*Asymptotic Solution of Linear Hyperbolic Partial Differential Equations**

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1. Introduction. In view of the difficulties in obtaining exact solutions of partial differential equations with prescribed initial and boundary conditions, attention has been given to the possibilities of obtaining solutions in the form of asymptotic series. The present paper presents a method of obtaining an asymptotic solution for some types of problems involving linear second order hyperbolic partial differential equations in n independent variables. Specifically, we shall consider the equation

$$(1.1) \quad L(u) \equiv \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k + cu = f_{tt},$$

where u is a function of $x_1, x_2, \dots, x_n$, the coefficients in $L(u)$ are functions of $x_1, x_2, \dots, x_{n-1}$, the superscripts serve merely for identification, and $u_{ij} = \partial^2 u / \partial x_i \partial x_j$. For brevity we shall write x for $x_1, x_2, \dots, x_{n-1}$ and t for x_n so that $u(x_1, x_2, \dots, x_{n-1}, x_n) = u(x, t)$. The source function $f(x, t)$ is assumed to be of the form $g(x)h(t)$, and for the most important results $h(t)$ will be limited to $e^{-i\omega t}$ where ω is the circular frequency of the source.

It will be shown that, apart from a transient which can be neglected for large t , the solution of equation (1.1) takes the form $u(x)e^{-i\omega t}$. We shall then obtain for $u(x)$ an asymptotic series whose terms contain the zero and positive integral powers of $1/\omega$. A method of obtaining the coefficients of this asymptotic series

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will be given. It is apparent from the form of this asymptotic series that for $\omega = \infty$ only the first term remains and this term gives the "geometrical optics" solution of the problem. Hence the theory of this paper offers an approximate solution to some types of problems which improves on geometrical optics approximations. It will also be shown that this asymptotic solution is a generalization to partial differential equations of existing theory for ordinary differential equations. The relevant asymptotic solutions of the latter equations contain as their first term the well known W K B approximation. Hence our theory is intimately related to the W K B method.

The basic ideas of this paper have already been applied to Maxwell's equations.¹ The present paper not only makes explicit the differences between the case of one second order partial differential equation and a special system of first order equations but yields a theory which should be simpler to apply. Moreover, the relationship of both theories to the W K B method is apparent from the results for the scalar case.

The procedure of this paper may be described briefly as follows. Certain problems involving second order partial differential equations of the form (1.1) are formulated. In connection with each problem a special solution, denoted by $U(x, t)$ and called the pulse solution, is introduced. Essentially $U(x, t)$ is the solution of equation (1.1) where $h(t)$ is replaced by the Heaviside unit function $\eta(t)$. A Duhamel integral then relates the desired solution $u(x, t)$ to the pulse solution $U(x, t)$. From the Duhamel integral the form of the asymptotic series representing $u(x)$ is derived by the process of integration by parts. The coefficients of the asymptotic series in $1/\omega$ prove to be discontinuities in $U(x, t)$ and in the successive time derivatives of $U(x, t)$. To show how to obtain these coefficients we study discontinuities of solutions of equation (1.1). This work involves recasting equation (1.1) into an integral equation and then studying the latter equation. It is shown that the discontinuities of $U(x, t)$ and its successive time derivatives satisfy a recursive system of first order linear ordinary differential equations, or transport equations, which in principle at least are immediately solvable. Initial conditions to fix the solutions of these transport equations are derived. Thus the above theory shows that the asymptotic solution of linear hyperbolic partial differential equations may be obtained by solving a recursive system of first order ordinary differential equations.

Some of the purely formal aspects of the above theory have been developed by earlier writers. By assuming the existence of an asymptotic series of the form treated in this paper and by substitution in a partial differential equation one can obtain the form of the transport equations for the coefficients.² The present

¹ KLINE, M.: *Asymptotic Solution of Maxwell's Equations*, Research Report No. EM-24 submitted by New York University to the Geophysics Research Directorate of the Air Force Cambridge Research Center and Communications on Pure and Applied Mathematics, V. IV, 1951, pp. 225-262.

² See, for example, BIRKHOFF, GEORGE D.: *Some Remarks Concerning Schrödinger's Wave Equation*, Proc. Nat'l. Acad. of Sciences, V. 19, 1933, pp. 339-344, and in *Collected*

paper differs from the earlier formal theory in that aside from deriving the series, the relationship of the coefficients to the discontinuities of the pulse solution is exploited to give a more general form to the series and to derive much mathematical information which now provides a method for the asymptotic solution of initial and boundary value problems. Also, from a purely mathematical point of view the theory of this paper shows that the asymptotic solution of linear hyperbolic partial differential equations is determined by the singularities of what we have called the pulse solution.

2. Formulation of the Problem. We shall consider the non-homogeneous linear hyperbolic partial differential equation

$$(2.1) \quad \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k + cu = f_{tt},$$

wherein u is a function of $x_1, x_2, \dots, x_n$; the coefficients a^{ij} , b^k , and c are functions of $x_1, x_2, \dots, x_{n-1}$ and possess all second order continuous partial derivatives with respect to these variables, the superscripts serving only to distinguish these functions; and $u_{ij} = \partial^2 u / \partial x_i \partial x_j$. We shall write equation (2.1) as

$$(2.2) \quad L(u) = f_{tt}.$$

Also, we shall often write x for $x_1, x_2, \dots, x_{n-1}$ and t for x_n so that $u(x_1, x_2, \dots, x_n) = u(x, t)$. The function f is, of course, given and we shall assume that its first and second partial derivatives are sectionally continuous. Moreover, we shall assume that $f(x, t)$ has the form

$$(2.3) \quad f(x, t) = g(x)h(t),$$

where $g(x) \neq 0$ in some closed and bounded $(n - 1)$ -dimensional region V of the space variables $x_1, x_2, \dots, x_{n-1}$ and $h(t) = 0$ for $t < 0$. The function f_{tt} in equations (2.1) and (2.2) is to be understood as the second partial derivative with respect to t of $g(x)h(t)\eta(t)$ where $\eta(t)$ is the Heaviside unit function, that is, $\eta(t)$ is 1 for $t > 0$ and 0 for $t < 0$. Thus at $t = 0$, f_{tt} contains a singularity of the order of $\delta(t)$ and one of the order of $\delta'(t)$ where $\delta(t)$ is the Dirac delta function.³

The solution $u(x, t)$ of the differential equation (2.1) is required to satisfy the initial conditions

$$(2.4) \quad u(x, t) = u_t(x, t) = 0 \quad \text{for } t < 0 \text{ and for all } x.$$

We require further that any bounded domain of (x, t) space be decomposable into a finite number of closed subdomains within which $u(x, t)$ possesses con-

Mathematical Papers, Amer. Math. Soc., 1950, V. II, pp. 813-818. Also, BIRKHOFF, GEORGE D.: *Quantum Mechanics and Asymptotic Series*, Amer. Math. Soc. Bulletin, V. 39, 1933, pp. 681-700 and in *Collected Mathematical Papers*, V. II, pp. 837-856.

³ See footnote 6.

tinuous second partial derivatives with respect to all the independent variables. Finally, we require that $u(x, t)$ be a limit of continuous solutions.⁴

To the solution $u(x, t)$ described above we associate the pulse solution $U(x, t)$ which satisfies the differential equation

$$(2.5) \quad L(U) = g(x) \delta'(t)$$

and the initial conditions (2.4). It is further required of this pulse solution that every bounded domain of (x, t) space be decomposable into a finite number of closed subdomains within which $U, U_t, U_{tt}, \dots$, possess continuous second derivatives with respect to all variables. This condition implies that the discontinuities in U and the various derivatives are finite jumps.⁵ Finally, we require of $U(x, t)$ as of $u(x, t)$ that it be a limit of continuous solutions.

The problem of this paper is to obtain an asymptotic series solution of the equation (2.1) for the case when $f(x, t)$ has the form $g(x)e^{-i\omega t}$. It will be shown later that, apart from a transient term, $u(x, t)$ takes the form $u(x)e^{-i\omega t}$. The asymptotic series will represent $u(x)$ and will contain $1/\omega$ as its basic variable. We shall show also how the coefficients of this series can be calculated.

The theory of this paper applies to problems in partial differential equations other than the one indicated. In particular we can replace $h(t)$ by $\int h(t) dt$. In this case one must replace $\delta'(t)$ in equation (2.5) by $\delta(t)$. If $h(t)$ is replaced by the twice repeated indefinite integral of $h(t)$ then $\delta'(t)$ in equation (2.5) must be replaced by $\eta(t)$.⁶ The treatment of some boundary value problems will be clear from the material in Section 8.

3. The Duhamel Integral. The pulse solution is due to the source having a δ' singularity introduced at $t = 0$ in the region V . The solution will therefore be discontinuous at $t = 0$ and for x in V . Moreover, these discontinuities will propagate through space and reach any given point x off V at some later time t . We shall suppose that these discontinuities lie on some hypersurface

⁴ This somewhat vague condition could be made more precise by specifying initial conditions at $t = 0+$. However it will be replaced by an integral condition in Section 5.

⁵ If only the first n time derivatives of $U(x, t)$ satisfy this condition, the theory of this paper will still hold. However, the asymptotic series to be derived will then contain $n + 1$ terms plus a remainder in the form of an integral. The condition under consideration is quite important in that it determines the fact that the asymptotic series to be derived will contain only the zero and positive integral powers of $1/\omega$. Such series are not adequate to represent all solutions of second order linear hyperbolic partial differential equations. In physical terms we have excluded diffraction phenomena and caustics.

⁶ The theory of this paper could have concerned itself with the differential equation $L(u) = g(x)h''(t)$, that is, where the non-homogeneous term has only an η function jump at $t = 0$ instead of the δ function and δ' function as well as the η function jump required by equation (2.1). In the former case both u and u_t would be continuous at $t = 0$ whereas equation (2.1) will lead to the fact that u and u_t will be discontinuous at $t = 0$ for those x for which $g(x) \neq 0$. The formulation of the paper is more general but leads to the same asymptotic expansion.

$\phi(x, t) = 0$ with $\phi_i \neq 0$. Further, we shall require that ϕ be explicitly solvable in the form $\phi \equiv t - \psi(x)$ where ψ possesses first and second continuous partial derivatives with respect to all x_i .

Under the conditions on $u(x, t)$ and $U(x, t)$ stated in Section 2 and under the further conditions⁷ that where $g(x) = 0$ then

$$(3.1) \quad \sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0 \quad \text{on } \phi = 0$$

and

$$(3.2) \quad \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k \right) [U(x, t)] + 2 \sum_{i,j=1}^n a^{ij} [U_i(x, t)] \phi_j = 0 \quad \text{on } \phi = 0,$$

where $[U(x, t)] = U(x, t+) - U(x, t-)$, the following assertion holds:

$$(3.3) \quad u(x, t) = \int_0^t U(x, t - \tau) h(\tau) d\tau, \quad t \geq 0.$$

Since $U(x, 0)$ is discontinuous, the upper limit of integration in (3.3) is understood to be⁸ $t+$. The assertion (3.3) is known as the Duhamel integral.

We shall verify that equation (3.3) holds.⁹ Let us suppose that for some x , $U(x, t)$ is discontinuous at $t = t_1$ where $0 < t_1 < t$. We substitute the value of $u(x, t)$ given by equation (3.3) into the differential equation (2.1) and obtain

$$(3.4) \quad \begin{aligned} & \sum_{i,j=1}^n a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau \\ & + \sum_{k=1}^n b^k \frac{\partial}{\partial x_k} \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau \\ & + c \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau = f_{it}(x, t). \end{aligned}$$

⁷ The full significance of these conditions will be apparent in Section 6. It will be shown there that both of these conditions necessarily hold. Equation (3.1) is known as the characteristic condition.

⁸ The device of interpreting the upper limit of the Duhamel integral to be $t+$ is merely a convenience in this section. The essential point is that $U(x, 0+)$ and $U_i(x, 0+)$, $i = 1, 2, \dots, n$, are generally not zero. If the Duhamel integral is interpreted as having the bounds 0 and $t-$ then additional terms arise in steps (3.5) to (3.10) involving the quantities just listed and these can be taken care of by the initial conditions on U and U_i derived in Section 6. Under this interpretation the integral in step (3.11) is zero. By interpreting the bounds as 0 and $t+$ we take care of the terms under discussion by step (3.11). This latter procedure is briefer and more convenient in this section because the initial conditions are not derived at this point.

⁹ A purely formal verification which applies to a somewhat different problem and which takes no account of discontinuities in $U(x, t)$ is given by COURANT, R. & HILBERT, D.: *Methoden der Mathematischen Physik*, V. II, J. Springer, Berlin, 1937, p. 184.

Since $U(x, t)$ is discontinuous at $t = t_1$, $U(x, t - \tau)$ is discontinuous at $t - \tau = t_1$ or $\tau = t - t_1$. We shall therefore break up the τ -interval of integration in (3.4) into 0 to $t - t_1 -$ and $t - t_1 +$ to t , where by $t - t_1 -$ we mean $t - t_1 - \epsilon$ and by $t - t_1 +$ we mean $t - t_1 + \epsilon$.

We consider the terms of equation (3.4) separately. First we consider those terms of the first summation for which i and j run independently from 1 to $n - 1$. For these terms

$$a^{ij} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x, t - \tau) h(\tau) d\tau = a^{ij} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ \int_0^{t-t_1-} U(x, t - \tau) h(\tau) d\tau + \int_{t-t_1+}^t U(x, t - \tau) h(\tau) d\tau \right\}.$$

We now differentiate each integral separately first with respect to x_j and then with respect to x_i . In carrying out these differentiations one must remember that $t_1 = \psi(x_1, x_2, \dots, x_{n-1})$. The result of straightforward differentiations after combining terms is

$$\begin{aligned} a^{ij} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U(x, t - \tau) h(\tau) d\tau &= a^{ij} \frac{\partial}{\partial t} \left\{ \int_0^t U_{ij}(x, t - \tau) h(\tau) d\tau \right. \\ &\quad + [U(x, t_1)] h'(t - t_1) \psi_i \psi_j - [U(x, t_1)] h(t - t_1) \psi_{ij} \\ &\quad - [U_i(x, t_1)] h(t - t_1) \psi_j - [U_j(x, t_1)] h(t - t_1) \psi_i \\ &\quad \left. - [U_i(x, t_1)] h(t - t_1) \psi_i \psi_j \right\}, \quad i, j = 1, 2, \dots, n - 1. \end{aligned} \quad (3.5)$$

We consider next those terms of the first summation in (3.4) for which $i = 1, 2, \dots, n - 1$ and $j = n$. Again breaking up the integral into two terms and differentiating under the integral sign we obtain

$$\begin{aligned} a^{it} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x_i \partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau &= a^{it} \frac{\partial}{\partial t} \left\{ \int_0^t U_{it}(x, t - \tau) h(\tau) d\tau \right. \\ &\quad \left. - [U(x, t_1)] h'(t - t_1) \psi_1 + [U_i(x, t_1)] h(t - t_1) \right\}, \\ &\quad i = 1, 2, \dots, n - 1, \quad j = n. \end{aligned} \quad (3.6)$$

In carrying out the differentiations involved in step (3.6) one encounters terms such as $U(x, 0-)$ and $U_i(x, 0-)$. All such terms are zero because of the initial conditions.¹⁰

Those terms of the first summation in (3.4) for which $i = n$ and $j = 1, 2, \dots, n - 1$ yield the same result as (3.6).

¹⁰ See footnote 8.

We now consider the term of the first summation in (3.4) for which $i = n$ and $j = n$. Proceeding as above we obtain

$$(3.7) \quad \begin{aligned} a'' \frac{\partial}{\partial t} \frac{\partial^2}{\partial t^2} \int_0^t U(x, t - \tau) h(\tau) d\tau = \\ a'' \frac{\partial}{\partial t} \left\{ \int_0^t U_{tt}(x, t - \tau) h(\tau) d\tau + [U(x, t_1)] h'(t - t_1) + [U_t(x, t_1)] h(t - t_1) \right\}. \end{aligned}$$

The terms of the second summation in equation (3.4) yield

$$(3.8) \quad \begin{aligned} b^k \frac{\partial}{\partial t} \frac{\partial}{\partial x_k} \int_0^t U(x, t - \tau) h(\tau) d\tau = \\ b^k \frac{\partial}{\partial t} \left\{ \int_0^t U_k(x, t - \tau) h(\tau) d\tau - [U(x, t_1)] h(t - t_1) \psi_k \right\} \end{aligned}$$

for $k = 1, 2, \dots, n - 1$;

$$(3.9) \quad \begin{aligned} b^t \frac{\partial}{\partial t} \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau \\ = b^t \frac{\partial}{\partial t} \left\{ \int_0^t U_t(x, t - \tau) h(\tau) d\tau + [U(x, t_1)] h(t - t_1) \right\}. \end{aligned}$$

Finally we have from equation (3.4) that

$$(3.10) \quad c \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau = c \frac{\partial}{\partial t} \int_0^t U(x, t - \tau) h(\tau) d\tau.$$

To verify that the Duhamel integral (3.3) is correct we must show that the sum of the terms obtained in steps (3.5) to (3.10) yield the right side of equation (3.4). We consider first those terms which contain $h'(t - t_1)$ as a factor. Neglecting this factor and the differentiation with respect to t for a moment, we find, since $t_1 = \psi(x)$ is an explicit form of $\phi(x, t) = 0$ and therefore $\psi_i = -\phi_i/\phi_t$, that these terms add up to zero because of equation (3.1).

We consider next those terms on the right sides of steps (3.5) to (3.10) which contain $h(t - t_1)$ as a factor. Neglecting this factor and the partial differentiation with respect to t and converting expressions in ψ to expressions in ϕ , we find, by virtue of equation (3.2), that these terms also total zero.

To complete the verification of the Duhamel integral (3.3) we consider the remaining terms on the right side of steps (3.5) to (3.10). Since the coefficients of $L(U)$ are independent of t we may write these terms as

$$(3.11) \quad \frac{\partial}{\partial t} \int_0^t L(U(x, t - \tau)) h(\tau) d\tau.$$

By equation (2.5) this expression equals

$$\frac{\partial}{\partial t} \int_0^t g(x) \delta'(t - \tau) h(\tau) d\tau$$

or

$$g(x) h''(t).^{11}$$

In view of equation (2.3) it is clear that the Duhamel integral satisfies the differential equation (2.1) for $t > 0$. Since, however, equation (3.3) holds for $t > 0$, it holds for $t = 0+$. Hence the verification of equation (3.3) for $t \geq 0$ is complete.

4. Derivation of the Asymptotic Expansion. We now restrict the $h(t)$ which appears in the non-homogeneous term of our problem to the function $e^{-i\omega t}$, ω being in physical application the circular frequency. From the Duhamel integral we shall prove first that the solution $u(x, t)$ contains a "transient" term, that is a term which approaches 0 as t becomes infinite, and a "steady state" term $u(x)e^{-i\omega t}$. We shall then derive the asymptotic series for $u(x)$ by the process of successive integration by parts applied to the Duhamel integral.

We therefore let $h(\tau)$ be $e^{-i\omega\tau}$ in equation (3.3), make the change of variable from τ to s by letting $\tau = t - s$ and obtain¹²

$$\begin{aligned} u(x, t) &= \frac{\partial}{\partial t} \int_0^t U(x, s) e^{-i\omega(t-s)} ds, \\ (4.1) \qquad &= U(x, t) - i\omega e^{-i\omega t} \int_0^t U(x, s) e^{i\omega s} ds. \end{aligned}$$

Adding and subtracting equivalent terms gives

$$\begin{aligned} u(x, t) &= U(x, t) - U(x, \infty) + e^{-i\omega t} \\ (4.2) \qquad &\left\{ U(x, \infty) - i\omega \int_0^t (U(x, s) - U(x, \infty)) e^{i\omega s} ds \right\}. \end{aligned}$$

The field $U(x, t)$ results from the sudden action at $t = 0$ of some source whose time behavior thereafter is constant. On physical grounds one expects that this field will establish itself readily and approach the static field $U(x, \infty)$ as t increases. One therefore expects further that $U(x, t) - U(x, \infty)$ will approach 0 rapidly as t becomes large. Equations (4.2) therefore show that with increas-

¹¹ For the treatment of the δ -function and its derivatives see VAN DER POL, B. & BREMMER, H.: *Operational Calculus*, Cambridge Univ. Press, London, 1950, Chap. V. See p. 82 in particular.

¹² In this section we interpret the bounds of the Duhamel integral (3.3) to be 0 and $t-$, remembering, however, that $U(x, 0+)$, $U_i(x, 0+)$, $i = 1, 2, \dots, n$, and higher derivatives at x and $0+$ need not be zero. Compare footnote 8.

ing t , $u(x, t)$ approach the form $u(x)e^{-i\omega t}$ where

$$(4.3) \quad u(x) = U(x, \infty) - i\omega \int_0^\infty (U(x, s) - U(x, \infty))e^{i\omega s} ds.$$

To obtain the asymptotic form of $u(x)$ we apply integration by parts to equation (4.3). One obtains first of all, writing τ for s ,

$$u(x) = U(x, \infty) - \{U(x, \tau) - U(x, \infty)\}e^{i\omega\tau} \Big|_0^\infty + \int_0^\infty U_\tau(x, \tau)e^{i\omega\tau} d\tau.$$

In evaluating the integrated term one must take into account the discontinuities in $U(x, t)$ with respect to t . We have assumed that these discontinuities are finite. There may, however, be an infinite number of them. Let $0, \tau_1, \tau_2, \dots, \tau_\alpha \dots$ be values of t at which $U(x, t)$ is discontinuous for a given x . Then

$$(4.4) \quad u(x) = U(x, 0+) + \sum_\alpha [U(x, \tau_\alpha)]e^{i\omega\tau_\alpha} + \int_0^\infty U_\tau(x, \tau)e^{i\omega\tau} d\tau.$$

Another integration by parts yields¹³

$$(4.5) \quad \begin{aligned} u(x) = U(x, 0+) + \sum_\alpha [U(x, \tau_\alpha)]e^{i\omega\tau_\alpha} - \frac{1}{i\omega} \sum_\alpha [U_\tau(x, \tau_\alpha)]e^{i\omega\tau_\alpha} \\ + \frac{1}{i\omega} \int_0^\infty U_{\tau\tau}(x, \tau)e^{i\omega\tau} d\tau. \end{aligned}$$

It is apparent that the process of integration by parts can be continued as long as the order of the derivative of U with respect to t has only finite jumps.¹⁴ If all the higher time derivatives of U contains only finite jumps then we have that

$$(4.6) \quad \begin{aligned} u(x) \sim U(x, 0+) + \sum_\alpha [U(x, \tau_\alpha)]e^{i\omega\tau_\alpha} - \frac{1}{i\omega} \sum_\alpha [U_\tau(x, \tau_\alpha)]e^{i\omega\tau_\alpha} \\ + \frac{1}{(i\omega)^2} \sum_\alpha [U_{\tau\tau}(x, \tau_\alpha)]e^{i\omega\tau_\alpha} \dots \end{aligned}$$

This infinite series in $1/i\omega$ is asymptotic to $u(x)$, for at any stage of the integration by parts the term neglected can be seen by another integration by

¹³ The values of t at which $U_\tau(x, \tau)$ are discontinuous need not be the same as those for which $U(x, \tau)$ is discontinuous. However, we shall use α as a general index which runs over all the τ values at which $U(x, \tau)$ or any one of its successive time derivatives is discontinuous. In each summation, then, some discontinuities will be zero. The second summation and all later summations (see equation (4.6)) must include a discontinuity at $\tau = 0+$. These summations also include the value at $\tau = \infty$. However, these terms, one in each summation except the first, should be 0 because $U(x, t)$ approaches a static field as $t \rightarrow \infty$.

¹⁴ See footnote 5.

parts to be of order one higher in $1/i\omega$ than the terms retained. The remainder will therefore approach zero to the proper order if the integral in it converges. This convergence may be expected on physical grounds because the higher time derivatives of $U(x, t)$ should approach zero sufficiently rapidly with increasing t .

Information leading to the determination of the τ_α in the series (4.6) will be given later.¹⁵ Here it need only be noted that if (x, τ_α) is a point at which $U(x, t)$ or any one of its successive time derivatives is discontinuous, then (x, τ_α) lies on the discontinuity hypersurface $\phi = 0$ which satisfies the characteristic condition (3.1). We have assumed that $\phi = 0$ can be solved explicitly for t . However, for any given x there may be many τ_α and hence the explicit solutions will be denoted by $\tau_\alpha = \psi_\alpha(x)$.

The determination of $\phi = 0$ involves the solution of a first order partial differential equation with the initial condition furnished by the fact that the surface of the volume V containing the source gives the shape of $\phi = 0$ at $t = 0$. There is much literature on the solution of equation (3.1).

In view of the fact that the τ_α in equation (4.6) are given by $\psi_\alpha(x)$ and view of the fact which will be proven¹⁶ in Section 8 that

$$U(x, 0+) = \frac{g(x)}{\alpha^{nn}(x)},$$

we may write equation (4.6) as

$$(4.7) \quad u(x) \sim \frac{g(x)}{\alpha^{nn}(x)} + \sum_{\alpha} [U(x, \psi_{\alpha})] e^{i\omega\psi_{\alpha}} - \frac{1}{i\omega} \sum_{\alpha} [U_{\tau}(x, \psi_{\alpha})] e^{i\omega\psi_{\alpha}} \\ + \frac{1}{(i\omega)^2} \sum_{\alpha} [U_{\tau\tau}(x, \psi_{\alpha})] e^{i\omega\psi_{\alpha}} - \dots$$

To utilize the series (4.7) one must know the discontinuities in the pulse solution $U(x, t)$ and its successive time derivatives. We shall therefore study discontinuous solutions of equation (2.1) and shall obtain conditions which the discontinuities of the solution and its successive time derivatives must satisfy. These conditions will then be transformed into a recursive system of first order ordinary differential equations whose solutions furnish the successive coefficients of the series (4.7).

5. The Integral Equation For Discontinuous Solutions. A solution of the differential equation (2.1) which is discontinuous at points (x, t) will, of course, not satisfy the equation at those points since the equation has no meaning there. Moreover, if solutions exist in two regions entirely separated by some discontinuity surface $\phi = 0$, the solution is uniquely determined only in that

¹⁵ See Sections 7 and 8.

¹⁶ See equation (8.4).

region in which the initial conditions apply. Some further condition or conditions must relate the solutions in the two regions so that one can speak of a unique solution for all x and t . One may require as a sufficient condition that a discontinuous solution of equation (2.1) be a limit of continuous solutions in some sense which could be made precise; it would then result that a unique solution is determined everywhere. However, an alternative and more useful procedure, namely, to convert the problem posed by the differential equation and associated conditions into an integral equation, will be employed. The integral equation method will also obviate the difficulties involved in the use of the delta function and will lead to the desired information on the discontinuities of $U(x, t)$ and its successive time derivatives.

Insofar as the derivation of the discontinuity conditions is concerned one could express a discontinuous solution of equation (2.1) in the form $u(x, t)\eta(t)$ and substitute in the equation. Using the δ -function and its derivatives as the successive derivatives of $\eta(t)$ one could obtain the desired discontinuity conditions by a purely formal procedure.¹⁷ The integral equation procedure we shall follow is not more complicated formally and will take care of questions of rigor involved in the use of the delta function, as well as the matter of uniqueness discussed above.¹⁸

Let us suppose first that $u(x, t)$ and all its first and second partial derivatives exist and are continuous in some closed bounded region G of (x, t) space and that $u(x, t)$ is a solution of equation (2.1). Let Ω be any function of x and t such that Ω possesses all continuous second order partial derivatives in G and such that Ω and all its first order derivatives vanish on the boundary Γ of G . Let $dv = dx_1 dx_2 \cdots dx_n$. From equation (2.2)

$$(5.1) \quad \int_G \Omega L(u) dv - \int_G \Omega f_{tt} dv = 0.$$

Let (dx_j) indicate that partial integration with respect to x_j has been performed. Then, considering first the second order terms of $L(u)$ and applying integration by parts we have

$$\int_G \Omega \sum_{i,j} a^{ij} u_{ij} dv = \int_{\Gamma} \sum_{i,j} (\Omega a^{ij}) u_i (dx_j) - \int_G \sum_{i,j} (\Omega a^{ij})_j u_i dv.$$

Since Ω vanishes on Γ the surface integral vanishes. Another integration by parts applied to the volume integral on the right plus the fact that Ω and Ω_j

¹⁷ For the case of the wave equation, the discontinuity condition on $U(x, t)$ to be derived in Section 6 was obtained according to the procedure just outlined by BREMMER, H.: *The Jumps of Discontinuous Solutions of the Wave Equation*, Communications on Pure and Applied Mathematics, Vol. IV, 1951, pp. 419-426.

¹⁸ The integral equation procedure we shall use is in essence the theory of distributions due principally to LAURENT SCHWARTZ. See his *Théorie des distributions*, 2 vols., Hermann & Co., Paris, 1951-52

vanish on Γ yields

$$\int_G \Omega \sum_{i,j} a^{ij} u_{ij} dv = \int_G \sum_{i,j} (\Omega a^{ij})_{ij} u dv.$$

The process of integration by parts when applied to all the terms of $L(u)$ yields

$$(5.2) \quad \int_G \Omega L(u) dv = \int_G u M(\Omega) dv$$

where

$$M(\Omega) = \sum_{i,j=1}^n (a^{ij} \Omega)_{ij} - \sum_{k=1}^n (b^k \Omega)_k + c\Omega.$$

$M(\Omega)$ is called the adjoint differential expression of $L(u)$.

With regard to the second term in equation (5.1), after two integrations by parts we obtain

$$-\int_G \Omega f_{tt} dv = -\int_{\Gamma} \Omega f_t(dt) + \int_{\Gamma} \Omega_t f(dt) - \int_G \Omega_{tt} f dv.$$

Since Ω and Ω_t vanish on Γ

$$(5.3) \quad -\int_G \Omega f_{tt} dv = -\int_G \Omega_{tt} f dv.$$

It follows from equations (5.1), (5.2), and (5.3) that if $u(x, t)$ satisfies the differential equation $L(u) - f_{tt}(x, t) = 0$, then

$$(5.4) \quad \int_G u M(\Omega) dv - \int_G \Omega_{tt} f dv = 0,$$

where Ω is any function subject to the conditions stated above.¹⁹

The converse is also true; that is, if u possesses all first and second continuous partial derivatives at any point within G and if u satisfies the integral equation (5.4) for Ω arbitrary to the extent indicated then u satisfies $L(u) - f_{tt} = 0$ at any point within G . To prove this we have but to take a sphere around any point in G , choose Ω so that it does not vanish and is of one sign within this sphere but such that Ω and its first derivatives vanish on the boundary of the sphere. Since the steps (5.1) to (5.4) are reversible, it then follows from equation (5.1) and the arbitrariness of Ω that $L(u) - f_{tt} = 0$ within this sphere.

Whereas the differential equation has meaning within G only when u has the necessary first and second partial derivatives, the integral equation (5.4) has meaning even for functions u which are discontinuous in G . This integral equation will be the starting point for our study of the discontinuities of u and its successive time derivatives.

¹⁹ Cf. COURANT-HILBERT, *loc. cit.*, pp. 469-470.

6. The First Discontinuity Condition and First Transport Equation. We consider functions $u(x, t)$ which satisfy the integral equation

$$(6.1) \quad \int_G u M(\Omega) dv - \int_G \Omega_{tt} f dv = 0$$

in some n -dimensional domain G of (x, t) space. We require of $u(x, t)$ that any bounded domain be decomposable into a finite number of closed subdomains within each of which u possesses continuous partial derivatives of all orders with respect to all of the independent variables.²⁰ Let Ω be any function of x and t such that $\Omega, \Omega_t, \Omega_{tt}, \dots$ possess all first and second order continuous partial derivatives in the interior and boundary Γ of G and such that $\Omega, \Omega_t, \Omega_{tt}, \dots$ and all their first and second partial derivatives vanish on Γ .

Let u, u_t or any of the successive time derivatives be discontinuous in G . Let $\phi(x_1, x_2, \dots, x_n) = 0$ be the surface on which these discontinuities occur. We require that $\phi_t \neq 0$ so that ϕ may be expressed in the form $t = \psi(x)$. Also, ϕ must have continuous first and second order partial derivatives. We suppose that $\phi = 0$ divides the domain G into two subdomains G_1 and G_2 . Let Γ_1 be that portion of Γ which forms part of the boundary of G_1 and Γ_2 , that portion of Γ which forms part of the boundary of G_2 . We shall denote the common boundary of G_1 and G_2 by Φ .

Under the above hypotheses we shall proceed to derive conditions on the discontinuities of u and of the successive time derivatives. We shall apply the

²⁰ The theory of Sections 6 and 7 is phrased so as to apply to any $u(x, t)$. In this paper the theory will be applied principally to $U(x, t)$. Hence it is the latter function which must satisfy the conditions imposed. See footnote 5. Insofar as the theory of Sections 6 and 7 is needed for the $u(x, t)$ of the other sections, u need possess only continuous second partial derivatives in each of the closed subdomains.

integral equation (6.1) to G_1 and G_2 separately using the fact that the sum of the integrals over G_1 and G_2 equals the integral over G . We consider therefore

$$(6.2) \quad \int_{G_1} u M(\Omega) dv - \int_{G_1} \Omega_{it} f dv.$$

We shall now apply integration by parts to each term of the expression (6.2). We treat first a typical second order term in $M(\Omega)$ for which $i, j = 1, 2, \dots, n - 1$. By integration by parts

$$\int_{G_1} u (a^{ij} \Omega)_{ij} dv = \int_{\Gamma_1 + \Phi} u (a^{ij} \Omega)_i (dx_j) - \int_{G_1} (a^{ij} \Omega)_i u_j dv.$$

Because the Ω_i are zero on Γ_1 we may ignore that part of the surface integral.

By another integration by parts applied to the volume integral on the right side we obtain

$$\int_{G_1} u (a^{ij} \Omega)_{ij} dv = \int_{\Phi} u (a^{ij} \Omega)_i (dx_j) - \int_{\Phi} a^{ij} \Omega u_j (dx_i) + \int_{G_1} a^{ij} \Omega u_{ij} dv.$$

We note that the direction cosines of the normal to $\phi = 0$ are $\pm \lambda \phi_i$, where $\lambda = 1/\sqrt{\sum_i \phi_i^2}$. We shall suppose that the plus sign indicates that the direction of the normal points outside of G_1 into G_2 . Also, let $\alpha_i, i = 1, 2, \dots, n$, denote the direction angles of this positive direction. Let us use the notation dw for an element of surface on Φ . We may therefore rewrite the preceding results thus:

$$\int_{G_1} u (a^{ij} \Omega)_{ij} dv = \int_{\Phi} u (a^{ij} \Omega)_i \lambda \phi_j dw - \int_{\Phi} a^{ij} \Omega u_j \lambda \phi_i dw + \int_{G_1} a^{ij} \Omega u_{ij} dv.$$

But

$$dw = dx_1 \cdots dx_{n-1} \sec \alpha_n = dx_1 \cdots dx_{n-1} \frac{1}{\lambda \phi_n}.$$

Since $\psi_j = -\phi_j/\phi_n$ we may state that

$$\int_{G_1} u (a^{ij} \Omega)_{ij} dv = - \int_{\Phi} u (a^{ij} \Omega)_i \psi_j (dx_n) - \int_{\Phi} a^{ij} \Omega u_j \lambda \phi_i dw + \int_{G_1} a^{ij} \Omega u_{ij} dv.$$

In the first surface integral x_n is understood to be replaced by its value $\psi(x_1, x_2, \dots, x_{n-1})$. However the partial derivative with respect to x_i applies only to the x_i present before the substitution.

To obtain integrands in which only Ω and Ω_i appear—the purpose will soon be apparent—we make a further transformation of the first surface integral. We note that

$$\frac{\partial}{\partial x_i} (a^{ij} \Omega) = (a^{ij} \Omega)_i + (a^{ij} \Omega)_i \psi_i.$$

We substitute for $(a^{ij}\Omega)_i$ in the first surface integral and integrate by parts the resulting integrand which contains $\partial/\partial x_i (a^{ij}\Omega)$. The first term which results from the integration by parts must be evaluated over the boundary of Φ , which boundary lies on Γ . Since the integrand also contains Ω , which vanishes on Γ , there results finally

$$\begin{aligned} \int_{G_1} u(a^{ij}\Omega)_{ij} dv &= \int_{\Phi} a^{ij}\Omega \frac{\partial}{\partial x_i} (u\psi_j)(dx_n) + \int_{\Phi} (a^{ij}\Omega)_i u\psi_i \psi_j(dx_n) \\ &\quad - \int_{\Phi} a^{ij}\Omega u_j \lambda \phi_i dw + \int_{G_1} a^{ij}\Omega u_{ij} dv. \end{aligned}$$

Simple changes yield

$$\begin{aligned} \int_{G_1} u(a^{ij}\Omega)_{ij} dv &= \int_{\Phi} a^{ij}\Omega \{ (u_i + u_i \psi_i) \psi_j + u\psi_{ij} \} \lambda \phi_i dw \\ (6.3) \quad &+ \int_{\Phi} (a^{ij}\Omega)_i u\psi_i \psi_j \lambda \phi_i dw - \int_{\Phi} a^{ij}\Omega u_j \lambda \phi_i dw \\ &+ \int_{G_1} a^{ij}\Omega u_{ij} dv, \quad i, j = 1, 2, \dots, n-1. \end{aligned}$$

We now return to the expression (6.2) and treat those second order terms of $M(\Omega)$ for which $i = 1, 2, \dots, n-1$ and $j = n$. Integration by parts first with respect to x_i and then with respect to x_n (or the reverse order with slightly different steps) gives

$$\begin{aligned} \int_{G_1} (\Omega a^{in})_{in} u dv &= \int_{\Phi} (a^{in}\Omega)_i u \lambda \phi_i dw - \int_{\Phi} a^{in}\Omega u_i \lambda \phi_i dw \\ (6.4) \quad &+ \int_{G_1} a^{in}\Omega u_{in} dv, \quad i = 1, 2, \dots, n-1. \end{aligned}$$

Since $a^{ij} = a^{ji}$ and the order of differentiation is immaterial we shall use the result of equation (6.4) for those terms of the expression (6.2) for which $i = n$ and $j = 1, 2, \dots, n-1$.

The remaining second order term in $M(\Omega)$ of expression (6.2), namely, $i = n$ and $j = n$ may be transformed thus:

$$\begin{aligned} \int_{G_1} (a^{nn}\Omega)_{nn} u dv &= \int_{\Phi} (a^{nn}\Omega)_n u \lambda \phi_n dw \\ (6.5) \quad &- \int_{\Phi} a^{nn}\Omega u_n \lambda \phi_n dw + \int_{G_1} a^{nn}\Omega u_{nn} dv. \end{aligned}$$

The first order terms of $M(\Omega)$ in (6.2) are treated as follows:

$$(6.6) \quad - \int_{G_1} (\Omega b^k)_k u dv = - \int_{\Phi} \Omega b^k u \lambda \phi_k dw + \int_{G_1} \Omega b^k u_k dv, \quad k = 1, 2, \dots, n.$$

The cu term in $M(\Omega)$ is not transformed. That is,

$$(6.7) \quad \int_{G_1} cu\Omega \, dv = \int_{G_1} cu\Omega \, dv.$$

Finally, the last term in expression (6.2) is transformed thus:

$$(6.8) \quad -\int_{G_1} \Omega_{ii} f \, dv = -\int_{\Phi} \Omega_i f \lambda \phi_i \, dw + \int_{\Phi} \Omega f_i \lambda \phi_i \, dw - \int_{G_1} \Omega f_{ii} \, dv.$$

All of the results contained in equation (6.3) to (6.8) apply to G_2 as well except that λ must be replaced by $-\lambda$. We shall therefore add the results of equations (6.3) to (6.8) for G_1 and G_2 and use equation (6.1). Before summing the two sets of results let us apply an earlier observation, namely, that if u satisfies equation (6.1) it satisfies $L(u) - f_{ii} = 0$ at each point within G_1 and G_2 . Since the values of u and its first and second partial derivatives are assumed to be continuous within the closed subdomains G_1 and G_2 the sum of the volume terms on the right sides of equations (6.3) to (6.8) is zero. Likewise the corresponding sum for G_2 is zero. In summing the surface integrals in equation (6.3) to (6.8) as well as the corresponding integrals obtained for G_2 we shall use the notation $[u]$, $[u_i]$, etc. to denote the discontinuity in u , u_i , etc. on $\phi = 0$. Also, we shall use the fact that the coefficients of $L(u)$ are independent of x_n .

We note that the surface integrals in equations (6.3) to (6.8), and the corresponding integrals obtained from G_2 involve either Ω or Ω_i as a factor. We argue that Ω and Ω_i may be chosen to be zero on $\phi = 0$ *independently* while retaining the otherwise arbitrary features of the Ω function. Hence the coefficient of Ω and the coefficient of Ω_i in the sum of equations (6.3) to (6.8) for G_1 and the corresponding equations for G_2 must each be zero in order that the sum be zero. We consider first the terms in Ω_i . Ignoring the factor $\Omega_i \lambda \, dw$ and using the fact that $\psi_i = -\phi_i/\phi_i$, we have that

$$(6.9) \quad \frac{[u]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - [f] \phi_i = 0 \quad \text{on } \phi = 0.$$

At points in (x, t) space where f is continuous and where $[u] \neq 0$ we have

$$(6.10) \quad \sum_{i,j=1}^n a^{ij} \phi_i \phi_j = 0 \quad \text{on } \phi = 0.$$

In other words, if $\phi = 0$ is a discontinuity surface for $u(x, t)$ and if the source function is continuous on $\phi = 0$ then ϕ must satisfy the characteristic condition on $\phi = 0$.

We now consider those terms in equations (6.3) to (6.8) which contain Ω as a factor. We ignore the factor $\Omega \lambda \, dw$; use the fact that $\phi \equiv t - \psi$; and sum these terms for the domains G_1 and G_2 . The result is

$$\begin{aligned}
(6.11) \quad & - \sum_{i,j=1}^n [u] a^{ij} \phi_{ij} - 2 \sum_{i,j=1}^{n-1} a^{ij} [u_i] \phi_j + \sum_{i,j=1}^{n-1} \frac{[u_t]}{\phi_t} a^{ij} \phi_i \phi_j \\
& - 2 \sum_{i=1}^{n-1} a^{in} [u_i] \phi_t - a^{nn} [u_n] \phi_t - \sum_{k=1}^n [u] b^k \phi_k + [f_t] \phi_t = 0 \quad \text{on} \quad \phi = 0.
\end{aligned}$$

If we allow the summation in the third term to run from 1 to n , subtract the terms added thereby, and combine with existing terms, we obtain

$$\begin{aligned}
(6.12) \quad & - \sum_{i,j=1}^n [u] a^{ij} \phi_{ij} - 2 \sum_{i,j=1}^n a^{ij} [u_i] \phi_j + \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j \\
& - \sum_{k=1}^n [u] b^k \phi_k + [f_t] \phi_t = 0 \quad \text{on} \quad \phi = 0.
\end{aligned}$$

This condition is called the *first discontinuity condition*.

The discontinuity condition (6.12) can be converted into a more significant and more useful form as follows. We introduce the concept of differentiation in the direction transverse to $\phi = 0$. The direction is given by²¹

$$(6.13) \quad \dot{x}_i = \frac{dx_i}{ds} = \sum_{j=1}^n a^{ij} \phi_j, \quad i = 1, 2, \dots, n,$$

where $x_i(s)$, $i = 1, 2, \dots, n$, is a family of curves determined by the directions (6.13) and s is any parameter along the curves. The curves $x_i(s)$, $i = 1, 2, \dots, n$ are called the characteristic rays; they are the rays of geometrical optics. The directional derivative of a function $u(x_1, x_2, \dots, x_n)$ in the transverse direction is given by

$$\frac{du}{ds} = \sum_{i=1}^n u_i \dot{x}_i = \sum_{i,j=1}^n u_i \phi_j.$$

We may therefore write the discontinuity condition (6.12) as

$$\begin{aligned}
(6.14) \quad & -[u] \left(\sum_{i,j=1}^n a^{ij} \phi_{ij} + \sum_{k=1}^n b^k \phi_k \right) \\
& - 2 \frac{d[u]}{ds} + \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + [f_t] \phi_t = 0 \quad \text{on} \quad \phi = 0.
\end{aligned}$$

We introduce the notation

$$\bar{L}(u) = \sum_{i,j=1}^n a^{ij} u_{ij} + \sum_{k=1}^n b^k u_k.$$

²¹ See COURANT-HILBERT, *loc. cit.*, p. 353. Since the normal to $\phi = 0$ has the direction numbers ϕ_i , $i = 1, 2, \dots, n$, the transverse direction is perpendicular to the normal to $\phi = 0$ and therefore lies in the surface $\phi = 0$.

Then equation (6.14) can be written as

$$(6.15) \quad -\bar{L}(\phi)[u] - 2 \frac{d[u]}{ds} + \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + [f]_t \phi_t = 0 \quad \text{on } \phi = 0.$$

Equation (6.15) is therefore an alternative form of the first discontinuity condition (6.12).

In case $[u] \neq 0$ we may use equation (6.9) to rewrite the first discontinuity condition (6.12) as

$$(6.16) \quad -\sum_{i,j=1}^n [u] a^{ij} \phi_{ij} - 2 \sum_{i,j=1}^n a^{ij} [u_i] \phi_j + [f]_t \phi_t \frac{[u_t]}{[u]} - \sum_{k=1}^n [u] b^k \phi_k + [f]_t \phi_t = 0 \quad \text{on } \phi = 0.$$

If $[u] \neq 0$ and if we consider points on $\phi = 0$ where f and its first time derivative are continuous then equation (6.16) becomes

$$(6.17) \quad \bar{L}(\phi)[u] + 2 \frac{d[u]}{ds} = 0 \quad \text{on } \phi = 0.$$

This linear first order ordinary differential equation for $[u]$ will be called the *first transport equation*. The use of this equation to obtain the first coefficient of the asymptotic expansion (4.7) will be discussed later.

7. The Higher Discontinuity Conditions and Transport Equations. Since the coefficients of the asymptotic expansion involve $[u_i]$, $[u_{it}]$, etc., we shall now seek the discontinuity condition and transport equation which each of these functions satisfies. Our starting point again will be the integral equation (6.1). Since Ω_t satisfies the same conditions as does Ω , we may replace Ω by Ω_t in equation (6.1) and require that $u(x, t)$ satisfy the integral equation

$$(7.1) \quad \int_G u M(\Omega_t) dv - \int_G f \Omega_{tt} dv = 0$$

for all Ω satisfying the conditions specified in Section 6. It follows²² from equation (7.1) that within any closed domain $u(x, t)$ satisfies the differential equation

$$(7.2) \quad L(u_t) - f_{tt} = 0.$$

We now treat the integral equation (7.1) as we treated (6.1), that is, we shall apply it to the domains G_1 and G_2 , apply integration by parts, and then use the fact that the integral over $G_1 + G_2$ vanishes because of equation (7.1). We consider the integral over G_1 . Since $M(\Omega_t) = M(\Omega)_t$ we have that

²² See Section 5 for the corresponding argument concerning $u(x, t)$ and see footnote 20.

$$(7.3) \quad \int_{\sigma_1} u M(\Omega_t) dv - \int_{\sigma_1} f \Omega_{tt} dv = \int_{\Phi} u M(\Omega)(dx_n) - \int_{\sigma_1} u_t M(\Omega) dv \\ - \int_{\Phi} f \Omega_{tt}(dx_n) + \int_{\sigma_1} f_t \Omega_{tt} dv.$$

We shall first transform the surface integrals. We consider individual terms of the first surface integral. Because of the integration by parts performed in step (7.3), t is understood to be replaced by $\psi(x)$ in u and $M(\Omega)$. Hence by virtue of

$$(a^{ij}\Omega)_{ij} = \frac{\partial}{\partial x_j} (a^{ij}\Omega)_i - (a^{ij}\Omega)_{it} \psi_j,$$

integration by parts, and the fact that Ω and Ω_i vanish on the boundary of Φ , we have that

$$\int_{\Phi} u(a^{ij}\Omega)_{ij}(dx_n) = - \int_{\Phi} (a^{ij}\Omega)_i \frac{\partial u}{\partial x_j}(dx_n) - \int_{\Phi} u(a^{ij}\Omega)_{it} \psi_j(dx_n), \\ i, j = 1, 2, \dots, n-1$$

In the first surface integral on the right side we use the identity

$$(a^{ij}\Omega)_i = \frac{\partial}{\partial x_i} (a^{ij}\Omega) - (a^{ij}\Omega)_t \psi_i.$$

In the second surface integral on the right side we use the identity

$$(a^{ij}\Omega)_{it} = \frac{\partial}{\partial x_i} (a^{ij}\Omega)_t - (a^{ij}\Omega)_{tt} \psi_i.$$

We substitute, integrate by parts with respect to x_i , use the fact that Ω and Ω_t vanish on the boundary of Φ , and obtain

$$(7.4) \quad \int_{\Phi} u(a^{ij}\Omega)_{ij}(dx_n) = \int_{\Phi} (a^{ij}\Omega) \frac{\partial^2 u}{\partial x_i \partial x_j}(dx_n) + \int_{\Phi} (a^{ij}\Omega)_t \psi_i \frac{\partial u}{\partial x_j}(dx_n) \\ + \int_{\Phi} (a^{ij}\Omega)_t \frac{\partial(u\psi_j)}{\partial x_j}(dx_n) + \int_{\Phi} u(a^{ij}\Omega)_{tt} \psi_i \psi_j(dx_n), \\ i, j = 1, 2, \dots, n-1.$$

To transform those terms in the first surface integral of equation (7.3) for which $i = 1, 2, \dots, n-1, j = n$, we use the identity

$$(a^{it}\Omega)_{it} = \frac{\partial}{\partial x_i} (a^{it}\Omega)_t - (a^{it}\Omega)_{tt} \psi_i,$$

integrate by parts with respect to x_i , use the fact that Ω_t is 0 on the boundary of Φ , and obtain that

$$(7.5) \quad \int_{\Phi} u(a^{it}\Omega)_{it}(dx_n) = -\int_{\Phi} (a^{it}\Omega)_i \frac{\partial u}{\partial x_i}(dx_n) - \int_{\Phi} (a^{it}\Omega)_{it} u\psi_i(dx_n),$$

$$i = 1, 2, \dots, n-1.$$

The same result is obtained for $\int_{\Phi} u(a^{tj}\Omega)_{ij}(dx_n)$, $j = 1, 2, \dots, n-1$.
For reference we list that

$$(7.6) \quad \int_{\Phi} u(a^{it}\Omega)_{it}(dx_n) = \int_{\Phi} u(a^{it}\Omega)_{it} dx_n.$$

By the processes used in steps (7.4) and (7.5) we obtain that

$$(7.7) \quad -\int_{\Phi} (b^k\Omega)_k u(dx_n) = \int_{\Phi} b^k\Omega \frac{\partial u}{\partial x_k}(dx_n)$$

$$+ \int_{\Phi} (b^k\Omega)_i u\psi_k(dx_n), \quad k = 1, 2, \dots, n-1;$$

and for reference we note that

$$(7.8) \quad -\int_{\Phi} (b^i\Omega)_i u(dx_n) = -\int_{\Phi} (b^i\Omega)_i u(dx_n)$$

and

$$(7.9) \quad \int_{\Phi} \Omega cu(dx_n) = \int_{\Phi} \Omega cu(dx_n).$$

Steps (7.3) to (7.9) hold for G_2 as well as for G_1 except that $(dx_n) = \lambda\phi_i dw$ for G_1 and $-\lambda\phi_i dw$ for G_2 .

We have therefore transformed the surface integrals in equation (7.3). The result of transforming the volume integrals on the right side of equation (7.3) may be obtained from Section 6. We note that these volume integrals are the *negative of expression (6.2) except that u_i replaces u and f_i replaces f* . Hence we shall accept the results of steps (6.3) to (6.8) with the appropriate changes just noted. The volume integrals which would result from the steps analogous to steps (6.3) to (6.8) would be zero by virtue of equation (7.2). Hence the contribution to the surface integrals over Φ of the volume integrals on the right side of equation (7.3) will be terms in Ω_i which may be read off from the left side of equation (6.9) and terms in Ω which may be read off from the left side of equation (6.15) with the changes noted in the italicized statement above.

We therefore sum the results obtained by transforming the left side of equation (7.3) and the corresponding equation for G_2 . It is convenient for what follows to group the terms according as they contain Ω , Ω_i , or Ω_{it} . We shall ignore the surface integral symbol and the factor λdw which occurs in every term when (dx_n) is replaced by $\pm\lambda\phi_i dw$. Then

$$(7.10) \quad \Omega_{tt} \rightarrow \frac{[u]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - [f] \phi_t.$$

The terms coming from equation (7.3) and involving Ω_t are

$$(7.11) \quad \begin{aligned} \Omega_t \rightarrow & \sum_{i,j=1}^{n-1} a^{ij} \psi_i \left[\frac{\partial u}{\partial x_j} \right] \phi_t + \sum_{i,j=1}^{n-1} a^{ij} \left[\frac{\partial(u\psi_j)}{\partial x_i} \right] \phi_t - 2 \sum_{i=1}^{n-1} a^{it} \left[\frac{\partial u}{\partial x_i} \right] \phi_t \\ & + \sum_{k=1}^{n-1} b^k [u] \psi_k \phi_t - b^t [u] \phi_t - \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + [f_t] \phi_t. \end{aligned}$$

We note for later reference that the last two terms come from the volume integrals on the right side of equation (7.3). Expansion of the various derivatives in the expression for Ω_t , remembering that $u = u(x_1, x_2, \dots, x_{n-1}, \psi)$, plus algebraic manipulations similar to those performed in Section 6 in connection with equation (6.12), yields

$$(7.12) \quad \begin{aligned} \Omega_t \rightarrow & -\bar{L}(\phi)[u] - 2 \frac{d[u]}{ds} + 2 \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j \\ & - \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + [f_t] \phi_t, \end{aligned}$$

the last two terms coming from the volume integrals on the right side of equation (7.3). Hence

$$(7.13) \quad \Omega_t \rightarrow -\bar{L}(\phi)[u] - 2 \frac{d[u]}{ds} + \frac{[u_t]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j + [f_t] \phi_t.$$

The terms coming from equation (7.3) and involving Ω are

$$(7.14) \quad \begin{aligned} \Omega \rightarrow & \sum_{i,j=1}^{n-1} a^{ij} \left[\frac{\partial^2 u}{\partial x_i \partial x_j} \right] + \sum_{k=1}^{n-1} b^k \left[\frac{\partial u}{\partial x_k} \right] + c[u] + \bar{L}(\phi)[u_t] \\ & + 2 \frac{d[u_t]}{ds} - \frac{[u_{tt}]}{\phi_t} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - [f_{tt}] \phi_t, \end{aligned}$$

the last four terms coming from the volume integrals on the right side of equation (7.3). It should be remembered that in expression (7.14) $u = u(x_1, x_2, \dots, x_{n-1}, \psi)$.

We may now determine how the coefficients of Ω , Ω_t , and Ω_{tt} are related to each other. Let us consider at this point surfaces $\phi = 0$ on which f and its successive time derivatives are continuous. If $[u] \neq 0$, then we know from equation (6.10) that ϕ satisfies the characteristic condition. Hence the coefficient of Ω_{tt} , as given by equation (7.10) is zero. Since $[u]$ satisfies the first transport equation (6.17), it follows from step (7.13) that the coefficient of Ω_t is zero. Further, since the surface integral obtained from equation (7.3) would now contain only terms in Ω and since Ω is arbitrary, the coefficient of Ω , as given by

step (7.14) must be zero. Hence we have that if $\phi = 0$ is a surface on which f and its successive time derivatives are continuous, then $[u_t]$ satisfies the first order ordinary non-homogeneous linear differential equation

$$(7.15) \quad \bar{L}(\phi)[u_t] + 2 \frac{d[u_t]}{ds} = - \sum_{i,j=1}^{n-1} a^{ij} \left[\frac{\partial^2 u}{\partial x_i \partial x_j} \right] - \sum_{k=1}^{n-1} b^k \left[\frac{\partial u}{\partial x_k} \right] - c[u],$$

wherein $u = u(x_1, x_2, \dots, x_{n-1}, \psi)$. Because the values of the independent variables in u lie on $\phi \equiv t - \psi = 0$, the derivatives on the right hand side of equation (7.15) imply differentiation along the surface $\phi = 0$. Since the discontinuities involved on the right side of equation (7.15) are functions defined only on $\phi = 0$ we can rewrite this equation as

$$(7.16) \quad \bar{L}(\phi)[u_t] + 2 \frac{d[u_t]}{ds} = - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [u]}{\partial x_k} - c[u],$$

wherein it is assumed that f and its successive time derivatives are continuous on $\phi = 0$. Equation (7.16) will be called the *second transport equation*.²³ The function $[u]$ on the right side is, of course, the solution of the first transport equation.

We now consider the case where $[u] = 0$ but $[u_t] \neq 0$ on $\phi = 0$, assuming as before that f and its successive time derivatives are continuous on $\phi = 0$. Since $[u] = 0$, we see that the coefficient of Ω_t , as given by step (7.12), is 0. It follows from the fact that Ω or Ω_t may be chosen to be zero independently while retaining the otherwise arbitrary nature of Ω that the coefficients of Ω_t and Ω , as given by steps (7.13) and (7.16), respectively, are zero. Since $[u] = 0$ it follows from step (7.13) that $\phi = 0$ must satisfy the characteristic condition. In other words, if $\phi = 0$ is a surface on which $u(x, t)$ is continuous but $u_t(x, t)$ is finitely discontinuous then ϕ must satisfy the characteristic condition. Since the coefficient of Ω is also zero, it follows that in this case too $[u_t]$ must satisfy the second transport equation. However, this equation is now homogeneous.

We can now see how to derive the transport equation for $[u_t]$ and the higher time derivatives. We start with the integral equation (compare equations (6.1) and (7.1))

$$(7.17) \quad \int_G u M(\Omega_t) dv - \int_G f \Omega_{ttt} dv = 0.$$

Again we regard the integral over the domain G as the sum of the integrals over the subdomains G_1 and G_2 . We then apply integration by parts to each of these integrals. We shall ignore the source term in what follows; that is, we shall treat discontinuity surfaces $\phi = 0$ on which f and its successive time de-

²³ Cf. COURANT-HILBERT, *loc. cit.*, V, II, pp. 362-363. See especially equation (12) on p. 363. The quantity J there is the right hand side of equation (7.16) above. Our derivation shows the relationship of equation (7.16) to the characteristic condition and the first transport equation.

derivatives are continuous. As in equation (7.3) we argue that

$$\int_{G_1} u M(\Omega_{ii}) dv = \int_{\Phi} u M(\Omega_i)(dx_n) - \int_{G_1} u_i M(\Omega_i) dv.$$

Another integration by parts gives

$$(7.18) \quad \int_{G_1} u M(\Omega_{ii}) dv = \int_{\Phi} u M(\Omega_i)(dx_n) - \int_{\Phi} u_i M(\Omega)(dx_n) + \int_{G_1} u_{ii} M(\Omega) dv.$$

The last two terms on the right side are *the negative of two terms on the right side of equation (7.3) except that u_i replaces u* . The first surface integral on the right side of equation (7.18) is the same as the first surface integral on the right side of equation (7.3) *except that Ω_i replaces Ω* . Hence we may use steps (7.10), (7.12), and (7.14) to write down the contributions of the terms on the right side of equation (7.18) to the final surface integral over Φ . We must be careful in using steps (7.10), (7.12), and (7.14) to note which terms in them come from the surface integral in equation (7.3) and which come from the volume integral. Again we group the terms in the final surface integral according to the derivative of Ω which they contain as a factor. We obtain the following groups:

$$(7.19) \quad \Omega_{iii} \rightarrow \frac{[u]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j;$$

$$(7.20) \quad \Omega_{ii} \rightarrow -\bar{L}(\phi)[u] - 2 \frac{d[u]}{ds} + 2 \frac{[u_i]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j - \frac{[u_i]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j;$$

$$(7.21) \quad \begin{aligned} \Omega_i \rightarrow & \sum_{i,j=1}^{n-1} a^{ij} \left[\frac{\partial^2 u}{\partial x_i \partial x_j} \right] + \sum_{k=1}^{n-1} b^k \left[\frac{\partial u}{\partial x_k} \right] + c[u] + \bar{L}(\phi)[u_i] \\ & + 2 \frac{d[u_i]}{ds} - \frac{[u_{ii}]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j; \end{aligned}$$

$$(7.22) \quad \begin{aligned} \Omega \rightarrow & - \sum_{i,j=1}^{n-1} a^{ij} \left[\frac{\partial^2 u_i}{\partial x_i \partial x_j} \right] - \sum_{k=1}^{n-1} b^k \left[\frac{\partial u_i}{\partial x_k} \right] - c[u_i] - \bar{L}(\phi)[u_{ii}] \\ & - 2 \frac{d[u_{ii}]}{ds} + \frac{[u_{iii}]}{\phi_i} \sum_{i,j=1}^n a^{ij} \phi_i \phi_j. \end{aligned}$$

We may now argue that if $[u] \neq 0$ on $\phi = 0$ then from previous considerations we know that the coefficients of Ω_{iii} , Ω_{ii} , and Ω_i in the final surface integral over ϕ must vanish. Hence, in view of the arbitrariness of Ω , the coefficient of Ω must vanish and we obtain the third transport equation:

$$(7.23) \quad \bar{L}(\phi)[u_{ii}] + 2 \frac{d[u_{ii}]}{ds} = - \sum_{i,j=1}^{n-1} a^{ij} \frac{\partial^2 [u_i]}{\partial x_i \partial x_j} - \sum_{k=1}^{n-1} b^k \frac{\partial [u_i]}{\partial x_k} - c[u_i],$$

which holds on $\phi = 0$ where f and its successive time derivatives are continuous. On the right side u_t must be understood to mean $u_t(x_1, x_2, \dots, x_{n-1}, \psi)$.

If $[u] = 0$, then the coefficient of Ω_{tt} and part of the coefficient of Ω_{tt} vanishes. However we noted earlier that if $[u_t] \neq 0$, then the characteristic condition must hold and $[u_t]$ satisfies the second transport equation. Hence the entire coefficients of Ω_{tt} and Ω_t vanish and again we get the third transport equation. If $[u]$ and $[u_t]$ both vanish but $[u_{tt}] \neq 0$, then arguing from the fact that Ω and Ω_t can be chosen zero independently of each other while Ω is otherwise arbitrary on $\phi = 0$, we conclude that the characteristic condition must hold and so must the third transport equation.

It is apparent that we can obtain the higher transport equations from our basic integral equation (6.1).

We must now relate the general theory of discontinuity conditions to the asymptotic solution (4.7) of our hyperbolic partial differential equation. The theory of Sections 5, 6, and 7 applies to any solution u of the partial differential equation. Hence in particular it applies to the pulse solution whose discontinuities furnish the coefficients of the asymptotic expansion. Thus the transport equations provide ordinary first order differential equations for the successive coefficients. It will be noted, however, that each coefficient of the asymptotic expansion (4.7) is a sum of discontinuities. Each of these discontinuities belongs to one branch $t = \psi_\alpha(x)$ of the discontinuity surface $\phi = 0$. Each transport equation, say the equation for $[U]$, furnishes the value of $[U]$ along a curve which lies entirely in that one branch. Hence one must solve the family of transport equations for each branch $t = \psi_\alpha(x)$ of the total discontinuity surface $\phi = 0$. Suitable initial conditions, to be discussed in the next section, determine $[U]$, $[U_t]$, etc. on each branch.

Physically speaking, any one branch $t = \psi_\alpha(x)$ of the surface $\phi = 0$ may be regarded as a family of wave fronts in x -space, each value of t determining one surface of the family. One such branch may correspond to the family of wave fronts emanating from the source. Another branch may correspond to the family of wave fronts generated when the first family strikes an obstacle.

8. Solution of The Transport Equations. To obtain the coefficients of the asymptotic expansion at least along one branch $t = \psi_\alpha(x)$ of the characteristic surface $\phi = 0$ we must solve the system of first, second, and higher transport equations. Each equation is a first order ordinary linear differential equation, either homogeneous or nonhomogeneous; hence each is immediately reducible to quadratures. However, it will be noted that the non-homogeneous term of the second transport equation (7.16) involves the solution $[u]$ of the first transport equation. Similarly each of the higher transport equations involves the solution of the next lower one. Hence the system is a recursive one.

There is a further point to be considered in the solution of this recursive system. The expression $\bar{L}(\phi)$ in the first transport equation contains functions of $x_1, x_2, \dots, x_n$. Since the equation itself is expressed in terms of the parameter s , it is necessary to replace each x_i by its value in terms of s . These values are

obtained by solving the system of equations for the rays, namely (see equation (6.13)),

$$(8.1) \quad \frac{dx_i}{ds} = \sum_{j=1}^n a^{ij} \phi_j, \quad i = 1, 2, \dots, n.$$

This system of n first order differential equations can be solved at least theoretically and would yield

$$(8.2) \quad x_i = x_i(s, \alpha_1, \alpha_2, \dots, \alpha_{n-1})$$

where the α_i are constants of integration. (Only $n - 1$ parameters are needed because the $x_i(s)$ satisfy $\phi = 0$.)

If the equations (8.2) are substituted for the x_i in the first transport equation we obtain a differential equation for $[u]$ in terms of the independent variable s and the $n - 1$ parameters α_i . Solution of this transport equation then gives $[u]$ as a function of s and the $n - 1$ parameters α_i . However, to calculate the right hand side of the second transport equation we need $[u]$ as a function of $x_1, x_2, \dots, x_{n-1}$ or, alternatively, as a function of $x_1, x_2, \dots, x_n$ with $x_n = \psi_\alpha(x_1, \dots, x_{n-1})$. It would then be necessary to replace each x_i in the resulting expression by its value in equation (8.2) in order to solve the second transport equation.

It is at least theoretically possible to invert the system of equations (8.2). It is then possible to replace s and the $n - 1$ parameters α_i in $[u]$ as functions of $x_1, x_2, \dots, x_n$. Hence it is at least theoretically possible to determine the right side of equation (7.16) and to solve the second transport equation.

One more important consideration enters into the solution of the transport equations and this is the matter of initial conditions. There are several situations in which initial conditions are known. In the first of these, initial conditions for the transport equations are obtained from the initial conditions for the original differential equation.

Let us suppose that in the equations (8.2) for the rays, it is required that $t = 0$ when $s = 0$. This requirement enables us to express all derivatives with respect to s as derivatives with respect to t . Further, let us consider the asymptotic expansion along the rays issuing from the source. Then we need $[U(x, 0)]$, $[U_i(x, 0)]$, etc. to set the initial conditions for the transport equations which hold along this family of rays. Let the discontinuity surface $\phi = 0$ be the surface $t = 0$ and let the normal to this surface point into the region $t < 0$. Then equation (6.9) tells us that

$$(8.3) \quad [U(x, 0)]a^{nn} = [f].$$

For the pulse solution $U(x, t)$, $f = g(x)\eta(t)$. Hence

$$(8.4) \quad [U(x, 0)] = \frac{g(x)}{a^{nn}}.$$

Of course $U(x, t) = 0$ for $t < 0$; hence equation (8.4) gives $U(x, 0)$.

To obtain $[U_i(x, 0)]$ we use the first discontinuity condition (6.12). For the surface $\phi \equiv t$, we obtain that

$$(8.5) \quad -2 \sum_{i=1}^n a^{in} [u_i(x, 0)] + a^{nn} [u_i(x, 0)] - b^n [u(x, 0)] + [f_i(x, 0)] = 0.$$

We may apply equation (8.5) to the pulse solution $U(x, t)$. For this solution $[f_i(x, 0)] = 0$ because this quantity must be interpreted as $f_i(x, 0+) - f_i(x, 0-) = 0 - 0$. The functions $U_i(x, 0-)$ are zero in view of the initial conditions. The functions $U_i(x, 0+)$, $i = 1, 2, \dots, n-1$, can be calculated from $U(x, 0+)$. Hence equation (8.5) applied to $U(x, t)$ gives us the value of $[U_i(x, 0)]$ or $U_i(x, 0+)$, namely,

$$(8.6) \quad [U_i(x, 0)] = -2 \sum_{i=1}^{n-1} \frac{a^{in}}{a^{nn}} [U_i(x, 0)] - \frac{b^n}{(a^{nn})^2} g(x).$$

Equation (8.6) furnishes the initial value for the second transport equation.

To obtain an initial condition for $[U_{ii}(x, t)]$, expression (7.14), which is shown in the text to be zero, can be used. However, this condition can also be obtained from the differential equation $L(u) - f_{ii} = 0$ since the differential equation is required to hold for $t = 0+$ and all terms in it vanish for $t = 0-$. One obtains that

$$(8.7) \quad [U_{ii}(x, 0)] = - \sum_{i,j=1}^{n-1} \frac{a^{ij}}{a^{nn}} [U_{ij}(x, 0)] - 2 \sum_{i=1}^{n-1} \frac{a^{in}}{a^{nn}} [U_{ii}(x, 0)] \\ - \sum_{k=1}^n \frac{b^k}{a^{nn}} [U_k(x, 0)] - \frac{c}{a^{nn}} [U(x, 0)].$$

The quantities on the right hand side may be obtained from equations (8.4) and (8.6) directly or by differentiation. Initial conditions for $[U_{iii}]$ and higher time derivatives may also be obtained from the original differential equation by differentiating with respect to t .

The above discussion of initial conditions concerns that part of the field or solution which originates at the source. If the source is in an unbounded medium then the asymptotic expansion obtained may very well be the asymptotic expansion of the entire solution.²⁴ If however there is a discontinuity in the medium due, for example, to the presence of an obstacle, then a new family of wave fronts $t = \psi(x)$ originates from the discontinuity and the transport equations must be solved along the rays of this family. Initial conditions for these transport equations can often be obtained. Let us suppose, for concreteness, that a plane wave of the form e^{ikx} impinges on a surface such as the exterior of

²⁴ If a source exists in a non-homogeneous medium then there may be internal reflections which will add to the terms of the asymptotic expansion of the directly transmitted field. However, these reflections will generally be of higher order in $1/\omega$ than first term of our series. The subject of internal reflections will be discussed in a later paper.

a parabolic cylinder and let us suppose that the boundary condition to be satisfied on the surface is $u = 0$. One can readily show that the sum of the incident asymptotic and reflected asymptotic solutions must be zero on the surface. Since e^{ikx} is the first and only term of its own asymptotic expansion, the initial values of the coefficients of the reflected asymptotic solution are -1 for the first coefficient and zero for the others. The same remarks apply if the boundary condition is $\partial u / \partial n = 0$.

Some special considerations are needed to treat the dipole as a source. The dipole of three-dimensional space may be subsumed under the theory of this paper by letting

$$g(x) = \begin{cases} \frac{3}{4\pi} \frac{1}{\epsilon^3} & \text{for } \sqrt{x_1^2 + x_2^2 + x_3^2} \leq \epsilon \\ 0 & \text{for } \sqrt{x_1^2 + x_2^2 + x_3^2} > \epsilon, \end{cases}$$

and then considering the limit of the solution obtained as ϵ approaches 0.

Alternatively we may treat the dipole in a non-homogeneous three-dimensional space, when

$$L(u) = \Delta u - n^2(x)u_{tt}$$

as follows. The scalar potential for a dipole field in a homogeneous medium is e^{ikr}/r , where r is the distance from the dipole to the observation point. To obtain the asymptotic expansion of the dipole field in a non-homogeneous medium we can require that each coefficient of this expansion approach at $r = 0$, the corresponding coefficient of the asymptotic expansion of e^{ikr}/r . However, the asymptotic expansion of the latter function consists of just the function itself. To avoid infinities in the procedure just recommended we note that the analogue of r in a non-homogeneous medium is $\sqrt{K}$ where K is the expansion coefficient or the reciprocal of the Gauss curvature of the wave front at any point in space. We therefore multiply the n -th term of the desired asymptotic expansion by $(\sqrt{K})^n$ and require that this term approach the value at $r = 0$ of the product of r^n and the n -th term of the expansion e^{ikr}/r . Of course these products will have the limit zero except for $n = 1$.

9. The Relationship of The Asymptotic Series to The W K B Approximation.

Let us consider the general second order linear hyperbolic partial differential equation in one space variable and the time t :

$$(9.1) \quad a^{11} u_{xx} + a^{12} u_{xt} + a^{22} u_{tt} + b^1 u_x + b^2 u_t + cu = 0.$$

The theory of this paper gives the asymptotic solution of this equation when the time behavior of $u(x, t)$ is $e^{-i\omega t}$. If we therefore assume that $u(x, t) = u(x) e^{-i\omega t}$ and substitute in equation (9.1) we obtain a linear ordinary differential equation in $u(x)$. Our asymptotic series (4.7) represents this $u(x)$.

The theory of asymptotic solution of the n -th order linear ordinary differential equation

$$(9.2) \quad \frac{d^n u}{dx^n} + \omega a^{n-1}(x, \omega) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + \omega^n a^n(x, \omega) u = 0$$

for large ω has already been derived.²⁵ In particular, a recursive system of transport equations²⁶ in this theory enables us to calculate the successive coefficients in the asymptotic series representing each of the n solutions of equation (9.2). The first term of each of these n asymptotic series is the well known W K B approximation to the solution.

Comparison of the theory of this paper when applied to equation (9.1) for solutions of the form $u(x)e^{-i\omega t}$ with the asymptotic theory for ordinary differential equations shows complete agreement. Hence the theory of this paper, which could readily be generalized to higher order linear hyperbolic partial differential equations, is a generalization to partial differential equations of the theory of asymptotic solution of ordinary differential equations and, in particular, generalizes the W K B approximation.²⁷ Our generalization confirms the widely known fact that the W K B approximation gives the "geometrical optics" solution.

10. Conclusion. This paper offers a theory of asymptotic solution of certain partial differential equations and a method of calculating the coefficients of the asymptotic series. Under the hypotheses of this paper we are led to a series in the zero and positive integral powers of $1/\omega$. It is known on mathematical and physical grounds that such series do not represent the solution to all problems. In the field of electromagnetic phenomena problems involving caustics or diffraction should lead to series containing fractional powers of $1/\omega$. The theory of asymptotic solution required to treat such problems is not at present adequate.

²⁵ See BIRKHOFF, GEORGE D.: *On the Asymptotic Character of the Solutions of Certain Linear Differential Equations Containing a Parameter*, Amer. Math. Soc. Trans., V. 9, 1908, pp. 219-231.

²⁶ See equation (16) of the reference cited in footnote 25.

²⁷ In many papers, see, for example, BREMMER, H.: *The W K B approximation as the First Term of a Geometric-Optical Series*, Communications on Pure and Applied Mathematics, V. IV, 1951, pp. 105-115, the W K B approximation appears as the first term of a convergent series. Such series are not the same, of course, as asymptotic series.