New Boundary Value Problems for the Wave Equation and Equations of Mixed Type

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1. Introduction. For hyperbolic equations in two independent variables,

$$u_{xy} + au_x + bu_y + cu + f = 0, \quad (1)$$

it is well known that certain boundary value problems, frequently called “problems of Goursat type,” are correctly set. In these problems values of the unknown function $u(x, y)$ are prescribed along two intersecting curves, thereby determining a solution in some “influence domain”. For example let γ_1 , given by $y = f_1(x)$, $0 \leq x \leq x_0$, and γ_2 : $y = f_2(x)$, $0 \leq x \leq x_0$, be two monotone increasing curves each starting from the origin and extending into the first quadrant. We assume that γ_1 and γ_2 have no points in common except the origin and for definiteness suppose γ_1 lies above γ_2 . Let $\varphi_1(x)$, $\varphi_2(x)$, $0 \leq x \leq x_0$, be two given smooth (say twice differentiable) functions. The problem of Goursat consists in the determination of a solution $u(x, y)$ of (1) such that

$$u[x, f_1(x)] = \varphi_1(x), \quad u[x, f_2(x)] = \varphi_2(x), \quad 0 \leq x \leq x_0.$$

The solution will be determined in the rectangle $0 \leq x \leq x_0$, $0 \leq y \leq y_0$, where $y_0 = f_1(x_0)$. This rectangle is the “influence domain” for the above problem. The results remain valid if either γ_1 or γ_2 (or both) coincides with a characteristic, *i.e.* is a segment of one of the coordinate axes.

It is not immediately evident how one would generalize a problem of Goursat type to hyperbolic equations in more than two variables. With two independent variables the quantities x and y are interchangeable, *i.e.* either one may be treated as the time axis. For three or more variables however the time axis is distinguished and the problem becomes more complicated. Progress toward the

solution of problems of Goursat type for equations in more than two variables has been made only in the case of the wave equation

$$u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = u_{tt}. \quad (2)$$

Soboleff [7] has discussed the following boundary value problem for this equation. Let C be a retrograde characteristic cone of (2) with vertex at a point P . Let C_1 be a conical surface with vertex at P and otherwise lying interior to C . This cone is to have the property that any line parallel to the t -axis intersects it in at most one point. Soboleff showed that if boundary values are assigned on the surface C_1 a solution of (2) is determined in its interior. The same problem has been treated recently by Garding [1], who employed a somewhat different method.

It is the purpose of this paper to show how problems of Goursat type other than those considered by Soboleff and Garding may be stated and solved for equation (2). These results lead in a natural way to certain generalizations of the Tricomi problem for equations of mixed elliptic-hyperbolic type.

2. The wave equation. We restrict ourselves to the wave equation in three variables although the results may easily be extended to the wave equation in any number of variables. Thus we consider the equation

$$u_{xx} + u_{yy} = u_{zz}. \quad (3)$$

The two characteristic cones

$$\begin{aligned} x^2 + y^2 &= (z - z_0)^2, & z_0 > 0 \\ (x - x_0)^2 + (y - y_0)^2 &= z^2 \end{aligned}$$

and the plane

$$z = 0$$

enclose a domain D whenever the numbers x_0, y_0, z_0 satisfy the condition $x_0^2 + y_0^2 < z_0^2$. We denote by S_1 that part of the cone $(x - x_0)^2 + (y - y_0)^2 = z^2$ coinciding with the boundary of D , by S_2 the part of $x^2 + y^2 = (z - z_0)^2$ coinciding with the boundary of D , and by S_3 the circle $x^2 + y^2 \leq z_0^2$ lying in the plane $z = 0$. Let $F(x, y), G(x, y)$ be given continuously differentiable functions defined for $0 \leq x^2 + y^2 \leq z_0^2$. We seek a solution of (3) in D satisfying the boundary conditions

$$\begin{aligned} u(x, y, 0) &= F(x, y) & \text{for } x, y \text{ on } S_3, \\ u(x, y, z_0 - \sqrt{x^2 + y^2}) &= G(x, y) & \text{for } x, y \text{ on } S_2. \end{aligned} \quad (4)$$

It is assumed that F and G coincide on the circle common to S_2 and S_3 . This problem we designate *Problem A*. The determination of a solution of (3) in D

subject to the boundary conditions

$$\begin{aligned} u(x, y, 0) &= F(x, y) && \text{for } x, y \text{ on } S_3, \\ u(x, y, \sqrt{(x - x_0)^2 + (y - y_0)^2}) &= G(x, y) && \text{for } x, y \text{ on } S_1 \end{aligned} \quad (5)$$

we call *Problem B*. It is clear that Problems A and B are generalizations of the two-dimensional problem in which the value of the unknown function is prescribed along one characteristic and one non-characteristic curve.

Let D_1 be the domain enclosed by the three surfaces: the cone (not characteristic)

$$x^2 + y^2 = \alpha^2 \left(z - \frac{z_0}{\alpha} \right)^2, \quad z_0 > 0, \alpha > 1;$$

the characteristic cone $(x - x_0)^2 + (y - y_0)^2 = z^2$; and the plane $z = 0$. Again we must impose the condition $x_0^2 + y_0^2 < z_0^2$. We denote by T_1 that part of the cone $(x - x_0)^2 + (y - y_0)^2 = z^2$ coinciding with the boundary of D_1 , by T_2 the part of $x^2 + y^2 = \alpha^2(z - z_0/\alpha)^2$ coinciding with the boundary of D_1 , and by T_3 the circle $x^2 + y^2 \leq z_0^2$ lying in the plane $z = 0$. *Problem C* is that of determining a solution of (3) in D_1 satisfying the boundary conditions

$$\begin{aligned} u(x, y, 0) &= F(x, y) && \text{for } x, y \text{ on } T_3 \\ u\left(x, y, \frac{z_0}{\alpha} - \frac{1}{\alpha} \sqrt{x^2 + y^2}\right) &= G(x, y) && \text{for } x, y \text{ on } T_2. \end{aligned} \quad (6)$$

A fourth problem that could be considered is one that arises from the domain D_2 formed by the cones

$$\begin{aligned} (x - x_0)^2 + (y - y_0)^2 &= \beta^2 z^2, && \beta > 1 \\ x^2 + y^2 &= (z - z_0)^2, && z_0 > 0 \end{aligned}$$

and the plane $z = 0$. In analogy with the preceding problems we have the bounding surfaces U_1, U_2, U_3 . *Problem D* corresponds to finding a solution in D_2 satisfying the boundary conditions

$$\begin{aligned} u(x, y, 0) &= F(x, y) && \text{for } x, y \text{ on } U_3, \\ u\left(x, y, \frac{1}{\beta} \sqrt{(x - x_0)^2 + (y - y_0)^2}\right) &= G(x, y) && \text{for } x, y \text{ on } U_1. \end{aligned}$$

It should be remarked that the surfaces considered in Problems A–D are merely typical. We could equally well take in place of the plane $z = 0$ a general space-like surface. The non-characteristic circular cones may be replaced by more general conical surfaces, and the characteristic cones by characteristic surfaces.

A solution of Problem A has been obtained by the author [4] by use of a gener-

alization of Riemann's method and an application of a mean value theorem of Asgeirsson. Although it was not stated explicitly in [4] the same methods can be employed to obtain solutions of Problems B, C, and D. Since the mean value theorem of Asgeirsson is limited to equations with constant coefficients the proof employed in [4] cannot be extended to equations with variable coefficients.

Theorem 1 below gives a new proof for the uniqueness of the solution to Problem A. Theorem 2 is a uniqueness theorem for Problem C. The proofs of these theorems have the virtue that they can be extended to hyperbolic equations with variable coefficients. The methods are then applied to a class of equations with variable coefficients and new results are obtained for equations of mixed elliptic-hyperbolic type.

Theorem 1. *There exists at most one solution of (3) in D satisfying the boundary conditions (4).*

Theorem 2. *There exists at most one solution of (3) in D satisfying the boundary conditions (6).*

Proof of Theorem 1. For convenience we select $x_0 = y_0 = 0$ and $z_0 = 1$. We wish to show that if F and G vanish identically then $u(x, y, z)$ vanishes identically. We consider an expression of the form

$$2 \iiint_D (au + bu_x + cu_y + du_z)(u_{xx} + u_{yy} + u_{zz}) dV \quad (7)$$

where a, b, c , and d are functions yet to be determined. If u satisfies (3) then the above expression will vanish. We have the identities

$$\begin{aligned} 2auu_{xx} &= 2(auu_x)_x - 2au_x^2 - (a_x u^2)_x + a_{xx} u^2 \\ 2auu_{yy} &= 2(auu_y)_y - 2au_y^2 - (a_y u^2)_y + a_{yy} u^2 \\ 2auu_{zz} &= 2(auu_z)_z - 2au_z^2 - (a_z u^2)_z + a_{zz} u^2 \\ 2bu_x u_{xx} &= (bu_x^2)_x - b_x u_x^2 \\ 2bu_x u_{yy} &= 2(bu_x u_y)_y - 2b_y u_x u_y - (bu_x^2)_x + b_x u_y^2 \\ 2bu_x u_{zz} &= 2(bu_x u_z)_z - 2b_z u_x u_z - (bu_x^2)_x + b_x u_z^2 \\ 2cu_y u_{xx} &= 2(cu_x u_y)_x - 2c_x u_x u_y - (cu_x^2)_y + c_y u_x^2 \\ 2cu_y u_{yy} &= (cu_y^2)_y - c_y u_y^2 \\ 2cu_y u_{zz} &= 2(cu_y u_z)_z - 2c_z u_y u_z - (cu_y^2)_y + c_y u_z^2 \\ 2du_z u_{xx} &= 2(du_x u_z)_x - 2d_x u_x u_z - (du_x^2)_z + d_x u_x^2 \\ 2du_z u_{yy} &= 2(du_y u_z)_y - 2d_y u_y u_z - (du_y^2)_z + d_y u_y^2 \\ 2du_z u_{zz} &= (du_z^2)_z - d_z u_z^2. \end{aligned}$$

We denote the boundary of D by B and let $x_n = \partial x / \partial n$, $y_n = \partial y / \partial n$, $z_n = \partial z / \partial n$ be the normal derivatives of x , y , z , on B with the outward normal direction considered positive. Inserting the above identities in (7), using (3), and applying Green's theorem we find

$$\begin{aligned}
 0 &= \iiint_D (a_{xx} + a_{yy} - a_{zz}) u^2 dV + \iiint_D \{[-2a - b_x + c_y + d_z] u_x^2 \\
 &\quad + [-2a + b_x - c_y + d_z] u_y^2 + [2a - b_x - c_y + d_z] u_z^2 \\
 &\quad + 2[-b_y - c_x] u_x u_y + 2[b_z - d_x] u_x u_z + 2[c_z - d_y] u_y u_z\} dV \\
 &\quad + \iint_B \{[2auu_x - a_x u^2] x_n + [2auu_y - a_y u^2] y_n + [-2auu_z + a_z u^2] z_n\} dS \\
 &\quad + \iint_B \{[bx_n - cy_n - dz_n] u_x^2 + [-bx_n + cy_n - dz_n] u_y^2 + [bx_n + cy_n - dz_n] u_z^2 \\
 &\quad + 2(cx_n + by_n) u_x u_y + 2(dx_n - bz_n) u_x u_z + 2(dy_n - cz_n) u_y u_z\} dS \\
 &= I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

We shall insert the condition $u \equiv 0$ on S_2 and S_3 in the above integrals and then select a , b , c and d so that all the integrands become positive definite. This in turn will imply that u vanishes throughout D . We make the following selections:

$$a = \frac{1}{2}, \quad b = x, \quad c = y, \quad d = z.$$

Then I_1 and I_2 vanish identically. From the fact that u vanishes on S_2 and S_3 and that $a \equiv \frac{1}{2}$, we have

$$I_3 = \iint_{S_1} (uu_x x_n + uu_y y_n - uu_z z_n) dS.$$

However the integrand is merely the derivative of $\frac{1}{2}u^2$ in the direction of one of the generators of the cone S_1 and hence one of the integrations can be performed. Since u vanishes at the upper and lower limits of S_1 we see that $I_3 = 0$. Thus there remains only the integral I_4 . From the fact that $u = 0$, $x_n = y_n = 0$ and $z = 0$ on S_3 we find that the portion of I_4 taken over S_3 vanishes. We now consider the portion of I_4 evaluated over S_2 . We write this integral in the form

$$\begin{aligned}
 &\iint_{S_2} (bu_x + cu_y + du_z)(u_x x_n + u_y y_n - u_z z_n) dS \\
 &\quad - \iint_{S_2} [(u_x y_n - u_y x_n)(-bu_y - cu_x) + (u_x z_n - u_z x_n)(bu_z + du_x) \\
 &\quad + (u_y z_n - u_z y_n)(cu_z + du_y)] dS. \tag{8}
 \end{aligned}$$

Since S_2 is characteristic, the vector $(x_n, y_n, -z_n)$ is parallel to S_2 . Also since $du = u_x dx + u_y dy + u_z dz$ on S_2 , the vector (u_x, u_y, u_z) is normal to S_2 . Hence the first integral above vanishes. Since (u_x, u_y, u_z) and (x_n, y_n, z_n) are parallel vectors their vector product vanishes and hence the second integral above also vanishes. We note that this holds for arbitrary b, c, d . Thus there only remains the integral I_4 taken along the surface S_1 : $x^2 + y^2 = z^2$. We have

$$\begin{aligned} I_4 &= -\frac{2}{\sqrt{2}} \iint_{S_1} \frac{1}{z} [x^2 u_x^2 + y^2 u_y^2 + z^2 u_z^2 + 2xyu_x u_y + 2xz u_x u_z + 2yz u_y u_z] dS \\ &= -\frac{2}{\sqrt{2}} \iint_{S_1} \frac{1}{z} [xu_x + yu_y + zu_z]^2 dS = 0. \end{aligned}$$

Hence we find that $xu_x + yu_y + zu_z = 0$ on S_1 . This implies that the derivative of u in the direction of each of the generators of the cone S_1 vanishes, or that $u = \text{const.}$ along each such generator. Since $u = 0$ at the ends of these line segments we conclude that u vanishes on S_1 . From this it follows, by solving the characteristic initial value problem for S_1 , that u vanishes in the direct cone determined by S_1 , and hence on the entire surface of the cone $x^2 + y^2 = (z - 1)^2$, $0 \leq z \leq 1$. Solving in turn the characteristic initial value problem for this cone, we conclude that u vanishes in the retrograde cone determined by this surface and therefore in D .

Proof of Theorem 2. We make the same choices for a, b, c , and d . In this case however an additional argument is required for the first integral in (8) since the vectors (u_x, u_y, u_z) and $(x_n, y_n, -z_n)$ are no longer normal. Taking into account the form of $(x_n, y_n, -z_n)$ we have the following expression for this integral:

$$\begin{aligned} - \iint_{T_2} (xu_x + yu_y + zu_z) \left(xu_x + yu_y + \alpha^2 \left(z - \frac{1}{\alpha} \right) u_z \right) \\ \cdot (1 + \alpha^2)^{-\frac{1}{2}} \left(z - \frac{1}{\alpha} \right)^{-1} \frac{1}{\alpha} dS, \end{aligned}$$

which may be written

$$\begin{aligned} - \iint_{T_2} \left[xu_x + yu_y + \left(z - \frac{1}{\alpha} \right) u_z + \frac{1}{\alpha} u_z \right] \\ \cdot \left[xu_x + yu_y + \left(z - \frac{1}{\alpha} \right) u_z + (\alpha^2 - 1) \left(z - \frac{1}{\alpha} \right) u_z \right] (1 + \alpha^2)^{-\frac{1}{2}} \left(z - \frac{1}{\alpha} \right)^{-1} \frac{1}{\alpha} dS. \end{aligned}$$

Since the vectors (u_x, u_y, u_z) and $(x, y, z - 1/\alpha)$ are orthogonal the last expression reduces to

$$- \iint_{T_2} \frac{1}{\alpha^2} (\alpha^2 - 1) (1 + \alpha^2)^{-\frac{1}{2}} u_z^2 dS.$$

Thus the only effect is that an additional non-positive integral is to be taken into account. Therefore u vanishes on T_1 and u together with its first derivatives vanishes on T_2 . Since T_2 is a space-like surface we see that u vanishes identically.

3. Equations of mixed type. In this section we consider equations of the form

$$K(z)(u_{xx} + u_{yy}) + u_{zz} = 0 \quad (9)$$

where $K(z)$ is a twice continuously differentiable monotone increasing function of z with $K(0) = 0$. The results of this section are stated for three independent variables although in fact they hold equally well for equations of the form

$$K(t)(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}) + u_{tt} = 0.$$

Equation (9) is elliptic for $z > 0$, hyperbolic for $z < 0$ and parabolic on the plane $z = 0$. For equations of this type there arises the problem: what kind of boundary value problem is solvable for domains which include both elliptic and hyperbolic regions of the equation? In the case of two independent variables it is well known that for equations of the form

$$K(y)u_{xx} + u_{yy} = 0 \quad (10)$$

with $K(y) \gtrless 0$ according as $y \gtrless 0$, the *Tricomi problem* is correctly set.

FIGURE 1

Consider a domain consisting of a simple arc Γ_1 lying in the upper half-plane with endpoints on the x -axis (Figure 1). From these endpoints we draw the characteristics Γ_2 , Γ_3 of (10) which intersect. The curves Γ_1 , Γ_2 , Γ_3 enclose a domain D . The Tricomi problem consists in determining the unique solution of

(10) in D which assumes prescribed values on Γ_1 and Γ_2 . The Tricomi problem has been shown to be correctly set for a wide class of domains D and functions $K(y)$. Existence and uniqueness theorems have been obtained by Tricomi [8] for $K(y) = y$, Gellerstedt [2] for $K(y) = y^n$, and the author [5, 6] for more general $K(y)$. Uniqueness theorems under somewhat different assumptions on the boundary of the domain have been obtained by Morawetz [3]. In the uniqueness proof given in [5] the dependence on the domain revolved about the size of y_M , the maximum ordinate of Γ_1 . The dependence on $K(y)$ hinged on the convexity of the function $(-K)^{-\frac{1}{2}}$ (i.e. depended only on the negative values of $K(y)$).

A problem somewhat more general than the Tricomi problem has also been solved. Let Γ_1 be as before and P be an arbitrary point on the x -axis between the endpoints of Γ_1 . Draw the characteristics $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ of (10) as shown in Figure 2. The problem concerns the determination of a solution of (10) with values prescribed on Γ_1, Γ_2 , and Γ_5 .

FIGURE 2

While the Tricomi problem does not seem to lend itself to a generalization to more than two variables, the above two-dimensional problem may be generalized to equations of the form (9). Consider a simple smooth surface S_1 lying in the upper half-space, $z > 0$, which has for its boundary a simple, closed rectifiable curve C in the plane $z = 0$ (Figure 3). Examples would be a hemisphere, semi-ellipsoid or similar surface. The term simple is used to mean not self-intersecting. We denote by C^0 the plane domain in $z = 0$ consisting of the interior of C . The characteristic surfaces of (9) (which we call curvilinear cones, or briefly, cones)

FIGURE 3

are obtained by revolving the curves

$$\frac{dz}{dx} = \pm \frac{1}{\sqrt{-K}}, \quad y = 0$$

about lines parallel to the z -axis. Let P be a point in the half-space $z < 0$ and construct the characteristic retrograde cone K_P having P as its vertex. In this connection we take "retrograde" in the direction of increasing z . The cone K_P will intersect the plane $z = 0$ in a circle C_P . Let D_2 be the set of all points P such that the corresponding circle C_P is in C^0 , and let D'_2 denote its boundary (excluding C^0). The three dimensional domain bounded by S_1 and C^0 we call D_1 . Then the domain $D = D_1 + D_2 + C^0$ is a generalization of the Tricomi domain shown in Figure 1. Such a domain is shown in Figure 3. This domain is bounded by S_1 , D'_2 , and C and the question arises concerning the portion of this boundary

FIGURE 4

on which values of u should be prescribed in order to have a correctly set problem. It is apparent that in analogy with the two-dimensional case we would like to prescribe u on S_1 , C and on part of D'_2 . To determine the portion of D'_2 over which we prescribe u we make the following construction. We select any point P_0 in C^0 and construct the characteristic direct cone K_{P_0} having P_0 as its vertex (Figure 4). This cone will intersect D'_2 in a closed curve Γ . The points on D'_2 between C^0 and Γ form a surface which we denote by S_3 ; the part of K_{P_0} between P_0 and Γ we denote by S_4 . The domain we discuss is the portion of D (denoted D^*) situated above the cone K_{P_0} . This domain is a generalization of the domain shown in Figure 2.

In the two dimensional case the auxiliary function

$$F(z) = 1 + 2 \left(\frac{K}{K'} \right)'$$

was used in [5] to determine the dependence of the uniqueness result upon the function $K(z)$ and the domain. The proof in [5] separated into two cases depending on whether or not F is positive for negative values of the argument. This positiveness is equivalent to the convexity of the function $(-K)^{-\frac{1}{2}}$.

Theorem 3. *Let $K(z)$ be a monotone increasing function of z with a continuous second derivative, $K(0) = 0$, $K'(z) \neq 0$ for $z < 0$. Let D^* be the domain described above. If $F(z) > 0$ for $z \leq 0$ then there exists at most one solution of (9) in D^* assuming prescribed values on S_1 , C and S_3 .*

Proof. We consider the expression

$$J = 2 \iiint_{D^*} (au + bu_x + cu_y + du_z)(Ku_{xx} + Ku_{yy} + u_{zz}) dV.$$

Proceeding as in the proof of Theorem 1 we obtain

$$\begin{aligned} 0 &= \iiint_{D^*} [K(a_{xx} + a_{yy}) + a_{zz}]u^2 dV + \iiint_{D^*} \{[-2aK - Kb_x + Kc_y + (Kd)_z]u_x^2 \\ &\quad + [-2aK + Kb_x - Kc_y + (Kd)_z]u_y^2 + [-2a + b_x + c_y - d_z]u_z^2 \\ &\quad + 2[-Kb_y + Kc_x]u_x u_y + 2[-b_z - Kd_x]u_x u_z + 2[-c_z - Kd_y]u_y u_z\} dV \\ &\quad + \iint_B \{(2aKuu_x - Ka_x u^2)x_n + (2aKuu_y - Ka_y u^2)y_n + (2auu_z - a_z u^2)z_n\} dS \\ &\quad + \iint_B \{[Kbx_n - Kcy_n - Kdz_n]u_x^2 + [-Kbx_n + Kcy_n - Kdz_n]u_y^2 \\ &\quad + [-bx_n - cy_n + dz_n]u_z^2 + 2[Kby_n + Kcx_n]u_x u_y + 2[bz_n + Kdx_n]u_x u_z \\ &\quad + 2[Kdy_n + cz_n]u_y u_z\} dS \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

As before B denotes the boundary of D^* . If $(-K)^{-\frac{1}{2}}$ is a convex function of z for $z \leq 0$ (that is, if $F(z)$ is non-negative) we may select $a \equiv -1$. With this value for a we have $J_1 = 0$. However J_3 is not necessarily zero as in the case of the wave equation. The integral J_3 vanishes on S_1 and S_3 since $u = 0$ on these surfaces. Thus we have

$$J_3 = -2 \iint_{S_4} (K u u_x x_n + K u u_y y_n + u u_z z_n) dS.$$

However the vector (x_n, y_n, z_n) is orthogonal to the vector $(x_n, y_n, z_n/K)$ as this surface is characteristic. We therefore find

$$\begin{aligned} J_3 &= -2 \iint_{S_4} K u \left(u_x x_n + u_y y_n + u_z \frac{z_n}{K} \right) dS \\ &= -2 \iint_{S_4} K u \frac{\partial u}{\partial \nu} dS, \end{aligned}$$

where $\partial/\partial \nu$ is the derivative in the direction of one of the generators of K_{P_0} . Since $K(z)$ vanishes at P_0 and u vanishes on Γ , an integration by parts yields

$$J_3 = \iint_{S_4} u^2 K'(z) dS.$$

We thus obtain a positive definite expression for J_3 .

The quantities b , c , and d are selected as follows:

$$\begin{aligned} d &= -4 \frac{K}{K'}, & b = c &= \frac{1}{\sqrt{2}} \sqrt{-K} d, & z &\leq 0 \\ b = c = d &= 0, & & & z &\geq 0. \end{aligned}$$

With this choice it is easy to verify that J_2 is positive definite. The integral J_4 is taken over the three surfaces S_1 , S_3 , S_4 . From the choice for b , c , d , the integral over S_1 vanishes. Since u vanishes on S_3 the integral taken over this characteristic surface is seen to vanish by the same argument as that given for the wave equation. The integral over S_4 , from the above choice for b , c , d represents a perfect square in u_x , u_y , and u_z and hence is positive definite. Since the sum of these positive integrals is zero the integrand in J_2 vanishes identically and therefore all the first partial derivatives of u are identically zero.

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