 + |y - x|^2]^{\frac{1}{2}}.$$

Substituting, integrating and making λ tend to zero and then $\epsilon \rightarrow 0$, we deduce the assertion.

Now we consider the integral operator (3.2). To show that it represents a completely continuous operator in $\mathfrak{L}^\alpha$, $1 \leq \alpha \leq \infty$, we approximate $\mathbf{\Gamma}(y, x)$ by means of a polynomial $\mathbf{\Gamma}_a(y, x)$ with vector coefficients in such a way that their difference $\mathbf{\Gamma}_d(y, x)$ has the property that

$$|[\mathbf{\Gamma}_d(y, x) \times \mathbf{k}(x)] \times \mathbf{n}_y| \leq G(y, x) |\mathbf{k}(x)|,$$

where $G(y, x)$ is a function satisfying

$$\int_{\Sigma} G(y, x) dx < \epsilon, \quad \int_{\Sigma} G(y, x) dy < \epsilon.$$

Now calling I_a and I_d the operators obtained from I by replacing $\mathbf{\Gamma}$ by $\mathbf{\Gamma}_a$ and $\mathbf{\Gamma}_d$ respectively, we have that I_a has finite dimensional range and I_d has a norm in $\mathfrak{L}^\alpha$ not exceeding ϵ . In fact if $\beta = \alpha/(\alpha - 1)$ and $\mathbf{k}(y)$ is any tangential vector,

then

$$\begin{aligned} \left| \int_{\Sigma} \mathbf{k} \cdot I_d(\mathbf{h}) \, dy \right| &\leq \int_{\Sigma} \int_{\Sigma} |\mathbf{k}(y)| G(y, x) |\mathbf{h}(x)| \, dx \, dy \\ &\leq \left[\int_{\Sigma} |\mathbf{k}(y)|^{\beta} G(y, x) \, dx \, dy \right]^{1/\beta} \left[\int_{\Sigma} G(y, x) |\mathbf{h}(x)|^{\alpha} \, dx \, dy \right]^{1/\alpha} \\ &\leq \epsilon \left[\int_{\Sigma} |\mathbf{k}(y)|^{\beta} \, dy \right]^{1/\beta} \left[\int_{\Sigma} |\mathbf{h}(x)|^{\alpha} \, dx \right]^{1/\alpha}, \end{aligned}$$

which shows that the norm of I_d in $\mathfrak{L}^{\alpha}$ does not exceed ϵ . Hence I is completely continuous.

Next we consider a magnetic current distribution $\mathbf{K}$ such that $0 = \int_{\Sigma} [\mathbf{h} - I_t(\mathbf{h})] \cdot d\mathbf{K}$ for all $\mathbf{h}$. Let I^3 denote the operator I iterated three times. Then the kernel $\mathbf{\Gamma}^{(3)}(y, x)$ of I^3 is a tensor of rank 2 mapping vectors tangent to Σ at x into vectors tangent at y . It is readily seen that $\mathbf{\Gamma}^{(3)}(y, x)$ is continuous in both x and y and that $I^3(d\mathbf{K})$ is well defined as an integral and is equal to a continuous function of y . Now on account of the assumptions we have that

$$\int_{\Sigma} [\mathbf{h} - I_t^3(\mathbf{h})] \cdot d\mathbf{K} = \int_{\Sigma} (1 - I_t)(1 + I_t + I_t^2)(\mathbf{h}) \cdot d\mathbf{K} = 0,$$

and therefore

$$\int_{\Sigma} [\mathbf{h} - I_t^3(\mathbf{h})] \cdot d\mathbf{K} = \int_{\Sigma} \mathbf{h} \cdot d\mathbf{K} - \int_{\Sigma} \mathbf{h} \cdot I^3(d\mathbf{K}) \, dy = 0$$

for all $\mathbf{h}$. Hence $\mathbf{K}$ is absolutely continuous and its derivative is equal to the continuous function $I^3(d\mathbf{K})$.

Now if $\mathbf{k}(x)$ is continuous then $I(\mathbf{k})$ satisfies a Lipschitz condition of order β for any $\beta < 1$. In fact the integrands in

$$\left| \int_{\Sigma} [\mathbf{\Gamma}(y', x) \times \mathbf{k}(x)] \times \mathbf{n}_{y'} \, dx - \int_{\Sigma} [\mathbf{\Gamma}(y, x) \times \mathbf{k}(x)] \times \mathbf{n}_y \, dx \right|$$

are of order $|y - x|^{-1}$ and $|y' - x|^{-1}$ respectively and the integrals over $|y - x| \leq |y - y'|^{\beta}$ are therefore of order $|y - y'|^{\beta}$. Over $|y - x| > |y - y'|^{\beta}$ the difference of the integrands is of order $|y - y'| |y - x|^{-2}$ from which it follows that the difference of the integrals is of order $|y - y'| \log |y - y'|$.

Next let $\mathbf{K}$ be a magnetic current distribution whose field

$$\mathbf{E}(p) = \int_{\Sigma} \mathbf{\Gamma}(p, x) \times d\mathbf{K}$$

vanishes identically in $\mathfrak{R}$. Let $\mathbf{h}(y)$ be an arbitrary continuous tangential vector

field in Σ . Then the integral

$$-\int_{\Sigma} \mathbf{h}(y) \cdot \left[\int_{\Sigma} \Gamma(y - \lambda \mathbf{n}_y, x) \times d\mathbf{K} \right] \times \mathbf{n}_y dy$$

is zero for small positive λ . Interchanging the order of integration and letting λ tend to zero we find that

$$\int_{\Sigma} [\mathbf{h} - I_t(\mathbf{h})] \cdot d\mathbf{K} = 0$$

for all continuous $\mathbf{h}$. Hence $\mathbf{K}$ is absolutely continuous and its derivative satisfies a Lipschitz condition of any order β , $\beta < 1$. Now we compute the energy radiated by $\mathbf{E}(p)$. First we observe that $\nabla \times \mathbf{E}(p) = 0$ in $\mathfrak{R}$ and since

$$|[\nabla \times \mathbf{E}(y + \lambda \mathbf{n}_y) - \nabla \times \mathbf{E}(y - \lambda \mathbf{n}_y)] \times \mathbf{n}_y| \rightarrow 0$$

as $\lambda \rightarrow 0$ it follows that $[\nabla \times \mathbf{E}(p)] \times \mathbf{n} \rightarrow 0$ as p approaches Σ along an exterior normal. Hence the energy radiated by $\mathbf{E}(p)$, namely

$$\text{Im} \left[\int_{\Sigma} \mathbf{E} \times (\nabla \times \bar{\mathbf{E}}) \cdot \mathbf{n}_x dx \right]$$

vanishes and $\mathbf{E}$ is also zero in $\mathfrak{D}$. Since the derivative $\mathbf{k}(x)$ of $\mathbf{K}$ is equal to $\frac{1}{2}[\mathbf{e}_s(x) - \mathbf{e}_i(x)]$, it follows that $\mathbf{k}(x) = 0$, which proves our assertion.

Finally let $\mathbf{k}(x)$ be a magnetic current distribution in $\mathfrak{R}^1$ such that $\mathbf{e}_s(y) = 0$. Then $\mathbf{k}(x)$ is continuous and satisfies a Lipschitz condition of any order β , $\beta < 1$. Hence if $p = y + \lambda \mathbf{n}_y$, $\lambda > 0$,

$$|\mathbf{E}(p) \times \mathbf{n}_y - \mathbf{e}_s(y)| = |\mathbf{E}(p) \times \mathbf{n}_y| = O(\lambda^{\beta})$$

and

$$|\nabla \times \mathbf{E}(p)| = O(\lambda^{\beta-1}).$$

Therefore we have that

$$|\mathbf{E}(p) \times [\nabla \times \mathbf{E}(p)] \cdot \mathbf{n}_y| = O(\lambda^{2\beta-1}).$$

Integrating and letting λ tend to zero shows that the energy radiated by the field of $\mathbf{k}(x)$ is zero. Hence the field vanishes identically in $\mathfrak{D}$.

4. The approximation theorems. Let $\mathbf{k}(x)$ be a continuous tangential vector field on Σ and $\mathbf{E}(p)$ the field of a finite collection of dipoles located in $\mathfrak{R}$. We shall show that *by choosing the dipoles properly we can make the magnitude of the difference between the tangential component of $\mathbf{E}$ at Σ and $\mathbf{k}(x)$ less than ϵ for any preassigned positive ϵ* . In other words, *any continuous tangential $\mathbf{k}(x)$ on Σ can be approximated uniformly by the tangential component of the field of a finite collection of dipoles in $\mathfrak{R}$* .

On the other hand, given a set of points p_i , one in each component of $\mathfrak{R}$, there exist multipoles located at the points p_i such that the tangential component of their field along Σ approximates $\mathbf{k}(x)$ within a vector of magnitude less than ϵ for any given $\epsilon > 0$.

Let $\mathcal{C}$ be the Banach space of all continuous tangential vector distributions $\mathbf{k}$ on Σ with the norm $\|\mathbf{k}\| = \sup_x |\mathbf{k}(x)|$. Consider the closed subspace $\mathfrak{N}$ spanned by all functions of the form

$$\Gamma(p, x) \times \mathbf{a} - [\Gamma(p, x) \times \mathbf{a} \cdot \mathbf{n}_x] \mathbf{n}_x$$

where p is a point in $\mathfrak{R}$, $x \in \Sigma$ and $\mathbf{a}$ is any constant vector. If $\mathfrak{N}$ is a proper subspace of $\mathcal{C}$, then by the Hahn-Banach theorem there exists a non-vanishing continuous linear functional in $\mathcal{C}$ which vanishes identically in $\mathfrak{N}$. By a theorem of F. Riesz this linear functional can be expressed as the integral of $\mathbf{k}(x)$ with respect to a tangential distribution $\mathbf{K}$. Hence

$$\int_{\Sigma} \Gamma(p, x) \times \mathbf{a} \cdot d\mathbf{K} = 0$$

for all $p \in \mathfrak{R}$ and all constant vectors $\mathbf{a}$. This implies that

$$\int_{\Sigma} \Gamma(p, x) \times d\mathbf{K} = 0$$

for all $p \in \mathfrak{R}$ and this in turn implies that $\mathbf{K}$ is zero, which contradicts our previous conclusion. Hence $\mathfrak{N}$ must coincide with $\mathcal{C}$ and our first assertion is proved.

Next let us consider the fields

$$\Gamma(p_i, x) \times \mathbf{b}_j, \quad x \in \Sigma$$

the p_i being the points given in the components of $\mathfrak{R}$ and the $\mathbf{b}_j$, $j = 1, 2, 3$, being three constant linearly independent vectors, and all the derivatives of these fields with respect to the coordinates of the points p_i . The tangential components of these fields along Σ span a subspace $\mathfrak{N}$ in $\mathcal{C}$. If $\mathfrak{N}$ is a proper subspace of $\mathcal{C}$ it follows as above that there exists a tangential distribution such that the field

$$\int_{\Sigma} \Gamma(p, x) \times d\mathbf{K}$$

and all its derivatives vanish at the points p_i . Now this field is an analytic function of the coordinates of p and thus by analytic continuation it follows that

$$\int_{\Sigma} \Gamma(p, x) \times d\mathbf{K} = 0$$

for $p \in \mathfrak{R}$. Now this implies that $\mathbf{K} = 0$, which contradicts our assumption that $\mathbf{K} \neq 0$. Hence $\mathfrak{N}$ coincides with $\mathcal{C}$ and the second assertion is established.

5. The existence theorems. Let $\mathbf{e}(x)$ be a given tangential vector field on Σ such that

$$\|\mathbf{e}\|_\alpha = \left[\int_\Sigma |\mathbf{e}(x)|^\alpha dx \right]^{1/\alpha} < \infty, \quad 1 \leq \alpha < \infty.$$

Then there exists a unique vector field $\mathbf{E}(p)$ satisfying (2.1) in $\mathfrak{D}$ and such that

$$\int_\Sigma |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)|^\alpha dx \rightarrow 0$$

as λ tends to zero. The field $\mathbf{E}(p)$ can be obtained as the sum of the field of a magnetic current distribution $\mathbf{k}(x)$ in $\mathfrak{R}^\alpha$ and the fields of either a finite number of dipoles located in $\mathfrak{R}$ or a finite number of multipoles located at preassigned points p_i , one in each component of $\mathfrak{R}$. These dipoles or multipoles can be chosen (once for all) so as to yield the field $\mathbf{E}(p)$ for any given $\mathbf{e}(x)$ and any α , $1 \leq \alpha < \infty$.

If $\mathbf{e}(x)$ is continuous then the magnetic current $\mathbf{k}(x)$ is continuous and

$$\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)$$

tends to zero as $\lambda \rightarrow 0$ uniformly in x . Finally we shall also prove the following result. If $\mathbf{e}_m$ is a sequence such that $\|\mathbf{e}_m - \mathbf{e}\|_\alpha \rightarrow 0$ as m tends to infinity then the corresponding fields $\mathbf{E}_m(p)$ and their derivatives tend to $\mathbf{E}(p)$ and its corresponding derivatives uniformly outside any Σ_λ .

Let $\mathfrak{X}^1$ be the Banach space of all $\mathbf{e}(x)$ such that $\|\mathbf{e}\|_1 < \infty$ and consider the linear operation $\mathbf{k} + I(\mathbf{k})$ on elements $\mathbf{k}$ of $\mathfrak{X}^1$. Since $I(\mathbf{k})$ is a completely continuous operator in $\mathfrak{X}^1$ the range $\mathfrak{R}$ of $\mathbf{k} + I(\mathbf{k})$ is closed and of finite codimension. By the approximation theorems there exist either dipoles located in $\mathfrak{R}$ or multipoles at preassigned points p_i , one in each component of $\mathfrak{R}$, whose fields $\mathbf{E}_n(p)$ are such that the $\mathbf{h}_n = \mathbf{E}_n(x) \times \mathbf{n}_x$ are linearly independent modulo $\mathfrak{R}$ and together with $\mathfrak{R}$ span all of $\mathfrak{X}^1$. Now take continuous linear functionals l_m in $\mathfrak{X}^1$ vanishing on $\mathfrak{R}$ and such that $l_m(\mathbf{h}_n) = \delta_{nm}$, and set $\mathbf{e}' = P(\mathbf{e}) = \mathbf{e} - \sum l_n(\mathbf{e})\mathbf{h}_n$. It is easy to see that $\mathbf{e}' \in \mathfrak{R}$, and thus there exists a $\mathbf{k}$ such that $\mathbf{k} + I(\mathbf{k}) = \mathbf{e}'$. Then $\mathbf{e} = \mathbf{k} + I(\mathbf{k}) + \sum l_n(\mathbf{e})\mathbf{h}_n$, and this implies that the field

$$\mathbf{E}(p) = \sum l_n(\mathbf{e})\mathbf{E}_n(p) + \int_\Sigma \Gamma(p, x) \times \mathbf{k}(x) dx$$

has the property that

$$\int_\Sigma |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)| dx \rightarrow 0$$

as $\lambda \rightarrow 0$. This establishes the existence in the case $\mathbf{e}(x) \in \mathfrak{X}^1$. Since all spaces $\mathfrak{X}^\alpha$, $\alpha > 1$, and the space $\mathfrak{C}$ of all continuous $\mathbf{e}$ are subspaces of $\mathfrak{X}^1$, in order to complete the proof of existence we only have to show that if $\mathbf{e}(x)$ belongs to $\mathfrak{X}^\alpha$ or $\mathfrak{C}$ the same is true for $\mathbf{k}(x)$ and that $\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x$ tends to $\mathbf{e}(x)$ in the

corresponding stronger metrics. If $\mathbf{e}(x) \in \mathfrak{L}^\alpha$ or $\mathbf{e}(x) \in \mathfrak{C}$ then $\mathbf{k} + I(\mathbf{k}) = \mathbf{e} - \Sigma l_n(\mathbf{e})\mathbf{h}_n = \mathbf{h}$ and $\mathbf{h}$ belongs to $\mathfrak{L}^\alpha$ or $\mathfrak{C}$ respectively. From this it follows that

$$\mathbf{k} = \mathbf{h} - I(\mathbf{h}) + I^2(\mathbf{h}) - I^3(\mathbf{k}).$$

Now the first three terms on the right belong to $\mathfrak{L}^\alpha$ (or to $\mathfrak{C}$), and the last, as we have shown in section 3, is equal to a continuous function. Hence $\mathbf{k}(x)$ belongs to $\mathfrak{L}^\alpha$ or to $\mathfrak{C}$ respectively, and from this it follows that

$$\int_{\Sigma} |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)|^\alpha dx \rightarrow 0$$

if $\mathbf{e}(x) \in \mathfrak{L}^\alpha$ or that

$$|\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)| \rightarrow 0$$

as $\lambda \rightarrow 0$, uniformly in x , if $\mathbf{e}(x) \in \mathfrak{C}$, and this concludes the proof of the existence theorem.

Before passing to the proof of uniqueness we observe that if $\mathbf{e}(x) = \mathbf{E}'(x) \times \mathbf{n}_x$ where $\mathbf{E}'(p)$ is a field having continuous first derivatives in $\mathfrak{D} + \Sigma$ then the field $\mathbf{E}$ such that $\mathbf{E}(x) \times \mathbf{n}_x = \mathbf{e}(x)$ given by the preceding proof of existence coincides with $\mathbf{E}'(p)$. In fact $\mathbf{E}(x) \times \mathbf{n}_x = \mathbf{e}(x) = \mathbf{k} + I(\mathbf{k}) + \Sigma l_n(\mathbf{e})\mathbf{h}_n$ where $\mathbf{k}(x)$ is continuous. Therefore, as we have shown in section 3, $I(\mathbf{k})$ satisfies a Lipschitz condition of any order $\beta < 1$, and thus the same is true for $\mathbf{k}(x)$. Hence

$$|\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)| = O(\lambda^\beta)$$

and

$$|\nabla \times \mathbf{E}(x + \lambda \mathbf{n}_x)| = O(\lambda^{\beta-1}),$$

and since $\mathbf{E}'(p)$ is continuously differentiable in $\mathfrak{D} + \Sigma$

$$|[\mathbf{E}'(x + \lambda \mathbf{n}_x) - \mathbf{E}'(x + \lambda \mathbf{n}_x)] \times \mathbf{n}_x| = O(\lambda^\beta)$$

$$|\nabla \times \mathbf{E}'(x + \lambda \mathbf{n}_x) - \nabla \times \mathbf{E}'(x + \lambda \mathbf{n}_x)| = O(\lambda^{\beta-1}).$$

From this it follows that the energy radiated by the field $\mathbf{E}'(p) - \mathbf{E}(p)$ is zero and thus $\mathbf{E}'(p) = \mathbf{E}(p)$.

Now we pass to the proof of the uniqueness of the radiation field. The difficulty here is the lack of information about the behavior of $\nabla \times \mathbf{E}(p)$ in the vicinity of Σ , so that although we know that $\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x$ tends to zero as $\lambda \rightarrow 0$ we cannot conclude immediately that the energy radiated by $\mathbf{E}(p)$ vanishes. Let $\mathbf{E}$ be such that

$$\int_{\Sigma} |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x| dx \rightarrow 0$$

as $\lambda \rightarrow 0$ and denote by I_λ the integral operator associated with Σ_λ the same way I is associated with Σ . In order to be able to consider functions on Σ and Σ_λ as functions defined on a common domain we identify the points $x + \lambda \mathbf{n}_x$ and x on Σ_λ and Σ respectively. Then it is readily seen that $\|I_\lambda - I\| \rightarrow 0$ as $\lambda \rightarrow 0$ in the norm topology of linear operators in $\mathfrak{L}^1$. If $\mathbf{E}_n(p)$ are the fields of dipoles or multipoles of the preceding argument, denote by $\mathbf{h}_n(\lambda)$ the functions $\mathbf{E}_n(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x$ and again by $\mathfrak{R}$ the range of $\mathbf{k} + I(\mathbf{k})$. Since $\mathbf{h}_n(\lambda) \rightarrow \mathbf{h}_n$ as $\lambda \rightarrow 0$ and the latter are linearly independent modulo $\mathfrak{R}$, the same is true for $\mathbf{h}_n(\lambda)$ if λ is sufficiently small. Now let $l_{n\lambda}(\mathbf{k})$ be continuous linear functionals in $\mathfrak{L}^1$ such that $l_{n\lambda}[\mathbf{h}_m(\lambda)] = \delta_{nm}$ and $l_{n\lambda}(\mathfrak{R}) = 0$. Finally let us denote $\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x$ by $\mathbf{e}(\lambda)$, by $\mathfrak{N}$ the null space of $\mathbf{k} + I(\mathbf{k})$ and by $\mathfrak{M}$ a closed subspace of $\mathfrak{L}^1$ complementary to $\mathfrak{N}$, that is such that $\mathfrak{N} \cap \mathfrak{M} = 0$ and $\mathfrak{N} \oplus \mathfrak{M} = \mathfrak{L}^1$. If we set $P_\lambda(\mathbf{k}) = \mathbf{k} - \Sigma l_{n\lambda}(\mathbf{k}) \mathbf{h}_n(\lambda)$, then the range of P_λ is $\mathfrak{R}$ and P_λ tends to $P(\mathbf{k}) = \mathbf{k} - \Sigma l_n(\mathbf{k}) \mathbf{h}_n$ as λ tends to zero. Therefore the range of $P_\lambda + P_\lambda I_\lambda$ is contained in $\mathfrak{R}$ and $\|(P_\lambda + P_\lambda I_\lambda) - (P + PI)\| \rightarrow 0$ as $\lambda \rightarrow 0$. Now $P + PI$ restricted to $\mathfrak{M}$ is a one-to-one linear transformation from $\mathfrak{M}$ onto $\mathfrak{R}$ and therefore, by a theorem of Banach [1], it has a continuous inverse mapping $\mathfrak{R}$ onto $\mathfrak{M}$. Hence for sufficiently small λ , $P_\lambda + P_\lambda I_\lambda$ has a continuous inverse mapping $\mathfrak{R}$ onto $\mathfrak{M}$ and of norm bounded in λ . From this it follows that since $P_\lambda[\mathbf{e}(\lambda)] \in \mathfrak{R}$ and $\|\mathbf{e}(\lambda)\|_1 \rightarrow 0$ as $\lambda \rightarrow 0$ there exists a $\mathbf{k}(\lambda)$ in $\mathfrak{M}$ such that

$$P_\lambda[\mathbf{k}(\lambda)] + P_\lambda I_\lambda[\mathbf{k}(\lambda)] = P_\lambda[\mathbf{e}(\lambda)] = \mathbf{e}(\lambda) - \Sigma l_{n\lambda}[\mathbf{e}(\lambda)] \mathbf{h}_n(\lambda)$$

and that $\|\mathbf{k}(\lambda)\|_1 \rightarrow 0$ as $\lambda \rightarrow 0$. Now if we set

$$\mathbf{k}(\lambda) + I_\lambda[\mathbf{k}(\lambda)] = \mathbf{g}(\lambda),$$

then

$$P_\lambda[\mathbf{g}(\lambda)] = \mathbf{g}(\lambda) - \Sigma l_{n\lambda}[\mathbf{g}(\lambda)] \mathbf{h}_n(\lambda)$$

and $\|\mathbf{g}(\lambda)\|_1 \rightarrow 0$. Substituting above we get

$$\mathbf{e}(\lambda) = \Sigma l_{n\lambda}[\mathbf{e}(\lambda) - \mathbf{g}(\lambda)] \mathbf{h}_n(\lambda) + \mathbf{k}(\lambda) + I_\lambda[\mathbf{k}(\lambda)],$$

and thus we have represented our field $\mathbf{E}(p)$ as

$$\mathbf{E}(p) = \Sigma l_{n\lambda}[\mathbf{e}(\lambda) - \mathbf{g}(\lambda)] \mathbf{E}_n(p) + \int_{\Sigma_\lambda} \mathbf{\Gamma}(p, x) \times \mathbf{k}(\lambda, x) dx$$

for p outside Σ_λ . Now the norms of $\mathbf{e}(\lambda)$, $\mathbf{k}(\lambda)$ and $\mathbf{g}(\lambda)$ tend to zero as $\lambda \rightarrow 0$, and thus it follows that $\mathbf{E}(p) = 0$ in $\mathfrak{D}$. Thus the vanishing of $\mathbf{E}(p)$ is established under the weakest of our assumptions, namely, that

$$\int_{\Sigma} |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x| dx \rightarrow 0$$

as $\lambda \rightarrow 0$.

Finally let us consider a sequence of tangential fields $\mathbf{e}_m(x)$ converging in the mean of order 1 to $\mathbf{e}(x)$, that is, such that

$$\int_{\Sigma} |\mathbf{e}_m(x) - \mathbf{e}(x)| dx \rightarrow 0$$

as $m \rightarrow \infty$. We wish to show that the corresponding fields $\mathbf{E}_m(p)$ and $\mathbf{E}(p)$ defined by

$$\int_{\Sigma} |\mathbf{E}_m(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}_m(x)| dx \rightarrow 0$$

as $\lambda \rightarrow 0$, and similarly for $\mathbf{E}(p)$, are such that $\mathbf{E}_m(p)$ and their derivatives converge to $\mathbf{E}(p)$ and the corresponding derivatives uniformly outside every Σ_λ . We know that $\mathbf{E}_m(p)$ is the field of a magnetic current distribution $\mathbf{k}_m(x)$ satisfying the equation

$$\mathbf{e}_m = \sum l_n(\mathbf{e}_m) \mathbf{h}_n + \mathbf{k}_m + I(\mathbf{k}_m).$$

Since the operator $\mathbf{k} + I(\mathbf{k})$ maps bi-continuously the subspace $\mathfrak{M}$ introduced above onto the range $\mathfrak{R}$ of $\mathbf{k} + I(\mathbf{k})$, and since $\mathbf{e}_m - \sum l_n(\mathbf{e}_m) \mathbf{h}_n$ belongs to $\mathfrak{R}$, if we impose on $\mathbf{k}_m(x)$ the condition that it belong to $\mathfrak{M}$, $\mathbf{k}_m(x)$ will converge to a $\mathbf{k}(x)$ such that $\mathbf{k} + I(\mathbf{k}) = \mathbf{e} - \sum l_m(\mathbf{e}) \mathbf{h}_m$. Hence

$$\mathbf{E}_m(p) = \int_{\Sigma} \Gamma(p, x) \times \mathbf{k}_m(x) dx$$

$$\mathbf{E}(p) = \int_{\Sigma} \Gamma(p, x) \times \mathbf{k}(x) dx$$

and

$$\int_{\Sigma} |\mathbf{k}_m(x) - \mathbf{k}(x)| dx \rightarrow 0$$

as $m \rightarrow \infty$. By differentiating under the integral sign we derive our assertion.

6. The expansion of a field in fields of multipoles. Let p_i be points, one in each component of $\mathfrak{R}$, and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be constant, linearly independent vectors. Consider the fields

$$\mathbf{E}_{ij}(p) = \Gamma(p_i, p) \times \mathbf{a}_j$$

and all their derivatives with respect to the coordinates of p_i . Then according to the approximation theorem their tangential components on Σ span the space of all $\mathbf{e}$ such that $\|\mathbf{e}\|_2 < \infty$; that is, the smallest closed subspace of $\mathfrak{X}^2$ containing all those tangential components is $\mathfrak{X}^2$ itself. Hence we can normalize these fields and obtain a sequence $\mathbf{E}_n^*(p)$ such that $\mathbf{E}_n^*(x) \times \mathbf{n}_x$, $x \in \Sigma$, is a complete

orthonormal system of tangential vector fields on Σ . Thus given any $\mathbf{e}$ in $\mathfrak{L}^2$ we have that

$$\mathbf{e}(x) = \sum_1^\infty c_n \mathbf{E}_n^*(x) \times \mathbf{n}_x, \quad c_n = \int_\Sigma \mathbf{e}(x) \cdot \bar{\mathbf{E}}_n^*(x) \times \mathbf{n}_y \, dx,$$

where the series converges to $\mathbf{e}(x)$ in the mean of order 2. By our previous results it follows that the series

$$\sum_1^\infty c_n \mathbf{E}_n^*(p)$$

converges to the field $\mathbf{E}(p)$ defined by

$$\int_\Sigma |\mathbf{E}(x + \lambda \mathbf{n}_x) \times \mathbf{n}_x - \mathbf{e}(x)|^2 \, dx \rightarrow 0$$

uniformly at a positive distance from Σ . Furthermore the series can be differentiated term by term any number of times. This solves the boundary value problem for $\mathbf{e}(x) \in \mathfrak{L}^2$.

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