*On the Asymptotic Stability of Solutions of Parabolic Differential Equations**

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Communicated by D. GILBARG

1. Introduction. We consider in this paper the stability of solutions of a wide class of nonlinear parabolic differential equations having the following general form:

$$(1.1) \quad \frac{\partial u_\nu}{\partial t} = \sum_{i,j} a_{ij}(x) \frac{\partial^2 u_\nu}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u_\nu}{\partial x_i} + F(x, t, u_\mu),$$
$$i, j = 1, \dots, n; \quad \nu, \mu = 1, \dots, m; \quad x \in G; \quad t > 0.$$

The domain of definition G is a bounded open region in n -dimensional x -space. Suitable conditions are to be imposed on the coefficients $a_{ij}(x)$, $b_i(x)$ and on the nonlinear part F . The solution vector $U(x, t)$ besides satisfying certain obvious continuity and differentiability conditions is assumed to be zero on the boundary Γ of G for $t \geq 0$. It is then shown that for suitably "small" initial values the solution vector tends to zero as t tends to infinity, *i.e.*,

$$\lim_{t \rightarrow \infty} |u_\nu(x, t)| = 0 \quad \text{uniformly in } x, \quad \nu = 1, \dots, m.$$

These details will be made more precise subsequently in the actual statement of the theorems.

The notion of stability of solutions of partial differential equations as used here is a generalisation of the classical definition of stability as applied to solu-

* This paper is based on part of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Indiana University, February 1954. The author wishes to express his sincere thanks to Professor DAVID GILBARG for his invaluable help and guidance in the preparation of this paper.

tions of systems of ordinary differential equations. To recall briefly the latter definition consider for example the following nonlinear system (expressed in vector notation):

$$(1.2) \quad \frac{dY}{dt} = AY + F(Y, t), \quad t \geq 0, \quad F(0, t) \equiv 0$$

where

$$\begin{aligned} Y(t) &= [y_1(t), \dots, y_n(t)] \\ A(t) &= (a_{ij}(t)) \\ F(Y, t) &= [f_1(Y, t), \dots, f_n(Y, t)]. \end{aligned}$$

The solution vector $Y \equiv 0$ is said to be stable if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that every solution whose initial values satisfy the condition $\|Y(0)\| \equiv \sum_i |y_i(0)| \leq \delta$ satisfies the inequality $\|Y(t)\| \equiv \sum_i |y_i(t)| \leq \epsilon$ for $t > 0$. If in addition $\|Y(t)\| \rightarrow 0$ as $t \rightarrow \infty$, then the zero solution is said to be asymptotically stable.

Following the usual practice (*cf.* Bellman [1]) we shall say that a solution u of a partial differential equation is stable with respect to a given class S of solutions if any other solution v of the class whose initial and boundary values are sufficiently "close" to those of u remains "close" to u for all values of the independent variables. The term "close" has of course to be defined more precisely with reference to a suitable metric. In the discussion that follows we shall be concerned only with the stability of the zero solution of equation (1.1) and we shall adopt the norm

$$\|U(x, t)\| = \max_{x \in G} \sum_v |u_v| \quad \text{for each fixed } t.$$

The principal results to date concerning this problem are due to R. Bellman and G. Prodi. Bellman [1] has considered the rather specialised nonlinear parabolic differential equation

$$(1.3) \quad u_t = u_{xx} + u_{yy} + u_{zz} + F(x, y, z, t, u), \quad F(x, y, z, t, 0) \equiv 0, \quad t > 0.$$

The domain of definition is a bounded closed convex region R of the 3-dimensional Euclidean space and it is assumed that the solutions vanish on the boundary of R for all values of t . Specifically, Bellman proves the following stability theorem when R is the cube $0 \leq x, y, z \leq \pi$. *Consider equation (1.3) with the boundary condition mentioned above. Let $F(x, y, z, t, u) = o(u)$ as $u \rightarrow 0$ uniformly in x, y, z . Then $u \equiv 0$ is a stable solution.* (For details see [1], p. 24.)

His method of proof consists in reducing the differential equation to an integral equation by means of known results from the theory of multiple Fourier series. This approach is rather closely analogous to his proofs of similar stability theorems in the theory of nonlinear ordinary differential equations.

Improving on certain aspects of Bellman's result G. Prodi [5] has proved the following theorem in a very elegant manner. Consider equation (1.3) in any bounded closed region R of x - y - z space. If the solution is continuous in and on the boundary of R then it is stable with respect to variations of its initial values provided the following conditions are satisfied:

- a. $F(x, y, z, t, u)$ is continuous in all its arguments for $-\infty < u < \infty$.
- b. $F(x, y, z, t, u_2) - F(x, y, z, t, u_1) \leq \lambda(u_2 - u_1)$ in the region considered whenever $u_2 > u_1$. $\lambda > 0$ is a suitably small, given constant.

Actually to establish his result it is quite unnecessary, as we shall see in the sequel, to impose the global restriction on F implied by condition b. Prodi's proof of his theorem makes use of an interesting comparison theorem based on differential inequalities (Lemma 1 of this work); a brief outline of his proof of the comparison theorem is included in §3 below.

The main theorem (Theorem 1.1 of §4) of this paper proves the asymptotic stability of the zero solution of equation (1.1) for the case $m = 1$ assuming that F satisfies the following condition: there exists a constant $K > 0$ such that for $|u| < K$ we have $uF \leq \lambda u^2$ where $\lambda > 0$ is some suitably prescribed small constant. We note that this result includes as special cases those of Bellman and Prodi described earlier. Our method of proof, which successfully exploits Prodi's comparison theorem, is rather closely related to the second method of Liapounoff in the stability theory of systems of nonlinear ordinary differential equations. (See Malkin, AMS Translation No. 20.)

In the remarks following the proof of Theorem 1.1 we shall indicate how that theorem can be extended under rather special assumptions on F to include the case $F = F(x, t, u, u_x)$. In §5, Theorem 1.2 extends the results to systems of parabolic equations such as (1.1).

2. Notation. Let G be a bounded open domain in n -dimensional x -space and let Γ be its boundary. We shall denote by x a point of $G + \Gamma$; $(x_1, \dots, x_n)$ are its coordinates in some fixed cartesian coordinate system. Let $\bar{B}$ be the $(n + 1)$ -dimensional domain in x - t -space with $x \in G + \Gamma$ and $0 \leq t \leq T$. Denote by E the interior points of this hypercylinder and by B_T the set of points (x, T) where $x \in G$. Let B be the remainder of the boundary of $\bar{B}$. Thus $\bar{B} = E + B_T + B$.

3. Prodi's lemma.* The comparison theorem referred to in §1, and on which the proof of the main theorem is based, is concerned with the general nonlinear parabolic differential equation

$$(3.1) \quad u_t = F(x, t, u, p_i, p_{ij}), \quad i, j = 1, \dots, n,$$

* This lemma in its original form seems to have been introduced by WESTPHAL [6] in his work on the estimation of solutions of parabolic differential equations. For a somewhat more generalised formulation of the lemma see NAGUMO & SIMODA [3].

where $x \in G$, $0 < t < T$ and p_i and p_{ij} represent the first and second partial derivatives $\partial u / \partial x_i$, $\partial^2 u / \partial x_i \partial x_j$ respectively. We shall assume that F is continuous in all its arguments, $\partial F / \partial p_{ij}$ are continuous for all i, j and that the quadratic form $\sum_{i,j} (\partial F / \partial p_{ij}) \xi_i \xi_j$, ξ_i, ξ_j real, is positive definite. Under the above assumptions we prove

Lemma 1 (Westphal-Prodi). *Let $u(x, t)$, $v(x, t)$ be functions continuous in $\bar{B}$ and with the partial derivatives u_t , $p_i \equiv u_{x_i}$, $p_{ij} \equiv u_{x_i x_j}$, v_t , $q_i \equiv v_{x_i}$, and $q_{ij} \equiv v_{x_i x_j}$, continuous in $E + B_T$. Moreover, in $E + B_T$ let the following inequalities hold:*

$$(3.2) \quad \begin{aligned} v_t &\leq F(x, t, v, q_i, q_{ij}) \\ u_t &> F(x, t, u, p_i, p_{ij}), \end{aligned}$$

where the function F has the properties enumerated above. Then, if $v(x, t) < u(x, t)$ in B , the inequality holds throughout $\bar{B}$.

Proof. Let $u - v = g$. Suppose that the set of points of $\bar{B}$ where $g(x, t) \leq 0$ is not null. They by the continuity of g and the assumptions about g on the boundary B it follows that there exists a point $(\bar{x}, \bar{t})$ such that $g(\bar{x}, \bar{t}) = 0$ and $g(x, t) > 0$ for all $t < \bar{t}$ and all $x \in G$. From the assumptions of the lemma it is evident that $(\bar{x}, \bar{t})$ is an interior point of $\bar{B}$. Also from the definition of the point $(\bar{x}, \bar{t})$ it follows that $g(x, \bar{t}) \geq 0$ for $x \in G$. Thus $g(x, \bar{t})$ considered as a function of the n variables $(x_1, \dots, x_n)$ attains an interior minimum at $(\bar{x}, \bar{t})$. Hence at the point $(\bar{x}, \bar{t})$

- a) the first partials derivatives $g_i \equiv \partial g / \partial x_i = 0$,
- b) the quadratic form $\sum_{i,j} g_{ij} \xi_i \xi_j \geq 0$; $i = 1, \dots, n$;
- c) $g_t(\bar{x}, \bar{t}) \leq 0$.

Thus at the point $(\bar{x}, \bar{t})$ we have the following results:

- 1) $v = u$
- 2) $q_i = p_i$, $i = 1, \dots, n$
- 3) $v_t \geq u_t$
- 4) $\sum_{i,j} (p_{ij} - q_{ij}) \xi_i \xi_j \geq 0$.

Making use of 4) and a well known lemma of Th. Moutard on quadratic forms, we see immediately that $\sum_{i,j} F_{ij}(p_{ij} - q_{ij}) \geq 0$, where $p_{ij} = p_{ij}(\bar{x}, \bar{t})$ and $q_{ij} = q_{ij}(\bar{x}, \bar{t})$.

From the theorem of mean values and the above result we have for the function $F(\bar{x}, \bar{t}, \bar{u}, \bar{p}_i, \alpha_{ij})$ —where the barred quantities are to be evaluated at the point $(\bar{x}, \bar{t})$ —

$$F(\bar{x}, \bar{t}, \bar{u}, \bar{p}_i, \bar{p}_{ij}) \geq F(\bar{x}, \bar{t}, \bar{v}, \bar{q}_i, \bar{q}_{ij}) \quad \text{since} \quad \bar{u} = \bar{v} \quad \text{and} \quad \bar{p}_i = \bar{q}_i.$$

Therefore from equation (3.2) we get

$$\bar{u}_i > F(\bar{x}, \bar{t}, \bar{u}, \bar{p}_i, \bar{p}_{ij}) \geq F(\bar{x}, \bar{t}, \bar{v}, \bar{q}_i, \bar{q}_{ij}) \geq v_i,$$

which contradicts 3) above. Hence the set of points of $\bar{B}$ where $g(x, t) \leq 0$ is null. This completes the proof of the lemma.

4. Main theorem. In applying the above lemma we shall be concerned with the differential form

$$(4.1) \quad L[u(x, t)] \equiv \sum_{i,j} a_{ij}(x) \frac{\partial^2 u(x, t)}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u(x, t)}{\partial x_i}, \quad i, j = 1, \dots, n$$

where $x \in G$; the coefficients $a_{ij}(x)$, $b_i(x)$ are continuous in $G + \Gamma$; and the quadratic form $\sum_{i,j} a_{ij} \xi_i \xi_j$ is positive definite for $x \in G + \Gamma$ and ξ_i, ξ_j real. The main theorem may now be stated as

Theorem 1.1. Consider the nonlinear parabolic differential equation

$$(4.2) \quad u_t = L[u] + F(x, t, u), \quad x \in G, \quad t > 0,$$

where $L[u]$ is the differential form defined in (4.1). Let $F(x, t, 0) \equiv 0$.

Assume

(a) The real valued function $u(x, t)$ is a solution of equation (4.2) with the following properties:

1. $u, u_t, u_{x_i}, u_{x_i x_j}$ are continuous functions of x and t for $x \in G + \Gamma$ and $t \geq 0$;
2. Boundary condition: $u(x, t) = 0$ for $x \in \Gamma$ and $t \geq 0$.

(b) There exists a number $K > 0$ such that for $|u| < K$ we have

$$uF(x, t, u) \leq \lambda u^2, \quad x \in G, \quad t > 0,$$

where $\lambda > 0$ is a given constant (to be specified) depending only on the coefficients a_{ij} , b_i and the domain G .

Conclusion. There exists a number $\delta > 0$ such that if $|u(x, 0)| < \delta$ for $x \in G$, then

$$\lim_{t \rightarrow \infty} |u(x, t)| = 0 \quad \text{uniformly in } x \quad \text{for } x \in G.$$

In other words the theorem asserts that under conditions (a) and (b) $u \equiv 0$ is an asymptotically stable solution relative to the class of solutions satisfying condition (a). We remark that a sufficient condition for (b) to be satisfied is that $F = o(u)$ as $u \rightarrow 0$ uniformly in $x \in G$.

Proof of Theorem 1.1. Multiplying both sides of equation (4.2) by u we get

$$(4.3) \quad \frac{\partial u^2}{\partial t} = L[u^2] - 2 \sum_{i,j} a_{ij} u_{x_i} u_{x_j} + 2uF.$$

Making use of the positive definiteness of the quadratic form and assumption (b) of the theorem we have from equation (4.3) that as long as $|u| < K$ the following basic inequality holds:

$$(4.4) \quad \frac{\partial u^2}{\partial t} \leq L[u^2] + \lambda_1 u^2, \quad \text{where} \quad \lambda_1 = 2\lambda.$$

It is clear that by a proper choice of the initial value $u(x, 0)$ we can ensure the validity of inequality (4.4) at least in a neighbourhood of $t = 0$, $x \in G$. We wish now to show that for such a choice the inequality (4.4) holds in fact for all $t > 0$ and $x \in G$.

In order to do so we shall make use of the following comparison function:

$$\phi(x, t) = \frac{1}{N} (A - e^{Dx_1})e^{-\alpha t},$$

where N, A, D, α are positive constants to be specified. To begin with

$$-\frac{\partial \phi}{\partial t} + L[\phi] + \lambda_1 \phi = \frac{e^{-\alpha t}}{N} \{(\alpha + \lambda_1)(A - e^{Dx_1}) - e^{Dx_1}(D^2 a_{11} + Db_1)\}.$$

Since $a_{11} > 0$ and b_1 is bounded in $G + \Gamma$, it is evident that for arbitrary (but fixed) $\epsilon^2 > 0$ one can choose $D > 0$ sufficiently large so that

$$(D^2 a_{11} + Db_1) \geq \epsilon^2 > 0 \text{ in } G + \Gamma.$$

For definiteness let us assume without loss of generality that in $G + \Gamma$, $x_1 \geq 0$. Now let $A = e^{Dx_1} + y^2$ where $y^2 < 1$ and X_1 is the maximum value of x_1 in $G + \Gamma$. Then

$$(A - e^{Dx_1}) \geq y^2 > 0 \quad \text{in } G + \Gamma,$$

and

$$(A - e^{Dx_1}) < e^{Dx_1} \quad \text{in } G + \Gamma.$$

Finally select $\alpha, \lambda_1 > 0$ such that $(\alpha + \lambda_1) < \epsilon^2 e^{-Dx_1}$ and N so that $N^{-1} e^{Dx_1} < K$. Then it follows that

$$-\frac{\partial \phi}{\partial t} + L[\phi] + \lambda_1 \phi < 0$$

or in other words that

$$(4.5) \quad \frac{\partial \phi}{\partial t} > L[\phi] + \lambda_1 \phi.$$

Also

$$\phi > 0 \text{ for } x \in \Gamma \text{ and } t < \infty.$$

and

$$\phi \geq \delta^2 \equiv \frac{1}{N} y^2 \quad \text{for} \quad x \in G + \Gamma, \quad t = 0.$$

Thus if the initial value of u satisfies the relation $|u(x, 0)| < \delta$ for $x \in G$ then we have

$$(4.6) \quad \phi > u^2 \text{ in } B \text{ for every interval } 0 \leq t \leq T.$$

This, together with (4.4) and (4.5), implies by Lemma 1 that as long as $|u| < K$ we have $u^2 < \phi$ in $\bar{B}$. As remarked earlier, from the condition $|u(x, 0)| < \delta$ and the continuity of u it is clear that $u^2 < \phi$ in $\bar{B}$ for some $T > 0$ in the neighbourhood of $t = 0$. We assert now that in fact $u^2 < \phi$ for $t > 0$.

For, suppose that the assertion were false. Then there exists a $t = t_1 > 0$ such that

$$(4.7) \quad \begin{aligned} u^2 &< \phi \text{ for } 0 \leq t < t_1 \text{ and all } x \in G + \Gamma, \\ u^2 &= \phi \text{ for } t = t_1 \text{ and at least one } x \in G + \Gamma. \end{aligned}$$

Then for $x \in G + \Gamma$ we have

$$u^2(x, t_1) \leq \phi(x, t_1) \leq \phi(x, 0) < K.$$

Hence the inequality (4.4) remains true for $x \in G$ and $0 \leq t \leq t_1$. Thus if we take $T = t_1$, from (4.4), (4.5), (4.6) and lemma 1 it follows that $u^2 < \phi$ in $\bar{B}$. That is,

$$u^2(x, t_1) < \phi(x, t_1) \quad \text{for all } x \in G + \Gamma,$$

which contradicts (4.7). Thus no such t_1 can exist, or in other words

$$u^2(x, t) < \phi(x, t) \quad \text{for } x \in G + \Gamma, \quad t > 0.$$

But $\phi \rightarrow 0$ uniformly in x as $t \rightarrow \infty$, and consequently so does u^2 , i.e., $|u(x, t)| \rightarrow 0$ uniformly in x as $t \rightarrow \infty$ for $x \in G + \Gamma$, which proves the theorem completely.

Remark 1. From the proof of Theorem 1.1 it is readily inferred that equations of the form

$$(4.2a) \quad u_t = L[u] + q(x)u + F(x, t, u), \text{ where } q(x) \leq 0,$$

automatically satisfy conditions of the theorem if, e.g., $F = o(u)$ as $u \rightarrow 0$; consequently $u \equiv 0$ is a stable solution of equation (4.2a). It is of interest to note that equations of Sturm-Liouville type that Bellman discusses at some length in his paper (see p. 39–40 of [1]) are but special cases of equation (4.2a).

Remark 2. We shall not attempt here any complete generalisation of Theorem 1.1 to include the case where the nonlinear part of equation (4.2) contains terms

involving not only the function $u(x, t)$ but its first partial derivatives with respect to x as well; i.e., where $F = F(x, t, u, u_x)$. From the statement of Lemma 1 and the nature of the proof of the main theorem it is easily seen that to be able to carry through the generalisation imposing merely a condition on F analogous to (b) of Theorem 1.1 much more stringent smoothness properties would be required of the solution $u(x, t)$. Specifically we should need to know that the partial derivatives $u_{x_i}(x, t)$ remain "small" if the function $u(x, t)$ remains "small". However, for certain special classes of F the generalisation can be accomplished as indicated below without placing any explicit restrictions on the partial derivatives $u_{x_i}(x, t)$. We shall limit ourselves to the case where $F(x, t, u, u_x)$ contains terms only of the form $f(x, t, u)$, $g(x, t, u)u_x$, or $h(x, t, u)u_{x_i}u_{x_j}$, where h is uniformly bounded and f and g satisfy conditions analogous to (b) of Theorem 1.1 for arbitrarily small λ (for example, the case where f, g are $o(u)$ as $u \rightarrow 0$ uniformly in $x \in G$).

To begin with, it is clear that if u_{x_i} occurs in F only in the form $g(x, t, u)u_{x_i}$ then, from the above assumption on g , there exists a number $K > 0$ such that for $|u| < K$ we have

$$gu_{x_i} \leq \lambda(u^2 + u_{x_i}^2) \text{ for arbitrary but fixed } \lambda > 0.$$

By an obvious extension of the argument it is evident that if F contains only linear and quadratic terms in u_{x_i} multiplied by functions of u as mentioned earlier then again one can find a $K > 0$ such that for $|u| < K$ we have

$$uF \leq \lambda(u^2 + \sum_i u_{x_i}^2) \text{ for arbitrary but fixed } \lambda > 0.$$

With these remarks we shall now state the result as follows.

Corollary to Theorem 1.1. *Consider the nonlinear parabolic differential equation*

$$(4.8) \quad u_t = L[u] + F(x, t, u, u_x), \quad o \in G, t > 0,$$

where $L[u]$ is the differential form defined in (4.1).

Assume

(a) *The quadratic form $\sum_{i,j} a_{ij} \xi_i \xi_j$ is strongly elliptic in $G + \Gamma$; i.e., there exists a number $\mu > 0$ such that $\sum_{i,j} a_{ij} \xi_i \xi_j \geq \mu \sum \xi_i^2$ for $x \in G + \Gamma$ and ξ_i, ξ_j real.*

(b) *The solution $u(x, t)$ satisfies condition (a) of Theorem 1.1.*

(c) *There exists a number $K > 0$ such that for $|u| < K$ we have*

$$uF \leq \lambda(u^2 + \sum_i u_{x_i}^2), \quad x \in G, \quad t > 0,$$

where $\lambda > 0$ is a given constant (to be specified) depending only on the coefficients a_{ij}, b_i , the constant μ and the domain G .

Conclusion. *The conclusion of Theorem 1.1 remains true.*

Proof. From equation (4.8) and assumption (c) we obtain for $|u| < K$

$$\frac{\partial u^2}{\partial t} \leq L[u^2] - 2 \sum_{i,j} a_{ij} u_{x_i} u_{x_j} + \lambda_1 (u^2 + \sum_i u_{x_i}^2)$$

where $\lambda_1 = 2\lambda$. Using assumption (a) and assuming $\lambda_1 < 2\mu$, we conclude that as long as $|u| < K$ the following basic inequality holds:

$$(4.9) \quad \frac{\partial u^2}{\partial t} \leq L[u^2] + \lambda_1 u^2.$$

Replacing inequality (4.4) by (4.9), we see clearly that the proof of this theorem can now be completed by duplicating that of Theorem 1.1 except that now λ_1 will have to be prescribed so that

$$(\alpha + \lambda_1) < \text{Min} (\epsilon^2 e^{-Dx_1}, 2\mu).$$

5. Extension to systems. The extension to the case of systems is now almost immediate. We assume that the differential form $L[u]$ is the same in each component equation of the system.

Theorem 1.2. *Consider the system of nonlinear parabolic differential equations*

$$(5.1) \quad \begin{aligned} \frac{\partial u_\nu}{\partial t} &= L[u_\nu] + F_\nu(x, t, u_\mu), & F_\nu(x, t, 0) &\equiv 0, \\ \nu, \mu &= 1, \dots, m; & x &\in G; & t > 0. \end{aligned}$$

Assume

(a) *Each component of the solution vector $U(x, t)$ satisfies condition (a) of Theorem 1.1.*

(b) *There exists a number $K > 0$ such that for $|u_\nu| < K$, $\nu = 1, \dots, m$, we have*

$$\sum_\nu u_\nu F_\nu \leq \lambda \sum_\nu u_\nu^2,$$

where $\lambda > 0$ is a given constant (to be specified) depending only on the coefficients a_{ij} , b_i and the domain G .

Conclusion. *There exists a number $\delta > 0$ such that if $|u_\nu(x, 0)| < \delta/m$ for $\nu = 1, \dots, m$ and for $x \in G$, then*

$$\lim_{t \rightarrow \infty} |u_\nu(x, t)| = 0 \quad \text{uniformly in} \quad x \in G, \quad \nu = 1, \dots, m.$$

Proof. From equation (5.1) we get

$$\frac{\partial u_\nu^2}{\partial t} = 2u_\nu L[u_\nu] + 2u_\nu F_\nu;$$

summing over ν and making use of assumption (b), we get for $|u_\nu| < K$

$$\sum_\nu \frac{\partial u_\nu^2}{\partial t} \leq 2 \sum_\nu u_\nu L[u_\nu] + \lambda_1 \sum_\nu u_\nu^2 \quad \text{where} \quad \lambda_1 = 2\lambda.$$

Let $v(x, t) = \sum_\nu u_\nu^2(x, t)$. Then we obtain the basic inequality as

$$(5.2) \quad \frac{\partial v}{\partial t} \leq L[v] + \lambda_1 v.$$

The rest of the proof now duplicates that of Theorem 1.1 with the inequality (4.4) replaced by (5.2). Thus we obtain the result that

$$\lim_{t \rightarrow \infty} v(x, t) = 0 \quad \text{uniformly in } x \quad \text{for} \quad x \in G.$$

But

$$v(x, t) = \sum_\nu u_\nu^2(x, t)$$

Hence $|u_\nu(x, t)| \rightarrow 0$ as $t \rightarrow \infty$, uniformly in x for $x \in G$ which completes the proof of Theorem 1.2.

Remark 1. It is evident that the extension indicated in Remark 2 of §4 applies *mutatis mutandis* to Theorem 1.2 also. Only, here one would require in the place of assumption (c) of Corollary to Theorem 1.1 the following condition:

$$\sum_\nu u_\nu F_\nu \leq \lambda \left\{ \sum_\nu u_\nu^2 + \sum_\nu \left[\sum_i \left(\frac{\partial u_\nu}{\partial x_i} \right)^2 \right] \right\}$$

for some suitably small constant $\lambda > 0$.

Remark 2. Except under special circumstances (e.g., see Remark 1 of §4) the stability theorems formulated in this paper do not apply to the case where terms linear in u are present.

6. Concluding remarks. It is of interest to note that in the proofs of neither of the Theorems 1.1 or 1.2 has any mention been made of the nature of the stability of solutions of the linear part

$$(6.1) \quad u_t = L[u].$$

It is well known that in the case of systems of nonlinear ordinary differential equations, the stability of solutions of the nonlinear system is intimately connected with that of the linear part of the system, or more precisely, with the eigenvalues of the matrix A in the vector equation

$$(6.2) \quad \dot{Y} = AY + F(t, Y), \quad F(t, 0) \equiv 0.$$

In fact it has been shown (see Perron [4]) that if the matrix A has *one* characteristic root with positive real part then $Y \equiv 0$ is an unstable solution; *i.e.*, for some $\epsilon > 0$, there are solutions $Y(t)$ of equation (6.2) with arbitrarily small initial values such that $\|Y(t)\| \geq \epsilon$ for some t .

It may be wondered whether an analogous statement is true of equation (4.2) and its linear part (6.1). Theorem 1.1 of course does not answer this question, although it ensures that in the case treated there the eigenvalues of (6.1) do in fact all have negative real parts.

It is of interest to note that the above result can be established quite independently of Theorem 1 by making use of the well known maximum principle for functions satisfying linear elliptic differential inequalities with the differential form $L[u]$ defined in (4.1). (See Courant & Hilbert [2], p. 274–275.) Since the proof is quite elementary and straightforward we shall omit the details here and merely state the result as

Theorem 2. Let $u(x, t) \equiv [v(x) + iw(x)]e^{(\alpha+i\beta)t}$ (where α, β are constants and $w \equiv 0$ if $\beta = 0$) be a solution of the linear parabolic differential equation

$$(6.1) \quad u_t = L[u], \quad x \in G, t > 0,$$

satisfying conditions (a) of Theorem 1.1.

Conclusion. If $\alpha \geq 0$, then $v(x) \equiv w(x) \equiv 0$ for $x \in G$.

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