*Parabolic Cylinder Functions of Large Order*¹

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1. Introduction. In this paper we shall investigate the solutions of the differential equation

$$(1.1) \quad \frac{d^2 y}{dx^2} - 4\nu^2(x^2 - 1)y = 0$$

when ν is a large (real or complex) parameter, and x is an unrestricted complex variable. This equation and closely related differential equations occur in problems of wave motion inside or outside a parabolic cylinder, electromagnetic waves in an inhomogeneous atmosphere, diffraction problems, vibration problems, and as the Schrödinger equation of the harmonic oscillator in quantum theory. In several of these problems it is of importance to know the behavior of the solutions for large ν . Moreover, (1.1) appears to be the simplest differential equation of the second order with two transition points and would seem the natural comparison equation for studying the behavior for large ν of solutions of an equation of the form

$$(1.2) \quad \frac{d^2 y}{dx^2} - 4\nu^2 \phi(x)y = 0$$

in a region which contains two simple zeros, or one double zero, of $\phi(x)$ very much in the same way as the Airy equation (4.1) is the comparison equation when the region includes only one transition point.

The differential equation (1.1) with a large parameter ν has been studied by a number of writers among whom it will be sufficient to mention Watson (1918), Kienast (1936), and Schwid (1935). When $\nu = n + \frac{1}{2}$, where n is a non-negative

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integer, certain solutions of (1.1) may be expressed in terms of Hermite polynomials of degree n . These have been studied (for large n) by Adamoff (1906), Plancherel & Rotach (1929), Van Veen (1931), and others.

The most important papers from our point of view are those by Watson and Schwid. Watson applies the method of steepest descents to an integral representation of $D_n(z)$ and obtains results for complex n and z . Schwid applies Langer's method (of comparison with the Airy equation) to obtain asymptotic formulas for those solutions of (1.1) which are even or odd functions of x . Our work differs from Schwid's in two respects. We investigate solutions determined by their behavior at infinity, rather than by their behavior at $x = 0$; and we use the comparison technique in the form developed by Cherry (1950).

The lines $\operatorname{Re} (x\sqrt{\nu}) \pm \operatorname{Im} (x\sqrt{\nu})$ divide the complex x -plane into four sectors. For each of these four sectors the differential equation (1.1) has a solution, determined up to a constant factor, which vanishes exponentially as $x \rightarrow \infty$ in that sector. These four solutions may be expressed in terms of parabolic cylinder functions. Any three of the four solutions are linearly connected, and the connection formulas are known explicitly.

We have investigated these four solutions as $|\nu| \rightarrow \infty$, $\arg \nu = \text{const.}$ while x is unrestricted, comparing them with corresponding solutions of the Airy equation. Since the Airy equation has only one transition point, the validity of this comparison is restricted to a closed region of the x -plane which contains only one of the two transition points ($x = \pm 1$) of (1.1). By using the connection formulas for solutions of (1.1), and the corresponding formulas for solutions of the Airy equation, we obtained asymptotic representations of each of the four solutions of (1.1) in a variety of regions in the x -plane which, between them, cover the entire plane. Each of these regions extends to infinity; each contains a transition point on its boundary; in each region the asymptotic representation holds uniformly in x as $\nu \rightarrow \infty$ along a line $\arg \nu = \text{const.}$ It seems likely that our asymptotic representations hold uniformly in $\arg \nu$, in certain sectors of the ν -plane, but we have made no effort to establish this fact.

An unwelcome feature of our work is a tedious, if elementary, discussion of certain many-valued functions and their Riemann surfaces. As far as we can see, this cannot be avoided. Although the solutions of (1.1) are single-valued analytic functions, in fact, entire functions of x , the comparison functions are many-valued functions of x and force us to discuss the solutions of (1.1) on (a portion of) a Riemann surface rather than in the schlicht x -plane. The comparison itself is carried out by means of the integral equation technique developed by Liouville (1837), Stekloff (1907), Langer (1932, 1934, 1949 and other papers), and Cherry (1949, 1950), using Cherry's version of this technique. The behavior of the coefficients of (1.1) at infinity precludes a direct application of Cherry's results², but it turns out that the technique can be carried out with minor modifi-

² CHERRY assumes that the coefficients of the differential equation are asymptotically linear in each of the several infinite sectors which his region may contain. We find that corresponding results hold if the coefficients are merely linearly bounded, and that an adaptation to coefficients not so bounded is feasible.

cations and adjustments. By far the most difficult part is the determination of the regions of validity, and the discussion of the many-valued functions on which they depend.

In Section 2 of this paper we summarize those properties of parabolic cylinder functions which are needed for the present investigation, and in Section 3 we do the same for Airy functions. In Section 4 we give an outline of our method, in Section 5 discuss the many-valued functions, and in Section 6 the various domains in the x -plane. In Section 7 we solve the integral equation in a typical case, in Section 8 we give some asymptotic forms of parabolic cylinder functions, and in Section 9 we state our results.

The results of this paper will be applied to a general asymptotic theory of differential equations of the second order with two transition points and containing a large parameter.

2. Parabolic cylinder functions. The differential equation

$$(2.1) \quad \frac{d^2 y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0$$

in which z is a complex variable³ and n is an unrestricted complex parameter, is known as *Weber's equation*. Certain solutions of (2.1), called *parabolic cylinder functions*, have been investigated in great detail. For an account of these investigations, and for further references to the literature, see, for instance, Whittaker and Watson, (1927, Chapter XVI, especially p. 347ff.), Magnus and Oberhettinger (1948, Chapter VI, especially p. 120ff.), Buchholz (1953, especially p. 38ff.), Erdélyi *et al.* (1953, Chapter VIII). The following properties of parabolic cylinder functions are extracted from the above works.

The parabolic cylinder function of order n is defined by the expansion

$$(22) \quad \begin{aligned} D_n(z) &= 2^{\frac{1}{2}n} e^{-\frac{1}{4}z^2} \sqrt{\pi} \left[\frac{1}{\Gamma(\frac{1}{2} - \frac{1}{2}n)} \sum_{r=0}^{\infty} \frac{(-\frac{1}{2}n)_r}{(2r)!} (2z^2)^r \right. \\ &\quad \left. - \frac{1}{\Gamma(-\frac{1}{2}n)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}n)_r}{(2r+1)!} (2z^2)^{r+\frac{1}{2}} \right] \\ &= \frac{2^{-\frac{1}{2}n-1}}{\Gamma(-n)} e^{-\frac{1}{4}z^2} \sum_{s=0}^{\infty} (-1)^s \frac{\Gamma(\frac{1}{2}s - \frac{1}{2}n)}{s!} (2^{\frac{1}{2}}z)^s \end{aligned}$$

where

$$(2.3) \quad (\alpha)_0 = 1, \quad (\alpha)_r = \alpha(\alpha+1) \cdots (\alpha+r-1), \quad r = 1, 2, 3, \dots$$

The expansion (2.2) converges for all n and z and shows that $D_n(z)$ is an entire function of both n and z . If n is a non-negative integer, the first of the two forms of $D_n(z)$ in (2.2) shows that $D_n(z)$ is $\exp(-\frac{1}{4}z^2)$ times a polynomial of degree n in z . The polynomial involved is a Hermite polynomial, and in this case $D_n(z)$ is an even or odd function of z , according to the parity of n .

³ This variable must not be confused with the variable z to be introduced in (4.3).

If n is fixed we have, for each $N = 1, 2, \dots$,

$$(2.4) \quad D_n(z) = z^n e^{-\frac{1}{2}z^2} \left[\sum_{r=0}^{N-1} \frac{(-n)_{2r}}{r!(-2z^2)^r} + O(|z|^{-2N}) \right], \quad z \rightarrow \infty, \quad -\frac{3}{4}\pi < \arg z < \frac{3}{4}\pi.$$

This asymptotic form shows that $D_n(z)$ vanishes exponentially in the sector $\operatorname{Re} z > |\operatorname{Im} z|$, and increases exponentially in the two sectors $|\operatorname{Im} z| > |\operatorname{Re} z|$. In the remaining quadrant, $\operatorname{Re} z < -|\operatorname{Im} z|$, the behavior of $D_n(z)$ is more complex. If n is a non-negative integer, it can be shown that the O -term in (2.4) vanishes identically when $2N > n$. Thus, for $n = 0, 1, 2, \dots$, (2.4) furnishes an explicit formula for $D_n(z)$ showing that this function vanishes exponentially when $|\operatorname{Re} z| > |\operatorname{Im} z|$. On the other hand, the connection formulas (2.6) show that $D_n(z)$ increases exponentially as $z \rightarrow \infty$ in the quadrant $\operatorname{Re} z < -|\operatorname{Im} z|$ whenever $n \neq 0, 1, 2, \dots$.

The two pairs of functions

$$(2.5) \quad D_n(z), \quad D_n(-z); \quad D_{-n-1}(iz), \quad D_{-n-1}(-iz)$$

are pairs of solutions of Weber's equation. In general (*i.e.*, for non-integral values of n) exactly one of the solutions (2.5) vanishes exponentially as $z \rightarrow \infty$ in any given one of the four sectors discussed above. In general any of the four solutions may be expressed in terms of any two of the remaining three. The *connection formulas* furnishing these expressions are

$$\begin{aligned} D_n(z) &= \frac{\Gamma(n+1)}{(2\pi)^{\frac{1}{2}}} [e^{\frac{1}{2}n\pi i} D_{-n-1}(iz) + e^{-\frac{1}{2}n\pi i} D_{-n-1}(-iz)] \\ (2.6) \quad &= e^{-n\pi i} D_n(-z) + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-n)} e^{-\frac{1}{2}(n+1)\pi i} D_{-n-1}(iz) \\ &= e^{n\pi i} D_n(-z) + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-n)} e^{\frac{1}{2}(n+1)\pi i} D_{-n-1}(-iz) \end{aligned}$$

and those obtained by changing the sign of z throughout these formulas. If n is an integer, some of these formulas become nugatory, and one of the two pairs of solutions (2.5) degenerates into a single solution.

The substitution

$$(2.7) \quad n = \nu - \frac{1}{2}, \quad z = 2x\sqrt{\nu}$$

(or any of three other substitutions obtained by changing the signs of x or ν or both) transforms Weber's equation (2.1) into

$$(1.1) \quad \frac{d^2 y}{dx^2} - 4\nu^2(x^2 - 1)y = 0.$$

Four solutions of (1.1) are

$$(2.8) \quad \begin{aligned} &D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}), \quad D_{\nu-\frac{1}{2}}(-2x\sqrt{\nu}), \\ &D_{-\nu-\frac{1}{2}}(2ix\sqrt{\nu}), \quad D_{-\nu-\frac{1}{2}}(-2ix\sqrt{\nu}). \end{aligned}$$

Any two of these four solutions are linearly independent in general, *i.e.*, with the possible exception of certain integral values of $\nu - \frac{1}{2}$. Some of our formulas, for instance the first part of (2.6), become meaningless for such exceptional values. The four solutions (2.8) vanish exponentially if $x \rightarrow \infty$ in the quadrants

$$(2.9) \quad \begin{aligned} \operatorname{Re} (x\sqrt{\nu}) &> |\operatorname{Im} (x\sqrt{\nu})|, & \operatorname{Re} (x\sqrt{\nu}) &< -|\operatorname{Im} (x\sqrt{\nu})| \\ \operatorname{Im} (x\sqrt{\nu}) &< -|\operatorname{Re} (x\sqrt{\nu})|, & \operatorname{Im} (x\sqrt{\nu}) &> |\operatorname{Re} (x\sqrt{\nu})| \end{aligned}$$

respectively. All four solutions are entire functions of $\sqrt{\nu}$ and x . Between the four solutions (2.8) we have connection formulas which may be written down by means of (2.6).

3. The Airy function. The differential equation

$$(3.1) \quad \frac{d^2 U}{dt^2} - tU = 0,$$

where t is a complex variable, is known as *Airy's equation*. The general solution of (3.1) may be expressed in terms of a particular solution, called the *Airy function*, which in its turn may be expressed in terms of Bessel functions of order $\pm \frac{2}{3}$. All properties of the Airy function given below follow from known properties of Bessel functions.

The Airy function is defined by the relations

$$(3.2) \quad \begin{aligned} Ai(t) &= \frac{\sqrt{t}}{\pi\sqrt{3}} K_{\frac{2}{3}}(\tfrac{2}{3}t^{\frac{3}{2}}) \\ &= \tfrac{1}{3}t^{\frac{1}{3}}[I_{-\frac{2}{3}}(\tfrac{2}{3}t^{\frac{3}{2}}) - I_{\frac{2}{3}}(\tfrac{2}{3}t^{\frac{3}{2}})] \\ &= \frac{2}{3^{\frac{2}{3}}} \sum_{r=0}^{\infty} \frac{\sin [\frac{2}{3}\pi(r+1)]}{\Gamma(\frac{r+2}{3}) \Gamma(\frac{r+3}{3})} \left(\frac{t}{3^{\frac{2}{3}}}\right)^r, \end{aligned}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind and $K_\nu(z)$ is the modified Bessel function of the third kind (see Watson 1922, p. 77, 78). Clearly, $Ai(t)$ is an entire function of t ,

$$(3.3) \quad Ai(0) = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})}, \quad Ai'(0) = \frac{-3^{-\frac{1}{3}}}{\Gamma(\frac{4}{3})}$$

and from the asymptotic expansion of $K_\nu(z)$ (see Watson 1922, p. 202),

$$(3.4) \quad Ai(t) = \tfrac{1}{2}\pi^{-\frac{1}{2}}t^{-\frac{1}{2}}\exp(-\tfrac{2}{3}t^{\frac{3}{2}})[1 + O(|t|^{-\frac{3}{2}})], \quad t \rightarrow \infty, \quad -\pi < \arg t < \pi.$$

The Airy function has a sequence of zeros on the negative real axis, and it has no other zero (Watson, 1922, p. 511 ff.).

We shall use throughout this paper the notation

$$(3.5) \quad \omega = \exp(\tfrac{2}{3}\pi i).$$

For any integer m ,

$$(3.6) \quad U_m(t) = Ai(t\omega^m)$$

is a solution of (3.1). Three "consecutive" solutions are connected by the relation

$$(3.7) \quad Ai(t\omega^m) + \omega Ai(t\omega^{m+1}) + \omega^2 Ai(t\omega^{m+2}) = 0$$

which may be verified by means of (3.2). This connection formula enables us to write down asymptotic expressions for the Airy function when $\arg t$ is outside the range indicated in (3.4), in particular, to write down an asymptotic expression for a sector containing the negative real axis,

$$(3.8) \quad Ai(t) = \frac{1}{2}\pi^{-\frac{1}{2}}t^{-\frac{1}{2}}\{\exp(-\frac{2}{3}t^{\frac{3}{2}})[1 + O(|t|^{-\frac{1}{2}})] + i \exp(\frac{2}{3}t^{\frac{3}{2}})[1 + O(|t|^{-\frac{1}{2}})]\},$$

$$t \rightarrow \infty, \quad \frac{\pi}{3} < \arg t < \frac{5\pi}{3}.$$

From (3.2), (3.4), (3.8) it follows that

$$(3.9) \quad (1 + |t|^{\frac{1}{2}}) \exp(\frac{2}{3}t^{\frac{3}{2}})Ai(t), \quad -\pi \leq \arg t \leq \pi$$

is bounded in the entire t -plane; and this expression is also bounded away from zero if the zeros of the Airy function are suitably excluded. In this paper the exclusion of the zeros will always be effected by excluding a sector around the negative real t axis. Other methods, for instance excluding the zeros by means of small circles placed around them, are feasible but will not be used here.

In the actual comparison it will be useful to transform Airy's equation by the substitution

$$(3.10) \quad z = \frac{t^{\frac{1}{2}}}{3\nu\sqrt{2}}, \quad W(z) = \left(\frac{3}{2}z\right)^{\frac{1}{2}} U(t)$$

where z is the new independent variable, and ν is a (real or complex) parameter. The differential equation becomes

$$(3.11) \quad \frac{d^2 W}{dz^2} + \left(-8\nu^2 + \frac{5}{36z^2}\right) W = 0$$

thus showing clearly the relationship to Bessel's equation. For any integer m ,

$$(3.12) \quad W_m(z) = \left(\frac{3}{2}z\right)^{\frac{1}{2}} Ai[2\left(\frac{3}{2}\nu z\right)^{\frac{1}{2}} \omega^m]$$

is a solution of (3.11). Between three "consecutive" solutions we have the connection formula

$$(3.13) \quad W_m(z) + \omega W_{m+1}(z) + \omega^2 W_{m+2}(z) = 0.$$

We shall always take $m = -1, 0, 1$.

The Wronskian of any two solutions of (3.11) is constant. In particular,

$$(3.14) \quad W_m(z)W'_{m-1}(z) - W'_m(z)W_{m-1}(z) = \frac{\nu^{\frac{3}{2}}\omega^{m+\frac{1}{2}}}{\pi}.$$

The function $W_m(z)$ has a sequence of zeros on the lines

$$(3.15) \quad \arg z = (3r + \tfrac{3}{2} - m)\pi - \arg \nu, \quad r \text{ integral}$$

and is otherwise free of zeros.

$$(3.16) \quad (|\nu|^{\frac{1}{2}} + |z|^{-\frac{1}{2}}) \exp[(-1)^m 2^{\frac{1}{2}} \nu z] W_m(z)$$

is bounded if

$$(3.17) \quad -\tfrac{3}{2}\pi \leq \arg \nu z + m\pi \leq \tfrac{3}{2}\pi$$

and is bounded away from zero if

$$(3.18) \quad -\tfrac{3}{2}\pi + \epsilon \leq \arg \nu z + m\pi \leq \tfrac{3}{2}\pi - \epsilon, \quad \epsilon > 0.$$

The quotient

$$(3.19) \quad \frac{W_{m-1}(z)}{W_m(z)} \exp[(-1)^{m-1} 2^{\frac{1}{2}} \nu z]$$

is bounded if

$$(3.20) \quad -\tfrac{1}{2}\pi \leq \arg(\nu z) + m\pi \leq \tfrac{3}{2}\pi - \epsilon, \quad \epsilon > 0$$

and bounded away from zero if

$$(3.21) \quad -\tfrac{1}{2}\pi + \epsilon \leq \arg(\nu z) + m\pi \leq \tfrac{3}{2}\pi, \quad \epsilon > 0.$$

In the sector (3.18) we have the asymptotic representation

$$(3.22) \quad W_m(z) = 2^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \nu^{-\frac{1}{2}} \exp[(-1)^{m-1} 2^{\frac{1}{2}} \nu z - \tfrac{1}{8} m \pi i] \left[1 + O\left(\frac{1}{|\nu z|}\right) \right]$$

$$z \rightarrow \infty, \quad \epsilon - \tfrac{3}{2}\pi \leq \arg(\nu z) + m\pi \leq \tfrac{3}{2}\pi - \epsilon, \quad \epsilon > 0.$$

All bounds mentioned in these statements, including the one involved in the O symbol are uniform bounds, that is independent of both z and ν .

4. Outline of the method. We shall study the differential equation

$$\frac{d^2 y}{dx^2} - 4\nu^2(x^2 - 1)y = 0$$

for large (real or complex) values of the parameter ν . Since the differential equation depends on ν^2 , we may take $-\frac{1}{2}\pi \leq \arg \nu \leq \frac{1}{2}\pi$ without restriction on the

generality of our considerations. The coefficient of $\nu^2 y$ in (1.1) has two zeros, $x = \pm 1$, and these are *transition points* of the differential equation. In any region of the x -plane in which $x^2 - 1$ is bounded away from zero, that is to say in any region whose closure does not contain a transition point, asymptotic representations of the solutions of (1.1) for large ν may be obtained by Liouville's method (1837). This method is based on the comparison of (1.1) with a differential equation with constant coefficients: the asymptotic representations involve elementary functions only, and are thus incapable of rendering the rather complex behavior of solutions of (1.1) near one of the transition points.

In order to obtain asymptotic forms valid in a closed region of the x -plane containing the transition point $x = 1$ of the solutions of (1.1), we shall compare the given equation with

$$(4.1) \quad \frac{d^2 \eta}{d\xi^2} - 8\nu^2 \xi \eta = 0.$$

The comparison equation (4.1) is the simplest equation with one transition point, and the coefficients have been chosen in such a manner that (1.1) near $x = 1$ and (4.1) near $\xi = 0$ are similar.

For reasons to be indicated later, it is advisable to transform both the comparison equation (4.1) and the given equation (1.1) to a form in which the transition point appears as a singularity and the coefficients of the differential equation are approximately constant if ν is sufficiently large and the variable is bounded away from transition points. Cherry (1950, p. 227) calls this form the *Bessel canonical form*.

Equation (4.1) is transformed by

$$(4.2) \quad \xi = \left(\frac{3}{2}z\right)^{\frac{2}{3}}, \quad \eta = \left(\frac{d\xi}{dz}\right)^{\frac{1}{3}} W$$

to the form (3.11) and solutions of this equation are $W_m(z)$, given by (3.12). Equation (1.1) is transformed by

$$(4.3) \quad z = f(x) = \int_1^x \left(\frac{t^2 - 1}{2}\right)^{\frac{1}{2}} dt, \quad y = \left(\frac{dx}{dz}\right)^{\frac{1}{2}} w$$

to the form

$$(4.4) \quad \frac{d^2 w}{dz^2} + \left(-8\nu^2 + \frac{3x^2 + 2}{2(x^2 - 1)^{\frac{3}{2}}}\right) w = 0$$

which may be written as

$$(4.5) \quad \frac{d^2 w}{dz^2} + \left(-8\nu^2 + \frac{5}{36z^2}\right) w = G(z)w.$$

where

$$(4.6) \quad G(z) = \frac{5}{36z^2} - \frac{3x^2 + 2}{2(x^2 - 1)^3}, \quad z = f(x).$$

The transformation $z = f(x)$, together with certain domains in which all the many-valued functions occurring in our work may be defined uniquely, will be discussed in Sections 5, 6.

We could have transformed (4.1) directly into a form similar to (1.1), or vice versa, by putting

$$(4.7) \quad \xi = [\tfrac{3}{2}f(x)]^{\frac{1}{3}}, \quad \eta = \left(\frac{d\xi}{dx}\right)^{\frac{1}{3}} y$$

and making the comparison in the x or in the ξ -plane, but the auxiliary variable z seems to offer certain advantages. For one thing, $[f(x)]^{\frac{1}{3}}$ has branch-points at all zeros of $f(x)$ and is a more difficult transformation than $f(x)$; and moreover, the asymptotic behavior of the solutions of (4.1) is most simply expressed in terms of the variable z , thus enabling us to describe "shadow zones" in a simple fashion. The first of these advantages is partly off-set by the necessity of taking into consideration the zeros of $f(x)$, when dealing with certain many-valued functions.

For the comparison of (3.11) and (4.5) by means of an integral equation we use Cherry's lemma 2 (1950, p. 239). If the integral equation

$$(4.8) \quad w_m(z) = W_m(z) - \frac{1}{\Delta_m} \int_{a_m}^z [W_m(z)W_{m-1}(\zeta) - W_m(\zeta)W_{m-1}(z)] \\ \times G(\zeta)w_m(\zeta) d\zeta,$$

where

$$\Delta_m = \text{Wronskian}(W_m, W_{m-1}) = \pi^{-1} \nu^{\frac{1}{3}} \omega^{m+\frac{1}{2}},$$

for a given $m = 0, 1, -1$, and a given a_m (possibly infinite), has a solution $w_m(z)$ for which the integral in (4.8) may be differentiated twice under the integral sign, then such a solution satisfies the differential equation (4.5), and it also satisfies at $z = a_m$ the same initial conditions as $W_m(z)$. In our case a_m will always be a point at infinity, and the solution $w_m(z)$ will be identified by its asymptotic behavior rather than by initial conditions.

Restricting z to a region in which $W_m(z)$ never vanishes, we write

$$(4.9) \quad \psi_m(z) = \frac{w_m(z)}{W_m(z)}$$

and rewrite the integral equation (4.8) in the form

$$(4.10) \quad \psi_m(z) = 1 - \int_{a_m}^z K_m(z, \zeta) G(\zeta) \psi_m(\zeta) d\zeta,$$

where

$$(4.11) \quad K_m(z, \zeta) = \frac{\pi}{\nu^{\frac{1}{2}} \omega^{m+\frac{1}{2}}} \left[W_m(\zeta) W_{m-1}(\zeta) - W_m^2(\zeta) \frac{W_{m-1}(z)}{W_m(z)} \right].$$

We note that, for each m , $K_m(z, \zeta)$ is a well-determined combination of solutions of the comparison equation, and further that

$$F(z) = W_m(z) K_m(z, \zeta) / W_m(\zeta)$$

being that solution of (3.11) which vanishes, and has derivative 1, at $z = \zeta$ is independent of m .

In discussing the integral equation (4.10) we have to take account of the fact that the functions $W_m(z)$, $W_{m-1}(z)$ and $G(z)$ are many-valued functions. For this reason $\psi_m(z)$ will be discussed not in the schlicht z -plane but rather on a portion of a Riemann surface $\mathfrak{Z}$.

We shall take $m = 0, 1, -1$ in (4.11), choosing a_m at infinity so that $\arg(\nu a_m) = -m\pi$. We shall consider (4.10) in a closed region on the Riemann surface $\mathfrak{Z}$ which will depend on $\arg \nu$, but will be independent of $|\nu|$, and will be chosen so as to ensure the following properties:

- (i) $z = f(x)$ is a (1-1) mapping of a region in the x -plane onto the region on $\mathfrak{Z}$.
- (ii) $W_m(z)$, $W_{m-1}(z)$ are single-valued, and the zeros of $W_m(z)$ are excluded by sectors as in (3.18).
- (iii) Every point z of the region may be joined by a path to a_m in such a manner that for every point ζ on the path

$$|K_m(z, \zeta)| \leq \frac{C}{|\nu|}$$

where C is independent of $z, \zeta, |\nu|$. This is the condition excluding "shadow zones" (see Cherry, 1950, p. 241).

- (iv) If integration is extended over the path mentioned in (iii), the integral

$$\int_z^{a_m} |G(\zeta)| |d\zeta|$$

exists for each z in the region, and is a bounded function of z .

It will be found possible to include the principal image of $x = 1$ (more precisely defined in Section 5) as a boundary point of our region.

Restricting z to such a region, the existence of a unique solution $\psi_m(z)$ will be proved, and it will be established that this solution is asymptotic to 1.

The results so obtained may be expressed in terms of x by the transformation (4.3). This will lead to certain solutions $y_m(x)$ of (1.1) whose behavior as $x \rightarrow \infty$, ν being fixed, will enable us to identify them as multiples of parabolic cylinder functions. Thus asymptotic representations for large $|\nu|$ of three of the four solutions (2.8) are obtained in a variety of regions in the x -plane. In each part of the x -plane we have asymptotic forms for two distinct solutions, and the other solutions may be reconstructed by means of the connection formulas.

FIGURE 1

5. The transformation $z = f(x)$ and the function $G(z)$. The function $f(x)$ is defined by analytic continuation from its values on $1 < x < \infty$, given by

$$f(x) = \int_1^x \left(\frac{t^2 - 1}{2} \right)^{\frac{1}{2}} dt$$

where the integrand is taken to be real and positive.

We first consider the mapping $z = f(x)$ of a region in the first quadrant of the x -plane bounded by $\operatorname{Re} x = 0$, $\operatorname{Im} x = 0$, a semi-circular indent at $x = 1$, and the circle $|x| = R$ (Fig. 1).

The mapping of the section $FOABCD$ of the contour is easily seen to be such that no value of z is assumed twice on it. We now show that this is true for the whole contour $FOABCDEF$. Consider the point E with affix x on the arc DEF . Let $\arg x = \phi$ and $\arg(x^2 - 1)^{\frac{1}{2}} = \lambda(\phi) = \text{slope of the internal bisector of the angle at } E \text{ subtended by the points } -1 \text{ and } 1$. As x describes the arc DEF , $\lambda(\phi)$ increases steadily from 0 to $\frac{1}{2}\pi$ and therefore $\phi + \lambda(\phi)$ increases steadily from 0 to π . Thus for any point E on $|x| = R$ in the first quadrant we have

$$\operatorname{Re} z = -R \int_0^\phi F(R, \tau) \sin(\tau + \lambda(\tau)) d\tau,$$

$$\operatorname{Im} z = R \int_0^\phi F(R, \tau) \cos(\tau + \lambda(\tau)) d\tau,$$

where

$$F(R, \tau) = \left| \frac{R^2 e^{2i\tau} - 1}{2} \right|^{\frac{1}{2}}.$$

As ϕ varies from 0 to $\frac{1}{2}\pi$, $\operatorname{Re} z$ decreases monotonely. $\operatorname{Im} z$ increases monotonely until $\phi + \lambda(\phi) = \frac{1}{2}\pi$ and then decreases monotonely. As will be indicated below, $f(x) = O(x^2)$ as $x \rightarrow \infty$. These considerations show that in the map of the con-

tour $OABCDEFO$ in the x -plane onto the z -plane, no value of z occurs twice. Hence $f(x)$ is schlicht inside the contour. Thus letting $R \rightarrow \infty$ we have shown that $z = f(x)$ is a schlicht mapping of the first quadrant (with indent at $x = 1$) onto a region of the z -plane.

Since OF maps onto $O'F'$ and CD maps onto $C'D'$, we may by repeated applications of Schwarz's reflection principle, prove that $f(z)$ is a *schlicht mapping of a Riemann surface $\mathfrak{X}$ over the x -plane with branch-points over $x = \pm 1$ onto a Riemann surface $\mathfrak{Z}$ over the z -plane whose branch-points are the images of the branch-points in the x -surface.*

Since

$$(5.1) \quad z = f(x) = \frac{1}{2\sqrt{2}} [x(x^2 - 1)^{\frac{1}{2}} - \log (x + (x^2 - 1)^{\frac{1}{2}})]$$

it is easy to see that

$$(5.2) \quad z = f(x) = \pm \frac{1}{2\sqrt{2}} x^2 \left(1 + O\left(\frac{1}{x}\right) \right)$$

on any sheet over the x -plane.

The point $z = 0$ which is the image of $x = 1$ will play a special role in subsequent work. *It will be denoted by Θ* in order to distinguish this point from the other points on $\mathfrak{Z}$ for which $z = 0$, and it will be called *the principal image of $x = 1$* . By the properties of $f(x)$ just established we now have that

$$G(z) = \frac{5}{36z^2} - \frac{3x^2 + 2}{2(x^2 - 1)^{\frac{3}{2}}}, \quad z = f(x)$$

is one-valued on $\mathfrak{Z}$, regular except at branch-points and all points for which $z = f(x) = 0$.

Near $x = 1$ the principal determination of z is

$$(5.3) \quad z = f(x) = \frac{2}{3}(x - 1)^{\frac{3}{2}} + \frac{1}{15}(x - 1)^{\frac{5}{2}} + \dots$$

Substituting in (4.6), the leading terms cancel and we obtain the result

$$(5.4) \quad z^{\frac{1}{2}} G(z) \text{ tends to a finite limit as } z \rightarrow \Theta.$$

$|G(z)|$ is not integrable on paths terminating at branch-points or at points for which $z = 0$, other than the point Θ . For this reason we shall always exclude these points by circles.

Substituting (5.2) in (4.6), we have that for any closed region of $\mathfrak{Z}$ with the branch-points other than Θ excluded,

$$(5.5) \quad z^2 G(z) \text{ is a bounded analytic function,}$$

provided that the closed region considered is contained on a finite number of sheets of the Riemann surface.

TABLE I

Domain	Arg ν is in	The domain is	The complementary domain consists of the interiors of
$\mathfrak{G}_0$	0	$M[-\frac{3}{2}\pi + \epsilon, \frac{3}{2}\pi - \epsilon]$	
	$(0, \frac{1}{2}\pi]$		$C^*, M[\pi\frac{3}{2} - \arg \nu - \epsilon, \frac{3}{2}\pi - \arg \nu + \epsilon],$ $M^*[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon]$
	$[-\frac{1}{2}\pi, 0)$		$C^*, M[-\frac{3}{2}\pi - \arg \nu - \epsilon, -\frac{3}{2}\pi - \arg \nu + \epsilon],$ $M^*[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon]$
$\mathfrak{G}_1$	$[0, \frac{1}{2}\pi]$		$C^*, M[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon],$ $M^*[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon]$
	$[-\frac{1}{2}\pi, 0]$		$C^*, M[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon],$ $M^*[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon]$
	$[0, \frac{1}{2}\pi]$		$C^*, M[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon],$ $M^*[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon]$
$\mathfrak{G}_{-1}$	$[0, \frac{1}{2}\pi]$		$C^*, M[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon],$ $M^*[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon]$
	$[-\frac{1}{2}\pi, 0]$		$C^*, M[-\frac{1}{2}\pi - \arg \nu - \epsilon, -\frac{1}{2}\pi - \arg \nu + \epsilon],$ $M^*[\frac{1}{2}\pi - \arg \nu - \epsilon, \frac{1}{2}\pi - \arg \nu + \epsilon]$
	$[-\frac{1}{2}\pi, \frac{1}{2}\pi]$		$C^*, M[-\arg \nu - \frac{1}{2}\pi - \epsilon, -\arg \nu + \frac{1}{2}\pi + \epsilon],$ $M^*[-\arg \nu - \frac{1}{2}\pi - \epsilon, -\arg \nu + \frac{1}{2}\pi + \epsilon]$
$\mathfrak{M}_1$	$[0, \frac{1}{2}\pi]$	$M[-\frac{3}{2}\pi + \epsilon, \frac{1}{2}\pi - \arg \nu - \epsilon]$	
	$[-\frac{1}{2}\pi, 0]$	$M[-\frac{3}{2}\pi - \arg \nu + \epsilon, \frac{1}{2}\pi - \arg \nu - \epsilon]$	
	$[0, \frac{1}{2}\pi]$	$M[-\frac{1}{2}\pi - \arg \nu + \epsilon, \frac{3}{2}\pi - \arg \nu - \epsilon]$	
$\mathfrak{M}_{-1}$	$[-\frac{1}{2}\pi, 0]$	$M[-\frac{1}{2}\pi - \arg \nu + \epsilon, \frac{3}{2}\pi - \epsilon]$	

6. Description of domains in the x -plane. In this section the domains of validity of asymptotic formulas to be found in later sections are described. In order to do this we introduce some notation.

For real positive ρ let C_ρ denote the circle with center at $x = 1$ and radius ρ , and let C_ρ^* denote the circle with center at $x = -1$ and radius ρ .

For $-\frac{3}{2}\pi < \theta < \frac{3}{2}\pi$, let L_θ denote the curve issuing from $x = 1$ along which $\arg f(x) = \theta$. A qualitative idea of the shape of the curve L_θ can be deduced as follows. If $0 < \psi < \pi$ then

$$f(1 + re^{i\psi}) = e^{\frac{3}{2}i\psi} \int_0^r \rho^{\frac{1}{2}} (1 + \frac{1}{2}\rho e^{i\psi})^{\frac{1}{2}} d\rho.$$

Hence as r increases from 0 to ∞ with ψ fixed, $\arg f(1 + re^{i\psi})$ increases steadily from $\frac{3}{2}\psi$ to 2ψ . It follows that if $0 < \theta < \frac{3}{2}\pi$ then the curve L_θ lies in the sector $\frac{1}{2}\theta \leq \arg(x - 1) \leq \frac{2}{3}\theta$, is tangent to the line $\arg(x - 1) = \frac{2}{3}\theta$ at $x = 1$, and has the asymptotic direction $\arg(x - 1) = \frac{1}{2}\theta$ at infinity. Furthermore L_θ does not cross any of the lines $\arg(x - 1) = \text{constant}$ more than once.

For $-\frac{3}{2}\pi < \theta < \frac{3}{2}\pi$ let L_θ^* denote the curve issuing from $x = -1$ which is obtained by reflecting L_θ in the origin.

For $-\frac{3}{2}\pi < \alpha < \beta < \frac{3}{2}\pi$, we denote by $M[\alpha, \beta]$ the closed curvilinear sector of the x -plane which is covered by the lines L_θ , $\alpha \leq \theta \leq \beta$, and by $M^*[\alpha, \beta]$ the closed curvilinear sector covered by the lines L_θ^* , $\alpha \leq \theta \leq \beta$.

In the tables I and II, a number of domains are described. In each case the dependence of the domain on the (fixed) value of $\arg \nu$ and on the small positive parameter ϵ , has been suppressed in the notation. In each case it is assumed that ϵ is small enough to make the definition meaningful. When it is stated that $\arg \nu$ is in a certain interval the symbols (a, b) , $[a, b)$, $[a, b]$ denote respectively the intervals $a < \arg \nu \leq b$, $a \leq \arg \nu < b$, $a \leq \arg \nu \leq b$.

TABLE II

Domain	m	Arg ν is in	The domain is the closure of the smaller of the two curvilinear sectors bounded by the lines
$\mathfrak{S}_m$	0	$[-\frac{1}{2}\pi, \frac{1}{2}\pi]$	$C_\epsilon, L_{\frac{1}{2}\pi - \arg \nu}, L_{-\frac{1}{2}\pi - \arg \nu}$
	1, -1	$[-\frac{1}{2}\pi, \frac{1}{2}\pi]$	$C_{2+\epsilon}, L_{-\frac{1}{2}m\pi - \arg \nu}, L_{\frac{1}{2}m\pi - \arg \nu}^*$

Let $\mathfrak{E}$ be a generic symbol denoting one of the domains $\mathfrak{G}_m, \mathfrak{R}, \mathfrak{M}_m, \mathfrak{S}_m$. We will denote by $\mathfrak{E}^*$ the domain obtained from $\mathfrak{E}$ by reflection in the origin. That is

$$\mathfrak{E}^* = \{x; -x \in \mathfrak{E}\}.$$

When ϵ is sufficiently small $\mathfrak{E}$ is a closed region containing $x = 1$ as a boundary point.

7. Discussion of the integral equation. As was explained in Section 4 we shall solve the integral equation

$$\psi_m(z) = 1 + \int_z^{a_m} K_m(z, \zeta) G(\zeta) \psi_m(\zeta) d\zeta$$

where $\arg \nu$ is a fixed constant, where z lies in a suitable domain $\mathfrak{D}_m$ on the covering surface $\mathfrak{J}$ over the z -plane and where a_m is the point at infinity in the direction $\arg z = -m\pi - \arg \nu$. For the domain $\mathfrak{D}_m$ we will select a *closed simply connected region* lying entirely on a finite number of sheets of the surface $\mathfrak{J}$, so restricted that conditions (i), (ii), (iii), and (iv) of Section 4 are valid.

According to conditions (ii) the functions

$$W_k(z) = \left(\frac{3}{2}z\right)^{\frac{1}{3}} Ai[2\left(\frac{3}{2}\nu z\right)^{\frac{2}{3}} \omega^k]$$

must be one-valued in $\mathfrak{D}_m$. This will always be the case if we require that any point of $\mathfrak{D}_m$ which covers $z = 0$ should be a boundary point of the (simply connected) closed region $\mathfrak{D}_m$. Under this requirement it will also make sense to speak of $\arg z$ as a function defined in $\mathfrak{D}_m$.

Now consider condition (iii). Using (3.16), (3.19), (4.11) one finds that

$$(7.1) \quad |K_m(z, \zeta)| \leq \frac{A + B |\exp \{2^{\frac{1}{3}} \nu e^{m\pi i} (z - \zeta)\}|}{|\nu|^{\frac{1}{3}} (|\nu|^{\frac{1}{3}} + |\zeta|^{-\frac{1}{3}})^2}$$

where A, B are constants independent of ν, z, ζ , if z, ζ are in $\mathfrak{D}_m$ and

$$-\frac{1}{2}\pi \leq \arg(\nu t e^{m\pi i}) \leq \frac{3}{2}\pi - \epsilon, \quad \epsilon > 0,$$

is valid both for $t = z$ and $t = \zeta$. We can express W_{m-1} in terms of W_m and W_{m+1} by means of (3.13), and if this expression is substituted in (4.11) we have

$$K_m(z, \zeta) = \frac{-\pi}{\nu^{\frac{1}{3}} \omega^{m+\frac{1}{2}}} \left[W_m(\zeta) W_{m+1}(\zeta) - W_m^2(\zeta) \frac{W_{m+1}(z)}{W_m(z)} \right],$$

and this together with (3.16), (3.19) shows that (7.1) is also valid if z, ζ are in $\mathfrak{D}_m$ and

$$-\frac{3}{2}\pi + \epsilon \leq \arg(\nu t e^{m\pi i}) \leq \frac{1}{2}\pi, \quad \epsilon > 0,$$

is valid both for $t = z$ and $t = \zeta$. Thus in the first place we will require that all points t in $\mathfrak{D}_m$ satisfy

$$(7.2) \quad -\frac{3}{2}\pi + \epsilon \leq \arg(\nu t e^{m\pi i}) \leq \frac{3}{2}\pi - \epsilon.$$

We next make a further restriction on $\mathfrak{D}_m$ so that condition (iii) becomes a consequence of (7.1). We assume that each point z of $\mathfrak{D}_m$ may be joined by a path to a_m such that either

$$-\frac{1}{2}\pi \leq \arg(\nu\zeta e^{m\pi i}) \leq \frac{3}{2}\pi - \epsilon$$

for all ζ on the path or

$$-\frac{3}{2}\pi + \epsilon \leq \arg(\nu\zeta e^{m\pi i}) \leq \frac{1}{2}\pi$$

for all ζ on the path, and in addition

$$(7.3) \quad \operatorname{Re} \{ \nu e^{m\pi i} (z - \zeta) \} \leq 0$$

for all ζ on the path.

The domain $\mathfrak{D}_m$ must be further restricted so that conditions (i) and (iv) are satisfied. In particular one sees from the discussion of the function $G(z)$ in Section 5 that the closed region $\mathfrak{D}_m$ must not contain any branch-point of the surface $\mathfrak{Z}$, nor any point covering $z = 0$, with the exception of the point Θ which may be part of the boundary of $\mathfrak{D}_m$.

In summary we require that the domain $\mathfrak{D}_m$ in which the integral equation will be studied should satisfy the following conditions.

(A) $\mathfrak{D}_m$ is a closed simply connected region which contains no branch-points nor any points covering $z = 0$, with the exception of Θ which may be part of the boundary of $\mathfrak{D}_m$.

(B) For all points z in $\mathfrak{D}_m$

$$(7.2) \quad | \arg(\nu z e^{m\pi i}) | \leq \frac{3}{2}\pi - \epsilon, \quad \epsilon > 0.$$

(C) Each point z in $\mathfrak{D}_m$ can be joined by a path in $\mathfrak{D}_m$ to a_m such that either

$$-\frac{1}{2}\pi \leq \arg(\nu\zeta e^{m\pi i}) \leq \frac{3}{2}\pi - \epsilon$$

for all ζ on the path or

$$-\frac{3}{2}\pi + \epsilon \leq \arg(\nu\zeta e^{m\pi i}) \leq \frac{1}{2}\pi$$

for all ζ on the path, and

$$(7.3) \quad \operatorname{Re} \{ \nu e^{m\pi i} (z - \zeta) \} \leq 0$$

for all ζ on the path. In addition conditions (i) and (iv) of Section 4 must be satisfied.

We will consider the integral equation in detail for the particular case with $m = 0$ and $\arg \nu = 0$. Here the point a_m is at infinity in the direction $\arg z = 0$ and we have to solve the equation

$$\psi_0(z) = 1 + \int_z^\infty K_0(z, \zeta) G(\zeta) \psi_0(\zeta) d\zeta$$

the path of integration extending to infinity in the direction of the positive real axis. On the covering surface $\mathfrak{Z}$ we select the sector

$$\left\{ \begin{array}{l} |z| \geq 0 \\ |\arg z| \leq \frac{3}{2}\pi - \epsilon \end{array} \right.$$

as the region $\mathfrak{D}_0$ in which to solve the equation. This is precisely the domain determined by condition (B). Clearly conditions (A) and (C) are satisfied. In fact a permissible path of integration for each point z may be taken as follows:

If $|\arg z| \leq \pi$ the path is the line through z parallel to the real axis in the direction of increasing $\operatorname{Re} \{z\}$, but if $\pi < |\arg z| \leq \frac{3}{2}\pi - \epsilon$ the path is the line joining z to the origin and thence along the positive real axis to infinity. One easily verifies that for this path

$$\int_z^\infty |G(\zeta)| \cdot |d\zeta| \quad \text{is uniformly bounded.}$$

The integral equation is now solved by the method of successive approximations according to the scheme

$$(7.4) \quad \psi_0(z) = \sum_{k=0}^{\infty} \psi_0^{(k)}(z)$$

where

$$(7.5) \quad \begin{cases} \psi_0^{(0)}(z) \equiv 1 \\ \psi_0^{(k)}(z) = \int_z^\infty K_0(z, \zeta) G(\zeta) \psi_0^{(k-1)}(\zeta) d\zeta, \quad k = 1, 2, \dots \end{cases}$$

Because of (5.5) one sees that if $\psi(z)$ is a continuous bounded function on $\mathfrak{D}_0$, regular in the interior of $\mathfrak{D}_0$, then the value of the integral

$$\int_z^\infty K_0(z, \zeta) G(\zeta) \psi(\zeta) d\zeta$$

remains unaltered when the path of integration is replaced by a straight line from z to ∞ in a direction given by that of $\overrightarrow{OP}$ or $\overrightarrow{PO}$ (P being the point z , O the origin) according as $|\arg z| \leq \frac{1}{2}\pi$ or $|\arg z| > \frac{1}{2}\pi$. This path is permissible under condition (C). Using this deformation of path and (5.5), one proves by induction that

$$|\psi_0^{(k)}(z)| \leq \left(\frac{4A}{\nu}\right)^k, \quad k = 0, 1, 2, \dots$$

for all z in $\mathfrak{D}_0$ and

$$|\psi_0^{(k)}(z)| \leq \frac{1}{k!} \left(\frac{A}{\nu|z|}\right)^k, \quad k = 0, 1, 2, \dots$$

if $|z| \geq 1$ and $|\arg z| \leq \frac{1}{2}\pi$. Here A is a positive constant depending on ϵ but not on z or on ν . Hence there is a positive number $N = N(\epsilon)$ such that the series

$$\psi_0(z, \nu) = \sum_{k=0}^{\infty} \psi_0^{(k)}(z)$$

is uniformly and absolutely convergent for z in $\mathfrak{D}_0$ and $\nu \geq N$. It follows that $\psi_0(z, \nu)$ is a regular function of z in the interior of $\mathfrak{D}_0$, and for each such ν is a continuous solution of the integral equation on $\mathfrak{D}_0$. Furthermore

$$(7.6) \quad |\psi_0(z, \nu) - 1| \leq \frac{B}{\nu} \quad \text{if } z \in \mathfrak{D}_0, \quad \nu \geq N$$

where $B = B(\epsilon)$ is independent of z and ν , and for any positive number ρ

$$(7.7) \quad |\psi_0(z, \nu) - 1| \leq \frac{B}{\nu|z|} \quad \text{if } |z| \geq 1, \quad |\arg z| \leq \frac{\pi}{2}, \quad \nu \geq \rho$$

where $B = B(\epsilon, \rho)$ is independent of z and ν .

The closed region $\mathfrak{D}_0$ is the image under the transformation $z = f(x)$ of the closed region $\mathfrak{G}_0$ (Section 6), in the x -plane. Using (4.3), (4.9), (3.12) and (5.2) one obtains from (7.6) and (7.7) two formulas which compare, in $\mathfrak{G}_0$, a solution of (1.1) with an expression involving a solution of (4.1). This result is given in (9.5) and (9.6).

A similar discussion can be carried out for the cases when $m \neq 0$ or $\arg \nu \neq 0$. We restrict our considerations to $-\frac{1}{2}\pi \leq \arg \nu \leq \frac{1}{2}\pi$. For each fixed value of $\arg \nu$ and each $m = -1, 0, 1$ there is a closed region $\mathfrak{D}_m$ on the surface $\mathfrak{Z}$, which depends as before on a small positive parameter ϵ . On $\mathfrak{D}_m$ (4.10) has a continuous solution $\psi_m(z) = \psi_m(z, \nu)$ such that

$$(7.8) \quad |\psi_m(z, \nu) - 1| \leq \frac{B}{|\nu|} \quad \text{if } z \in \mathfrak{D}_m, \quad |\nu| \geq N$$

where $B = B(\epsilon)$ and $N = N(\epsilon)$ are independent of z and $|\nu|$. These constants depend of course on m and on the fixed value of $\arg \nu$. Again as before there is a closed subregion $\mathfrak{F}_m$ of $\mathfrak{D}_m$ with the property that for any positive number ρ

$$(7.9) \quad |\psi_m(z, \nu) - 1| \leq \frac{B}{|\nu z|} \quad \text{if } z \in \mathfrak{F}_m, \quad |\nu| \geq \rho$$

where $B = B(\epsilon, \rho)$ is independent of z and $|\nu|$. The closed regions $\mathfrak{D}_m$ and $\mathfrak{F}_m$ are the images under the transformation $z = f(x)$ of the closed regions $\mathfrak{G}_m$ and $\mathfrak{S}_m$ of the x -plane, described in Section 6.

As we have seen, when $\arg \nu = 0$ the condition (7.2) completely determines the domain $\mathfrak{D}_0$. When $\arg \nu \neq 0$ or $m \neq 0$ the shadow condition (7.3) comes into play, and branch-points and zeros of $f(x)$ are encountered and must be excluded. Some of these complications may be avoided by taking domains which are smaller than the maximal domains of validity. The domains of Section 6 correspond to such unnecessarily restrictive but comparatively simple domains on $\mathfrak{Z}$.

8. Asymptotic forms of parabolic cylinder functions. In order to identify the solutions

$$y_m(x) = \left(\frac{dx}{dz}\right)^{\frac{1}{2}} w_m(z)$$

(see (4.3) and (4.8)) in terms of parabolic cylinder functions, it is convenient to introduce the notation

$$(8.1) \quad \begin{aligned} \phi_0(x) &= D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}), & \phi_1(x) &= D_{-\nu-\frac{1}{2}}(2ix\sqrt{\nu}), \\ \phi_{-1}(x) &= D_{-\nu-\frac{1}{2}}(-2ix\sqrt{\nu}). \end{aligned}$$

Since $y_m(x)$, $\phi_m(x)$ are solutions of (1.1) which for a fixed ν approach zero as x tends to infinity in a direction given by $\arg(\nu^{\frac{1}{2}}x) = -\frac{1}{2}m\pi$, it follows by the properties of $\phi_m(x)$ ($m = 0, 1, -1$) discussed in Section 2, that

$$(8.2) \quad y_m(x) = \alpha_m \phi_m(x).$$

Carrying out the integration in (4.3), the following result is easily obtained

$$(8.3) \quad z = f(x) = \frac{1}{2^{\frac{1}{2}}} \left[x^2 - \frac{1}{2} - \log 2x + o\left(\frac{1}{x^2}\right) \right],$$

$$|x| > 1, -\pi < \arg x < \pi.$$

Keeping ν fixed in (8.2) and letting $x \rightarrow \infty$ in the direction $\arg(\nu^{\frac{1}{2}}x) = -\frac{1}{2}m\pi$, we obtain by (2.4), and the substitution of (8.3) in (3.22), in conjunction with (7.7), that

$$(8.4) \quad \alpha_m = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{(-1)^m \frac{1}{2}\nu} \nu^{(-1)^m - \frac{1}{2}\nu + \frac{1}{2}} e^{-\frac{1}{2}m\pi i [(-1)^m \nu - \frac{1}{2}]}.$$

Hence

$$(8.5) \quad y_0(x) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}\nu} \nu^{-\frac{1}{2}\nu + \frac{1}{2}} D_{\nu-\frac{1}{2}}(2x\sqrt{\nu})$$

$$(8.6) \quad y_1(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\nu} \nu^{\frac{1}{2}\nu + \frac{1}{2}} e^{\frac{1}{2}\pi i(\nu + \frac{1}{2})} D_{-\nu-\frac{1}{2}}(2ix\sqrt{\nu})$$

$$(8.7) \quad y_{-1}(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\nu} \nu^{\frac{1}{2}\nu + \frac{1}{2}} e^{-\frac{1}{2}\pi i(\nu + \frac{1}{2})} D_{-\nu-\frac{1}{2}}(-2ix\sqrt{\nu}).$$

As explained in Section 7, we can express (7.6) in terms of x . Using (8.5), (8.6), (8.7) we obtain

$$(8.8) \quad \begin{aligned} D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}) &= (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\nu} \nu^{\frac{1}{2}\nu - \frac{1}{2}} \left(\frac{x^2 - 1}{2} \right)^{-\frac{1}{2}} \left(\frac{3}{2}f(x) \right)^{\frac{1}{2}} \\ &\quad \cdot Ai(2\nu^{\frac{1}{2}} \left(\frac{3}{2}f(x) \right)^{\frac{1}{2}}) \left(1 + O\left(\frac{1}{|\nu|}\right) \right), \end{aligned} \quad x \in \mathfrak{G}_0$$

$$(8.9) \quad \begin{aligned} D_{-\nu-\frac{1}{2}}(2ix\sqrt{\nu}) &= (2\pi)^{\frac{1}{2}} e^{\frac{1}{2}\nu} \nu^{-\frac{1}{2}\nu - \frac{1}{2}} e^{-\frac{1}{2}\pi i(\nu + \frac{1}{2})} \left(\frac{x^2 - 1}{2} \right)^{-\frac{1}{2}} \left(\frac{3}{2}f(x) \right)^{\frac{1}{2}} \\ &\quad \cdot Ai(2\nu^{\frac{1}{2}} \left(\frac{3}{2}f(x) \right)^{\frac{1}{2}} \omega) \left(1 + O\left(\frac{1}{|\nu|}\right) \right), \end{aligned} \quad x \in \mathfrak{G}_1$$

$$\begin{aligned}
 (8.10) \quad D_{-\nu-\frac{1}{2}}(-2ix\sqrt{\nu}) &= (2\pi)^{\frac{1}{2}} e^{\frac{1}{2}\nu} \nu^{-\frac{1}{2}\nu-\frac{1}{2}} e^{\frac{1}{2}i\pi(\nu+\frac{1}{2})} \left(\frac{x^2-1}{2}\right)^{-\frac{1}{2}} (\tfrac{3}{2}f(x))^{\frac{1}{2}} \\
 &\quad \cdot Ai(2\nu^{\frac{1}{2}}(\tfrac{3}{2}f(x))^{\frac{1}{2}}\omega^{-1}) \left(1 + O\left(\frac{1}{|\nu|}\right)\right), \quad x \in \mathfrak{G}_{-1}.
 \end{aligned}$$

Similarly, from (7.7) we have that $O(1/|\nu|)$ may be replaced by $O(1/|\nu x^2|)$ in these formulas if x is restricted to $\mathfrak{G}_m$. The many-valued functions, together with the O -symbols, occurring in above formulas are defined in Section 9.

By (3.3)

$$Ai(0) = \frac{1}{3^{\frac{1}{2}}\Gamma(\frac{2}{3})},$$

and thus we have for $x = 1$

$$(8.11) \quad D_{\nu-\frac{1}{2}}(2\sqrt{\nu}) = \frac{(2\pi)^{\frac{1}{2}}}{3^{\frac{1}{2}}\Gamma(\frac{2}{3})} e^{-\frac{1}{2}\nu} \nu^{\frac{1}{2}\nu-\frac{1}{2}} \left(1 + O\left(\frac{1}{|\nu|}\right)\right)$$

$$(8.12) \quad D_{-\nu-\frac{1}{2}}(2i\sqrt{\nu}) = \frac{(2\pi)^{\frac{1}{2}}}{3^{\frac{1}{2}}\Gamma(\frac{2}{3})} e^{\frac{1}{2}\nu} \nu^{-\frac{1}{2}\nu-\frac{1}{2}} e^{-\frac{1}{2}i\pi(\nu+\frac{1}{2})} \left(1 + O\left(\frac{1}{|\nu|}\right)\right).$$

9. Summary of results. The object of this section is to provide a comprehensive set of asymptotic approximations for the solutions $D_{\nu-\frac{1}{2}}(2x\sqrt{\nu})$, $D_{-\nu-\frac{1}{2}}(2ix\sqrt{\nu})$, $D_{-\nu-\frac{1}{2}}(-2ix\sqrt{\nu})$ of equation (1.1). These are given by the relations (9.5)–(9.12).

In what follows it will be understood that in each of the regions $\mathfrak{G}_m$, the functions

$$(9.1) \quad \left(\frac{x^2-1}{2}\right)^{-\frac{1}{2}}, \quad (\tfrac{3}{2}f(x))^{\frac{1}{2}}, \quad (\tfrac{3}{2}f(x))^{\frac{1}{2}}$$

are defined by analytic continuation from real and positive values on that part of the real axis to the right of $x = 1$. There are two exceptions to this rule. If $\arg \nu = \frac{1}{2}\pi$ (or if $\arg \nu + \epsilon > \frac{1}{2}\pi$) $\mathfrak{G}_1$ does not contain this segment, and if $\arg \nu = -\frac{1}{2}\pi$ (or if $\arg \nu - \epsilon < \frac{1}{2}\pi$) $\mathfrak{G}_{-1}$ does not contain this segment. In the first of these two cases we continue the functions (9.1) from real and positive values on $1 < x < \infty$ into the fourth quadrant, and in the second case into the first quadrant.

We define for $m = -1, 0, 1$ new functions $\Phi_m(x)$ in the domains $\mathfrak{G}_m$ by

$$(9.2) \quad \Phi_m(x) = \frac{1}{\alpha_m} \left(\frac{x^2-1}{2}\right)^{-\frac{1}{2}} (\tfrac{3}{2}f(x))^{\frac{1}{2}} Ai(2\nu^{\frac{1}{2}}(\tfrac{3}{2}f(x))^{\frac{1}{2}}\omega^m)$$

where $\omega = \exp \frac{2}{3}\pi i$. The α_m are given by (8.4).

We denote by $\Psi_1(x)$ the function which is the restriction of $\Phi_1(x)$ from $\mathfrak{G}_1$ to $\mathfrak{R}$ and by $\Psi_{-1}(x)$ the function which is the restriction of $\Phi_{-1}(x)$ from $\mathfrak{G}_{-1}$ to $\mathfrak{R}$.

If x is in $\mathfrak{G}_m^*$, $m = -1, 0, 1$ then $-x$ is in $\mathfrak{G}_m$ and we define a function Φ_m^* in $\mathfrak{G}_m^*$ by the formula

$$(9.3) \quad \Phi_m^*(x) = \Phi_m(-x).$$

Similarly if x is in $\mathfrak{R}^*$ then $-x$ is in $\mathfrak{R}$ and we define $\Psi_m^*(x)$, $m = 1, -1$ in $\mathfrak{R}^*$ by

$$(9.4) \quad \Psi_m^*(x) = \Psi_m(-x).$$

By means of (8.8), (8.9), (8.10), and the above formulas we are in a position to obtain the asymptotic approximations set out below. After they are stated, it will be indicated how they can be obtained. The domains $\mathfrak{G}_m$, $\mathfrak{G}_m^*$, $\mathfrak{S}_m$, $\mathfrak{R}$, $\mathfrak{M}_m$, $\mathfrak{M}_m^*$ have been described in Section 6.

We make the following convention with respect to the O -symbols $O(1/|\nu|)$, $O(1/|\nu x^2|)$. These always occur in formulas in which it is understood that $\arg \nu$ is fixed, and x lies in a domain $\mathfrak{E}$. The symbol $O(1/|\nu|)$ denotes a function $f(x, \nu)$ such that

$$|f(x, \nu)| \leq \frac{B}{|\nu|}$$

for all x in $\mathfrak{E}$ and all sufficiently large $|\nu|$, where B is a constant independent of x and $|\nu|$. The symbol $O(1/|\nu x^2|)$ denotes a function $f(x, \nu)$ such that given any positive number ρ , there is a constant B satisfying

$$|f(x, \nu)| \leq \frac{B}{|\nu x^2|}$$

for all x in $\mathfrak{E}$, all $|\nu| \geq \rho$.

We now list the results. In equations which contain double signs either all upper or all lower signs should be taken. Rough sketches of the various regions in the x -plane are also given. These sketches typify the various regions for $0 \leq \arg \nu \leq \frac{1}{2}\pi$ and are constructed for $\arg \nu = \frac{1}{6}\pi$. In addition the diagram of $\mathfrak{G}_0$ for $\arg \nu = 0$ is drawn since it has a special character.

$$(9.5) \quad D_{\nu-\frac{1}{4}}(2x\sqrt{\nu}) = \Phi_0(x) \left[1 + O\left(\frac{1}{|\nu x^2|}\right) \right], \quad x \in \mathfrak{S}_0$$

$$D_{-\nu-\frac{1}{4}}(\pm 2ix\sqrt{\nu}) = \Phi_{\pm 1}(x) \left[1 + O\left(\frac{1}{|\nu x^2|}\right) \right], \quad x \in \mathfrak{S}_{\pm 1}$$

$$(9.6) \quad D_{\nu-\frac{1}{4}}(2x\sqrt{\nu}) = \Phi_0(x) \left[1 + O\left(\frac{1}{|\nu|}\right) \right], \quad x \in \mathfrak{G}_0$$

$$D_{-\nu-\frac{1}{4}}(\pm 2ix\sqrt{\nu}) = \Phi_{\pm 1}(x) \left[1 + O\left(\frac{1}{|\nu|}\right) \right], \quad x \in \mathfrak{G}_{\pm 1}$$

$$(9.7) \quad D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu}) = \Phi_{\mp 1}^*(x) \left[1 + O\left(\frac{1}{|\nu|}\right) \right], \quad x \in \mathfrak{G}_{\mp 1}^*$$

$$D_{\nu-\frac{1}{2}}(-2x\sqrt{\nu}) = \Phi_0^*(x) \left[1 + O\left(\frac{1}{|\nu|}\right) \right], \quad x \in \mathfrak{G}_0^*$$

$$(9.8) \quad \begin{aligned} D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu}) &= e^{\mp \frac{1}{2}\pi i(\nu-\frac{1}{2})} \left[\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \Phi_0(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right. \\ &\quad \left. - e^{\mp \frac{1}{2}\pi i(\nu-\frac{1}{2})} \Phi_{\mp 1}(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right], \quad x \in \mathfrak{M}_{\mp 1} \end{aligned}$$

$$(9.9) \quad \begin{aligned} D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu}) &= e^{\pm \frac{1}{2}\pi i(\nu-\frac{1}{2})} \left[\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \Phi_0^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right. \\ &\quad \left. - e^{\pm \frac{1}{2}\pi i(\nu-\frac{1}{2})} \Phi_{\pm 1}^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right], \quad x \in \mathfrak{M}_{\pm 1}^* \end{aligned}$$

$$(9.10) \quad \begin{aligned} D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}) &= \frac{\Gamma(\nu+\frac{1}{2})}{(2\pi)^{\frac{1}{2}}} \left[e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})} \Psi_1(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right. \\ &\quad \left. + e^{-\frac{1}{2}\pi i(\nu-\frac{1}{2})} \Psi_{-1}(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right], \quad x \in \mathfrak{R} \end{aligned}$$

$$(9.11) \quad \begin{aligned} D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}) &= \frac{\Gamma(\nu+\frac{1}{2})}{(2\pi)^{\frac{1}{2}}} \left[e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})} \Psi_{-1}^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right. \\ &\quad \left. + e^{-\frac{1}{2}\pi i(\nu-\frac{1}{2})} \Psi_1^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \right], \quad x \in \mathfrak{R}^* \end{aligned}$$

$$(9.12) \quad \begin{aligned} D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}) &= e^{\pm \pi i(\nu-\frac{1}{2})} \Phi_0^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right) \\ &\quad + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu+\frac{1}{2})} e^{\pm \frac{1}{2}\pi i(\nu+\frac{1}{2})} \Phi_{\pm 1}^*(x) \left(1 + O\left(\frac{1}{|\nu|}\right) \right), \quad x \in \mathfrak{M}_{\pm 1}^*. \end{aligned}$$

The identity

$$D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu}) = D_{-\nu-\frac{1}{2}}(\mp 2i(-x)\sqrt{\nu})$$

enables us to obtain the values of $D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu})$ in $\mathfrak{E}^*$ from the values of $D_{-\nu-\frac{1}{2}}(\mp 2ix\sqrt{\nu})$ in $\mathfrak{E}_1$ where $\mathfrak{E}$ stands for any of the domains $\mathfrak{G}_{\pm 1}$, $\mathfrak{M}_{\pm 1}$ and $\mathfrak{R}$.

We conclude by indicating how formulas (9.5)–(9.12) were obtained:

(9.5) is (8.8), (8.9), (8.10) rewritten in a different notation and modified in accordance with the remark following these equations.

(9.6) is (8.8), (8.9), (8.10) rewritten in a different notation.

(9.7) is derived from (9.6). The second part is an auxiliary result used to obtain (9.12).

(9.8) is derived from (2.6) and (9.6).

(9.9) is derived from (9.8).

(9.10) is derived from (2.6) and (9.6).

(9.11) is derived from (2.6) and (9.7).

(9.12) is derived from (2.6) and second part of (9.7).

Domains in the x -plane $\mathcal{G}_0$ for $\arg v = 0$ $\mathcal{G}_0$ for $\arg v = \pi/6$ $\mathcal{G}_1$ for $\arg v = \pi/6$ $\mathcal{G}_{-1}$ for $\arg v = \pi/6$

Domains in the x -plane

ζ_0 for $\arg \nu = \pi/6$

ζ_1 for $\arg \nu = \pi/6$

ζ_{-1} for $\arg \nu = \pi/6$

Domains in the z -plane $\Re$ for $\arg v = \pi/6$ $\Re_1$ for $\arg v = \pi/6$ $\Re_{-1}$ for $\arg v = \pi/6$

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