*Generalizations of the Inversion Formula of Thin Airfoil Theory**

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1. In the theory of two-dimensional flow of an ideal fluid past a thin airfoil there arise two types of problem, which may be called the “thickness” problem and the “lifting” problem; they lead to two different types of boundary value problem for the complex velocity $w = u - iv$ in the complex plane of flow, $z = x + iy$. The problem in the “thickness” case is the determination of the flow past a symmetrical airfoil at zero angle of attack, leading to the following mathematical formulation: along a portion of the real axis (the chord: $a_1 \leq x \leq a_2$, say) values of the velocity component v , equal in magnitude and opposite in directions for the upper and lower sides of the chord, are prescribed: $v(x, +0) = -v(x, -0) (= v_0$, say). An analytic function w whose imaginary part satisfies this boundary condition and for which $w = 0$ at infinity is to be determined. Throughout this paper, the letter w represents only the “disturbance” velocity, which must vanish at infinity. Of special interest is the value of the u -component on the chord, $u(x, 0)$.

Since the boundary condition for v is antisymmetric with respect to the real axis, it is easily recognized that the harmonic function v is itself antisymmetric: $v(x, y) = -v(x, -y)$. It follows that for the portions of the real axis outside the chord we have $v(x, 0) = 0$. Thus, the problem may be formulated as a boundary value problem for the upper half plane (say), with v given along the whole real axis.

The solution is immediately obtained by distributing along the real axis sources whose strength per unit length equals twice the velocity v_0 on the upper

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border of the real axis, since this superposition of singularities produces the required boundary values of v . The complex velocity is thus

$$w(z) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{v(s, +0)}{s - z} ds, \quad (1)$$

where, in the present example,

$$\begin{aligned} v(x, +0) &= 0, & -\infty < x < a_1, \\ &= v_0, & a_1 < x < a_2, \\ &= 0, & a_2 < x < +\infty. \end{aligned}$$

For the components, it follows that

$$u(x, y) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(s - x)v(s, +0)}{(s - x)^2 + y^2} ds \quad (2a)$$

$$v(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{v(s, +0)}{(s - x)^2 + y^2} ds. \quad (2b)$$

In the limit $y \rightarrow 0$, (2b) reduces to an identity, as can be shown easily by introducing $\arctg [y/(s - x)]$ as variable of integration, and (2a) gives

$$u(x, 0) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{v(s, +0)}{s - x} ds, \quad (3)$$

where the principal value of the integral is understood.

Applying (1) to the function $iw = v + iu$, we obtain

$$iw(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(s, +0)}{s - z} ds,$$

which gives on the real axis

$$v(s, 0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(s, +0)}{s - x} ds. \quad (4)$$

This is the inversion of (3) due to Hilbert.

Proceeding now to the "lifting" problem, which considers the flow past an airfoil with zero thickness but with camber and with angle of attack, the boundary value problem may be stated as follows: values of v equal in magnitude and in direction for the upper and lower sides of the chord are prescribed:

$$v(x, +0) = v(x, -0) \quad (=v_0, \text{ say}).$$

Now, since the boundary conditions are symmetrical with respect to the real axis, the function v becomes symmetrical too: $v(x, y) = v(x, -y)$, and on the real axis, outside the chord, it follows that $\partial v/\partial y = 0$, but then also $\partial u/\partial x = 0$

and $u = 0$ (since $w = 0$ for $z = \infty$). Thus there results a "mixed" type boundary value problem for the upper half plane, since on the real axis the values of v and of $\partial v/\partial y$ or the values of v and u are prescribed alternately:

$$\begin{aligned} -\infty < x < a_1: & \quad u = 0 \\ a_1 < x < a_2: & \quad v = v_0 \\ a_2 < x < +\infty: & \quad u = 0. \end{aligned} \tag{5}$$

Neither (3) nor (4) now solves the problem, since both u and v are piecewise unknown, but these formulae provide integral equations for the unknown functions. From equation (4) follows the usual integral equation of the lifting problem [$u(x, +0) = u_0(x)$]:

$$v_0(x) = \frac{1}{\pi} \int_{a_1}^{a_2} \frac{u_0(s)}{s - x} ds. \tag{6}$$

If the boundary condition had consisted in prescribed values for u ("indirect" problem), the lifting case would be solved directly by (4) or by (6). Thus the "direct thickness" and the "indirect lifting" problems lead to quadratures, whereas the "indirect thickness" and the "direct lifting" cases lead to integral equations.

As is well known, in the first treatment of the lifting problem Birnbaum [1] solved (6) by applying it to a set of special functions u_0 ; any u_0 may be represented to a good approximation by a linear superposition of the members of this special set. Glauert [2] reduced this method to its simplest form by introducing a new variable in which the Birnbaum set becomes a Fourier series.

Although this solution can hardly be surpassed for simplicity, it is of great theoretical interest and also of practical importance to note that the integral equation (6) can be inverted. This inversion must be different from the pair of equations (3) and (4), since the integrals therein contain functions that must be known on the whole real axis. For the lifting problem of thin airfoil theory the inversion of (6), given by Munk [3] in 1924, leads to the formula

$$u_0(x) = -\frac{1}{\pi} \sqrt{\frac{a_2 - x}{x - a_1}} \int_{a_1}^{a_2} \sqrt{\frac{s - a_1}{a_2 - s}} \frac{v_0(s)}{s - x} ds, \tag{7}$$

which will be treated later.

The purpose of the present paper is to generalize the inversion formula (7) in two ways. First, we shall prescribe the two conjugate harmonics u and v alternately upon an arbitrary number of portions of the real axis or "slots." The first step in this direction, namely the treatment of two slots (the "bi-plane"), was given by Söhngen [4] in his paper on the integral equation (6). The second generalization is concerned with problems where u and v must be interpreted as the components of the first or higher derivatives of the complex

velocity. In the simplified and unified treatment which will be given here, this generalization also is easily obtained.

Both generalizations are suggested by the actual requirements for the solution of problems which arise in the theory of linearized conical supersonic flow past wings. The reduction of these problems to mixed-type boundary value problems for the complex half plane will not be carried out here; an excellent presentation of this procedure may be found in the paper by Goldstein & Ward [5]. The inversion formula for the conical supersonic flow past wings was established by the junior author in his thesis at Cornell University [6], furnishing the starting point for the present general investigation.

Concerning the early history of the problem, it must be mentioned that the integral equation (6) first appeared in connection with the Prandtl wing theory, and that the first series solutions were suggested by Prandtl and Betz. Birnbaum [1], in the paper mentioned above, orthogonalized a set of special solutions of equation (6).

The first inversion of (6) was given by Betz [7] in connection with the Prandtl wing theory, in a not readily accessible paper of the year 1919. Next, a paper of 1922 by Carleman [8] must be quoted. Carleman started from the mathematical problem of inverting an integral equation equivalent to (6) and found a solution by a method which may be characterized as analytic continuation to complex values. The basic idea of the present paper, which considers the mixed boundary value problem for an analytic function rather than the integral equation, is essentially contained in Carleman's work.* The solution of Munk [3] quoted above also uses the method of complex variables, which will be described and extended later.

The authors believe that a detailed presentation of the generalized solution following the line of thought initiated by Carleman and Munk may be of interest as it can be used as a powerful tool for solving many problems in aerodynamics and related fields. An application to problems where the slot geometry is a function of a parameter (which may be identified with time for unsteady problems, or with the streamwise ordinate in applications to slender wings) has recently been made by Mirels [10].

2. Before attacking the generalized problem, we shall discuss a feature of the solutions of the mixed-type boundary value problem which is present also in the simplest case of the thin airfoil theory. It is well known that the solution of the boundary value problem as stated in (5) is not unique, since it permits an arbitrary additive circulatory solution. This arbitrariness is eliminated in thin airfoil theory by the additional requirement of the Kutta condition, which requires that the velocity remain finite at the trailing edge, a_2 . The inversion formula (7) is written out for this case.

* The authors' attention to this reference was drawn by Prof. H. POLLARD of Cornell University. In a paper by KAC & POLLARD [9], (6) appears in connection with a probability problem.

We now discuss the circulatory solution in more detail. It is a solution of the *homogeneous* mixed boundary value problem; *i.e.*, the prescribed values of the conjugate harmonics are all zero:

$$\begin{aligned} -\infty < x < a_1: & \quad u = 0 \\ a_1 < x < a_2: & \quad v = 0. \\ a_2 < x < +\infty: & \quad u = 0. \end{aligned} \tag{8}$$

If the homogeneous problem has a solution, then the inhomogeneous solution with boundary values (5) is not unique, since two inhomogeneous solutions may differ by a homogeneous solution.

The reason for the existence of a homogeneous solution is physically clear and can be formulated mathematically in the following way. From the interpretation of w as a complex velocity it follows that w will have in general certain singularities, namely at the leading and the trailing edges, where, due to the idealized sharp corners, singularities of w of the type $(z - a_1)^{-\frac{1}{2}}$ and $(z - a_2)^{-\frac{1}{2}}$ may arise. The existence of the homogeneous solution is due to the admission of these singularities, as otherwise it would follow from the Liouville theorem that the only homogeneous solution is the trivial one $w \equiv 0$. With the admitted singularities it is easy to construct the homogeneous solution $H(z)$ for the boundary conditions (8): it is simply

$$H(z) = \frac{iA}{\sqrt{(z - a_1)(z - a_2)}}, \tag{9}$$

where A is a real constant. On the real axis, $H(x)$ is alternately (from left to right) purely imaginary, real, imaginary, and thus satisfies (8). The value of A is easily expressed by the circulation: $A = \Gamma/2\pi$, since for $z \rightarrow \infty$, $H(z) \rightarrow iA/z$. This procedure of constructing the homogeneous solution will be generalized later.

The situation in the case of thin airfoil theory may be summed up as follows: with singularities admitted both at a_1 and at a_2 , the solution is not unique (unless some other condition is given which fixes the circulation). If the singularity at a_2 is excluded, *i.e.*, with the Kutta condition, the solution is unique. If no singularity either at a_1 or at a_2 is allowed, the solution, naturally, is again unique but generally does not exist: it is known that "shock-free entry" and the Kutta condition cannot both be satisfied unless v_0 is subject to special conditions.

The generalized case will exhibit the same behavior. For uniqueness of solution there will be a maximum number of singularities admissible; if the number of singularities is greater, the homogeneous problem has a solution; if it is less, the solution in general does not exist.

If w has to be interpreted as the derivative of the velocity, the degree of admissible singularities will rise: at an edge-point a_n the solution will behave in

general like $(z - a_n)^{-\frac{1}{2}}$. Furthermore, if the velocity itself has a discontinuity prescribed on the real axis, a logarithmic singularity of the *velocity* and thus a pole of the *velocity derivative* will result. These singularities increase the indeterminacy of the mixed boundary value problem. It is clear that if the problem has been solved for the velocity with piecewise constant prescribed values on the real axis, then the derivative is a homogeneous solution for the velocity-derivative problem.

Finally, it is seen why w will be interpreted as velocity or velocity derivative of some order, but not as the complex potential. The potential must be regular everywhere; such a solution does not exist in general, but only if certain conditions for the boundary values are satisfied. This difficulty actually arises even in thin airfoil theory but may be seen much more clearly for the biplane problem. The (real) potential prescribed between the chords fixes the circulation, and the values of the stream function on the two chords fixes the quantity of flow between the two airfoils; it is physically clear that it is not possible to prescribe both circulation and flow quantity in an arbitrary manner.

Since w will be interpreted as velocity "at most," we impose the important condition on the solutions that for $z \rightarrow \infty$, $w = 0$; *viz.*, in our solutions there will be no flow at infinity. It must be required, naturally, that the boundary conditions permit such a solution, or that the prescribed boundary values vanish with a sufficient order at $x = \pm\infty$. For the simple boundary value problem which is solved by equation (1), we shall have $w(\infty) = 0$ if

$$\int_{-\infty}^{+\infty} v(s, +0) ds = \text{finite} \quad (10)$$

(*i.e.*, if this integral exists). For the mixed boundary-value problem, corresponding conditions will be found *a posteriori* from the inversion formulae.

In the next paragraphs we proceed to a systematic discussion of the homogeneous solutions of the mixed problem. It will be shown later that this discussion gives also the clue to the general inversion formula.

3. First, the uniqueness of the ordinary boundary value problem, which arises in the "direct thickness" (or the "indirect lifting") case, will be investigated. Suppose that v is prescribed along the whole real axis and that we are to determine a function $w = u - iv$ which is regular in the upper half plane and which vanishes for $z = \infty$. (It is supposed that the prescribed boundary values fulfill the condition (10).)

The uniqueness condition for the solution is obtained as follows: two possible solutions of the problem as stated above may differ only by a solution of the homogeneous problem with $v(x, 0) = 0$, *i.e.*, by functions $H(z)$ which are real on the real axis. By the principle of images, such a function $H(z)$ must have the same type of singularity $x - iy$ as at $x + iy$. Since our solutions are supposed to be regular in the upper half-plane, including infinity, their only singularities can be on the real axis. But the only singularities on the real axis possible for

an analytic function $H(z)$ which is real on the real axis are poles of integral order. (In the vicinity of such pole at a point $x = b$ (say), $H(z) \sim (z - b)^{-K} = \rho^{-K}(\cos K\vartheta - i \sin K\vartheta)$ where ρ, ϑ are polar coordinates with the origin $x = b, y = 0$. Such an expression can be real for $\vartheta = 0$ and $\vartheta = \pi$ only if K is an integer.)

The inhomogeneous problem becomes unique if the *singularities admitted on the real axis are weaker than a simple pole*, (i.e., “integrable”). For, in this case, the weakest possible singularity of the homogeneous solution $H(z)$, namely a simple pole ($K = 1$), is excluded; thus $H(z)$ must be regular in the whole plane and according to the Liouville theorem is a constant, whence, by the condition $H(\infty) = 0$, it follows that $H(z) \equiv 0$.

4. We proceed now to the investigation of possible non-trivial homogeneous solutions of the mixed-type boundary value problem. Suppose that N edge-points $a_1, a_2, \dots, a_n, \dots, a_N$ are given on the real axis; $a_1 < a_2 < \dots < a_N$. We are to determine a function $H(z)$ such that on successive portions of the real axis limited by these edge-points alternately $u(x, 0) = 0$ and $v(x, 0) = 0$. To fix ideas, suppose that on the portion $a_N < x < +\infty$, $u(x, 0) = 0$ is required. (Naturally, the meanings of u and v may be interchanged.) The behavior of $H(z)$ is fixed at the edge-points a_n by the requirement that $H(x)$ on the real axes changes from purely real to purely imaginary values upon passing through a_n ; thus

$$H(z) \sim (z - a_n)^{\frac{1}{2} + K_n}, \quad (11)$$

where K_n is any (positive or negative) integer, and a homogeneous solution may be written down immediately:

$$H(z) = iA \prod_{n=1}^N (z - a_n)^{\frac{1}{2} + K_n} \quad (12)$$

where A is a real constant. It is easily verified that real and imaginary parts alternately vanish on the real axis.

For applications where the solution is interpreted as complex velocity, singular behavior will be admitted only at the edge-points. In view of the generalization to velocity-derivatives, which we have in mind, a more general case will be treated. For this purpose it is convenient to start with the assumption that edge-points are all *regular*, but to admit a set of simple poles at some other points $b_1, b_2, \dots, b_n, \dots, b_M$ of the real axis. (Later the solution for the case of the singular edge-points will be found by confluence of the b_m with the a_n .)

The regularity requirement at the points a_n excludes negative values of K_n in (12); then $H(z)$ as defined by (12) violates the condition of no flow at infinity, since for $z \rightarrow \infty$ we have $H(z) \sim z^{\frac{1}{2}N}$ at least (for all $K_n = 0$). But the condition $H(\infty) = 0$ may be satisfied with the help of the poles admitted at the

points b_m by writing a new homogeneous solution

$$H(z) = iA \prod_{n=1}^N (z - a_n)^{\frac{1}{2} + K_n} \sum_{m=1}^M \frac{B_m}{z - b_m}, \quad (13)$$

where the B_m are real constants. (13) represents a homogeneous solution, since the sum is purely real on the real axis, so that the real and imaginary parts of $H(z)$ still vanish alternately. This function may also be written as

$$H(z) = iA \frac{\prod_{n=1}^N (z - a_n)^{\frac{1}{2} + K_n}}{\prod_{m=1}^M (z - b_m)} P_R(z) \quad (14)$$

where $P_R(z)$ is a polynomial in z of degree $R \leq M - 1$ and with coefficients all real. Let

$$A \prod_{n=1}^N (z - a_n)^{K_n} P_R(z) = P_r(z)$$

be a polynomial in z of any degree r with real coefficients and with zeros different from the b_m . Then

$$H(z) = \frac{i \prod_{n=1}^N (z - a_n)^{\frac{1}{2}}}{\prod_{m=1}^M (z - b_m)} P_r(z). \quad (15)$$

For $z \rightarrow \infty$ follows $H(z) \sim z^{\frac{1}{2}N + r - M}$, and no homogeneous solution exists if

$$\frac{1}{2}N + r - M \geq 0.$$

Since $r \geq 0$, the maximum number M^* for which no homogeneous solution exists, or for which the inhomogeneous solution is unique, is $M^* = \frac{1}{2}N$ if N is even or $M^* = \frac{1}{2}(N - 1)$ if N is odd; in conventional notation

$$M^* = [\tfrac{1}{2}N]. \quad (16)$$

Thus far only simple poles at points b_m have been admitted. It is immediately seen that the confluence of two or more points to one point yields a pole of order two or more at such a point. Finally, (15) shows that by confluence of a point b_m with an edge-point a_n a singularity of order $-\frac{1}{2}$ results. More generally, an admission of an edge singularity of the type $(z - a_n)^{-K + \frac{1}{2}}$ is equivalent to the admission of a pole of order K or of K simple poles. With this generalization of the counting of the admitted poles, it may be stated that M^* as given by (16) is the maximum number M for which the solution is unique; if $M < M^*$, the solution is still unique, if it exists, but existence is uncertain as we have fewer degrees of freedom than for $M = M^*$. If $M > M^*$, a homogeneous solu-

tion $H(z)$ exists and is given by (15), where in order to assure vanishing at infinity the polynomial $P_r(z)$ can have degree at most

$$\begin{aligned} r &= M - M^* - 1 \\ &= M - [\tfrac{1}{2}N] - 1 \end{aligned} \quad (17)$$

and hence $r + 1 = M - M^*$ arbitrary coefficients. The solution for $M > M^*$ can be made unique by additional requirements, *e.g.*

1) The requirement of a strong vanishing of $w(z)$ at infinity restricts the degrees of freedom. If, for instance $w(z) \sim z^{\frac{1}{2}(N-1)-M}$ is wanted at infinity, the solution is unique.

2) A prescribed zero of $H(z)$ and thus of $P_r(z)$ reduces the number of free coefficients, or the degrees of freedom. Confluence of a zero with an admitted pole will eliminate this pole.

3) Prescribed values of the circulation will give conditions for the free coefficients of P_r .

As an example, consider the case where at every edge a behavior of the type $(z - a_n)^{-\frac{1}{2}}$ is admitted. According to the general rule of counting, $M = N$ and by (17) the polynomial P_r has the degree

$$r = N - \left[\frac{N}{2} \right] - 1 = \begin{cases} \frac{N}{2} - 1 & (N \text{ even}) \\ \frac{N-1}{2} & (N \text{ odd}) \end{cases} \equiv \nu, \text{ say.}$$

$H(z)$ is the most general circulatory solution:

$$H(z) = \frac{i(A_0 + A_1 z + \cdots + A_\nu z^\nu)}{\sqrt{(z - a_1)(z - a_2) \cdots (z - a_N)}}. \quad (18)$$

If N is even, the algebraic sum of all circulations about the $\frac{1}{2}N$ slots will determine the value of the coefficient A_ν of the highest power, since for $z \rightarrow \infty$ we have $H(z) = iA_\nu/z$, so that $A_\nu = \Sigma \Gamma/(2\pi)$. But to determine the circulation about one slot alone, even for $N = 4$, an elliptic integral must be evaluated. Imposing the condition of regularity (the Kutta condition) at an edge-point a_n is equivalent to the requirement of a zero in the numerator of (18) at $z = a_n$; if the factor $z - a_n$ is put in evidence in the numerator, the remaining number of arbitrary coefficients is reduced by one. Generally, if there are N_1 points ($n = n_1$) where a Kutta condition is applied and $N_2 = N - N_1$ points ($n = n_2$) which may be singular as before, write

$$H_0(z) = i \frac{\prod_{n_1=1}^{N_1} (z - a_{n_1})^{\frac{1}{2}}}{\prod_{n_2=1}^{N_2} (z - a_{n_2})^{\frac{1}{2}}} \quad (19)$$

and

$$H(z) = H_0(z)P_r(z), \quad (20)$$

where the degree r easily follows from the condition of no flow at infinity.

The definition of the special homogeneous solution $H_0(z)$ may be put in words as follows: it is that homogeneous solution which fulfills all requirements on the real axis and is of lowest possible degree at infinity.

5. Before considering examples of more general types of singularities, we shall give special attention to those homogeneous solutions which violate the condition of no flow at infinity. Consider first a simple example: for thin air-foil theory ($N = 2$) with the Kutta condition at a_2 , the solution is unique and the homogeneous solution

$$H(z) = iA_0 \sqrt{\frac{z - a_2}{z - a_1}} \quad (21)$$

violates condition of no flow at infinity, as $H(\infty) = iA_0 \neq 0$. But consider the function

$$G(z) = iA_0 \left(\sqrt{\frac{z - a_2}{z - a_1}} - 1 \right). \quad (22)$$

This function satisfies the condition of no flow at infinity but evidently does not satisfy the homogeneous boundary conditions on the real axis. Thus $G(z)$ is a special solution of the non-homogeneous mixed boundary value problem. That the boundary conditions are satisfied is easily checked: note that where the *real* part of $H(z)$ vanishes, the real part of $G(z)$ is also zero. Thus the solution $G(z) \equiv w(z) = u(x, y) - iv(x, y)$ satisfies the boundary condition

$$\begin{aligned} -\infty < x < a_1: & \quad u(x, 0) = 0 \\ a_1 < x < a_2: & \quad v(x, 0) = A_0 \\ a_2 < x < +\infty: & \quad u(x, 0) = 0. \end{aligned}$$

Thus, as is immediately recognized from (22), it yields the solution for a flat plate.

As a next step, consider

$$H(z) = i \sqrt{\frac{z - a_2}{z - a_1}} (A_0 + A_1 z). \quad (23)$$

An asymptotic expansion for large z gives

$$\begin{aligned}
 H(z) &\sim i \left(1 - \frac{a_2 - a_1}{2z} + \cdots \right) (A_0 + A_1 z) \\
 &= i \left[A_1 z + A_0 - A_1 \frac{a_2 - a_1}{2} + O\left(\frac{1}{z}\right) \right].
 \end{aligned}$$

Thus the following function vanishes for $z \rightarrow \infty$:

$$G(z) = i \sqrt{\frac{z - a_2}{z - a_1}} (A_0 + A_1 z) - i \left(A_1 z + A_0 - A_1 \frac{a_2 - a_1}{2} \right). \quad (24)$$

Upon verification it is seen that this solution corresponds to a linearly varying boundary value of v on the chord $a_1 < x < a_2$.

Generalizing this approach, a new method in thin airfoil theory may be constructed. Suppose the velocity $v(x, 0)$ which is prescribed on the chord is given as a polynomial:

$$v(x, 0) = \alpha_0 + \alpha_1 x + \cdots + \alpha_l x^l. \quad (25)$$

The complex velocity w satisfying the Kutta condition is then given by

$$\begin{aligned}
 w(z) &= i \sqrt{\frac{z - a_2}{z - a_1}} (A_0 + A_1 z + \cdots + A_l z^l) \\
 &\quad - i(\alpha_0 + \alpha_1 z + \cdots + \alpha_l z^l),
 \end{aligned} \quad (26)$$

where the $A_0, A_1, \cdots A_l$ are to be determined, with the help of an asymptotic expansion, in such a way that $w \rightarrow 0$ for $z \rightarrow \infty$. The first part of (26) satisfies the homogeneous boundary conditions on the real axis; the second part is purely imaginary on the real axis and satisfies the boundary condition (25) on the chord but does not affect the condition $u(x, 0) = 0$ off the chord. This *method of cancellation of the singularity at infinity* will certainly not replace the well-known Birnbaum-Glauert method in thin airfoil theory, since the algebra involved for the asymptotic expansion grows rapidly with increasing l . However, this new method may easily be generalized for an arbitrary number of slots, where no solutions so simple as those occurring in thin airfoil theory are known. In the notation of (19) the general formula for the case of *even* N is

$$w(z) = H_0(z)(A_0 + A_1 z + \cdots + A_l z^l) - i(\alpha_0 + \alpha_1 z + \cdots + \alpha_l z^l), \quad (27)$$

where the α_l are taken from a polynomial development of $v(x, 0)$ which represents the boundary values on all chords, but no restriction on the α_l arises from the behavior of this function between the chords, where $u(x, 0) = 0$ is required. If the number N is odd, we have to start from a representation of $v(x, 0)$ in the form

$$v(x, 0) = \alpha_0 x^{\frac{1}{2}} + \alpha_1 x^{\frac{3}{2}} + \cdots + \alpha_l x^{l+\frac{1}{2}}, \quad (25a)$$

and the solution is

$$w(z) = H_0(z)(A_0 + A_1z + \cdots + A_lz^l) - i\sqrt{z}(\alpha_0 + \alpha_1z + \cdots + \alpha_lz^l). \quad (27a)$$

As an example, consider the case $N = 4$. Let two plates, extending from a_1 to a_2 and a_3 to a_4 be translated and rotated rigidly, so that the velocity distribution on both plates can be described by the equation

$$v(x, 0) = \alpha_0 + \alpha_1 x \quad \begin{cases} a_1 < x < a_2 \\ a_3 < x < a_4. \end{cases}$$

With Kutta conditions at the edges a_2 and a_4 we get

$$H_0(z) = i \sqrt{\frac{(z - a_2)(z - a_4)}{(z - a_1)(z - a_3)}},$$

and with $l = 1$ in (27) the cancellation of the singularity at infinity leads to

$$A_1 = \alpha_1, \quad A_0 = \alpha_0 + \frac{1}{2}(a_2 - a_1 + a_4 - a_3)\alpha_1.$$

A Kutta condition may be replaced by the requirement of a stronger vanishing at infinity. For instance, with $N = 2$ and no Kutta condition,

$$H_0(z) = \frac{i}{\sqrt{(z - a_1)(z - a_2)}}.$$

It will be required that the resulting circulation vanishes; this means that $w(z)$ at infinity behaves like z^{-2} at most. In this case, one replaces (27) by

$$w(z) = H_0(z)(A_0 + A_1z + \cdots + A_lz^l + A_{l+1}z^{l+1}) - i(\alpha_0 + \cdots + \alpha_lz^l), \quad (28)$$

requiring in addition that the term in z^{-1} in the asymptotic expansion shall vanish. For linear variation of $v(x, 0)$ as before, the result (written out for $a_2 = -a_1 = a$) is:

$$A_2 = \alpha_1, \quad A_1 = \alpha_0, \quad A_0 = \frac{1}{2}\alpha_1 a^2$$

6. We proceed now to examples of homogeneous solutions for more general types of singularities. The procedure for constructing the homogeneous solution remains the same: first, from the prescribed behavior at the edge-points a_n and, as the case may be, from admitted singularities at certain points b_m , a product representation of a special solution $H_0(z)$ is established. If $H_0(z)$ does not satisfy the conditions at infinity, no homogeneous solution exists; otherwise the most general homogeneous solution is $H(z) = H_0(z)P_r(z)$, where $P_r(z)$ is an arbitrary polynomial in z with real coefficients, whose degree r is limited in such a way that $H(z)$ satisfies the condition of no flow at infinity.

Consider the case of two edge-points a_1 and a_2 , where the solution is to be interpreted as the velocity derivative

$$w'(z) = u_x - iu_y = -v_y - iv_x.$$

If only the (strongest possible) edge-singularities are admitted, the function $H_0(z)$ is

$$H_0(z) = i(z - a_1)^{-\frac{1}{2}}(z - a_2)^{-\frac{1}{2}},$$

and, in view of the fact that a velocity derivative behaves at infinity like z^{-2} at most, the most general homogeneous solution is

$$H(z) = i(z - a_1)^{-\frac{1}{2}}(z - a_2)^{-\frac{1}{2}}(A_0 + A_1 z).$$

With a Kutta condition at a_2 on the "velocity-level," the velocity derivative behaves like $(z - a_2)^{-\frac{1}{2}}$ and the homogeneous solution is

$$H(z) = iA_0(z - a_1)^{-\frac{1}{2}}(z - a_2)^{-\frac{1}{2}}. \quad (29)$$

This is the velocity derivative of the solution $w(z) = G(z)$ from (22), with $v(x, 0) = \text{const.}$ on the chord.

It was mentioned above that the derivatives of velocities with piecewise constant boundary values are also homogeneous solutions. At a point where a jump between the constant velocities on the boundary is given, a simple pole of the velocity derivative will be found. Thus, if in the thin airfoil case a jump of the velocity at a point b is present, the velocity derivative has a homogeneous solution with

$$H_0(z) = i(z - a_1)^{-\frac{1}{2}}(z - a_2)^{-\frac{1}{2}}(z - b)^{-1},$$

and the general homogeneous solution is

$$H(z) = i(A_0 + A_1 z)(z - a_1)^{-\frac{1}{2}}(z - a_2)^{-\frac{1}{2}}(z - b)^{-1}, \quad (30)$$

which may be also written in the form

$$H(z) = \frac{i}{\sqrt{(z - a_1)(z - a_2)}} \left(\frac{B_1}{z - a_1} + \frac{B_2}{z - b} \right) \quad (31)$$

(with $A_1 = B_1 + B_2$, $A_0 = -bB_1 - a_1B_2$). This expression is split into two terms, where the first is identical with the expression (29), and the corresponding velocity is given by (22). As to the second part, let

$$\frac{dw}{dz} = \frac{iB_2}{\sqrt{(z - a_1)(z - a_2)}} \frac{1}{z - b}, \quad (32)$$

whence by an elementary integration follows

$$w = \frac{iB_2}{\sqrt{(b-a_1)(b-a_2)}} \log \frac{\sqrt{b-a_2} \sqrt{z-a_1} - \sqrt{b-a_1} \sqrt{z-a_2}}{\sqrt{b-a_2} \sqrt{z-a_1} + \sqrt{b-a_1} \sqrt{z-a_2}} \cdot \frac{\sqrt{b-a_2} + \sqrt{b-a_1}}{\sqrt{b-a_2} - \sqrt{b-a_1}}. \quad (33)$$

The second factor under the log sign has been introduced to insure the vanishing of w for $z \rightarrow \infty$. For the interpretation of the solution, a certain distinction must be made between the cases where b lies on the chord: $a_1 < b < a_2$, and where b lies outside the chord. In the first case, there will be a jump in v on the chord, and w represents the solution for a flat plate with deflected flap, with piecewise constant values of v on the chord. If b is situated outside the chord, there will be a jump in u at b , which may be interpreted as a consequence of a vortex sheet of constant strength between the plate and the point b . Our further discussion will be restricted to the case $a_1 < b < a_2$. It is advantageous to make the numerator of the first factor and the denominator of the second factor in (33) rational; the following form results:

$$w = \frac{B_2}{\sqrt{(b-a_1)(a_2-b)}} \log \frac{(i\sqrt{a_2-b} + \sqrt{b-a_1})^2(b-z)}{(\sqrt{a_2-b} \sqrt{z-a_1} + \sqrt{b-a_1} \sqrt{a_2-z})^2}. \quad (33a)$$

The quantity under the log sign changes its sign if z passes through the point b on the real axis. Thus the imaginary part of the logarithm changes by $-\pi$ if z passes through b on the upper border of the real axis in the sense of increasing x , and the jump in x has the value

$$\Delta v = \frac{\pi B_2}{\sqrt{(b-a_1)(a_2-b)}} \quad (34)$$

or one-half of the residue of (32) at $z = b$, as it must be. In writing out the velocity distribution on the chord, the cases $x < b$ and $x > b$ must be distinguished; for $x < b$ the decomposition into real and imaginary part gives

$$u(x, 0) = \frac{B_2}{\sqrt{(a_2-b)(b-a_1)}} \log \frac{(b-x)(a_2-a_1)}{(\sqrt{a_2-b} \sqrt{x-a_1} + \sqrt{b-a_1} \sqrt{a_2-x})^2}$$

$$v(x, 0) = -\frac{2B_2}{\sqrt{(a_2-b)(b-a_1)}} \operatorname{arctg} \sqrt{\frac{a_2-b}{b-a_1}} (= \text{const.}).$$

This example shows the construction of homogeneous solutions for velocity-derivatives and its application to the solutions for the velocity with piecewise constant boundary values, which can be generalized immediately for arbitrary number of edge-points. For $N > 2$, however, the integration corresponding to the step from (32) to (33) leads to elliptic or hyperelliptic functions in general.

If the method is extended to the K -th derivative of the velocity, edge sin-

gularities of the maximal order $(z - a_n)^{-K-1}$ and singular points with the behavior $(z - b_m)^{-K}$ (at most) must be admitted.

7. Next, special attention will be given to homogeneous solutions, which will be interpreted again as velocities, but simple poles on the real axis will also be admitted. In the velocity interpretation such a simple pole represents a source or a vortex on the real axis. The importance of such a solution lies in the fact that it provides an elementary solution (Green's function) enabling solution of the general inhomogeneous problem by superposition.

Consider the homogeneous solution with one additional simple pole at a point $x = s$ of the real axis; it will be called $H^*(z)$. In the notation introduced in (19), we have

$$H^*(z) = H_0(z) \frac{A_0}{z - s}. \quad (35)$$

Even if the solution of the problem without the additional pole is unique, so that $H_0(z)$ does not vanish for $z \rightarrow \infty$, the solution $H^*(z)$ will tend toward zero for $z \rightarrow \infty$. For example, in the case of thin airfoil theory with the Kutta condition

$$H^*(z) = i \sqrt{\frac{z - a_1}{z - a_2}} \frac{A_0}{z - s}. \quad (36)$$

For the interpretation of this solution, again two cases must be distinguished: first, $a_1 < s < a_2$, or s lies on the chord; and second, s is situated outside the chord. In the first case, the solution in the neighborhood of $z = s$ may be written as

$$H^*(z) \approx \sqrt{\frac{a_2 - s}{s - a_1}} \frac{A_0}{z - s}, \quad (37)$$

where $z = s$ has been introduced for each factor which is not singular for $z = s$. Interpreted as velocity, (37) represents a source, since the quantity which multiplies $(z - s)^{-1}$ is purely real. The source strength Q_+ which will be found on the upper half of the chord is obviously one half of the magnitude of the residue at $z = s$:

$$Q_+ = \pi A_0 \sqrt{\frac{a_2 - s}{s - a_1}}. \quad (38)$$

For the discussion of the whole flow pattern, it must be recalled that outside the chord $u(x, 0) = 0$, or the streamlines are vertical. This means that the whole flow quantity which has been emitted by the source above the chord must flow to the lower half plane. Thus, if the singularity is approached on the lower side of the chord, it must be interpreted as a sink which receives the flow

quantity emitted on the upper side; the source strength, which will be called Q_- on the lower side, is therefore

$$Q_- = -Q_+. \quad (39)$$

A more formal verification of this change of sign may be obtained by following the proper branch of the multi-valued function (36). This point of view will be discussed later.

Actually, (36) can be recognized as the "elementary solution" used by Munk [3] in his original treatment: a source has been put at a point $x = s$ on the upper half of the chord and a sink on the lower side; the chord is a streamline ($v = 0$), but v is singular for $x = s$, where the flow has the same direction on the upper and the lower sides, as a consequence of the change of sign in source strength above and below the chord. The solution (36) also satisfies the Kutta condition and differs in this respect from Munk's original treatment; solutions not satisfying the Kutta condition will be considered in §9.

With the help of the solution (36), the inhomogeneous problem of thin airfoil theory as formulated in (5) may be solved. The velocity component is prescribed on the chord: $v(x, 0) = v_0(x)$. At a point $x = s$, $v_0(s)$ will be produced by a source situated at $x = s$ with the strength

$$Q_+ = v_0(s) ds \quad (40)$$

on the upper half of the chord, provided that a sink of equal and opposite strength is put on the lower side, which insures $u = 0$ outside the chord. Thus, (36) is the proper elementary solution, and A_0 follows from (38) and (40):

$$A_0 = \frac{1}{\pi} \sqrt{\frac{s - a_1}{a_2 - s}} v_0(s) ds. \quad (41)$$

The total solution $w(z)$ is obtained by superposition of solutions (36) with A_0 given by (41):

$$w(z) = \frac{i}{\pi} \sqrt{\frac{z - a_2}{z - a_1}} \int_{a_1}^{a_2} \sqrt{\frac{s - a_1}{a_2 - s}} \frac{v_0(s)}{z - s} ds. \quad (42)$$

On the chord, (42) reduces to (7). Clearly, the method employed to obtain (42) may be generalized so as to solve the most general mixed-type boundary value problem for the half plane, set in §1. The following steps are required:

1. If values of $u(x, 0)$ different from 0 outside the chord are necessary, (36) is still the proper elementary solution. For s outside the chord, the solution (36) represents a vortex situated at $x = s$ with a plate between a_1 and a_2 . This solution has been used by v. Kármán & Sears [11] in order to construct free vortex sheets on the real axis outside the chord.

2. For arbitrary number of slots, the proper $H_0(z)$ has to be chosen in (35).

3. Different types of singularities, again, will be taken into account by the proper function $H_0(z)$.

For the explicit treatment of the general case, however, we shall use a different approach, which, while more formal, is advantageous in other respects. The present treatment has primarily historical interest, as it shows how Munk's original idea may be extended. The drawback of this method lies in a certain difficulty in fixing the sign of the singularity, a difficulty which has been encountered also for the present example but becomes even more serious with increasing number of slots. Moreover, the interpretation of the solution is not so easily visualized (although it may be) for velocity derivatives. The following treatment, which uses essentially the approach of Carleman, will avoid these difficulties.

8. In the general case of the mixed-type boundary value problem for the half plane, the values of $u(x, 0)$ and $v(x, 0)$ are given alternately on portions of the real axis which are separated by the edge-points $a_1, a_2, \dots, a_N$. The first step towards the general solution is the construction of the homogeneous solution $H_0(z)$, in which the alternately given boundary values of u and v are put equal to zero. For the present purpose, $H_0(z)$ is built up as a product in which all requirements on the behavior at the edge-points a_n are expressed, together with other singularities on the real axis, if any, which are admitted at certain points b_m ; the behavior at infinity may be disregarded for the moment and will be discussed later. Thus, $H_0(z)$ is simply the homogeneous solution with the lowest possible (but unspecified) power in z at infinity, using all requirements on the real axis.

Let $w(z)$ be the solution of the inhomogeneous problem, and consider the function

$$F(z) = \frac{w(z)}{H_0(z)} \quad (43)$$

or the quotient of the inhomogeneous and the homogeneous solutions. For the decomposition into real and imaginary parts, we put

$$F(z) = p(x, y) - iq(x, y).$$

It may be proved easily that the imaginary part of $F(z)$ on the real axis, *i.e.* $q(x, 0) = q_0(x)$, is known on the whole real axis. Writing

$$\begin{aligned} w(z) &= u(x, y) - iv(x, y) \\ H_0(z) &= u_H(x, y) - iv_H(x, y), \end{aligned}$$

we have on the real axis

$$p(x, 0) - iq(x, 0) = \frac{u(x, 0) - iv(x, 0)}{u_H(x, 0) - iv_H(x, 0)}. \quad (44)$$

Consider now a portion where $u(x, 0)$ is prescribed. According to the definition of $H_0(z)$, $u_H(x, 0) = 0$ on such a portion; thus, equating the imaginary parts of (44), we have

$$q(x, 0) = -\frac{u(x, 0)}{v_H(x, 0)}, \quad (45a)$$

and, both $u(x, 0)$ and $v_H(x, 0)$ being known, $q(x, 0)$ is obtained. On a portion where $v(x, 0)$ is prescribed, $v_H(x, 0)$ will vanish, and taking again the imaginary part of (44) we get

$$q(x, 0) = \frac{v(x, 0)}{u_H(x, 0)}, \quad (45b)$$

so that again $q(x, 0)$ is obtained. The non-vanishing harmonic parts of $H_0(z)$ on the real axis may be written also in the form

$$u_H(x, 0) = H_0(x)$$

$$-iv_H(x, 0) = H_0(x)$$

on the appropriate portions of the real axis; $q(x, 0) \equiv q_0(x)$ may be put in the form

$$q_0(x) = \frac{v(x, 0)}{H_0(x)} \quad (46a)$$

when $H_0(x)$ is real, and

$$q_0(x) = \frac{iu(x, 0)}{H_0(x)} \quad (46b)$$

when $H_0(x)$ is imaginary; both formulas are joined in the expression

$$q_0(x) = -\Im \left\{ \frac{w(x)}{H_0(x)} \right\}. \quad (47)$$

Since the imaginary part of $F(z)$ is known on the whole real axis, we have an ordinary boundary value problem for this function, a problem which may be solved by (1):

$$F(z) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{q_0(s) ds}{s - z},$$

or by (43):

$$w(z) = -\frac{1}{\pi} H_0(z) \int_{-\infty}^{+\infty} \frac{q_0(s)}{s - z} ds \quad (48)$$

with

$$q_0(x) = \begin{cases} \frac{v(x, 0)}{H_0(x)} \\ \frac{i u(x, 0)}{H_0(x)} \end{cases}, \text{ respectively.} \quad (46)$$

(48) and (46) give the general solution of the inhomogeneous problem. In the application of this formula special attention must be given to the sign of the double-valued function H_0 . It is important to remain on the *same* branch on the upper border of the real axis both for $H_0(z)$ in (48) and $H_0(x)$ in (46). The result does not depend on the choice of the branch: a change in the sign of $H_0(z)$ and $H_0(x)$ applied everywhere cancels in the result; but the change in sign passing an edge-point a_n must be checked carefully by considering the change of the argument of $H_0(z)$ following the upper border of the x -axis.

Finally, the behavior at infinity will be verified: (48) gives the solution only if w vanishes at infinity.

First, consider the case when $H_0(z)$ has the maximum number of singularities for which $w(z)$ is unique. Then, $H_0(z)$ will not vanish at infinity, but $H^*(z) = H_0(z)(z - s)^{-1}$ will, so that $w(z)$ according to (48) satisfies the condition of no flow at infinity; it must only be required that

$$\int_{-\infty}^{+\infty} q_0(s) ds = \text{finite} \quad (49)$$

which is the application of the condition (10) to the present case.

Second, assume that the number of admitted singularities is less than in the previous case; then, the solution will not exist in general, since $H^*(z)$ and also $w(z)$ will not vanish at infinity. Under certain circumstances, though, the solution may exist and may be represented by (48). Let the integral in (48) be expanded formally in an asymptotic series for large z :

$$w(z) = \frac{1}{\pi} H_0(z) \sum_{K=0}^{\infty} \frac{1}{z^{K+1}} \int_{-\infty}^{+\infty} s^K q_0(s) ds. \quad (50)$$

Then $w(\infty) = 0$ if

$$\int_{-\infty}^{+\infty} s^K q_0(s) ds = 0, \quad K = 0, 1, 2, \dots, l-1 \quad (51)$$

and

$$\int_{-\infty}^{+\infty} s^l q_0(s) ds = \text{finite}, \quad (52)$$

where l is determined by the condition that

$$z^{-l-1} H_0(z) \big|_{z=\infty} \rightarrow 0. \quad (53)$$

(51) and (52) certainly represent necessary conditions for the existence of the solution if there exists an $l > 0$ for which (53) holds, but the previous formal development does not assure that these conditions are also sufficient. However, it is certainly true that if $w(z)$ in (48) vanishes at infinity, the solution is the required one, independently of the behavior of $H_0(z)$.

In the third and last case, let $H_0(z)$ have singularities sufficient that it vanishes at infinity. Then (48) always represents a solution, but not the most general one. The general solution is obtained if (48) is completed by adding a homogeneous solution.

$$w(z) = -\frac{1}{\pi} H_0(z) \int_{-\infty}^{+\infty} \frac{q_0(s)}{s-z} ds + H_0(z)(A_0 + \dots + A_r z^r) \quad (54)$$

with real coefficients A_0, A_1 etc.; the degree r is limited by the condition

$$z^r H_0(z) \big|_{z=\infty} \rightarrow 0. \quad (55)$$

As an example, consider the following problem: a vortex is situated at a point $z = z_0 = x_0 + iy_0$ off the real axis; determine the complex velocity in the presence of boundaries (or plates) on the real axis, which are situated along the portions $a_1 < x < a_2, a_3 < x < a_4, \dots, a_{N-1} < x < a_N$ (N even). The portions $-\infty < x < a_1, a_2 < x < a_3, \dots, a_N < x < +\infty$ are free.

For the complete solution, write

$$W(z) = \frac{i\Gamma}{2\pi(z-z_0)} + w(z),$$

where $w(z)$ is regular everywhere except on the real axis. The boundary conditions for $w(z)$ are determined by the requirement that $w(z)$ has to cancel the v -component of the vortex velocity on the plates:

$$-\Im\{W(x)\} = \frac{\Gamma y_0}{2\pi[(x-x_0)^2 + y_0^2]} + v(x, 0) = 0 \quad (56)$$

on the portions $a_1 < x < a_2$, etc. Evidently $v(x, 0) = v(x, -0)$, and it follows that $u(x, 0) = 0$ on the free portions: the solution $w(z)$ cannot produce any change in the u -component of the vortex on the free portions of the real axis. Thus a mixed type boundary value problem for the function $w(z)$ has been set.

No Kutta condition has been prescribed; the solution has to be interpreted as a velocity, so that the special homogeneous solution will be

$$H_0(z) = i[(z-a_1) \dots (z-a_N)]^{-\frac{1}{2}}. \quad (57)$$

By (46), $q_0(x)$ is given. Let us choose the branch of $H_0(z)/i$ which is positive for $-\infty < x < a_1$. When x passes edge a_1 on the upper border of the real axis in the positive x -direction, the argument of the product $(z - a_1) \cdots (z - a_N)$ decreases by π ; the argument of $H_0(z)$ increases by $\frac{1}{2}\pi$. Thus

$$H_0(x) = - \frac{1}{|\sqrt{(x - a_1) \cdots (x - a_N)}|}$$

on the plate $a_1 < x < a_2$. The same reasoning applies for each edge point: the argument of $H_0(z)$ increases by $\frac{1}{2}\pi$ at each passage in the positive direction. To arrive at the next plate we must pass two edge-points, so that the argument increases by π . Thus $H_0(x)$ changes its sign from plate to plate. The result may be expressed in general as

$$H_0(x) = (-1)^{\frac{1}{2}n} |H_0(x)|, \quad (58)$$

where n is the subscript referring to the right-hand edge point a_n of the plate.

After the signs have been fixed, q_0 is given by (46); $q_0 = 0$ on the free portions. Finally, $w(z)$ follows from (48) with $H_0(z)$ given by (57) with the branch of $H_0(z)$ chosen as above. The complete solution is obtained after adding to $w(z)$ the homogeneous solution $H_0(z)P_r(z)$, where P_r is a polynomial with real coefficients and of degree $r = \frac{1}{2}N - 1$.

If a Kutta condition had been prescribed, the solution could be obtained in two ways: either the solution is obtained as above and the Kutta condition is used as an equation for determining the coefficients of $P_r(z)$; or the Kutta condition is taken into account by selecting a proper function $H_0(z)$, different from (57). The two methods are equivalent; there is a certain arbitrariness in the procedure. This will be discussed in detail for the thin airfoil theory as an example.

Suppose the problem, as formulated by the boundary conditions (5), is to be solved without the Kutta condition. The complete solution is easily obtained by the results of this section:

$$w(z) = - \frac{1}{\pi \sqrt{(z - a_1)(a_2 - z)}} \int_{a_1}^{a_2} \sqrt{(s - a_1)(a_2 - s)} \frac{v_0(s)}{s - z} ds + \frac{A_0}{\sqrt{(z - a_1)(a_2 - z)}}. \quad (59)$$

This is the formula originally given by Munk; A_0 may be determined later so as to satisfy the Kutta condition. For vanishing circulation, $A_0 = 0$.

Next, the problem with the Kutta condition at the trailing edge is solved by (42); the solution is

$$w(z) = -\frac{1}{\pi} \sqrt{\frac{a_2 - z}{z - a_1}} \int_{a_1}^{a_2} \sqrt{\frac{s - a_1}{a_2 - s}} \frac{v_0(s)}{s - z} ds. \quad (60)$$

Finally, it is also possible to set up the problem for two Kutta conditions, one at a_1 and the other at a_2 , although a solution exists only under special circumstances:

$$w(z) = -\frac{1}{\pi} \sqrt{(z - a_1)(a_2 - z)} \int_{a_1}^{a_2} \frac{1}{\sqrt{(s - a_1)(a_2 - s)}} \frac{v_0(s)}{s - z} ds. \quad (61)$$

By (51), the solution exists only if

$$\int_{a_1}^{a_2} \frac{v_0(s) ds}{\sqrt{(s - a_1)(a_2 - s)}} = 0. \quad (62)$$

If $v_0(x)$ does not satisfy (62), it is possible to find the angle of attack α for shock-free entry by replacing v_0 by $v_0(x) - \alpha U$. Introducing this expression in (62), we get

$$\alpha = \frac{1}{\pi U} \int_{a_1}^{a_2} \frac{v_0(s) ds}{\sqrt{(s - a_1)(a_2 - s)}}, \quad (63)$$

a formula due to Munk.

The set of formulae (59), (60), (61) represents a sequence in the sense of increasingly restricted arbitrariness of the solution; passing from (59) to (60), the arbitrary constant is eliminated; in the further step to (61), the restriction (62) is introduced. The next section will give a further discussion of these steps.

9. Thus far, the homogeneous function H_0 with the lowest possible power at infinity was employed for the solution. We may try now to use the function

$$H_1(z) = H_0(z)(A_0 + A_1 z)$$

for the general solution; it is convenient to normalize H_1 in the form

$$H_1(z) = H_0(z)(z - c), \quad (64)$$

which puts in evidence the fact that H_1 has a zero at the point c of the real axis.

If we consider the quotient

$$F_1(z) = \frac{w(z)}{H_1(z)}$$

instead of $F(z)$ given by (43), the argument of the previous paragraph remains unchanged, and it leads to the formula

$$w(z) = -\frac{1}{\pi} H_1(z) \int_{-\infty}^{+\infty} \frac{q_1(s)}{s - z} ds \quad (65)$$

instead of (48) and to

$$q_1(x) = \frac{q_0(x)}{x - c} \quad (66)$$

with q_0 given by (46). In general q_1 has a non-integrable singularity at $x = c$, in which case (65) becomes meaningless. This is not surprising, as the use of (65) must be restricted to cases where $w(z)$ may really *have* a zero at $z = c$, which implies that the prescribed boundary values are zero at $x = c$. If this is the case, $q_1(c)$ remains finite, since $q_0(c) = 0$, and (65) gives the result with the additional requirement of a stagnation point at $x = c$.

Furthermore, (65) may be used to cancel a singularity of H_0 by letting c take the value a_n for a singular edge-point (or by putting $c = b_m$ for other singular points if any). Note that in view of (46) q_0 has a zero wherever H_0 is singular. Thus q_1 remains integrable if c is a singular point of H_0 . By putting first $c = a_2$, then $c = a_1$, we see that (65) provides the formal connection between (59), (60) and (61).

The restriction in arbitrariness in these steps may be understood also. By using the identity

$$\frac{1}{(s - c)(s - z)} = \frac{1}{z - c} \left(\frac{1}{s - z} - \frac{1}{s - c} \right)$$

we may write (65) as

$$w(z) = -\frac{1}{\pi} H_0(z) \left[\int_{-\infty}^{+\infty} \frac{q_0(s)}{s - z} ds - \int_{-\infty}^{+\infty} \frac{q_0(s)}{s - c} ds \right]. \quad (67)$$

The first term represents the same solution as (48); the second term represents a homogeneous solution with a specified constant given by the second integral. It follows that (65) may be used only if there are still free constants to be specified in the solution (48).

Surprisingly, the decomposition (67) is always possible, whether $q_0(c)$ is zero or not. But it must be recalled that the integrals (65) and (67) are *not* meant to be principal values. Therefore the second integral in (67) is meaningless unless $q_0(c) = 0$.

The formal way of making a solution more singular is described by the following formulae:

$$\begin{aligned} w(z) &= -\frac{1}{\pi} H_0(z) \int_{-\infty}^{+\infty} \frac{z - s + s - b}{z - b} \frac{q_0(s)}{s - z} ds \\ &= -\frac{1}{\pi} \frac{H_0(z)}{z - b} \left[\int_{-\infty}^{+\infty} \frac{(s - b)q_0(s)}{s - z} ds - \int_{-\infty}^{+\infty} q_0(s) ds \right]. \end{aligned} \quad (68)$$

The process is permissible if

$$\int_{-\infty}^{+\infty} (s - b) q_0(s) ds = \text{finite.}$$

Generality in (68) is increased if the second integral is replaced by an arbitrary real constant.

10. Finally, a further generalization will be indicated briefly. Let us consider the more general mixed-type boundary value problem where in the intervals between the edge-points $a_1, a_2, \dots, a_N$ on the real axis the following quantity is prescribed:

$$g_n u(x, 0) + h_n v(x, 0) = f_n(x) \quad (a_n < x < a_{n+1}) \quad \text{for } n = 0, 1, 2, \dots, N.$$

Here g_n and h_n are real constants which have fixed values in each interval; $f_n(x)$ is a given real function in the interval. In order to extend this prescription to the whole real axis, put $a_0 = -\infty$ and $a_{N+1} = +\infty$.

It evidently involves no loss of generality if the constants g_n and h_n are normalized in each interval such that

$$g_n^2 + h_n^2 = 1.$$

Then it is possible to put

$$g_n = \sin \beta_n, \quad h_n = \cos \beta_n,$$

and the boundary condition becomes

$$-\Im\{e^{-i\beta_n} w(x)\} = f_n(x) \quad (a_n < x < a_{n+1}) \quad (69)$$

for $n = 0, 1, 2, \dots, N$.

So as to obtain the general solution of this boundary value problem, we first construct a homogeneous solution $H(z)$ satisfying the condition

$$\Im\{e^{-i\beta_n} H(x)\} = 0 \quad (70)$$

in each interval. Introduce the angle

$$\gamma_n = \beta_{n-1} - \beta_n \quad n = 1, 2, \dots, N. \quad (71)$$

A homogeneous solution is given by the function

$$H(z) = C \prod_{n=1}^N (z - a_n)^{\gamma_n/\pi}. \quad (72)$$

where C is a complex number to be determined. To prove this statement, write out $H(x)$ in an interval $a_n < x < a_{n+1}$:

$$H(x) = C \prod_{i=1}^n (x - a_i)^{\gamma_i/\pi} \prod_{j=n+1}^N (x - a_j)^{\gamma_j/\pi}, \quad (73)$$

where the factors $x - a_i$ are all positive in the first product, whereas the factors $x - a_j$ are all negative in the second product. Writing

$$x - a_j = (a_j - x)e^{i\pi},$$

we put (73) into the form

$$H(x) = C \prod_{i=1}^N (x - a_i)^{\gamma_i/\pi} \prod_{j=n+1}^N (a_j - x)^{\gamma_j/\pi} e^{i(\gamma_{n+1} + \gamma_{n+2} + \dots + \gamma_N)}.$$

But in view of (71)

$$\gamma_{n+1} + \gamma_{n+2} + \dots + \gamma_N = \beta_n - \beta_N.$$

Putting

$$C = Ae^{i\beta_N},$$

where A is real, we get

$$H(x) = A \prod_{i=1}^n (x - a_i)^{\gamma_i/\pi} \prod_{j=n+1}^N (a_j - x)^{\gamma_j/\pi} e^{i\beta_n}. \quad (74)$$

If we select the branch where both products are real, (74) evidently satisfies the homogeneous boundary condition (70). A somewhat more general homogeneous solution may be written down immediately:

$$H(z) = Ae^{i\beta_N} \prod_{n=1}^N (z - a_n)^{\gamma_n/\pi + K_n}, \quad (75)$$

where K_n is some (positive or negative) integer. (75) corresponds to the former solution (12), to which it reduces if all $\gamma_n = \frac{1}{2}\pi$. More general homogeneous solutions are given by the product of the expression (75) with any function which is real on the real axis.

If the behavior at infinity and the admitted singularities on the real axis are specified the existence of homogeneous solutions and the uniqueness of the inhomogeneous solution may be discussed in a fashion analogous to our detailed treatment of the special case.

Again, the homogeneous solution provides the clue for the treatment of the general inhomogeneous problem. Consider the quotient

$$F(z) = \frac{w(z)}{H(z)}, \quad (76)$$

where $w(z)$ and $H(z)$ satisfy the boundary conditions (69) and (70), respectively. On the real axis,

$$F(x) = \frac{w(x)}{H(x)} = \frac{w(x)}{e^{i\beta_n} R(x)},$$

where in each interval $H(x)$ is to be given by (74), and where $R(x)$ stands for a function which is real on the real axis. The imaginary part of $F(x)$ is known along the whole real axis, since by (69) we have

$$\Im\{F(x)\} = \frac{1}{R(x)} \Im\{e^{-i\beta_n} w(x)\} = -\frac{f_n(x)}{R(x)}. \quad (77)$$

Thus for the function $F(z)$ there results an ordinary boundary value problem, which is solved by (1).

An example of the general case treated in this section was given by Ursell [12], who solved the "indirect thickness" problem arising in the middle section of an infinite symmetrical swept wing with the same profile in every streamwise section. Ursell started from the integral equation of the profile shape for a given pressure distribution; by analytic continuation to a complex variable (as in the original work of Carleman), a mixed type boundary value problem for the half plane was set up and solved. It is interesting to note that in the examples discussed before, the treatment of the integral equation would have been rather a detour; all problems could be formulated directly as boundary value problems for a function of a complex variable. In Ursell's example, however, which is originally a three-dimensional potential problem, the establishment of the integral equation and the subsequent analytic continuation are essential steps.

A further generalization of the integral equation (6) is obtained if the path of integration is replaced by an arbitrary curve in the complex plane. This problem was treated by Elliott [13], using Carleman's method.

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