The Simple Wave in Rotational Gas Flow

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1. Introduction. The object of this paper is to show how the Legendre contact transformation may be employed to construct a generalisation of the simple wave under rotational conditions. Flows for which the velocity vector reduces to a function of just one independent variable are now commonly referred to as "simple waves". The flows obtained by a theory which originally applies to rotational compressible plane flow when the motion is steady, inviscid, adiabatic and iso-energetic are shown to possess an exact analogue when axially symmetric phenonema are considered. The use of the Legendre transformation when applied to the stream function of Crocco in terms of a parameter λ , which remains unspecified, enables the relevant theory of all such possible motions to be developed without loss of generality, and the success of the method depends on the replacement of the normal stream function by that of Crocco. In certain regions of the generalised simple waves, which may assume a diversity of flow patterns, the solution is not mathematically unique and the singularities which occur in the streamlines indicate a local collapse of the continuous flow theory.

2. The equations of rotational gas flow. The equations governing the steady flow of an inviscid gas are

$$(1) \quad q \operatorname{grad} q - \mathbf{q} \times \mathbf{w} + \rho^{-1} \operatorname{grad} p = 0$$

$$(2) \quad \operatorname{div} (\rho \mathbf{q}) = 0,$$

p denoting the pressure, ρ the density, q the magnitude of the velocity vector $\mathbf{q}$, and $\mathbf{w} = \operatorname{curl} \mathbf{q}$ the vorticity. In addition there is the condition for adiabatic changes of state

$$(3) \quad \mathbf{q} \cdot \text{grad } S = 0,$$

where

$$(4) \quad S = C_v \log (p\rho^{-\gamma})$$

denotes the entropy, C_v the specific heat at constant volume and γ the ratio of specific heats.

The substitution $a^2 = \gamma p/\rho$ in (2), followed by use of (3) and (4) yields

$$(5) \quad \text{div} (\mathbf{q} a^{2/(\gamma-1)}) = 0.$$

Similarly (1) may be arranged, with the use of (4), in the form of Crocco,

$$(6) \quad \text{grad } H - \frac{a^2}{\gamma R} \text{grad } S = \mathbf{q} \times \mathbf{w},$$

where $R = (\gamma - 1)C_v$ and

$$H = \frac{1}{2}q^2 + \frac{a^2}{\gamma - 1}$$

denotes the total energy per unit mass. In steady plane flow, in virtue of (3), H reduces to a function of ψ only but attention will be confined to iso-energetic flows for which H is a constant. Then if c is a constant, the relation

$$(7) \quad \frac{1}{2}q^2 + \frac{a^2}{\gamma - 1} = \frac{1}{2}c^2$$

is valid throughout the entire field of flow. Under these conditions (6) reduces to

$$(8) \quad a^2 \text{grad } S = \gamma(\gamma - 1)C_v \mathbf{w} \times \mathbf{q},$$

which implies the necessary condition

$$(9) \quad \text{curl} (a^{-2} \mathbf{w} \times \mathbf{q}) = 0.$$

Equations (5), (9) together with (7), which gives a^2 directly in terms of q , are the general equations for $\mathbf{q}$ when the motion is steady, rotational and iso-energetic. (5) is automatically satisfied by the use of the stream function of Crocco defined by

$$(10) \quad \text{grad } \psi = Q\mathbf{n}$$

in terms of $Q = qa^{2/(\gamma-1)}$ and $\mathbf{n}$, the unit vector normal to the streamline. In the following theory Crocco's stream function is preferable to the stream density function given by

$$\text{grad } \psi = \rho q \mathbf{n}$$

in that Q is a simple function of q , whereas ρ cannot be so expressed without introducing explicitly either the entropy S or the stream function itself. In plane flow, let $\mathbf{k}$ be a unit vector perpendicular to the plane of motion, so that $\mathbf{w} = w\mathbf{k}$; then (10) inserted in (9) yields

$$\text{grad } \psi \times \text{grad } (wa^{-2\gamma/(\gamma-1)}) = 0,$$

implying the existence of a relation of the form

$$(11) \quad w = G(\psi)a^{2\gamma/(\gamma-1)},$$

where $G(\psi)$ is a function of ψ only. After use of (10), (11), (8) may now be integrated to give the corresponding entropy distribution:

$$(12) \quad S - S_0 = \gamma(\gamma - 1)C_v g(\psi),$$

where $G(\psi) = g'(\psi)$.

3. One parameter velocity fields. When the velocity vector $\mathbf{q}$ reduces to a function of just one independent parameter, denoted by $\lambda = \lambda(\mathbf{r})$, then it follows from (7), (10) that $\text{grad } \psi$ and a^2 must also be expressible in terms of λ only. Let the function Ψ be defined by means of the Legendre contact transformation:

$$\Psi + \mathbf{r} \cdot \mathbf{Q} = \psi,$$

where

$$\mathbf{Q}(\lambda) = \text{grad } \psi.$$

Then

$$d\Psi + (\mathbf{r} \cdot \mathbf{Q}_\lambda) d\lambda = 0,$$

so that Ψ is a function of λ only and

$$(13) \quad \Psi_\lambda + \mathbf{r} \cdot \mathbf{Q}_\lambda = 0.$$

Since Ψ , $\mathbf{Q}$ are functions of λ only, (13) represents a one parameter family of straight lines along each of which λ is constant. Because $\mathbf{q} = \mathbf{q}(\lambda)$, $a = a(\lambda)$ it follows that the acoustic speed and the magnitude and direction of the velocity are all constant along any such line.

The relation

$$(14) \quad \mathbf{Q}(\lambda) = Q\mathbf{n}$$

may be differentiated successively to give

$$(15) \quad a^{(2\gamma-4)/(\gamma-1)} Q_\lambda = (a^2 - q^2)q_\lambda \mathbf{n} - qa^2 \theta_\lambda \mathbf{t}$$

$$(16) \quad Q_{\lambda\lambda} + F(\lambda)Q_\lambda = QK(\lambda),$$

where, for the unit tangent vector to the streamline,

$$\mathbf{t} = (\cos \theta, \sin \theta)$$

$$F(\lambda) = -\frac{d}{d\lambda} (\log Q^2 \theta_\lambda)$$

$$\frac{2Q_\lambda K(\lambda)}{Q^3 \theta_\lambda^2} = \frac{d}{d\lambda} (Q_\lambda^2 Q^{-4} \theta_\lambda^{-2} + Q^{-2}).$$

From (13), (15) it is evident that the lines of constant λ , which may be called parameter lines, are parallel to the local vector

$$(a^2 - q^2)q_\lambda \mathbf{t} + qa^2 \theta_\lambda \mathbf{n}$$

and therefore make with the streamlines the angle Γ given by

$$(17) \quad \cot \Gamma = \frac{q_\lambda (a^2 - q^2)}{qa^2 \theta_\lambda}.$$

This angle, being a function of λ only, is constant along any parameter line.

4. The vorticity and velocity. Martin [1] has shown that for flows of the type considered the entropy distribution cannot be arbitrarily assigned but is restricted to two special forms. The function $G(\psi)$ occurring in (11) will now be demonstrated to fall into one of two forms A or A/ψ , where A is a constant. In reconciling the results however, it must be remembered that ψ refers to Crocco's stream function whereas Martin retains the original stream function defined in terms of the density.

By taking the gradient of (13) we obtain

$$(18) \quad (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda}) \text{grad } \lambda + \mathbf{Q}_\lambda = 0.$$

Since $\mathbf{q} = \mathbf{q}(\lambda)$,

$$(19) \quad \mathbf{w}(\lambda, \psi) = (\text{grad } \lambda) \times \mathbf{q}_\lambda,$$

so that (11), (19) give, after substituting for $\mathbf{Q}_\lambda$ from (15), $\text{grad } \lambda$ from (18),

$$(20) \quad (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})G(\psi) = D(\lambda),$$

where

$$a^4 D(\lambda) = (a^2 - q^2)q_\lambda^2 + a^2 q^2 \theta_\lambda^2.$$

Taking the gradient of (20) and using (16), (20) to substitute for $\mathbf{Q}_{\lambda\lambda}$, $\text{grad } \lambda$ gives

$$\begin{aligned} & (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})[(\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})G'(\psi) + K(\lambda)G(\psi)]\mathbf{Q} \\ & - [(\Psi_{\lambda\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda\lambda})G(\psi) + (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})G(\psi)F(\lambda) - D'(\lambda)]\mathbf{Q}_\lambda = 0. \end{aligned}$$

If we assume for the moment that the vectors $\mathbf{Q}$, $\mathbf{Q}_\lambda$ are not parallel and $\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} \neq 0$, then

$$(21) \quad (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})G'(\psi) + K(\lambda)G(\psi) = 0$$

$$(22) \quad (\Psi_{\lambda\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda\lambda})G(\psi) + (\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})G(\psi)F(\lambda) = D'(\lambda).$$

If (20) is used to eliminate $(\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda})$, (21) becomes, on the assumption that neither $G(\psi)$ nor $D(\lambda)$ are identically zero,

$$\frac{G'(\psi)}{G(\psi)^2} + \frac{K(\lambda)}{D(\lambda)} = 0.$$

Now ψ is not reducible to a function of λ only, otherwise the streamlines would coincide with the lines of constant λ which are known to be straight. This corresponds to the degenerate case $\theta_\lambda = 0$ which occurs when the vectors $\mathbf{Q}$, $\mathbf{Q}_\lambda$ are parallel. Therefore let

$$\begin{aligned} \frac{G'(\psi)}{G(\psi)^2} &= -\frac{1}{A}, \\ \frac{K(\lambda)}{D(\lambda)} &= +\frac{1}{A}, \end{aligned}$$

where A is a constant which determines the function for the vorticity in the form $G(\psi) = A/\psi$, after the constant of integration has been incorporated in ψ . The velocity is given by

$$(23) \quad AK(\lambda) = D(\lambda).$$

On the other hand if $G'(\psi) = 0$ then the vorticity and velocity are simply given by

$$(24) \quad G(\psi) = A, \quad K(\lambda) = 0.$$

It follows that, for flows of the type considered, the vorticity distribution must fall into one of the forms

$$Aa^{2\gamma/(\gamma-1)}, \quad (A/\psi)a^{2\gamma/(\gamma-1)},$$

where A is some constant.

5. Equations for $\Psi(\lambda)$. Consider first the case $G(\psi) = A/\psi$; then (20) gives

$$(25) \quad \Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = \psi K(\lambda)$$

since $D(\lambda) = AK(\lambda)$. Eliminating $\mathbf{Q}$, $\mathbf{Q}_\lambda$, $\mathbf{Q}_{\lambda\lambda}$ between (25) and the field equations

$$\begin{aligned} \Psi + \mathbf{r} \cdot \mathbf{Q} &= \psi \\ \Psi_\lambda + \mathbf{r} \cdot \mathbf{Q}_\lambda &= 0 \end{aligned}$$

with the aid of (16) gives the required linear equation for $\Psi(\lambda)$,

$$(26) \quad \Psi_{\lambda\lambda} + F(\lambda)\Psi_{\lambda} - K(\lambda)\Psi = 0.$$

Alternatively the second possibility (24) implies, from (20), (16),

$$\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = \frac{D(\lambda)}{A}$$

$$\mathbf{Q}_{\lambda\lambda} + F(\lambda)\mathbf{Q}_{\lambda} = 0,$$

and a similar procedure to eliminate $\mathbf{Q}_{\lambda}$, $\mathbf{Q}_{\lambda\lambda}$ by use of (13) gives

$$(27) \quad \Psi_{\lambda\lambda} + F(\lambda)\Psi_{\lambda} = \frac{D(\lambda)}{A}.$$

If desired it may be verified that equations (9), (22) do not impose any additional conditions on the function $\Psi(\lambda)$ and that, in fact, the relations (26), (27) are reproduced in differential form.

6. Degenerate cases. The possibility $\theta_{\lambda} = 0$ implies that $\theta(\lambda)$ reduces to a constant and the streamlines consist of parallel straight lines, that is, a shear flow.

The possibility $\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = 0$ implies from (20), (21), since $G(\psi) \neq 0$ for rotational flow, that $D(\lambda) = K(\lambda) = 0$ simultaneously. These equations do not possess a common solution, $D(\lambda) = 0$ being the differential equation of the hodograph of the simple wave of steady irrotational flow whilst the solution of $K(\lambda) = 0$ is given by (44).

7. Streamlines and parameter lines. It may now be shown that the lines (13), along which the velocity is constant, must either be concurrent or parallel corresponding to whether $G(\psi) = A/\psi$ or A . In no case does a non-degenerate envelope exist.*

Any possible envelope would be obtained from

$$(13) \quad \Psi_{\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda} = 0$$

$$(28) \quad \Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = 0.$$

When $G(\psi) = A/\psi$, $AK(\lambda) = D(\lambda) \neq 0$ therefore by (16) the vectors $\mathbf{Q}_{\lambda}$, $\mathbf{Q}_{\lambda\lambda}$ are not parallel. Equations (13), (28) may now be solved to yield the equation of the envelope in parametric form $\mathbf{r}(\lambda)$, with λ as the current parameter. According to (25) however ψ is zero at all points where (28) is satisfied. Therefore the envelope must be a streamline at all points of which $\mathbf{q}$ is perpendicular to $\mathbf{Q}_{\lambda}$, but reference to (15) shows that this is not at all possible ($\theta_{\lambda} \neq 0$) and indicates a degenerate envelope.

* These results are obtained by MARTIN [2] from a more general theory in which the pressure and stream function are taken as the fundamental variables.

The result of eliminating $F(\lambda)$, $K(\lambda)$ from (16), (26) is

$$\begin{vmatrix} \Psi & U & V \\ \Psi_\lambda & U_\lambda & V_\lambda \\ \Psi_{\lambda\lambda} & U_{\lambda\lambda} & V_{\lambda\lambda} \end{vmatrix} = 0,$$

where U , V are the components of $\mathbf{Q}$.

This however means that the three non-parallel lines

$$\Psi + \mathbf{r} \cdot \mathbf{Q} = 0, \quad \Psi_\lambda + \mathbf{r} \cdot \mathbf{Q}_\lambda = 0, \quad \Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = 0$$

are concurrent and implies the existence of a vector $\mathbf{r}_0(\lambda)$ such that

$$\Psi + \mathbf{r}_0 \cdot \mathbf{Q} = 0, \quad \Psi_\lambda + \mathbf{r}_0 \cdot \mathbf{Q}_\lambda = 0, \quad \Psi_{\lambda\lambda} + \mathbf{r}_0 \cdot \mathbf{Q}_{\lambda\lambda} = 0$$

identically in λ . Differentiating the first and using the second of these formulae gives

$$(\mathbf{r}_0)_\lambda \cdot \mathbf{Q} = 0,$$

whilst differentiating the second and using the third yields

$$(\mathbf{r}_0)_\lambda \cdot \mathbf{Q}_\lambda = 0,$$

so that, since the vectors $\mathbf{Q}$, $\mathbf{Q}_\lambda$ are neither zero nor parallel,

$$\frac{d\mathbf{r}_0}{d\lambda} = 0$$

showing that $\mathbf{r}_0$ reduces to a fixed vector, independent of λ .

It follows that the three lines in question all pass through the same fixed point for all values of λ . In particular the λ -lines, (13), are concurrent. By selecting as the origin the point through which they all pass we get $\Psi_\lambda = 0$, so that, by (26), $\Psi = 0$. Therefore the equations for the streamline ψ reduce to

$$\mathbf{r} \cdot \mathbf{Q} = \psi, \quad \mathbf{r} \cdot \mathbf{Q}_\lambda = 0.$$

When $G(\psi) = A$, the relevant equation is $K(\lambda) = 0$, so that

$$\mathbf{Q}_{\lambda\lambda} + F(\lambda)\mathbf{Q}_\lambda = 0,$$

which, after use of the relevant expression for $F(\lambda)$, may be integrated in terms of a constant vector $\mathbf{c}$ to give

$$\mathbf{Q}_\lambda = Q^2 \theta_\lambda \mathbf{c},$$

showing that the λ -lines constitute a system of lines parallel to the fixed vector $\mathbf{k} \times \mathbf{c}$.

The streamlines are now given by assigning a constant value to ψ in

$$(29) \quad \Psi + \mathbf{r} \cdot \mathbf{Q} = \psi$$

$$(13) \quad \Psi_\lambda + \mathbf{r} \cdot \mathbf{Q}_\lambda = 0,$$

where $\Psi(\lambda)$ is a solution of the linear equation (27); therefore the streamlines are obtained as the envelope, for fixed ψ , of the system (29) regarded as a one parameter (λ) family of straight lines.

8. Curvature of the streamlines. By letting $\mathbf{q} = \mathbf{q}(\lambda)$ in the general equations for the velocity and vorticity, *viz.*

$$a^2 \operatorname{div} \mathbf{q} = q \mathbf{q} \cdot \operatorname{grad} q$$

$$\operatorname{curl} \mathbf{q} = \mathbf{k} G(\psi) a^{2\gamma/(\gamma-1)},$$

we get

$$(a^2 \mathbf{q}_\lambda - q q_\lambda \mathbf{q}) \cdot \operatorname{grad} \lambda = 0$$

$$(\operatorname{grad} \lambda) \times \mathbf{q}_\lambda = \mathbf{k} G(\psi) a^{2\gamma/(\gamma-1)}.$$

Substituting $\mathbf{q}_\lambda = q_\lambda \mathbf{t} + q \theta_\lambda \mathbf{n}$ and eliminating the common term $\mathbf{n} \cdot \operatorname{grad} \lambda$ yields

$$[(a^2 - q^2) q_\lambda^2 + q^2 a^2 \theta_\lambda^2] \mathbf{t} \cdot \operatorname{grad} \lambda = q a^2 \theta_\lambda G(\psi) a^{2\gamma/(\gamma-1)},$$

or, since $q = q(\lambda)$, $\theta = \theta(\lambda)$,

$$(30) \quad (a^2 - q^2) \left(\frac{dq}{d\theta} \right)^2 + q^2 a^2 = q a^{(4\gamma-2)/(\gamma-1)} G(\psi) / \theta_s,$$

where the suffix s denotes differentiation with respect to s , the arc along a streamline. Equation (30) provides a convenient expression for the curvature of a streamline which avoids resorting to the general equations in terms of $\Psi(\lambda)$.

Since q and a are both functions of θ only, (30) shows that

$$(31) \quad \frac{G(\psi)}{\theta_s} = f'(\theta),$$

say, where $f(\theta)$ is a function of θ only. Since $\psi_s = 0$, the relation (31) may be integrated to give

$$f(\theta) = s G(\psi) + J(\psi),$$

where $J(\psi)$ is a function of integration. Since the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds$, the equations of the streamlines are then obtained in integral form:

$$(32) \quad G(\psi) \mathbf{r}(\psi, \theta) = \int f'(\theta) \mathbf{t}(\theta) d\theta,$$

apart from an additive function of ψ .

These equations reaffirm that, whatever the form of $G(\psi)$, the streamlines will all be "similar" curves. These curves all meet the lines of constant λ at the

same angle $\Gamma(\lambda)$, and, if any one is specified by $\psi = \psi_1$ and intrinsic equation $s = s_1(\theta)$, then the remainder are determined with intrinsic equations

$$s G(\psi) = s_1(\theta) G(\psi_1) - J(\psi),$$

apart from an additive constant.

9. Entropy distribution. If in place of Crocco's stream function the normal stream density function is used, then by comparing with Crocco's function (10) and using (4), (12) the formula of conversion is obtained,

$$(33) \quad \psi' = \gamma^{-(\gamma-1)^{-1}} \int \exp [-\gamma g(\psi)] d\psi.$$

To consider first the case of parallel λ -lines, then

$$g(\psi) = A(\psi - \psi_0),$$

ψ_0 being a constant of integration, so that (33) gives

$$-\gamma A(\psi - \psi_0) = \log (\psi'/\psi'_0),$$

ψ'_0 being the value of ψ' corresponding to $\psi = \psi_0$.

(12) may now be written explicitly in either of the forms

$$(34) \quad S - S_0 = \gamma R A (\psi - \psi_0) = -R \log (\psi'/\psi'_0).$$

When the parameter lines are concurrent, $g(\psi) = A \log (\psi/\psi_0)$, and the corresponding equations become

$$(35) \quad S - S_0 = \gamma R A \log \frac{\psi}{\psi_0} = \gamma R A (1 - \gamma A)^{-1} \log \frac{\psi'}{\psi'_0} \quad \gamma A \neq 1$$

$$(36) \quad S - S_0 = R \log \frac{\psi}{\psi_0} = R \gamma^{(\gamma-1)^{-1}} (\psi' - \psi'_0) \quad \gamma A = 1.$$

It is seen that, in the case when the λ -lines are concurrent and the ordinary stream function ψ' is used, two distinct forms of the entropy distribution function $S(\psi')$ are necessary to cover all possible values of the parameter A . This is avoided by the use of Crocco's function ψ . On the other hand, when the forms of the entropy distributions corresponding to both parallel and concurrent λ -lines as a whole are considered, the form of $S(\psi)$ is different according to (34), (35) for each of the two possibilities. By comparing (23), (24) the case of parallel lines may, in one sense, be regarded as being derived from that of concurrent lines by letting A approach infinity and it is clear that the second alternative in (34) may be obtained from that of (35) by the same procedure. Therefore the expression $S(\psi')$ takes the form (35) whether the lines are parallel or concurrent, the change in the form of the function occurring at $A = 1/\gamma$ for concurrent lines.

It follows that the effect of using the stream function of Crocco is to replace the location of the exceptional case, originally occurring at $A = 1/\gamma$, by that corresponding to A tending to infinity.

10. Concurrent λ -lines. The equation (23) for the velocity may be written

$$qa^2(a^2 - q^2)q_{\theta\theta} - (\gamma q^4 + 2a^4 - a^2 q^2)q_{\theta}^2 - a^4 q^2 = A^{-1} q^2[(a^2 - q^2)q^2 + q^2 a^2]$$

and integrated in terms of the variable $z = q^2/b^2$, where b is an arbitrary real constant, to give

$$(37) \quad \left(\frac{dz}{d\theta}\right)^2 \frac{(\lambda - z)^2}{4z^2} = (m\lambda - z)(z - z_0) \frac{\lambda}{mz_0} + Cz(m\lambda - z)^{-\rho},$$

in which λ now denotes the constant parameter given by $\lambda = c^2/mb^2$ where

$$m = \frac{\gamma + 1}{\gamma - 1}, \quad A\rho = \frac{2(1 - \gamma A)}{\gamma - 1}, \quad z_0 = \frac{\lambda(\rho + 1)}{\rho + m}.$$

If $\rho = -1$, (37) must be replaced by

$$m \left(\frac{dz}{d\theta}\right)^2 \frac{(\lambda - z)^2}{4z^3(m\lambda - z)} = (m - 1) \log(m\lambda - z) - \frac{\lambda}{z} + C.$$

When $\rho = 0$, (37) may be expressed

$$(38) \quad \left(\frac{dz}{d\theta}\right)^2 (\lambda - z)^2 = 4z^2(2z - z^2 - \lambda^2)$$

by selecting

$$(2 - C)(\gamma + 1)^2 b^2 = 2(\gamma^2 + 1)c^2,$$

C being the constant of integration.

The integral of (38) can be put into the form

$$(39) \quad k \cos \theta = \frac{z + \lambda}{z^{\frac{1}{2}}},$$

where

$$k^2 = 2(\lambda + 1).$$

(17) then gives

$$(40) \quad \cot \Gamma = \pm m(m\lambda - z)^{-1}(2z - z^2 - \lambda^2)^{\frac{1}{2}}.$$

The relevant range of z for real values of θ ,

$$(41) \quad 1 - (1 - \lambda^2)^{\frac{1}{2}} \leq z \leq 1 + (1 - \lambda^2)^{\frac{1}{2}},$$

shows that the positive parameter λ must not exceed unity.

To sketch the streamlines when $\gamma = \frac{7}{3}$, $m = 6$, use (29), (17),

$$(42) \quad \psi = rqa^5 \sin \Gamma;$$

then

$$(43) \quad r = \psi b^{-6} 5^{\frac{1}{3}} (6\lambda - z)^{-\frac{1}{3}} [12(6 - \lambda) - 35z]^{\frac{1}{3}}.$$

Since

$$5a^2 = b^2(6\lambda - z), \quad 5(a^2 - q^2) = 6b^2(\lambda - z)$$

and equation (30) gives for the curvature

$$\theta_s = (z - \lambda)(6\lambda - z)^{\frac{1}{3}}(5z - 12 + 7\lambda)^{-1} z^{-\frac{1}{3}} H(\psi).$$

The flows predicted by these equations are different according as λ exceeds the value $\frac{1}{3}\frac{2}{7}$ or not.

Case $0 < \lambda < \frac{1}{3}\frac{2}{7}$. Along the sonic lines, given by $z = \lambda$, the streamlines have zero curvature and the flow changes from subsonic to supersonic (Fig. 1). The value $z = 6\lambda$ also lies in the required range (41) when $\lambda < \frac{1}{3}\frac{2}{7}$ and the streamlines then diverge to infinity. The motion takes place in the angle $(\pi - 2\beta)$, between the straight lines corresponding to $z = 6\lambda$, the acute angle β being given by

$$\cos \beta = \frac{7}{2} \left[\frac{\lambda}{3(\lambda + 1)} \right]^{\frac{1}{3}}.$$

Case $\frac{1}{3}\frac{2}{7} < \lambda < 1$. The sonic lines $z = \lambda$ still occur, but the value $z = 6\lambda$ now lies outside the range (41). Therefore the streamlines never extend to infinity (Fig. 2). The curvature is infinite when $z = (12 - 7\lambda)/5$ and the streamlines are abruptly reflected along the lines corresponding to this value of z . Between these lines the physical plane is covered twice by the variations

$$1 - (1 - \lambda^2)^{\frac{1}{3}} \leq z \leq \frac{1}{5}(12 - 7\lambda)$$

$$\frac{1}{5}(12 - 7\lambda) \leq z \leq 1 + (1 - \lambda^2)^{\frac{1}{3}},$$

but only one must be selected to yield a possible flow.

Case $\lambda = \frac{1}{3}\frac{2}{7}$. In this case the lines corresponding to the values

$$\frac{1}{5}(12 - 7\lambda), \quad 6\lambda, \quad \frac{1}{5}\frac{2}{7}(6 - \lambda), \quad 1 + (1 - \lambda^2)^{\frac{1}{3}}$$

move together to coincide at $z = \frac{7}{3}\frac{2}{7}$, and the equations (39), (40) may be employed to give the radial angle ϕ ,

$$\cos 2\phi = \cos (2\theta + 2\Gamma) = \frac{3}{8}\frac{7}{5}(z - 1).$$

It follows from (42) by eliminating z that the polar equations of any streamline are of the form

$$r \sin^6 \phi = B\psi,$$

B being a constant. The corresponding flow covers the half plane $0 \leq \phi \leq \pi$ and is free from discontinuities.

Case $C = 0$. Another instance for which (37) permits direct integration occurs when $C = 0$ giving

$$\begin{aligned} \pm\theta &= \Gamma + \left(\frac{mz_0}{\lambda}\right)^{\frac{1}{2}} \tan^{-1} \left[\left(\frac{z_0}{m\lambda}\right)^{\frac{1}{2}} \cot \Gamma \right] \\ \cot \Gamma &= \left(\frac{m\lambda}{z_0}\right)^{\frac{1}{2}} \left[\frac{z - z_0}{m\lambda - z} \right]^{\frac{1}{2}}, \end{aligned}$$

where

$$m\lambda \geq z \geq z_0 > 0.$$

The polar equations of the streamlines may now be obtained in the form

$$r \left[\cos \frac{\lambda}{mz_0} \phi \right]^{\lambda} = B\psi.$$

These equations, which resemble closely the Prandtl-Meyer relations and reduce to them when $z_0 = \lambda$, corresponding to the condition $A = 0$ for zero vorticity, are treated exhaustively under [2]. When $\lambda < mz_0$ the flow is everywhere supersonic and there are no inflexions in the streamlines.

FIGURE 1

FIGURE 2

11. Parallel λ -lines. The relevant equations are now (24), (27) giving, if C is a constant of integration,

$$\left(\frac{dQ}{d\theta}\right)^2 = Q^2(C^2 Q^2 - 1),$$

and by further integration

$$(44) \quad CQ = \sec(\theta - \theta_0).$$

This may be inserted in (27) to yield the equation for $\Psi(\theta)$,

$$(45) \quad \Psi_\theta \cos^2(\theta - \theta_0) = \int h(\theta) \cos^2(\theta - \theta_0) d\theta,$$

where

$$Aa^4 h(\theta) = (a^2 - q^2) \left(\frac{dq}{d\theta}\right)^2 + a^2 q^2.$$

The λ -lines may be rewritten

$$(46) \quad \Psi_\theta \cos^2(\theta - \theta_0) + \mathbf{r} \cdot \mathbf{c} = 0$$

where $\mathbf{c}$ denotes the constant vector $-(\mathbf{i} \cos \theta_0 + \mathbf{j} \sin \theta_0)C^{-1}$.

The streamlines may be obtained by solving for $\mathbf{r}$ the equations (29) and using (44), (45):

$$C^{-1} \mathbf{r}(\theta) = \Psi_\theta \cos(\theta - \theta_0) \mathbf{t}(\theta) - (\Psi - \psi) \mathbf{n}(\theta_0).$$

In virtue of (45), this may be written,

$$C^{-1} \mathbf{r}(\theta) = \int h(\theta) \cos(\theta - \theta_0) \mathbf{t}(\theta) d\theta + \psi \mathbf{n}(\theta_0).$$

To construct the streamlines take $\theta_0 = 0$, $\gamma = \frac{7}{5}$ for convenience and use the variable z given by

$$a = cz^{\frac{1}{3}};$$

then

$$A\lambda^2 h(\theta) = (1 - 5z)^2(z^6 - \lambda^2)(6z - 1)^{-1} z^{-1}$$

so that

$$x = \lambda b \int (z^6 - \lambda^2) z^{-7} (z^5 - 5z^6 - \lambda^2)^{-\frac{1}{3}} dz$$

$$y = b(\log z + \lambda^2/6z^6) + C\psi,$$

where

$$\lambda = \frac{1}{Cc^6}, \quad b = -\frac{5C}{2A}.$$

The curvature follows from (30),

$$2b\theta_s = 5\lambda z^{\frac{1}{3}}(1 - 6z)(z^6 - \lambda^2)^{-1}(1 - 5z)^{-\frac{1}{3}}.$$

The condition $0 < \lambda < 6^{-3}$ is necessary to ensure the existence of a range of z for which $z^5 - 5z^6 - \lambda^2 > 0$. When this is satisfied the curvature is never infinite except at $z = \lambda^{\frac{1}{3}}, \frac{1}{3}$, the latter possibility however being of no consequence since $z = \frac{1}{3}$ lies outside the relevant range of z . The streamlines have inflexions along the lines corresponding to $z = \frac{1}{6}$ where the flow is sonic. If $z_1, z_2 (> z_1)$ denote the two real roots of $z^5 - 5z^6 - \lambda^2 = 0$ then only values of z for which $0 < z_1 \leq z \leq z_2$ need be considered. It follows that both x and y remain finite. By drawing the curves of x and y against z it becomes clear that for any given value of z there is a unique (x, y) but not conversely. If x is prescribed, in the required range for real z , there are in general two possibilities for z and y , corresponding to the sections

$$z_1 \leq z \leq \lambda^{\frac{1}{3}}, \quad \lambda^{\frac{1}{3}} \leq z \leq z_2.$$

FIGURE 3

In the physical plane there are two possible branches which meet in a cusp. It is not possible to interpret both branches simultaneously, since the streamlines of different branches intersect. The flow, which along the first branch is entirely supersonic, becomes subsonic on crossing the line $z = \frac{1}{8}$ of the second branch. The solution $CQ = \sec(-\theta)$ is also contained in (44) and corresponds to a reflection in the y -axis of the first solution. Additional solutions given by $CQ = \pm \sec(\pm\theta)$ are shown in the lower half plane of Fig. 3.

Subject to conditions of continuity of z and ψ at the end points (z_1, z_2) , it is possible to combine solutions of the same branch to yield further continuous flow patterns.

12. Axially symmetric phenomena. In the two dimensional analysis the case $G(\psi) = A/\psi$ corresponded to conditions constant along each of a family of concurrent straight lines which became parallel when $G(\psi)$ reduced to a constant. When the three dimensional phenomenon possesses axial symmetry a stream function of Crocco still exists and is given by

$$\text{grad } \psi = R \mathbf{n} q a^{2/(\gamma-1)},$$

where R denotes the distance from the axis and $\mathbf{n}$ the unit vector normal to $\mathbf{q}$ in the axial plane. It may be demonstrated that the relation (11) is replaced by

$$w = RG(\psi)a^{2\gamma/(\gamma-1)}$$

If the flow is conical then the equations (5), (9) reduce to

$$(47) \quad \frac{\partial}{\partial r} [r^2 \sin \theta u a^{2/(\gamma-1)}] + \frac{\partial}{\partial \theta} [r \sin \theta v a^{2/(\gamma-1)}] = 0$$

$$(48) \quad \frac{\partial}{\partial \theta} (w v a^{-2}) = 0,$$

where

$$(49) \quad r w = v - \frac{du}{d\theta}$$

and where (u, v) refer to the components of the velocity, in the axial plane, along and perpendicular to the radius vector from the origin, (r, θ) denoting the significant spherical polar coordinates ($R = r \sin \theta$). (47) may be written so as to show that the stream function takes the form

$$(50) \quad \psi = -\frac{r^2}{2} \sin \theta v a^{2/(\gamma-1)}.$$

By use of (49), (48) may be integrated in terms of a constant A ,

$$(51) \quad v - \frac{du}{d\theta} = \frac{A a^2}{v};$$

therefore

$$w = \frac{Aa^2}{vr} = -Ra^{2\gamma/(\gamma-1)} \frac{A}{2\psi},$$

or

$$G(\psi) = -A/2\psi.$$

This shows the correspondence between the conical flow and that of §10 in which the lines of constant q were also concurrent. Simplification of (47) gives the second equation for (u, v) , to be used in conjunction with (51),

$$(52) \quad 2u + v \cot \theta + \frac{dv}{d\theta} = va^{-2} q \frac{dq}{d\theta}.$$

Elimination of u , or v , between (51), (52) will provide a second order differential equation for $u(\theta)$, or $v(\theta)$, analogous to that of §10, which would represent a generalisation of the Taylor-Maccoll conical field equation, to which it reduces when $A = 0$. To avoid the general method we may seek particular solutions, one of which is

$$u = c \cos \theta, \quad v = \rho c \sin \theta,$$

where $\rho = -(\gamma - 1)/\gamma$, and the flow in any axial plane resembles those of §10 when $\lambda = \frac{1}{3}\frac{2}{\gamma}$.

In a similar way the equations of cylindrical flow, in which (u, v) assumed independent of z , the coordinate along the axis of symmetry, are the components of velocity in the axial plane referred parallel and perpendicular to the axis of symmetry Oz , reduce to

$$(53) \quad \frac{d}{dR} (Rva^{2/(\gamma-1)}) = 0$$

$$(54) \quad \frac{d}{dR} \left(\frac{v}{a^2} \frac{du}{dR} \right) = 0,$$

so that

$$w = -\frac{du}{dR} = -\frac{Ba^2}{v}$$

and

$$va^{2/(\gamma-1)} = \frac{A}{R}.$$

Therefore

$$w = -\frac{B}{A} Ra^{2\gamma/(\gamma-1)},$$

A and B being constants of integration. The function $G(\psi)$ now reduces to a constant, just as in §11, parallel lines being replaced by coaxial cylinders. Neither of the pairs of equations (51), (52) and (53), (54) yield to exact solution in the way used to treat their two dimensional analogues (23), (24); therefore a similar detailed examination of the corresponding possibilities cannot be carried out.

13. Singularities. Some of the examples discussed in detail possess lines, "limit lines", along which the streamlines have cusps. Such discontinuities, characterised along each streamline by the development of an infinite acceleration, represent a local breakdown of the theory on which the flow was constructed. These limit lines are the loci of singularities and will appear in the equations of the envelope of the system of streamlines

$$(55) \quad \Psi + \mathbf{r} \cdot \mathbf{Q} = \psi$$

$$(56) \quad \Psi_\lambda + \mathbf{r} \cdot \mathbf{Q}_\lambda = 0.$$

On solving these equations the streamline ψ will be obtained in the parametric form $\mathbf{r} = \mathbf{r}(\lambda, \psi)$ and the required envelope may be written

$$\frac{\partial(x, y)}{\partial(\lambda, \psi)} = 0,$$

or

$$(\mathbf{r}_\lambda) \times (\mathbf{r}_\psi) = 0.$$

Since Ψ and $\mathbf{Q}$ are functions of λ only, the result of differentiating (55) with respect to ψ is

$$\mathbf{Q} \cdot \mathbf{r}_\psi = 0,$$

whilst differentiation with respect to λ , using (56), gives

$$\mathbf{Q} \cdot \mathbf{r}_\lambda = 0.$$

These last three conditions show that $\mathbf{r}_\lambda$ vanishes on the envelope. It follows that λ is constant on the envelope, which is therefore degenerate and made up of certain of the parameter lines (56). The limit lines are then obtained as the members of the family (56) for which λ is given by $\mathbf{r}_\lambda(\lambda, \psi) = 0$, ψ being used as the current parameter along any such line.

Differentiation of (56) with respect to λ gives, since $\mathbf{r}_\lambda = 0$,

$$\Psi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{Q}_{\lambda\lambda} = 0,$$

so that, from (20), $D(\lambda) = 0$; that is,

$$(57) \quad (a^2 - q^2)q_\lambda^2 + a^2 q^2 \theta_\lambda^2 = 0.$$

Therefore, from (30), the curvature of each streamline is infinite on the limit lines. The streamlines meet the limit lines at the angle $\Gamma(\lambda)$ given by (17), which by (57) reduces to the local Mach angle of supersonic flow. Consequently any possible limit lines will occur only in regions where the flow is supersonic. The collapse of the continuous flow theory in the region of a limit line, where the acceleration

$$(\mathbf{q} \cdot \text{grad})\mathbf{q} = (\mathbf{q} \cdot \text{grad } \lambda)\mathbf{q}_\lambda$$

becomes infinite with $\text{grad } \lambda$, suggests that in the physical plane a shock would be developed. However any shock which meets the streamlines at the local Mach angle, appropriate to conditions either in front of or behind it, must have zero intensity. The cuspidal tangents intersect the limit lines at the local Mach angle which prevents their replacement by shock curves. The limit lines represent the boundary of multiple regions in the physical plane. In such regions there are two or more values of λ and of the velocity vector at each point, with a corresponding number of solutions mathematically possible but only one must be selected.

14. Conclusion. The method here employed may be extended by applying the Legendre transformation to the general class of flows for which the velocity vector is a function of two independent variables λ_1, λ_2 , but the linearisation which is achieved when potential flow is considered is not effective in the presence of vorticity. The theory shows how the Legendre transform of Crocco's function may be taken to be zero, except when the parameter λ is constant along a system of parallel lines. These are the flows of §11, which appear to elude treatment by other methods. Similar phenomena have been investigated and the class of flows which in the notation of §10 occurs when $C = 0$ was first studied in detail from different aspects by Prim [3] and Martin [1, 2], whilst more general spatial flows appear in [4].

These flows, representing rotational flow extending to infinity in the absence of discontinuities, approach closely what may be regarded as the direct generalisation of the Prandtl-Meyer flow round a sharp corner. The other flows bear no such resemblance. The more general velocity fields which, though dependent on just one independent parameter, admit a possible distribution of vorticity, are similar to the simple wave of isentropic flow in that the lines of constant velocity are straight. Unlike the original simple wave however, these lines may not be arbitrarily located to envelop a suitable convex curve so that the theory cannot be used to predict the flow over a given boundary. Similarly the vorticity distribution cannot be assigned arbitrarily. The present theory is confined to iso-energetic flows of the type $\mathbf{q}(\lambda)$. The general theory of rotational flows of the simple wave pattern which are not necessarily iso-energetic has not yet been developed. In plane flow the quantity c occurring in (7) will then be a function of ψ and Munk & Prim [5, 6] have obtained equations analogous to (5), (9) valid when $c = c(\psi)$ by replacing $\mathbf{q}$ by the "reduced velocity vector" $\mathbf{W} = \mathbf{q}/c$. In [3,

5] it is shown directly that flows of the type $W(\lambda)$ exists when λ is constant along a series of concurrent lines. These flows become "simple waves" for the velocity only when they are also iso-energetic.

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