Novel Uses of Functional Equations

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We derive functional equations related to certain topics in electromagnetic theory. Although many of the results are believed to be new, the interest centers chiefly in the method, which does not take Maxwell's equations as the point of departure.

PART I. STRATIFIED DIELECTRIC MEDIA

The equation for uniform dielectric media. If a plane sheet with complex reflection coefficient r and transmission coefficient t is immediately followed by a second sheet with coefficients R, T , then the coefficients for the combination are

$$(1) \quad \frac{r + (t^2 - r^2)R}{1 - rR}, \quad \frac{tT}{1 - rR}.$$

(Equation (1) may be established by superposition or by considering multiple reflection; see [2].)

Suppose, now, that a homogeneous sheet of thickness x is immediately followed by a similar sheet of thickness y . If they be made of the same material, the composite sheet is a homogenous sheet of thickness $x + y$. Applying (1) to this composite sheet, we obtain the functional equations

$$(2) \quad R(x + y) = \frac{R(x) + [T^2(x) - R^2(x)]R(y)}{1 - R(x)R(y)}$$

$$(3) \quad T(x + y) = \frac{T(x)T(y)}{1 - R(x)R(y)},$$

where $R(x)$ and $T(x)$ are the coefficients for a sheet of thickness x at the given angle of incidence.

Equation (3) shows that $T(x) \equiv 0$ if $T(0) = 0$ and (2) then gives $R(x) = R(0)$. Setting aside this trivial case, we deduce $R(0) = 0$, $T(0) = 1$ by letting $x = 0$ in (3). Of course, these values are to be expected physically. We likewise set aside the case $R(x) \equiv 0$, for which (3) becomes the Hamel equation in $\log T$.

Symmetry. If x and y are interchanged, the composite sheet is unaltered, a fact reflected in the obvious equalities

$$(4) \quad R(x + y) = R(y + x), \quad T(x + y) = T(y + x).$$

The second equation (4) is a consequence of (3), and yields nothing new. But the first combines with (2) to give

$$(5) \quad R(x) + [T^2(x) - R^2(x)]R(y) = R(y) + [T^2(y) - R^2(y)]R(x).$$

Equation (5), which does not reduce to a formal identity, may be written

$$(6) \quad \frac{T^2(x) - 1}{R(x)} - R(x) = \frac{T^2(y) - 1}{R(y)} - R(y)$$

when we divide by $R(x)R(y)$ and transpose. Since the variables are separated in (6), each side is constant:

$$(7) \quad T^2(x) - 1 - R^2(x) = cR(x), \quad c \text{ constant}$$

If $R(x) = 0$ we deduce (7) from (5) directly.

To determine the constant c , suppose that the medium has a small amount of dissipation. That is, suppose $\text{Im}(k) > 0$ where k is the propagation constant in the sense of [1]. Then

$$(8) \quad \lim R(x) = r, \quad \lim T(x) = 0, \quad x \rightarrow \infty,$$

where r is the interface reflection. Letting $x \rightarrow \infty$ in (7) now gives

$$(9) \quad c = -(r + 1/r).$$

Since the coefficients for $\text{Im}(k) = 0$ are defined (or ought to be defined) by taking limits as $\text{Im}(k) \rightarrow 0+$, our results hold even for $\text{Im}(k) = 0$; and we have

$$(10) \quad T^2 = 1 - R(r + 1/r) + R^2$$

where r is the interface reflection.

Equation (10) is useful in dealing with series of sheets; it reduces the number of variables from two to one. Also it shows that a dielectric sheet is in "shunt" if and only if $R = 0$ or $r = -1$. (This is the condition for $T = 1 + R$; cf. [4]). Finally, we can have $R = 0$ if and only if $T = \pm 1$; in fact

$$(11) \quad \lim (T - 1)/R = c, \quad R \rightarrow 0.$$

The differential equation. When T is given by (10), it can be shown that (3) is equivalent to (2). Equation (2) itself takes the form

$$(12) \quad \frac{R(x+y) - R(x)}{y} = \frac{[R(y)/y][1 + cR(x)R^2(x)]}{1 - R(x)R(y)}$$

when we eliminate T by (10), subtract $R(x)$, and divide by y . Since $R(0) = 0$,

$$(13) \quad \lim R(y)/y = R'(0) = a, \quad y \rightarrow 0$$

when the limit exists. We now *define* the reflection and transmission coefficients as the solution of (2), (3) which satisfies (8) and (13). Letting $y \rightarrow 0$ in (12) yields

Theorem I. The function $R(x)$ is the unique solution of the Riccati equation and boundary condition

$$(14) \quad R' = a(1 + cR + R^2), \quad R(0) = 0;$$

the function $T(x)$ is given by $T^2 = 1 + Rc + R^2$.

The complex parameters a , c depend on the medium, the wavelength, and the incidence angle. If the sheet is terminated by an arbitrary reflection R_1 , it suffices to take that solution of the Riccati equation for which $R(0)$ has the appropriate value.

Theorem I shows that

$$(15) \quad \frac{dR}{dx} = aT^2,$$

a fact which also follows directly from (13) and (2), without intervention of (10).

Solving the Riccati equation yields the well-known forms for R and T . These results have been obtained here without any use of Maxwell's equations or boundary conditions; the work was based, really, on the algebraic structure discussed in [4]. Of course one can proceed in the opposite direction and verify Theorem I by substituting T and R as given in [1]. Such a verification, however, is as troublesome as the deduction from first principles given here.

Inhomogeneous media. Suppose, now, that the medium is inhomogeneous, the complex dielectric constant and permeability being arbitrary functions of the distance, x , from one interface. We specify the material here by giving the $a = a(x)$ and $c = c(x)$ of the previous discussion. At normal incidence,

$$(16) \quad \begin{aligned} a &= (\pi i / \lambda)(\epsilon / \epsilon_0 - \mu / \mu_0) \\ c &= -2 \frac{\epsilon / \epsilon_0 + \mu / \mu_0}{\epsilon / \epsilon_0 - \mu / \mu_0} \end{aligned}$$

by a short calculation based on reference [1].

The reflection and transmission of a thin homogeneous sheet of thickness y satisfy

$$(17) \quad \begin{aligned} R(y) &= ay + o(y) \\ T^2(y) &= 1 + acy + o(y) \end{aligned}$$

as $y \rightarrow 0$, in view of Theorem I. If our inhomogeneous material has reflection $R(x)$ at thickness x , we may find the reflection at thickness $x + y$ by adding a layer of thickness y . When a and c are continuous, this added layer may be replaced by a homogeneous layer with suitable intermediate values for the parameters. The error so committed can be shown to be $o(y)$ as $y \rightarrow 0$; cf. [3].

Proceeding as in the derivation of (2), (3), and using (17) for the coefficients of the added layer, we obtain

$$(18) \quad \begin{aligned} R(x + y) &= a(x)y + \frac{1 + a(x)c(x)yR(x)}{1 - R(x)a(x)y} + o(y) \\ \rho(x + y) &= \rho(x) + \frac{T^2(x)a(x)y}{1 - R(x)a(x)y} + o(y) \\ T(x + y) &= \frac{T(x)[1 + a(x)c(x)y/2]}{1 - R(x)a(x)y} + o(y) \end{aligned}$$

for the right-hand reflection, R , the left-hand reflection ρ , and the transmission T . The term $o(y)$ arises from our use of (17). Retaining only the leading terms we may write

$$(19) \quad \begin{aligned} R(x + y) - R(x) &= ya(x)[1 + c(x)R(x) + R^2(x)] + o(y) \\ \rho(x + y) - \rho(x) &= yT^2(x)a(x) + o(y) \\ T(x + y) - T(x) &= yT(x)a(x)[c(x)/2 + R(x)] + o(y). \end{aligned}$$

If we divide by y and let $y \rightarrow 0$ we get

$$(20) \quad R' = a(1 + cR + R^2)$$

$$(21) \quad \rho' = aT^2$$

$$(22) \quad T' = aT(R + c/2)$$

where the ' means d/dx and $a = a(x)$, $c = c(x)$.

The Riccati equation (20) determines $R(x)$, when we use the boundary condition $R(0) = 0$. Equations (21) and (22) now determine both T and ρ explicitly, by quadratures, the constants of integration being obtained from $T(0) = 1$, $\rho(0) = 0$. Thus we have

Theorem II. Let the right-hand reflection of a dielectric medium be $R(x)$ when the thickness is x . Then the transmission is given by

$$\log T(x) = \int_0^x a(x)[R(x) + c(x)/2] dx$$

and the left-hand reflection by

$$\rho(x) = \int_0^x a(x)T^2(x) dx.$$

The function $R(x)$ itself is determined by the Riccati equation

$$R' = a(1 + cR + R^2), \quad R(0) = 0.$$

The differential equation (20) has been derived previously [5], though by an altogether different procedure. (So far as we know, (21) and (22) may be new.) The interest of the present approach lies, as before, in the use of a functional equation based on (1), rather than Maxwell's equations. The theory of inhomogeneous dielectric sheets is thus put into the algebraic framework of [4].

The dielectric sheet is completely specified by the two complex functions $a(x)$, $c(x)$; and yet there are three complex coefficients R , ρ , and T . It is to be expected, then, that these three functions are in some sense dependent. In fact, multiplying (20) by T^2 and (22) by T yields

$$\begin{aligned} T^2 R' &= \rho'(1 + cR + R^2) \\ (23) \quad TT' &= \rho'(R + c/2). \end{aligned}$$

Eliminating c gives

$$(24) \quad T^2 R' - 2TT'R = \rho'(1 - R^2).$$

Conversely, if (24) holds then (21) is a consequence of (20) and (22). When R and T are given functions one may determine a and c so that (20), (22) hold; and we are thus led to the following realizability theorem:

Theorem III. Let $T(x)$, $R(x)$, $\rho(x)$ be three differentiable, complex functions satisfying $T(0) = 1$, $R(0) = \rho(0) = 0$. A necessary and sufficient condition that T , R , ρ can be realized as coefficients of a stratified dielectric medium is

$$(25) \quad R^2(T^2/R)' + (1 - R^2)\rho' = 0.$$

In general, the medium will turn out to be active, since we assume nothing about the ingoing and outgoing powers. The additional conditions which the coefficients must satisfy when the medium is passive or lossless are given in reference [4].

PART II. TRANSFER OF ENERGY IN A CAVITY

The problem. We now consider a quite different problem in electromagnetic theory, which can also be discussed by means of functional equations. The problem concerns the transfer of energy to objects in a multimode cavity. This energy is measured by finding the temperature rise over a given fixed time interval. It is supposed that the modes are constantly changing with time, or that the objects are moved about in the cavity; in other words on the average the field acts as if homogeneous. Also the objects are assumed fairly small compared to the cavity.

In these circumstances the temperature rise is found experimentally to be nearly independent of the shape of the object, for a wide range of shapes; it depends only on the volume of the material. Thus one may write

$$(26) \quad P_r/P = f(v)$$

for a given material, where P_r is the power received, P the input power to the cavity, and v the volume.

Suppose that two separate volumes, v and v' , of the same material are placed simultaneously in the cavity. How will the power divide between them? An exact solution to such a problem can hardly be simple; it requires detailed knowledge of the geometry and of the cavity characteristics. Also it is likely to be complicated by the time-dependent character of the mode pattern. Here we seek an approximate solution based on the functional equations suggested by certain experimental facts.

The main functional equation. With unit power input to the cavity, let $f(v, v')$ denote the power absorbed by a volume v in the presence of a second volume v' , both of the same material. The notation is meaningful in view of the assumed independence of shape. This independence also assures that the total power entering v and v' equals that for a single volume $v + v'$ in the otherwise empty cavity. We are thus led to the functional equation

$$(27) \quad f(v, v') + f(v', v) = f(v + v', 0),$$

which plays a fundamental role in the sequel.

Analysis of a particular form. Empirical evidence supports the equation

$$(28) \quad f(v, 0) = \frac{av}{v + b}$$

where a, b are constant for a given material and cavity. We assume, now, that (28) is *universally* valid. In particular, it holds for the modified cavity consisting of the original cavity together with the fixed volume, v' . In this modified cavity the volume v is unaccompanied by any other material; if the subscript 1 refers to the modified cavity we have

$$(29) \quad f(v, v') = f_1(v, 0).$$

The universal validity of (28) accordingly yields a specific functional form:

$$(30) \quad f(v, v') = \frac{a(v')v}{v + b(v')}.$$

We inquire, now, whether the functional equation (27) admits a solution of form (30) and if so, whether that solution is unique. Certain trivial degeneracies are ruled out by the experimental situation; *e.g.*, we have $a(0) \neq 0$; and we pass over these without comment. It will be found that no assumption is required as to the continuity (or even measurability) of a , b . The equation (27) is assumed to hold for $0 \leq v \leq V$, $0 \leq v' \leq V$ where $V > 0$.

Substituting (30) into (27) gives

$$(31) \quad \frac{a(v')v}{v + b(v')} + \frac{a(v)v'}{v' + b(v)} = \frac{(v + v')a_0}{v + v' + b_0}$$

where $a_0 = a(0)$, $b_0 = b(0)$. Setting $v' = v$ gives

$$(32) \quad a(v) = a_0 \frac{v + b(v)}{2v + b_0}$$

so that (31) becomes

$$(33) \quad \frac{v' + b(v')}{v + b(v')} \frac{v}{2v' + b_0} + \frac{v + b(v)}{v' + b(v)} \frac{v'}{2v + b_0} = \frac{v + v'}{v + v' + b_0}.$$

If we give to v' a fixed value other than v , then equation (33) may be solved for $b(v)$. It is found that

$$(34) \quad b(v) = P(v)/Q(v)$$

where P and Q are polynomials of degree ≤ 3 . Hence if (33) holds over any interval, then (33) holds for all v, v' . In particular, the poles correspond.

The second term in (33) is

$$(35) \quad \frac{v'}{2v + b_0} \frac{vQ(v) + P(v)}{v'Q(v) + P(v)}$$

which has a pole at $v = -b_0/2$ unless $2v + b_0$ divides $vQ(v) + P(v)$. It is easy to show that $b(v) \neq b_0/2$; and hence one can so choose v' that neither of the other terms of (33) has a pole at $v = -b_0/2$. Thus, $2v + b_0$ divides $vQ(v) + P(v)$.

If either $P(v)$ or $Q(v)$ has degree 3, we can so choose v' in (35) that $v'Q(v) + P(v)$ has degree 3 and also has no factor in common with the numerator. The term (35) then has three poles; but the other terms in (33), considered together, have only two. Thus, P and Q have degree ≤ 2 .

The two poles in the first and last term of (33) are distinct except when v' is so chosen that $b(v') = v' + b_0$. If this holds for many v' , then

$$(36) \quad b(v') \equiv v' + b_0.$$

Otherwise the poles are distinct for many v' , and therefore

$$(37) \quad v'Q(v) + P(v) \equiv k[v + b(v')][v + v' + b_0]$$

when we make the poles (35) correspond with the others in (33). Solving for $b(v')$ with v held fast we see that P and Q have degree ≤ 1 . With v' fixed we deduce $k = 0$ in (37), since otherwise the right-hand side has a term kv^2 . It is impossible that $k = 0$, however, since v' is arbitrary on the left. Hence (36) holds. Returning to the original expression (30) by way of (32) we obtain the following:

Theorem IV. Suppose the fraction of power absorbed by a given material in a cavity depends only on the volume v , and always has the form $av/(v + b)$ where a, b are constant for a given material and cavity. Then the fraction of power entering into a volume v when a second volume v' is present is given by

$$(38) \quad f(v, v') = \frac{a_0 v}{v + v' + b_0}$$

where a_0 and b_0 are constant. This form satisfies the two conditions of the hypothesis, both conditions being taken in the strong sense of the text.

Discussion. Experimentally, one finds that the difference in temperatures of v and v' is proportional to the difference in volumes, provided the sum of the volumes is constant. Since the heat input in calories is proportional to the volume times the temperature rise, the temperature is proportional to $f(v, v')/v$. The foregoing remark yields a new functional equation

$$(39) \quad f(v, v')/v - f(v', v)/v' = \lambda(v + v')(v - v').$$

Equation (39) is satisfied by the form (38) with $\lambda \equiv 0$. To obtain a nonzero λ we must abandon one of the assumptions leading to Theorem IV. Since independence of shape is essential to the mere use of the notation $f(v, v')$, we retain (27).

A necessary and sufficient condition that $f(v, v')$ satisfy both (39) and (27) is that $f(v, v')$ have the form

$$(40) \quad f(v, v') = v\phi(v + v') + \frac{vv'(v - v')}{v + v'} \lambda(v + v');$$

we omit the trivial proof. Setting $v' = 0$ yields

$$(41) \quad v\phi(v) = f(v, 0).$$

Assuming that (28) holds but not (30) we obtain

$$(42) \quad f(v, v') = \frac{a_0 v}{v + v' + b_0} + \frac{c_0 vv'(v - v')}{(v + v')^2(d_0 + v + v')}$$

upon taking $v\lambda(v) = c_0/(d_0 + v)$. Equation (42) has been experimentally verified, with $a_0 = 1$, $b_0 = 40$, $c_0 = \frac{1}{4}$, $d_0 = 20$. If $v + v' = s$ the second term (42) does not exceed

$$\frac{\sqrt{3}}{12} \frac{c_0 s}{d_0 + s}$$

which is less than $0.002s$ in the specific case cited. Since the first term (42) coincides with the result of Theorem IV, the second term measures the incompatibility of the assumptions.

REFERENCES

- [1] STRATTON, J. A., "Electromagnetic Theory", McGraw-Hill Book Co., Inc. N. Y., 1941.
- [2] Radiation Laboratory Series: "The Measurement of Dielectric Constants", Technique of Microwave Measurement, Ch. X, McGraw-Hill Book Co., Inc. N. Y., 1947.
- [3] REDHEFFER, R. M., "Equivalences and Other Relations for Dielectric Media", Research Lab. of Electronics, M.I.T., Technical Report No. 29, 1946.
- [4] REDHEFFER, R. M., "Remarks on the Basis of Network Theory", JI. of Math. and Phys., Vol. **28**, 1950.
- [5] BARRAR, R. & REDHEFFER, R. M., "Inhomogeneous Dielectric Media", submitted to the Journal of Applied Physics.