*On the Phragmén-Lindelöf Principle for Elliptic Differential Equations**

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This paper supplements and extends the conclusions of Gilbarg [1] and Hopf [2] concerning the Phragmén-Lindelöf theorem for linear elliptic partial differential equations. We consider especially the equation

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0,$$

where the domain of definition of the coefficients is the upper half plane $y > 0$. We suppose that L is *uniformly elliptic*; that is, there exists a positive constant k such that at every point $z = (x, y)$ in the upper half plane and for all real λ, μ ,

$$(1) \quad \frac{a\lambda^2 + 2b\lambda\mu + c\mu^2}{a + c} \geq k(\lambda^2 + \mu^2).$$

Moreover d and e are to tend to zero as $r \rightarrow \infty$, $r^2 = x^2 + y^2$, in such a way that there is a decreasing function $p(r)$ satisfying

$$(2) \quad \int_0^\infty p(r) \, dr < \infty, \quad \frac{(d^2 + e^2)^{\frac{1}{2}}}{a + c} \leq p(r).$$

For the purposes of this paper it will not be necessary to make any further assumptions regarding the coefficients; for example, they may be discontinuous or unbounded. The Gilbarg-Hopf theorem, which is an analogue for elliptic equations of the classical Phragmén-Lindelöf theorem for harmonic functions, states that if a function $u = u(z)$ satisfies the inequality $L(u) \geq 0$ in some sub-

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region $\mathfrak{D}$ of the upper half plane, if $u \leq 0$ on the boundary of $\mathfrak{D}$, and if there exists an infinite sequence $R_\nu \rightarrow \infty$ such that

$$\lim_{\nu \rightarrow \infty} \frac{M(R_\nu)}{R_\nu} = 0, \quad M(R) = \max_{r=R} u(z),$$

then $u \leq 0$ throughout $\mathfrak{D}$.

Thus for the validity of the Phragmén-Lindelöf theorem it is not the harmonic character of the function in question which is relevant. Now the Nevanlinna [3], and later Heins [5], have shown that the classic theorem is subject to certain refinements when the behavior of the ratio $M(R)/R$ is left unspecified. The major purpose of this paper is to show that these refinements also carry over to elliptic equations; in addition we are able to attain a previously unnoted symmetry in the formulation of the results. The main theorem can perhaps best be understood in its relation to previous work if we consider first a special case, of interest in itself.

Theorem 1. *Let $u = u(z)$ satisfy the elliptic differential inequality*

$$L'(u) = au_{xx} + 2bu_{xy} + cu_{yy} \geq 0$$

in the upper half plane $y > 0$, and let u be continuous and ≤ 0 on $y = 0$. Then

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R}$$

exists (finite or infinite); moreover α is non negative and

$$u \leq \alpha y$$

for $y > 0$. If the equality holds at any point in $y > 0$ then it holds everywhere (i.e. $u \equiv \alpha y$).

This result bears essentially the same relation to Gilbarg's theorem as the Nevanlinna-Heins result does to the classical Phragmén-Lindelöf theorem. Indeed Theorem 1 is the strict analogue for elliptic equations of Heins' result for harmonic functions, in the sense that Theorem 1 becomes precisely Heins' result when $L'(u)$ is replaced by $\Delta u = u_{xx} + u_{yy}$.

Let us now return to the operator $L(u)$ and the main Theorem 2. Let $\mathfrak{D}$ be an open subregion of the upper half plane such that the boundary $\bar{\mathfrak{D}}$ of $\mathfrak{D}$ lies in a strip $0 \leq y \leq \sigma$ ($\mathfrak{D}$ thus contains the half plane $y > \sigma$). We consider a pair of functions $u(z)$ and $v(z)$ defined and twice continuously differentiable in $\mathfrak{D}$, and we set

$$M(R) = \max_{r=R} u(z),$$

$$m(Y) = \min_{y=Y} v(z).$$

With these agreements, and under the conditions (1) and (2) already stated for the operator $L(u)$, we have

Theorem 2. *Let $u = u(z)$ satisfy the inequalities*

$$L(u) \geq 0 \text{ in } \mathfrak{D}, \quad u \leq 0 \text{ on } \overline{\mathfrak{D}},^1$$

and let $v = v(z)$ satisfy

$$L(v) \leq 0 \text{ in } \mathfrak{D}, \quad v \geq 0 \text{ in } \mathfrak{D}.$$

Then the limits

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R}, \quad \beta = \lim_{Y \rightarrow \infty} \frac{m(Y)}{Y}$$

exist; moreover α is non-negative, β is finite (obviously $\beta \geq 0$) and

$$(3) \quad \beta u \leq \alpha v$$

throughout $\mathfrak{D}$. In (3) either the strict inequality or the strict equality holds. If equality occurs and α and β are non-zero, then $u = \alpha\Psi$ and $v = \beta\Psi$, where $\Psi = \Psi(z)$ is uniquely defined by its properties $L(\Psi) = 0$, $\Psi = 0$ on $\overline{\mathfrak{D}}$,

$$\lim_{y \rightarrow \infty} \frac{\Psi}{y} = 1 \quad \text{uniformly in } x,$$

and the condition that it be bounded in any strip $0 \leq y \leq \Sigma$.

Extensions of Theorem 2 to the case of n independent variables and to equations without a growth condition (2) are discussed following the proof of Theorem 2. The latter extension applies in particular to a class of equations whose prototype is the well known equation of generalized axially symmetric potential theory.

Remarks. It is understood that equality for non-zero α and β can occur only if there exists a function Ψ having the properties listed. Actually if it is assumed or known that Ψ exists, then Theorem 2 can be expressed in a somewhat neater form. For in this case both $u = \Psi$ and $v = \Psi$ satisfy the hypotheses of Theorem 2 (to show that $\Psi \geq 0$, note that $\alpha = 0$ for the function $u = -\Psi$), and $\alpha(\Psi) = \beta(\Psi) = 1$. It follows that, for any functions u and v satisfying the hypotheses of Theorem 2,

$$(3') \quad \frac{u}{\alpha} \leq \Psi \leq \frac{v}{\beta},$$

¹ If $f(z)$ is any function defined in $\mathfrak{D}$ we shall mean by the inequality, $f \leq 0$ on $\overline{\mathfrak{D}}$, that $\lim_{z \rightarrow \overline{\mathfrak{D}}} f(z) \leq 0$.

² Division by zero is avoided if (3') is written in the form

$$(3'') \quad u \leq \alpha\Psi, \quad \beta\Psi \leq v.$$

where the first equality holds if and only if $u = \alpha\Psi$, the second if and only if $v = \beta\Psi$. We emphasize that (3') depends only on the existence of Ψ .

Since the first inequality in (3') provides a sharp upper bound on u/α , it therefore represents the appropriate extension to elliptic equations of the Nevanlinna-Heins result. It is nevertheless desirable to obtain another upper bound on u/α , because, except for special cases such as Theorem 1, the explicit form of Ψ is unknown and even its existence is open. To this end we set

$$\bar{v}(y) = \int_0^y \exp [P(t)] dt, \quad P(t) = \int_t^\infty \frac{p(s)}{k} ds.$$

Then $\bar{v} = \bar{v}(y) \geq 0$,

$$(4) \quad \lim_{y \rightarrow \infty} \frac{\bar{v}}{y} = 1,$$

and $L(\bar{v}) = \exp [P](e - k^{-1}pc)$. Putting $\lambda = 0, \mu = 1$ in (1) we get

$$c \geq k(a + c);$$

hence using (2) it follows that $L(\bar{v}) \leq 0$. The function $\bar{v}$ thus satisfies the hypotheses of Theorem 2 and $\beta = 1$; hence $u \leq \alpha\bar{v}$ as desired. We note the inequality

$$(5) \quad y \leq \bar{v} \leq Ky, \quad \log K = k^{-1} \int_0^\infty p(s) ds,$$

for future use.

The second inequality in (3') is related to a different principle of function theory. For let $L(u) = \Delta u$ and take $\mathfrak{D}$ to be the upper half plane. Then if $w = w(z) = u + iv$ is an analytic function of $z = x + iy$ mapping $\mathfrak{D}$ into itself (*i.e.* $v \geq 0$), we have by (3'),

$$(6) \quad \beta y \leq v$$

throughout $\mathfrak{D}$, equality holding if and only if $w = \beta z + v$, where v is a real constant. On the other hand, the Julia-Carathéodory theorem ([4], pp. 52-55) states, in one of its forms, that the quotient w/z approaches a real, finite, non-negative limit β^* as $z \rightarrow \infty$ through points in any angle $y \geq \delta |x|$, ($\delta > 0$), and that

$$(6') \quad \beta^* y \leq v$$

throughout $\mathfrak{D}$, equality holding if and only if $w = \beta^* z + v$. Now from the reality of β^* and (6') we have

$$\beta^* = \lim_{y \rightarrow \infty} \frac{w(iy)}{iy} = \lim_{y \rightarrow \infty} \frac{v(iy)}{y} \geq \beta \geq \beta^*.$$

Therefore $\beta = \beta^*$ and (6) is identified with (6').

It is thus shown that Theorem 2 implies a part of the Julia-Carathéodory theorem (the remaining part, namely that $w/z \rightarrow \beta$, is obviously not a consequence of Theorem 2). Under the minimum conditions of Theorem 2 this is as much as we can expect, and therefore the second inequality of (3') may be considered a legitimate extension of the Julia theorem to elliptic equations. If the hypotheses on v are strengthened, however, it should be possible to obtain a result in exact accord with function theory.

Indeed, *we conjecture that if the condition $L(v) \leq 0$ is replaced by $L(v) = 0$, then the quotient v/y will tend uniformly to a limit β as $y \rightarrow \infty$ in any angle $y \geq \delta |x|$, $\delta > 0$. Assuming this to be the case, we can show that w/z also tends uniformly to β in the angle.*³

Proof of Theorem 2. In the following demonstration we shall adopt the basic proof plan of Heins, although most of the details must be managed differently. We shall make extensive use of Hopf's form of the maximum principle for elliptic differential equations; namely, a non-constant function $u(z)$ satisfying the inequality $L(u) \geq 0$ cannot take on a maximum in the interior of its domain of definition [6]. Now set

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R}.$$

Our main effort will be devoted to showing that $\alpha < 0$ cannot occur and that $\alpha \geq 0$ implies

$$(7) \quad u \leq \alpha \bar{v}.$$

With the methods developed it is then easy to prove that the limit β exists and is finite.

This will prove the theorem. For existence of the remaining limit follows from the chain

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R} \leq \overline{\lim}_{R \rightarrow \infty} \frac{M(R)}{R} \leq \alpha \lim_{R \rightarrow \infty} \frac{\bar{v}(R)}{R} = \alpha,$$

where the second inequality is a consequence of (7). To get (3) fix ϵ , $0 < \epsilon < 1$, and let v be any function satisfying the hypotheses of Theorem 2. In virtue of (4) and the existence and finiteness of β there exists a positive constant y^* such that, for all z with $y > y^*$,

$$\frac{\bar{v}}{y} < 1 + \epsilon, \quad \frac{v}{y} \geq \beta(1 - \epsilon).$$

Hence by (7),

$$\beta u \leq \alpha \beta (1 + \epsilon) y \leq \alpha \frac{1 + \epsilon}{1 - \epsilon} v, \quad y > y^*.$$

³ The proof is patterned on an argument of LANDAU and VALIRON ([4]; p. 55).

⁴ It is evident that we need not consider $\alpha = +\infty$, the part of the theorem involving u being trivial in this case.

For the function

$$U^* = \beta u - \alpha \frac{1 + \epsilon}{1 - \epsilon} v,$$

we have

$$U^* \leq \begin{cases} \alpha \beta \bar{v}(y^*), & y \leq y^*, \\ 0, & y > y^*. \end{cases}$$

Therefore U^* has the properties:

$$\begin{aligned} i) & \quad L(U^*) \geq 0 && \text{in } \mathfrak{D}; \\ ii) & \quad U^* \leq 0 && \text{on } \overline{\mathfrak{D}}; \\ iii) & \quad \lim_{R \rightarrow \infty} \frac{M^*(R)}{R} = 0, \quad M^*(R) = \max_{r=R} U^*(z). \end{aligned}$$

It follows from Gilbarg's theorem that $U^* \leq 0$ throughout $\mathfrak{D}$, *i.e.*

$$\beta u \leq \alpha \frac{1 + \epsilon}{1 - \epsilon} v.$$

Since ϵ may be arbitrarily small, this proves (3). To obtain the final statements of the theorem note that in $\mathfrak{D}$

$$\beta u - \alpha v \leq 0, \quad L(\beta u - \alpha v) \geq 0;$$

thus, by the maximum principle in its strong form, either $\beta u < \alpha v$ throughout $\mathfrak{D}$ or $\beta u \equiv \alpha v$. Now assume that the equality holds and that α and β are non-zero. We define $\Psi = \Psi(z)$ by the relation $u/\alpha = v/\beta = \Psi$. Then, in view of the hypotheses of Theorem 2, $\Psi = 0$ on $\overline{\mathfrak{D}}$, $L(\Psi) = 0$ in $\mathfrak{D}$. Moreover, by (7),

$$0 \leq v/\beta = \Psi = u/\alpha \leq \bar{v},$$

and therefore Ψ is bounded in any strip $0 \leq y \leq \Sigma$; dividing by y and letting $y \rightarrow \infty$ one easily finds

$$\lim_{y \rightarrow \infty} \frac{\Psi}{y} = 1 \quad \text{uniformly in } x.$$

The desired properties of the function Ψ are thus established. The uniqueness of Ψ follows most easily from (3'). Thus, if Ψ_1 is another function with the listed properties, we can set $u = \Psi_1$, $v = \Psi_1$ in (3'). There arises $\Psi_1 \leq \Psi \leq \Psi_1$, and so $\Psi_1 = \Psi$. Theorem 2 thus depends on the italicized statement above, which will be shown with the help of three lemmas.

Lemma 1. *There exists a constant K_1 , depending only on L , such that $u \leq K_1 \text{Max}(0, \alpha)y$.*

Proof. Let $\mathcal{G}_R$ denote the intersection of $\mathcal{D}$ with the circle $r < R$. Gilbarg [1] and Hopf [2] have shown that there exists a function $V_R(z)$, $z \in \mathcal{G}_R$, with the following properties:

- i) $L(V_R) \leq 0$;
- ii) $V_R \geq 0$ on the common boundary of $\mathcal{G}_R$ and $\mathcal{D}$, and $V_R = 1$ when $r = R$;
- iii) for fixed z ,

$$\lim_{R \rightarrow \infty} R V_R(z) = K_1 y,$$

where $K_1 > 1$ is a constant depending only on L .⁵

Set

$$\mu_R = \text{Max}[0, M(R)] \geq 0,$$

$$U_R(z) = u(z) - \mu_R V_R(z).$$

Then U_R is evidently ≤ 0 on $r = R$, and also, since $V_R > 0$ throughout $\mathcal{G}_R$, $U_R \leq 0$ on the common boundary of $\mathcal{G}_R$ and $\mathcal{D}$. Moreover $L(U_R) \geq 0$, and we may therefore conclude from the maximum principle that $U_R < 0$ in $\mathcal{G}_R$. Hence

$$u < \mu_R V_R = (\mu_R/R) R V_R.$$

Letting $R \rightarrow \infty$ through values of R such that

$$M(R)/R < \alpha + \epsilon, \quad \epsilon > 0,$$

(this is possible, of course, since $\lim_{R \rightarrow \infty} M(R)/R = \alpha$) we derive

$$u \leq K_1 \text{Max}(0, \alpha + \epsilon)y.$$

Since ϵ is arbitrary it follows that

$$u \leq K_1 \text{Max}(0, \alpha)y. \quad \text{Q.E.D.}$$

If $\alpha = 0$ we have $u \leq 0$ which is the desired relation (7) in this case. If $\alpha > 0$, however, the conclusion cannot be reached so quickly. We shall make use of

Lemma 2. *If $0 < \alpha < \kappa$ and if*

$$u \leq \kappa \bar{v},$$

then there exists a number $\epsilon > 0$ such that

$$u \leq (\kappa - \epsilon)\bar{v}.$$

⁵ The function V_R corresponds to the harmonic measure used in the NEVANLINNAS' proof of the Phragmén-Lindelöf theorem. The third property is not explicitly stated in [1] or [2], but follows easily from the representation of $R V_R$ on p. 422 of [2].

Remark. Assuming that Lemma 2 is true, we easily obtain (7) for $\alpha > 0$. For by Lemma 1 and (5)

$$u \leq K_1 \alpha y \leq K_1 \alpha \bar{v}.$$

Now let $\mathfrak{R}$ denote the set of all numbers κ such that

$$u \leq \kappa \bar{v}.$$

Evidently $\mathfrak{R}$ is non-empty and closed so that it contains a smallest number κ' . If $\kappa' > \alpha$ we contradict Lemma 2, hence $\kappa' = \alpha$ and (7) is thereby proved.

Proof of Lemma 2. Let γ and δ be any numbers obeying

$$\alpha < \gamma < \delta < \kappa,$$

and let

$$\sin \phi = \delta / \kappa.$$

Choose a sequence $R_\nu \rightarrow \infty$ such that

$$\frac{M(R_\nu)}{R_\nu} < \gamma, \quad \nu = 1, 2, 3, \dots,$$

and finally introduce new variables $\zeta = (\xi, \eta)$ by the relations

$$x = R_\nu \xi, \quad y = R_\nu \eta.$$

Let $\mathfrak{D}_\nu$ be the image of $\mathfrak{D}$ in the ζ -plane; for $\zeta \in \mathfrak{D}_\nu$ define

$$W_\nu(\zeta) = \frac{\kappa \bar{v}(y) - u(z)}{R_\nu},$$

Then W_ν obeys the differential inequality

$$(8) \quad L_\nu(W) = a_\nu W_{\xi\xi} + 2b_\nu W_{\xi\eta} + c_\nu W_{\eta\eta} + R_\nu d_\nu W_\xi + R_\nu e_\nu W_\eta \leq 0,$$

where $a_\nu = a_\nu(\zeta) = a(z)$, etc. Moreover by hypothesis

$$(9) \quad W_\nu \geq 0 \quad \text{in } \mathfrak{D}_\nu.$$

Also if $\overline{\mathfrak{D}}_\nu$ denotes the image of $\overline{\mathfrak{D}}$, and $\mathfrak{C}$ is the arc $\phi \leq \theta \leq \pi - \phi$ of the unit circle in the ζ -plane ($\theta = \arg \zeta$), then

$$(10) \quad \begin{aligned} W_\nu &\geq \kappa \eta && \text{on } \overline{\mathfrak{D}}_\nu, \\ W_\nu &= \kappa(\bar{v}/y) \sin \theta - u/R_\nu \geq \delta - \gamma && \text{on } \mathfrak{C}. \end{aligned}$$

We now consider the behavior of W_ν for small ζ , using a method inspired by Hopf's proof of the maximum principle [6]. Let $\mathfrak{E}$ denote the ellipse with center

(0, 2), vertical major axis of length 4, and minor axis of length $2 \cos \phi$. Set

$$\eta' = \eta - 2, \quad \rho^2 = 4\xi^2 + \eta'^2 \cos^2 \phi,$$

so that

$$\rho^2 = 4 \cos^2 \phi \quad \text{on } \mathcal{E}.$$

The arc $\mathcal{C}$ divides the interior of $\mathcal{E}$ into two parts; let $\mathcal{E}_l$ be the lower part so that

$$(11) \quad \cos \phi < \rho < 2 \cos \phi \quad \text{in } \mathcal{E}_l.$$

We seek a function $h(\xi)$ defined in $\mathcal{E}_l$ and having the properties

$$(12) \quad h = 0 \text{ on } \mathcal{E}, \quad 0 < h < 1$$

$$(13) \quad L_\nu(h) \geq 0.^6$$

Such a function is immediately available for the operator $L'(u)$, namely

$$h = \exp[-A\rho^2] - \exp[-4A \cos^2 \phi],$$

provided A is a number $\geq 2/k \cos^4 \phi$. In the more general case this is not suitable, but we shall see that the desired function may still be taken in the form $h = h(\rho^2)$. Indeed with this choice of h we have

$$\begin{aligned} L_\nu(h) = & 4h''[a_\nu(4\xi)^2 + 2b_\nu(4\xi)(\eta' \cos^2 \phi) + c_\nu(\eta' \cos^2 \phi)^2] \\ & + 2h'[4a_\nu + c_\nu \cos^2 \phi + 4R_\nu d_\nu \xi + R_\nu e_\nu \eta' \cos^2 \phi], \end{aligned}$$

where the primes on h denote differentiation with ρ^2 . Denoting the first and second brackets on the right by B_1 and B_2 respectively, we get from (1) and (11),

$$\begin{aligned} (14) \quad B_1 \geq & (a_\nu + c_\nu)k[(4\xi)^2 + (\eta' \cos^2 \phi)^2] \\ \geq & (a_\nu + c_\nu)k \cos^4 \phi, \end{aligned}$$

and using Schwarz's inequality, (2), and (11), we get

$$\begin{aligned} (15) \quad B_2 \leq & 4(a_\nu + c_\nu) + R_\nu(d_\nu^2 + e_\nu^2)^{\frac{1}{2}}[(4\xi)^2 + (\eta' \cos^2 \phi)^2]^{\frac{1}{2}} \\ \leq & 4(a_\nu + c_\nu)[1 + R_\nu p(r)]. \end{aligned}$$

Now it is evident that

$$r \geq y = R_\nu(2 + \eta') \geq R_\nu\left(2 - \frac{\rho}{\cos \phi}\right);$$

⁶ The operator L_ν is clearly defined in the entire half plane $\eta > 0$ and *a fortiori* in $\mathcal{E}_l$.

hence in the bound for B_2 we can replace $p(r)$ by

$$Q(\rho) = p\left(2R_\nu - \frac{\rho R_\nu}{\cos \phi}\right).$$

In order to satisfy $L_\nu(h) \geq 0$ we assume

$$(16) \quad h' < 0, \quad h'' > 0,$$

and choose $h(\rho^2)$ as a solution of the ordinary differential equation

$$\frac{h''}{h'} = -K_2[1 + R_\nu Q(\rho)], \quad K_2 = \frac{2}{k \cos^4 \phi}.$$

A solution fulfilling the requirements (12) and (16) is easily obtained:

$$h = \frac{1}{3 \cos^2 \phi} \int_{\rho^2}^{4 \cos^2 \phi} \exp \left[-K_2 s - K_2 R_\nu \int_{\cos^2 \phi}^s Q(\sqrt{t}) dt \right] ds.$$

The conditions (12) and (16) make it clear that

$$(17) \quad 0 < h < \eta \quad \text{in } \mathcal{E}_l.$$

Now let $\mathcal{E}_\nu$ be the region common to $\mathcal{D}_\nu$ and $\mathcal{E}_l$. Using (12) and (17) on the one hand, and (9) and (10) on the other we see that

$$W_\nu - (\delta - \gamma)h \geq 0$$

on the boundary of $\mathcal{E}_\nu$, while from (8) and (13),

$$L_\nu(W_\nu - (\delta - \gamma)h) \leq 0$$

in $\mathcal{E}_\nu$. By the maximum principle, therefore,

$$(18) \quad W_\nu \geq (\delta - \gamma)h \quad \text{in } \mathcal{E}_\nu.$$

Now the function h depends on ν and we desire an estimate of W_ν independent of ν . To this end we note that, in the interval $\cos^2 \phi < s < 4 \cos^2 \phi$,

$$\begin{aligned} \int_{\cos^2 \phi}^s Q(\sqrt{t}) dt &\leq \frac{4 \cos^2 \phi}{R_\nu} \int_0^\infty p(t') dt' \quad \left(t' = R_\nu \left(2 - \frac{\sqrt{t}}{\cos \phi} \right) \right), \\ &= K_3/R_\nu, \quad \text{say.} \end{aligned}$$

If we define

$$g(\zeta) = \frac{1}{3 \cos^2 \phi} \int_{\rho^2}^{4 \cos^2 \phi} \exp [-K_2 s - K_2 K_3] ds,$$

we see that g is independent of ν and $g \leq h$ throughout $\mathcal{E}_\nu$. Consequently inequality (18) may be replaced with

$$(19) \quad W_\nu \geq (\delta - \gamma)g \quad \text{in } \mathcal{E}_\nu,$$

as desired. From a straight-forward differentiation

$$\left. \frac{\partial g}{\partial \eta} \right|_{\xi=0} = \frac{4}{3} \exp[-K_2 K_3 - 4K_2 \cos^2 \phi] = 3K_4, \quad \text{say.}$$

It follows that

$$\begin{aligned} g &= 3K_4 \eta + O(\xi^2) \\ &\geq 3K_4 \eta - K_5(\xi^2 + \eta^2) \end{aligned}$$

for small ξ , where K_5 is a constant depending only on the function g , and thus, like K_2 , K_3 , and K_4 , ultimately depending on ϕ and the gross properties of L . Now in the circle $\mathfrak{J}$:

$$\xi^2 + (\eta - \tau)^2 \leq \tau^2,$$

of radius $\tau = K_4/K_5$, we have $g \geq K_4 \eta$, and this together with (19) gives the required behavior of W_ν , namely

$$(20) \quad W_\nu \geq (\delta - \gamma)K_4 \eta \quad \text{in } \mathfrak{J}.$$

We assert that $\epsilon = (\delta - \gamma)K_4 K^{-1}$ fulfills the conditions of the lemma. For consider a fixed point z in $\mathfrak{D}$. If R_ν is large enough the corresponding point ζ is in $\mathfrak{J}$. Then from the definition of W_ν , (20), and (5),

$$\kappa \tilde{v} - u = R_\nu W_\nu \geq (\delta - \gamma)K_4 y \geq \epsilon \tilde{v},$$

and Lemma 2 is proved.

We next prove that $\alpha < 0$ cannot occur. In analogy with $\tilde{v}$ define

$$\tilde{u}(y) = \int_0^y \exp[-P(t)] dt;$$

obviously $0 \leq \tilde{u} \leq y$ and $L(\tilde{u}) \geq 0$.

Lemma 3. *If $\alpha < 0$ and if*

$$u \leq \kappa(\tilde{u} - \sigma), \quad \kappa \leq 0,$$

then there exists a number $\epsilon > 0$ such that

$$u \leq (\kappa - \epsilon)(\tilde{u} - \sigma).$$

Proof. Let γ and δ be any numbers satisfying

$$0 < \delta < \gamma < |\alpha|;$$

and let R_ν be a sequence, $R_\nu \rightarrow \infty$, such that

$$\frac{M(R_\nu)}{R_\nu} < -\gamma, \quad \nu = 1, 2, 3, \dots$$

Introduce the variables $\zeta = (\xi, \eta)$ as in Lemma 2 and set

$$W_\nu(\zeta) = \frac{\kappa(\tilde{u}(y) - \sigma) - u(z)}{R_\nu}$$

for $\zeta \in \mathfrak{D}_\nu$. Then W_ν obeys (8) and (9). Let $\mathfrak{C}'$ be the arc $0 \leq \theta \leq \phi$ of the unit circle in the ζ -plane, where

$$\begin{aligned} \phi &= 2\pi & \text{if } |\kappa| \leq \delta, \\ \sin \phi &= \delta/|\kappa| & \text{if } |\kappa| > \delta. \end{aligned}$$

Then (10) is replaced by

$$(10a) \quad \begin{aligned} W_\nu &\geq 0 & \text{on } \overline{\mathfrak{D}_\nu}, \\ W_\nu &\geq \kappa(\tilde{u}/y) \sin \theta - u/R_\nu \geq \gamma - \delta & \text{on } \mathfrak{C}'. \end{aligned}$$

We should like to continue as in Lemma 2, but unfortunately the first inequality in (10a) cannot play the same role as the corresponding first inequality in (10). However we do have

$$W_\nu + \frac{(\gamma - \delta)\sigma}{R_\nu} \geq (\gamma - \delta)\eta \quad \text{on } \overline{\mathfrak{D}_\nu},$$

using *for the first time* the hypothesis that $\overline{\mathfrak{D}}$ lies in a strip $0 \leq y \leq \sigma$. Thus we can conclude

$$(20a) \quad W_\nu + \frac{(\gamma - \delta)\sigma}{R_\nu} \geq (\gamma - \delta)K'_4 \eta \quad \text{in } \mathfrak{I}',$$

by reasoning almost exactly as in Lemma 2. The only notable difference in the argument is in the construction of the ellipse $\mathfrak{E}$. In the present case $\mathfrak{E}$ must be inclined from the vertical so that the arc $\mathfrak{C}'$ will divide it into two parts. The construction of the required ellipse is best illustrated by figure 1, opposite.

Setting $\epsilon = (\gamma - \delta)K'_4$ we can derive

$$u \leq (\kappa - \epsilon)(\tilde{u} - \sigma) + \sigma(\gamma - \delta)(1 - K'_4)$$

as in Lemma 2. To eliminate the additive constant in this inequality consider the function $U_* = u - (\kappa - \epsilon)(\tilde{u} - \sigma)$. By using Gilbarg's theorem it is easy to conclude that $U_* \leq 0$, and this proves Lemma 3.

Now suppose $\alpha < 0$. By Lemma 1, $u \leq 0$. Let $\mathfrak{R}$ denote the set of all numbers $\kappa \leq 0$ such that

$$u \leq \kappa(\tilde{u} - \sigma).$$

Evidently $\mathfrak{R}$ is non-empty and closed; it is also bounded from below by, say

$$\frac{u(0, y_*)}{\tilde{u}(y_*) - \sigma},$$

where y_* is any number large enough to make the denominator positive. Thus $\mathfrak{R}$ contains a smallest number κ' . But this contradicts Lemma 3; hence $\alpha < 0$ cannot occur.

It remains only to show that the limit β exists and is finite. Put

$$\beta = \overline{\lim}_{Y \rightarrow \infty} \frac{m(Y)}{Y}.$$

We can omit proof for the following lemma.

Lemma 4. *If $0 \leq \kappa < \beta$ and if*

$$v \geq \kappa(\tilde{u} - \sigma),$$

then there exists a constant $\epsilon > 0$ such that

$$v \geq (\kappa + \epsilon)(\tilde{u} - \sigma).$$

FIGURE 1

Now from Lemma 4 and arguments already given it follows that

$$v \geq \beta(\tilde{u} - \sigma).$$

Clearly, then, β is finite; also

$$\beta = \overline{\lim}_{Y \rightarrow \infty} \frac{m(Y)}{Y} \geq \lim_{Y \rightarrow \infty} \frac{m(Y)}{Y} \geq \beta \lim_{Y \rightarrow \infty} \frac{\tilde{u}(Y) - \sigma}{Y} = \beta,$$

so that the limit in question exists. Q.E.D.

Theorem 2 may be extended to equations with n independent variables without any particular difficulty; indeed we have restricted the preceding treatment

to $n = 2$ mainly in order to use the familiar terminology of plane geometry, but the method is perfectly general (see, for example, [2]).

As an example of equations for which a growth condition (2) does not hold we may take

$$\Delta(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + \left(e - m \frac{c}{y}\right) u_y = 0$$

where the domain of definition of the coefficients is again the upper half plane, and where m is a positive constant. We suppose that Δ is uniformly elliptic and that d, e are subject to a condition (2). Actually, the effect of the terms $du_x + eu_y$ has already been explored, so that it is sufficient to confine our discussion to the simpler equation

$$\Delta'(u) = au_{xx} + 2bu_{xy} + c \left(u_{yy} - m \frac{u_y}{y}\right) = 0,$$

whose salient behavior is typical. We prove the following result corresponding to Theorem 1.

Theorem 3. *Let $u = u(z)$ satisfy the inequality $\Delta'(u) \geq 0$ in the upper half plane $y > 0$, and let u be continuous and ≤ 0 on $y = 0$. Then*

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R^{m+1}}$$

exists (finite or infinite); moreover α is non-negative and

$$u \leq \alpha y^{m+1}$$

for $y > 0$. If the equality holds at any point in $y > 0$ then it holds everywhere.

A similar result has been obtained by Huber [7] for the equation

$$\Delta u - m \frac{u_y}{y} = 0.$$

Huber considered an extended range of values of m , namely $m > -1$, but his methods are probably not applicable to variable coefficients or to regions other than the half plane.

The proof of Theorem 3 closely follows the proof of Theorem 2, and therefore many similar arguments can be omitted or radically condensed. As before we set

$$\alpha = \lim_{R \rightarrow \infty} \frac{M(R)}{R^{m+1}};$$

in order to prove Theorem 3 it is sufficient to show that $\alpha < 0$ cannot occur and that $\alpha \geq 0$ implies $u \leq \alpha y^{m+1}$. This will be done with the aid of three lemmas.

Lemma 5. *There exists a constant M_1 , depending only on Δ' , such that*

$$u \leq M_1 \max(0, \alpha) y^{m+1}.$$

Proof. Let G_R denote the intersection of $\mathfrak{D}$ with the circle $r < R$. As in Lemma 1 the crux of the proof is the existence of a function $V_R(z)$ with the properties:

- i) $\Delta'(V_R) \leq 0$;
- ii) $V_R \geq 0$ on the common boundary of $\mathfrak{G}_R$ and $\mathfrak{D}$, and $V_R = 1$ when $r = R$;
- iii) for fixed z

$$\lim_{R \rightarrow \infty} R^{m+1} V_R(z) = M_1 y^{m+1},$$

where M_1 is a constant depending only on Δ' .

As a first step in the construction of V_R we define, for $r < R$, $y > 0$,

$$H_R(z) = \frac{2}{\pi} \left(\arctan \frac{y}{R+x} + \arctan \frac{y}{R-x} \right) = \frac{2}{\pi} \operatorname{Im} \log \frac{R+z}{R-z};$$

This is the harmonic majorant used in the proof of the classical Phragmén-Lindelöf theorem. It is constant on arcs of circles passing through $(R, 0)$ and $(-R, 0)$, and

$$\begin{aligned} H &= 0, & r < R, & & y = 0, \\ (21) \quad H &= 1, & r = R, & & y > 0, \\ &0 < H < 1 & & & \text{in } \mathfrak{G}_R, \end{aligned}$$

where we have dropped the subscript R for convenience.

The desired function V_R is taken in the form

$$V_R = \int_0^H t^m f(t) dt$$

where $f(t)$ remains to be chosen; if we assume

$$(22) \quad f > 0, \quad \int_0^1 t^m f(t) dt = 1,$$

then V_R has the properties (21) in common with H and so satisfies ii). Moreover

$$RH \rightarrow \frac{4y}{\pi}, \quad R^2 \frac{dH}{dR} \rightarrow -\frac{4y}{\pi} \quad \text{as } R \rightarrow \infty,$$

hence by L'Hospital's rule

$$\begin{aligned}\lim_{R \rightarrow \infty} R^{m+1} V_R &= \lim_{R \rightarrow \infty} \frac{\int_0^H t^m f(t) dt}{R^{-(m+1)}} \\ &= \lim_{R \rightarrow \infty} \frac{H^m f(H) \frac{dH}{dR}}{-(m+1)R^{-(m+2)}} = \frac{f(0)}{m+1} \left(\frac{4}{\pi}\right)^{m+1} y^{m+1};\end{aligned}$$

thus *iii*) is satisfied. Finally

$$\begin{aligned}\Lambda'(V_R) &= H^m \left\{ f'[aH_x^2 + 2bH_x H_y + cH_y^2] + f[(aH_{xx} + 2bH_{xy} + cH_{yy}) \right. \\ &\quad \left. + \frac{mH_x}{H} (aH_x + 2bH_y) + \frac{mCH_y}{yH} (yH_y - H)] \right\}.\end{aligned}$$

It is convenient at this point to obtain several bounds. From the complex form of H ,

$$\begin{aligned}(H_x^2 + H_y^2)^{\frac{1}{2}} &= \frac{2}{\pi} \left| \frac{2R}{z^2 - R^2} \right| \\ (H_{xx}^2 + 2H_{xy}^2 + H_{yy}^2)^{\frac{1}{2}} &= \frac{2\sqrt{2}}{\pi} \left| \frac{4Rz}{(z^2 - R^2)^2} \right|, \\ |H_x| &= \frac{2}{\pi} \left| \frac{4Rxy}{(z^2 - R^2)^2} \right|.\end{aligned}$$

From these results we get, setting $G = (H_x^2 + H_y^2)^{\frac{1}{2}}$,

$$(23) \quad G^{-2}(H_{xx}^2 + 2H_{xy}^2 + H_{yy}^2)^{\frac{1}{2}} = \frac{\pi}{\sqrt{2}} \left| \frac{z}{R} \right| < \frac{\pi}{\sqrt{2}};$$

also, assuming that z is in the first quadrant and letting $\alpha = \arctan y/(R - x)$,

$$(24) \quad (GH)^{-1} |H_x| = H^{-1} \left| \frac{2xy}{z^2 - R^2} \right| \leq \frac{\pi}{\alpha} \left| \frac{x \sin \alpha}{R} \right| \leq \pi,$$

and obviously the same inequality holds in the second quadrant. Finally since $H_{yy} \leq 0$ we have

$$(25) \quad yH_y - H \leq 0.$$

In the expression for $\Lambda'(V_R)$ denote the coefficient of f' by B_1 and the three terms in the coefficient of f by B_2 , B_3 , and B_4 respectively. These are bounded as follows:

$$B_1 \geq k(a + c)G^2,$$

$$B_2 \leq (a^2 + 2b^2 + c^2)^{\frac{1}{2}}(H_{xx}^2 + 2H_{xy}^2 + H_{yy}^2)^{\frac{1}{2}} \leq \frac{\pi}{\sqrt{2}}(a + c)G^2,$$

$$B_3 \leq \left| \frac{mH_x}{H} \right| (a^2 + 4b^2)^{\frac{1}{2}} G \leq \sqrt{2}\pi m(a + c)G^2,$$

$$B_4 \leq 0,$$

by (1), Schwarz's inequality, and (23), (24) and (25). If f is now chosen as a solution of the ordinary differential equation

$$\frac{f'}{f} = -\frac{\pi}{k\sqrt{2}}(1 + 2m) = -M_2, \quad \text{say,}$$

and if the conditions (22) are fulfilled, then $\Lambda'(V_R) < 0$. The required solution is

$$f = M_3 \exp [-M_2 t], \quad M_3^{-1} = \int_0^1 t^m \exp [-M_2 t] dt.$$

The function V_R being obtained, the remainder of the proof is the same as in Lemma 1.

Lemma 6. *If $0 < \alpha < \kappa$ and if*

$$u \leq \kappa y^{m+1},$$

then there exists a number $\epsilon > 0$ such that

$$u \leq (\kappa - \epsilon)y^{m+1}.$$

Lemma 7. *If $\alpha < 0$ and if*

$$u \leq \kappa y^{m+1}, \quad \kappa \leq 0,$$

then there exists a number $\epsilon > 0$ such that

$$u \leq (\kappa - \epsilon)y^{m+1}.$$

The proof of Theorem 3 is based on Lemmas 5, 6, and 7 as the proof of Theorem 2 was based on Lemmas 1, 2, and 3. The details, which are omitted, are handled in a manner corresponding exactly to the remarks following Lemma 2 and at the close of Lemma 3.

As for the proof of Lemmas 6 and 7 there is again very little to change from the preceding work. The only important alteration occurs in Lemma 6, where now there arises

$$(8a) \quad \Lambda'_\nu(W) = a_\nu W_{\xi\xi} + 2b_\nu W_{\xi\eta} + c_\nu \left(W_{\eta\eta} - \frac{m}{\eta} W_\eta \right) \leq 0$$

in place of (8). The ellipse $\mathcal{E}$ is constructed as before but the auxiliary function h must be altered in view of the change in (8) just noted. Preserving the notation of Lemma 2 we choose h in the form

$$h = \eta^m f(\rho^2),$$

where f is to be determined later. The following conditions are however imposed on f :

$$(26) \quad \begin{aligned} f(4 \cos^2 \phi) &= 0, & f(\cos^2 \phi) &< 1, \\ f' &< 0, & f'' &> 0, \end{aligned}$$

where primes denote differentiation with ρ^2 . The conditions (12) on h are therefore satisfied and it remains only to determine f so that

$$(13a) \quad \Lambda'_\nu(h) \geq 0.$$

A short computation shows that

$$\begin{aligned} \Lambda'_\nu(h) = \eta^m \bigg\{ & 4f''[a_\nu(4\xi)^2 + 2b_\nu(4\xi)(\eta' \cos^2 \phi) + c_\nu(\eta' \cos^2 \phi)^2] \\ & + 2f'[4a_\nu + 8m\xi\eta^{-1}b_\nu + (1+m)c_\nu \cos^2 \phi] - \frac{mc_\nu}{\eta^2} [4\eta f' \cos^2 \phi + f] \bigg\}. \end{aligned}$$

Now in $\mathcal{E}_\nu$ we have $|\xi| \leq \eta^{\frac{1}{2}} \cos \phi$, so that

$$(27) \quad |\xi\eta^{-1}| \leq \eta^{-\frac{1}{2}} \cos \phi \leq (2 \cos \phi - \rho)^{-\frac{1}{2}} \cos^{\frac{1}{2}} \phi.$$

Also

$$(28) \quad B \equiv 4\eta f' \cos^2 \phi + f \leq 0.$$

To prove (28), note that in view of (26) we have $B \leq 0$ on $\mathcal{E}$ and

$$B_\eta = 8\eta(\eta - 2) \cos^4 \phi f'' + 2\eta \cos^2 \phi f' \leq 0.$$

From (26), (27), (28), and bounds similar to (14) and (15) we obtain

$$\Lambda'_\nu(h) \leq 4(a_\nu + c_\nu)\eta^m \left[k \cos^4 \phi f'' + 2f' \left(1 + m + \frac{m \cos^{\frac{1}{2}} \phi}{(2 \cos \phi - \rho)^{\frac{1}{2}}} \right) \right].$$

In order to satisfy (13a) we can therefore choose f as a solution of

$$\frac{f''}{f'} = -K_2 \left[1 + m + \frac{m}{(2 \cos \phi - \rho)^{\frac{1}{2}}} \right], \quad K_2 = \frac{2}{k \cos^4 \phi}.$$

A solution fulfilling (26) is easily found:

$$f = \frac{1}{3 \cos^2 \phi} \int_{\rho^2}^{4 \cos^2 \phi} \exp \left[-K_2(1+m)s - 2K_2 m \int_{\cos \phi}^{\sqrt{s}} \frac{t dt}{(2 \cos \phi - t)^{\frac{1}{2}}} \right] ds.$$

This completes the determination of h . The proofs of Lemmas 6 and 7 may now be carried over without difficulty from Lemmas 2 and 3, and Theorem 3 is thus established.

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