*Singular Perturbations of Non-linear Systems of
Differential Equations and an Associated
Boundary Layer Equation**

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1. Introduction. We are concerned with showing the relationship of the solutions of a certain “full” system of differential equations (1.1), as $\epsilon \rightarrow 0+$, to a particular solution of a related “degenerate” system (1.2). The systems are

$$(1.1) \quad \begin{aligned} \frac{dx}{dt} &= f(x, y, t, \epsilon) \\ \epsilon^r \frac{dy}{dt} &= g(x, y, t, \epsilon) \end{aligned}$$

$$(1.2) \quad \begin{aligned} \frac{dx}{dt} &= f(x, y, t, 0) \\ 0 &= g(x, y, t, 0) \end{aligned}$$

where x and f are real vectors of dimension $n_0 \geq 1$, y and g are real vectors of dimension $n \geq 1$, $\epsilon > 0$, and $r > 0$ is a fixed constant.

* This paper was written in the course of work sponsored by the Office of Naval Research.

Under the assumption of three hypotheses, H1, H2, and H3, we shall prove Theorem 1.1 and its immediate consequence Theorem 1.2. H1 concerns itself with the existence of a solution of (1.2), H2 with smoothness conditions on f and g , and H3 with the nature of the characteristic roots of a certain critical matrix.

Before giving these hypotheses we introduce some conventional notation that is used throughout. If $Z = \text{col}(Z_1, Z_2, \dots, Z_\alpha)$, where the Z_i are the scalar components of an α dimensional complex vector Z , then $|Z| = \sum_{i=1}^{\alpha} |Z_i|$. If $A = (a_{ij})$, where the a_{ij} are the scalar components of an α by β complex matrix, then $|A| = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} |a_{ij}|$.

The following terminology is used also. Let $h = h(x, y, t, \epsilon)$ be a real valued vector function of dimension $\alpha \geq 1$, where x and y are real vectors of dimension $\beta \geq 1$ and $\gamma \geq 1$ respectively, and where t and ϵ are real scalars. Then " $h \in C$ in all variables" means that (in some suitable region) each component of h is a continuous function of (x_i, y_j, t, ϵ) where $i = 1, 2, \dots, \beta$ and $j = 1, 2, \dots, \gamma$. By h_x we mean the α by β matrix

$$h_x = \left(\frac{\partial h_i(x, y, t, \epsilon)}{\partial x_j} \right); \quad i = 1, 2, \dots, \alpha; \quad j = 1, 2, \dots, \beta.$$

Similarly, by h_y we mean the α by γ matrix

$$h_y = \left(\frac{\partial h_i(x, y, t, \epsilon)}{\partial y_j} \right); \quad i = 1, 2, \dots, \alpha; \quad j = 1, 2, \dots, \gamma.$$

A statement such as " $h_x \in C$ in all variables" means that (in some suitable region) h_x exists and that each component of h_x is a continuous function of all the variables.

H1: *The degenerate system (1.2) is assumed to possess a solution $x = \phi(t)$, $y = \psi(t)$, which we shall henceforth denote as $\phi(t)$, $\psi(t)$, which is continuously differentiable on the closed finite interval $a \leq t \leq b$.*

In order to state H2 we first define a region, R , in which we shall work throughout.

$$R: \quad |x - \phi(t)| < \delta_0, \quad |y - \psi(t)| < \delta_0, \quad a \leq t \leq b, \quad 0 \leq \epsilon < \delta_0, \\ \text{where } \delta_0 > 0.$$

H2: This is merely a smoothness hypothesis on f and g .

- (i) *In R : f, f_x , and $f_y \in C$ with respect to (x, y, t, ϵ) .*
- (ii) *Same as (i) with g replacing f .*

Remark. It will be evident from the proof of Theorem 1.1 that some of the differentiability hypotheses on f and g can be weakened to Lipschitz conditions without any change in the proof.

From H2 we have that g_y exists in R . We define the n by n matrix $g_\psi = g_\psi(t)$ as $g_\psi = g_\psi(\phi(t), \psi(t), t, 0)$; thus

$$g_\psi = \left(\frac{\partial g_i}{\partial y_j}(\phi(t), \psi(t), t, 0) \right), \quad i, j = 1, 2, \dots, n.$$

We note that H1 and H2 insure us that this matrix is continuous on $a \leq t \leq b$.

H3: This is the crucial hypothesis. *We assume that the real parts of all of the characteristic roots of g_ψ are $\leq -\mu < 0$ on $a \leq t \leq b$.*

Theorem 1.1. *H1, H2, and H3 imply that if*

$$\epsilon + |\Phi(a, \epsilon) - \phi(a)| + |\Psi(a, \epsilon) - \psi(a)|$$

is sufficiently small, then (1.1) has a unique solution $x(t, \epsilon)$, $y(t, \epsilon)$ satisfying the initial condition

$$x(a, \epsilon) = \Phi(a, \epsilon), \quad y(a, \epsilon) = \Psi(a, \epsilon).$$

Furthermore, $x(t, \epsilon)$, $y(t, \epsilon)$ exists over the entire interval $a \leq t \leq b$ and

$$|x(t, \epsilon) - \phi(t)| + |y(t, \epsilon) - \psi(t)| \rightarrow 0$$

uniformly over $a \leq t \leq b$ as

$$\epsilon + |\Phi(a, \epsilon) - \phi(a)| + |\Psi(a, \epsilon) - \psi(a)| \rightarrow 0.$$

Using Theorem 1.1 and certain inequalities obtained in its proof, we are able to consider the case in which the initial point for the full system does not approach the initial point for the degenerate system. The result is Theorem 1.2 which is given in Section 3.

In Section 4 we consider the "boundary layer equation" associated with (1.1). We use this equation both to improve Theorem 1.2 (to Theorem 4.1) and to obtain more information about the solution of (1.1) in the neighborhood of the boundary.

Analogs of Theorems 1.1 and 1.2 are later obtained for more general systems; see (5.1).

Problems of this type have been discussed previously in [1, 3, 4, 5]. Friedrichs & Wasow in [1] discuss the special case of (1.1) and (1.2) in which y is a vector of dimension one, *i.e.* a scalar. Our crucial hypothesis, H3, coincides with their hypothesis in this special case. Levinson in [3] discusses the case in which the solution of the degenerate system is discontinuous.

Closely related to our work are the results of Tihonov [5]. Corresponding to our really crucial hypothesis, H3, Tihonov uses the condition that a certain function be negative definite in a neighborhood of the surface on which lie the solutions of the degenerate system. Tihonov's negative definite function is

actually a quadratic form (whose associated matrix corresponds to our g_ψ) plus higher order terms. Now, in Tihonov's work, the quadratic form can be identically zero provided that his function, which in this case has only higher order terms, is negative definite—this is an advantage of his results over ours (*i.e.*, the ability to consider higher order terms). However, if the quadratic form is not identically zero, then Tihonov must have it negative definite, which by a result from algebra implies that the real parts of all the characteristic roots of the associated matrix are all negative. Thus, if the quadratic form is not identically zero, our results include Tihonov's. However, Tihonov's results do not include ours, for by very elementary examples one can show that having the real parts of all the characteristic roots of a matrix negative does not imply that the associated quadratic form is negative definite (it need not, in fact, be even negative). For example, if

$$A = \begin{pmatrix} -1 & 0 \\ 10 & -1 \end{pmatrix}$$

the characteristic roots of A are obviously both -1 ; however, the quadratic form associated with A is $-x^2 + 10xy - y^2$, which is not negative.

2. Preliminary lemmas. The following lemmas will be used repeatedly in our proofs.

Lemma 2.1. *Let $f(t)$ and $a(t)$ be real integrable functions of t on $t_0 \leq t \leq t_1$, and let b be any real positive number. If*

$$(2.1) \quad f(t) \leq a(t) + b \int_{t_0}^t f(\tau) d\tau, \quad t_0 \leq t \leq t_1,$$

then

$$(2.2) \quad f(t) \leq a(t) + b \int_{t_0}^t a(\tau) e^{b(t-\tau)} d\tau, \quad t_0 \leq t \leq t_1.$$

Proof. Define $H(t) = \int_{t_0}^t f(\tau) d\tau$, $t_0 \leq t \leq t_1$. Then

$$\frac{dH}{dt} - bH \leq a(t),$$

and

$$\frac{d}{dt} \{ H e^{-bt} \} \leq a(t) e^{-bt}$$

Hence,

$$H(t) \leq \int_{t_0}^t a(\tau) e^{b(t-\tau)} d\tau.$$

This is easily seen to yield the result.

A corollary of Lemma 2.1, which we shall also use, is the following lemma.

Lemma 2.2. *Let $\zeta(t)$ be a real vector function of t possessing an integrable derivative on $t_0 \leq t \leq t_1$. Let $a(t)$ be a real integrable function of t on $t_0 \leq t \leq t_1$, and let b be any real positive number. If*

$$(2.3) \quad \left| \frac{d\zeta}{dt} \right| \leq a(t) + b |\zeta|, \quad t_0 \leq t \leq t_1,$$

then

$$(2.4) \quad |\zeta(t)| \leq |\zeta(t_0)| e^{b(t-t_0)} + \int_{t_0}^t a(\tau) e^{b(t-\tau)} d\tau, \quad t_0 \leq t \leq t_1.$$

Proof. Integrating (2.3) we obtain

$$\int_{t_0}^t \left| \frac{d\zeta}{dt} \right| d\tau \leq \int_{t_0}^t a(\tau) d\tau + b \int_{t_0}^t |\zeta(\tau)| d\tau.$$

Hence,

$$|\zeta(t)| \leq |\zeta(t_0)| + \int_{t_0}^t a(\tau) d\tau + b \int_{t_0}^t |\zeta(\tau)| d\tau, \quad t_0 \leq t \leq t_1.$$

Applying Lemma 2.1 to this last inequality we obtain, after an elementary calculation, the desired result.

Lemma 2.3 (Variation of Constants Formula). *Given the two systems (2.5) and (2.6), where A is a constant n by n matrix, and $b(t) \in C(t_0 \leq t \leq t_1)$, $b = \text{col}(b_1, b_2, \dots, b_n)$.*

$$(2.5) \quad \frac{dy}{dt} = Ay + b(t),$$

$$(2.6) \quad \frac{dx}{dt} = Ax.$$

Then $e^{A(t-t_0)}$ is a fundamental matrix for (2.6), and

$$y(t) = e^{A(t-t_0)} y_0 + \int_{t_0}^t e^{A(t-\tau)} b(\tau) d\tau, \quad t_0 \leq t \leq t_1$$

is the solution of (2.5) for which $y(t_0) = y_0$.

Proof. See [2], page 63.

Lemma 2.4. Let A be any n by n constant (complex) matrix satisfying the hypothesis:

(i) The characteristic roots λ_i , $i = 1, 2, \dots, n$, of A are such that $\Re \lambda_i \leq -\mu < 0$.

(ii) $|A| \leq M$.

Then, if σ is any constant satisfying $0 < \sigma < \mu$, it follows that

$$|e^{At}| \leq Ke^{-\sigma t} \text{ for } t \geq 0,$$

where K is a positive constant depending only on μ , σ , and M .

Proof. We first remark that this lemma is a generalization of the well-known fact that under the same hypothesis $|e^{At}| \leq Ke^{-\sigma t}$ for $t \geq 0$, where K depends only on μ , σ , and A .

Suppose the lemma is false. Then there exists a sequence $\{A_k\}$, $k = 1, 2, \dots$, satisfying (i) and (ii) such that

$$(2.7) \quad \max_{0 \leq t < \infty} |e^{\sigma t} e^{A_k t}| \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Because $|A_k| \leq M$ it follows that there exists a convergent subsequence which can again be denoted by $\{A_k\}$. That is, there exists an A_0 such that $|A_k - A_0| \rightarrow 0$ as $k \rightarrow \infty$. Clearly A_0 satisfies (ii) and since the characteristic roots of A_k must tend to those of A_0 , it follows that A_0 satisfies (i).

Let σ_1 be chosen so that $\mu > \sigma_1 > \sigma$. Then by the remark at the beginning of the proof there exists K_1 such that

$$(2.8) \quad |e^{A_0 t}| \leq K_1 e^{-\sigma_1 t}, \quad t > 0.$$

Let δ be chosen so that $K_1 \delta < \sigma_1 - \sigma$, and let the matrix B satisfy $|B - A_0| < \delta$. Then e^{Bt} is a solution of $x' = A_0 x + (B - A_0)x$. Thus

$$e^{Bt} = e^{A_0 t} + \int_0^t e^{A_0(t-\tau)}(B - A_0)e^{B\tau} d\tau,$$

and using (2.8) we obtain

$$|e^{Bt}| \leq K_1 e^{-\sigma_1 t} + \delta K_1 \int_0^t e^{-\sigma_1(t-\tau)} |e^{B\tau}| d\tau.$$

Let $u(t) = |e^{Bt}| e^{\sigma_1 t}$. Then

$$u(t) \leq K_1 + \delta K_1 \int_0^t u(\tau) d\tau.$$

Thus by Lemma 2.1, $u(t) \leq K_1 e^{\delta K_1 t}$ and

$$|e^{Bt}| \leq K_1 e^{-(\sigma_1 - \delta K_1)t}, \quad t > 0.$$

Since $K_1 \delta < \sigma_1 - \sigma$, we obtain

$$|e^{Bt}| \leq K_1 e^{-\sigma t}, \quad t > 0.$$

By putting $B = A_k$ and taking k large enough so that $|A_k - A_0| < \delta$, we see that (2.7) is impossible.

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1: Let $(\Phi(a, \epsilon), \Psi(a, \epsilon), a, \epsilon)$ belong to R . Then it is clear from the fundamental existence and uniqueness theorems that there exists on some interval, whose length may depend on ϵ , to the right of $t = a$, a unique solution $x(t, \epsilon), y(t, \epsilon)$ of (1.1) with $x(a, \epsilon) = \Phi(a, \epsilon), y(a, \epsilon) = \Psi(a, \epsilon)$ as initial condition. It is part of the proof to show that if $(\Phi(a, \epsilon), \Psi(a, \epsilon), a, \epsilon)$ is sufficiently close to $(\phi(a), \psi(a), a, 0)$ then this interval extends all the way to $t = b$. We shall not be able to show this in one step.

Suppose t_0 satisfies $a \leq t_0 < b$, and let $(\Phi(t_0, \epsilon), \Psi(t_0, \epsilon), t_0, \epsilon)$ belong to R . Then it is clear, as in the preceding paragraph, that there exists on some interval to the right of $t = t_0$ (also to the left of $t = t_0$, but we make no use of this), whose length may depend on t_0 and on ϵ , a unique solution $x(t, \epsilon), y(t, \epsilon)$ of (1.1) with $x(t_0, \epsilon) = \Phi(t_0, \epsilon), y(t_0, \epsilon) = \Psi(t_0, \epsilon)$ as initial condition.

The plan of the proof is to prove Theorem 1.1 for a particular closed subinterval, $I(t_0)$, of $a \leq t \leq b$ with t_0 as left-hand end point. It will be seen from the proof that $l(t_0)$, the length of $I(t_0)$, does not depend on t_0 or on ϵ , so that we can span $a \leq t \leq b$ with a finite number of such intervals $I(t_0)$. Once we have this the proof is (essentially) completed.

In order to avoid the appearance of a large number of unnecessary constants we shall adopt the following convention. The letter K shall always denote a finite positive real number which does not depend on t_0 or on ϵ . Furthermore, while K may represent different constants in different formulae, K will be a well-determined number at each stage of the proof (this is exceedingly important).

Also in the interest of simplicity of notation, we shall adopt still another convention (which is similar to the use of K just described). The function $\omega = \omega(\epsilon)$ shall always denote a non-negative continuous function of ϵ , defined for ϵ sufficiently small (exactly how small will always be stated, and will be seen never to depend on t_0), such that $\omega(0) = 0$. Also $\omega(\epsilon)$ will never depend on t_0 ; furthermore, $\omega(\epsilon)$ will be a well-determined function at each stage of the proof.

Let t_0 be any fixed point of $a \leq t < b$. Further, let $(\Phi(t_0, \epsilon), \Psi(t_0, \epsilon), t_0, \epsilon)$ belong to R . As above we denote by $x(t, \epsilon), y(t, \epsilon)$ the solution of (1.1) existing on some closed subinterval of $a \leq t \leq b$ with t_0 as left end point and satisfying $x(t_0, \epsilon) = \Phi(t_0, \epsilon), y(t_0, \epsilon) = \Psi(t_0, \epsilon)$. Define $\xi = \xi(t, \epsilon)$ and $\eta = \eta(t, \epsilon)$ by

$$(3.1) \quad \begin{aligned} x &= \phi + \xi, \\ y &= \psi + \eta. \end{aligned}$$

It is easily seen that ξ, η satisfy the system

$$(3.2) \quad \begin{aligned} \frac{d\xi}{dt} &= f(\phi + \xi, \psi + \eta, t, \epsilon) - f(\phi, \psi, t, 0), \\ \epsilon^r \frac{d\eta}{dt} &= g(\phi + \xi, \psi + \eta, t, \epsilon) - \epsilon^r \frac{d\psi}{dt}. \end{aligned}$$

The notation $A = g_\psi(t_0)$ is used in the following. We rewrite the second equation of (3.2) as

$$(3.3) \quad \epsilon^r \frac{d\eta}{dt} = A\eta + (g_\psi - A)\eta + G_1 + G_2$$

where

$$G_1 = g(\phi + \xi, \psi + \eta, t, \epsilon) - g(\phi, \psi + \eta, t, 0) - \epsilon^r \frac{d\psi}{dt}$$

$$G_2 = g(\phi, \psi + \eta, t, 0) - g(\phi, \psi, t, 0) - g_\psi \eta.$$

We have used $g(\phi, \psi, t, 0) = 0$. Dividing both sides of (3.3) by ϵ^r and applying the variation of constants formula to the resulting equation yields

$$(3.4) \quad \begin{aligned} \eta(t, \epsilon) &= e^{A(t-t_0)/\epsilon^r} \eta(t_0, \epsilon) + \frac{1}{\epsilon^r} \int_{t_0}^t e^{A(t-\tau)/\epsilon^r} (g_\psi - A)\eta \, d\tau \\ &\quad + \frac{1}{\epsilon^r} \int_{t_0}^t e^{A(t-\tau)/\epsilon^r} (G_1 + G_2) \, d\tau. \end{aligned}$$

Equation (3.4) is, of course, valid as long as $x(t, \epsilon), y(t, \epsilon)$ remains in R . Let σ satisfy $0 < \sigma < \mu$; hold σ fixed for the remainder of the proof. Applying H3 and Lemma 2.4 to (3.4) we easily obtain

$$(3.5) \quad \begin{aligned} |\eta(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/\epsilon^r} |\eta(t_0, \epsilon)| + \frac{K}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |g_\psi - A| |\eta| \, d\tau \\ &\quad + \frac{K}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} (|G_1| + |G_2|) \, d\tau. \end{aligned}$$

Inequality (3.5) is valid for $t \geq t_0$ as long as $x(t, \epsilon), y(t, \epsilon)$ remains in R . From H2 we know that there exists a constant $M > 0$ such that $|g_\psi(t)| \leq M$, $a \leq t \leq b$. Therefore, from Lemma 2.4, the constant K appearing in (3.5) is independent of t_0 .

We shall now appraise the various terms on the right-hand side of (3.5). In order to obtain the estimates we need, we shall have to restrict t to a sub-interval, $I(t_0)$ (of as yet undetermined length, $l(t_0)$) of $a \leq t \leq b$ with t_0 as left-hand end point. We shall also have to restrict the size of $|\xi|$, $|\eta|$, and ϵ . It will, of course, have to be demonstrated that taking $|\xi(t_0, \epsilon)|$, $|\eta(t_0, \epsilon)|$, and ϵ small insures that $|\xi|$ and $|\eta|$ remain small.

Let $I(t_0)$ be a closed subinterval of $a \leq t \leq b$ with t_0 as left-hand end point and of, as yet undetermined, length $l(t_0)$. Let ϵ_1 satisfy $0 < \epsilon_1 \leq 1$. We shall eventually choose ϵ_1 sufficiently small so that $\sigma - K\epsilon_1 \geq \frac{1}{2}\sigma$ for an appropriate K that arises in the computation. Let δ_1 satisfy $0 < \delta_1 < \delta_0$; hold δ_1 fixed throughout the entire proof. It is clear from H2 that there exists a $K > 0$ and an $\omega(\epsilon)$ such that $|G_1| \leq K(|\xi| + \omega(\epsilon))$ if $|\xi| \leq \delta_1$, $|\eta| \leq \delta_1$, $0 \leq \epsilon \leq \delta_1$, $a \leq t \leq b$. There exists a $\delta_2 = \delta_2(\epsilon_1) > 0$ such that $|g_\psi - A| \leq \epsilon_1$ for $t \in I(t_0)$ if $l(t_0) \leq \delta_2$. From the uniform continuity of $g_\psi(t)$ on $a \leq t \leq b$, δ_2 does not depend on t_0 .

Because of our smoothness hypothesis on g , an elementary application of the mean value theorem yields $|G_2| \leq \epsilon_1 |\eta|$ if $|\eta| \leq \delta_3$ and $a \leq t \leq b$ for some $\delta_3 = \delta_3(\epsilon_1)$ satisfying $0 < \delta_3 \leq \delta_1$. K will not depend on δ_2 or δ_3 , which have not yet been fixed. Thus, we obtain from (3.5)

$$(3.6) \quad \begin{aligned} |\eta(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/\epsilon^r} |\eta(t_0, \epsilon)| + \frac{K}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} (|\xi| + \omega(\epsilon)) d\tau \\ &\quad + \frac{K\epsilon_1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |\eta| d\tau \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

Until now our attention has been confined mainly to the second equation of (3.2). At this point, however, we employ the first equation of (3.2) to simplify (3.6). We shall in fact now prove

$$(3.7) \quad \begin{aligned} \frac{1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |\xi(\tau, \epsilon)| d\tau &\leq \frac{1}{\sigma} |\xi(t, \epsilon)| + K\epsilon^r \\ \text{if } t_0 \leq t \leq b, \quad |\xi| &\leq \delta_1, \quad |\eta| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1. \end{aligned}$$

From the first equation of (3.2) and H2 it is clear that there exists a $K > 0$ such that

$$(3.8) \quad \begin{aligned} \left| \frac{d\xi}{dt} \right| &\leq K \\ \text{if } a \leq t \leq b, \quad |\xi| &\leq \delta_1, \quad |\eta| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1. \end{aligned}$$

A simple application of the mean value theorem together with (3.8) yields

$$(3.9) \quad \begin{aligned} |\xi(t, \epsilon) - \xi(\tau, \epsilon)| &\leq K |t - \tau| \\ \text{if } a \leq t, \tau \leq b, \quad |\xi| &\leq \delta_1, \quad |\eta| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1 \end{aligned}$$

Define

$$(3.10) \quad J = \frac{1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |\xi(\tau, \epsilon)| d\tau.$$

Then

$$(3.11) \quad J \leq J_1 + J_2$$

where

$$(3.12) \quad \begin{aligned} J_1 &= \frac{|\xi(t, \epsilon)|}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} d\tau \\ J_2 &= \frac{1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |\xi(\tau, \epsilon) - \xi(t, \epsilon)| d\tau. \end{aligned}$$

An elementary integration yields

$$(3.13) \quad J_1 \leq \frac{1}{\sigma} |\xi(t, \epsilon)|.$$

From (3.9) we have

$$(3.14) \quad J_2 \leq \frac{K}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} (t - \tau) d\tau.$$

The change of variables $s = \sigma(t - \tau)/\epsilon^r$ yields

$$(3.15) \quad J_2 \leq K\epsilon^r.$$

However, from (3.13) and (3.15) we see that we have proven (3.7).

An elementary integration yields

$$(3.16) \quad \frac{1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} \omega(\epsilon) d\tau \leq \frac{1}{\sigma} \omega(\epsilon), \quad t \geq t_0.$$

Substituting (3.7) and (3.16) into (3.6) yields

$$(3.17) \quad \begin{aligned} |\eta(t, \epsilon)| &\leq Ke^{-\sigma(t-t_0)/\epsilon^r} |\eta(t_0, \epsilon)| + K(|\xi(t, \epsilon)| + \omega(\epsilon)) \\ &\quad + \frac{K\epsilon_1}{\epsilon^r} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^r} |\eta| d\tau \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

Multiplying both sides of (3.17) by $e^{\sigma t/\epsilon^r}$ and letting $F = F(t, \epsilon) = e^{\sigma t/\epsilon^r} |\eta(t, \epsilon)|$ yields

$$F(t, \epsilon) \leq Ke^{\sigma t_0/\epsilon^r} |\eta(t_0, \epsilon)| + Ke^{\sigma t/\epsilon^r} (|\xi(t, \epsilon)| + \omega(\epsilon)) + \frac{K\epsilon_1}{\epsilon^r} \int_{t_0}^t F(\tau, \epsilon) d\tau.$$

Applying Lemma 2.1 to this last inequality we obtain

$$F(t, \epsilon) \leq K e^{\sigma t_0 / \epsilon^r} |\eta(t_0, \epsilon)| + K e^{\sigma t / \epsilon^r} (|\xi(t, \epsilon)| + \omega(\epsilon)) + \frac{K \epsilon_1}{\epsilon^r} \int_{t_0}^t e^{K \epsilon_1 (t-\tau) / \epsilon^r} \cdot \{K e^{\sigma t_0 / \epsilon^r} |\eta(t_0, \epsilon)| + K e^{\sigma \tau / \epsilon^r} (|\xi(\tau, \epsilon)| + \omega(\epsilon))\} d\tau,$$

from which we easily obtain

$$(3.18) \quad \begin{aligned} |\eta(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/\epsilon^r} |\eta(t_0, \epsilon)| + K (|\xi(t, \epsilon)| + \omega(\epsilon)) \\ &\quad + \frac{K \epsilon_1}{\epsilon^r} \int_{t_0}^t e^{K \epsilon_1 (t-\tau) / \epsilon^r} e^{-\sigma(t-t_0)/\epsilon^r} |\eta(t_0, \epsilon)| d\tau \\ &\quad + \frac{K \epsilon_1}{\epsilon^r} \int_{t_0}^t e^{-(\sigma-K \epsilon_1)(t-\tau)/\epsilon^r} (|\xi(\tau, \epsilon)| + \omega(\epsilon)) d\tau \end{aligned}$$

$$\text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3.$$

We now choose $\epsilon_1 > 0$ sufficiently small so that $\sigma - K \epsilon_1 \geq \frac{1}{2} \sigma$ (K here stands for the one in (3.18) and is, of course, perfectly well determined); ϵ_1 is held fixed from now on at this value. This determines a δ_2 and a δ_3 which are held fixed from now on. An elementary integration together with an inequality similar to (3.7) yields

$$(3.19) \quad \begin{aligned} |\eta(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/2\epsilon^r} |\eta(t_0, \epsilon)| + K (|\xi(t, \epsilon)| + \omega(\epsilon)) \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

Returning to the first equation of (3.2), we observe that $H2$ implies that there exists a $K > 0$ and an $\omega(\epsilon)$ such that

$$(3.20) \quad \left| \frac{d\xi}{dt} \right| \leq K (|\xi| + |\eta| + \omega(\epsilon))$$

$$\text{for } t_0 \leq t \leq b \text{ if } |\xi| \leq \delta_1, \quad |\eta| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1.$$

Substituting (3.19) into (3.20) yields

$$(3.21) \quad \left| \frac{d\xi}{dt} \right| \leq K |\xi| + K (e^{-\sigma(t-t_0)/2\epsilon^r} |\eta(t_0, \epsilon)| + \omega(\epsilon))$$

$$\text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3.$$

Applying Lemma 2.2 to (3.21) yields

$$(3.22) \quad \begin{aligned} |\xi(t, \epsilon)| &\leq |\xi(t_0, \epsilon)| e^{K(t-t_0)} \\ &\quad + K \int_{t_0}^t \{e^{-\sigma(r-t_0)/2\epsilon^r} |\eta(t_0, \epsilon)| + \omega(\epsilon)\} e^{K(t-r)} d\tau \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3, \end{aligned}$$

which easily yields

$$(3.23) \quad \begin{aligned} & |\xi(t, \epsilon)| \leq |\xi(t_0, \epsilon)| e^{K(t-t_0)} + K\omega(\epsilon) \\ & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

From (3.19) and (3.23) we obtain

$$(3.24) \quad \begin{aligned} & |\xi(t, \epsilon)| \leq |\xi(t_0, \epsilon)| e^{K(t-t_0)} + K\omega(\epsilon) \\ & |\eta(t, \epsilon)| \leq K e^{-\sigma(t-t_0)/2\epsilon^r} |\eta(t_0, \epsilon)| + K(|\xi(t_0, \epsilon)| + \omega(\epsilon)) \\ & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta| \leq \delta_3, \quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

Recall that δ_2 and δ_3 have already been determined. Now we simply take $l(t_0) = \delta_2$. It is easily seen that (3.24) proves Theorem 1.1 for the interval $I(t_0)$. As $l(t_0)$ does not depend on t_0 or on ϵ we can span $a \leq t \leq b$ with a finite number of such intervals, $I(t_0)$. An elementary argument now gives us the result for $a \leq t \leq b$. This completes the proof of Theorem 1.1.

Theorem 1.2. H1, H2, and H3 imply that there exists a $\delta > 0$, which is independent of ϵ , such that if $|\Psi(a) - \psi(a)| \leq \delta$ (where $\Psi(a)$ is a constant vector), then for sufficiently small ϵ (1.1) has a unique solution $x(t, \epsilon)$, $y(t, \epsilon)$ satisfying the initial condition $x(a, \epsilon) = \phi(a)$, $y(a, \epsilon) = \Psi(a)$. Furthermore, $x(t, \epsilon)$, $y(t, \epsilon)$ exists over the entire interval $a \leq t \leq b$ and $|x(t, \epsilon) - \phi(t)| + |y(t, \epsilon) - \psi(t)| \rightarrow 0$ as $\epsilon \rightarrow 0+$ uniformly over $c \leq t \leq b$ for any $c > a$.

Proof of Theorem 1.2. The proof follows directly from Theorem 1.1 and (3.24). We need only observe that for $t \in I(a)$ we have from (3.24)

$$(3.25) \quad \begin{aligned} & |\xi(t, \epsilon)| \leq K\omega(\epsilon), \\ & |\eta(t, \epsilon)| \leq K |\Psi(a) - \psi(a)| e^{-\sigma(t-a)/2\epsilon^r} + K\omega(\epsilon). \end{aligned}$$

We now apply Theorem 1.1 to the interval extending from the right-hand end point of $I(a)$ to b . The remainder of the proof is evident.

4. Boundary layer equation. We return to the systems (1.1) and (1.2), which are

$$(4.1) \quad \begin{aligned} \frac{dx}{dt} &= f(x, y, t, \epsilon) \\ \epsilon \frac{dy}{dt} &= g(x, y, t, \epsilon) \end{aligned}$$

$$(4.2) \quad \begin{aligned} \frac{dx}{dt} &= f(x, y, t, 0) \\ 0 &= g(x, y, t, 0). \end{aligned}$$

All notation is as previously introduced.

We are interested in the case in which the initial point for (4.1) is fixed. If $x(t, \epsilon)$, $y(t, \epsilon)$ is a solution of (4.1) then the initial value is again given by

$$(4.3) \quad \begin{aligned} x(a, \epsilon) &= \phi(a) \\ y(a, \epsilon) &= \Psi(a), \end{aligned}$$

where $\Psi(a)$ is a constant vector. Theorem 1.2 is of course applicable to the present case. However, for Theorem 1.2 to be valid $|\Psi(a) - \psi(a)|$ will in general have to be quite small. One of the results of this section, namely Theorem 4.1, is an improvement on Theorem 1.2 in the direction of having a more natural bound on $|\Psi(a) - \psi(a)|$. We shall also investigate the boundary layer equation, namely equation (4.5), for (4.1) which gives us more information on $y(t, \epsilon)$ in the neighborhood of $t = a$ than is obtainable from Theorems 1.1 and 1.2. The boundary layer equation will be used to establish Theorem 4.1.

Besides H1, H2, and H3 we shall also use the hypothesis H4 which will be given shortly. In the following $h(t - a)$ will always denote a well-determined continuous non-negative function of $t - a$ for $t \geq a$ but sufficiently close to a and $h(0) = 0$; however, in different inequalities $h(t - a)$ may denote different functions. It might be mentioned that if we strengthened H2, namely if we put more stringent smoothness conditions on f and g we could obtain somewhat more precise results; in particular, we could obtain the form of the continuous functions $h(t - a)$ and $\omega(\epsilon)$.

H4: The unique solution, $\gamma = \gamma(s)$, of

$$(4.4) \quad \frac{d\gamma}{ds} = g(\phi(a), \psi(a) + \gamma, a, 0)$$

satisfying the initial condition $\gamma(0) = \Psi(a) - \psi(a)$ exists for all $s \geq 0$ and $|\gamma(s)| \rightarrow 0$ as $s \rightarrow \infty$. There is a $\delta_0 > 0$ and an $\bar{\epsilon}_0 > 0$ such that in the (x, y, t, ϵ) region given by $|x - \phi(a)| < \delta_0$, $|y - \gamma(s) - \psi(a)| < \delta_0$ where $0 \leq s < \infty$, $|t - a| < \delta_0$, $0 \leq \epsilon \leq \bar{\epsilon}_0$, f and g and f_x, g_x, f_y, g_y are of class C in (x, y, t, ϵ) .

The solution $\gamma \equiv 0$ of (4.4) is asymptotically stable because the real parts of all of the characteristic roots of $g_\psi(\phi(a), \psi(a), a, 0)$ are $\leq -\mu < 0$. It follows from a well-known theorem on stability that once $|\gamma(s)|$ gets small then $|\gamma(s)| \rightarrow 0$ as $s \rightarrow \infty$. Thus part of H4 is a consequence of H3.

Clearly (4.1) has a solution $x(t, \epsilon)$, $y(t, \epsilon)$ satisfying the initial condition $x(a, \epsilon) = \phi(a)$, $y(a, \epsilon) = \Psi(a)$ and existing on some interval (whose length may depend on ϵ) to the right of $t = a$. In what follows this is what is meant by $x(t, \epsilon)$, $y(t, \epsilon)$.

It is intuitively evident, and will later be proven, that in the neighborhood of $t = a$, $y(t, \epsilon)$ is close to the solution $\bar{y}(t, \epsilon)$ of

$$(4.5) \quad \epsilon^r \frac{d\bar{y}}{dt} = g(\phi(a), \bar{y}, a, 0)$$

satisfying the initial condition $\bar{y}(a, \epsilon) = \Psi(a)$. Equation (4.5) is called the "boundary layer equation."

Define $\gamma = \gamma(t, \epsilon)$ by

$$(4.6) \quad \bar{y} = \psi(a) + \gamma.$$

Then γ satisfies the equation

$$(4.7) \quad \epsilon^\tau \frac{d\gamma}{dt} = g(\phi(a), \psi(a) + \gamma, a, 0)$$

and the initial condition

$$(4.8) \quad \gamma(a, \epsilon) = \Psi(a) - \psi(a).$$

Making the change of variables

$$(4.9) \quad s = (t - a)/\epsilon^\tau$$

we reduce equation (4.7) to the form

$$(4.10) \quad \frac{d\gamma}{ds} = g(\phi(a), \psi(a) + \gamma, a, 0)$$

which is the equation of hypothesis H4. Thus, for $t > a$ it follows that $|\bar{y}(t, \epsilon) - \psi(a)| \rightarrow 0$ as $\epsilon \rightarrow 0$.

Define $\beta = \beta(t, \epsilon)$ by

$$(4.11) \quad y = \bar{y} + \beta.$$

Thus

$$(4.12) \quad y = \psi(a) + \gamma + \beta.$$

It is clear that β satisfies the equation

$$(4.13) \quad \epsilon^\tau \frac{d\beta}{dt} = g(\phi(a), \psi(a) + \gamma + \beta, a, 0) - g(\phi(a), \psi(a) + \gamma, a, 0) + H,$$

where

$$(4.14) \quad H = g(x, \psi(a) + \gamma + \beta, t, \epsilon) - g(\phi(a), \psi(a) + \gamma + \beta, a, 0),$$

and the initial condition

$$(4.15) \quad \beta(a, \epsilon) = 0.$$

The crux of the entire situation is the behavior of $\beta(t, \epsilon)$ in the neighborhood of $t = a$ for small ϵ . By using (4.9), we may write (4.13) as

$$(4.16) \quad \frac{d\beta}{ds} = g(\phi(a), \psi(a) + \gamma + \beta, a, 0) - g(\phi(a), \psi(a) + \gamma, a, 0) + H.$$

The following lemma, which is essentially the continuity of the solutions of a differential equation with respect to a parameter, is required.

Lemma 4.1. *Let $s_1 > 0$ be given. Let $\tilde{t} = \tilde{t}(s_1, \epsilon)$ be defined by*

$$(4.17) \quad \tilde{t} = a + s_1 \epsilon^r.$$

Let $\nu > 0$. Then $|\beta| \leq \nu$ for $0 \leq s \leq s_1$ (or equivalently for $a \leq t \leq \tilde{t}$) for all sufficiently small $\epsilon > 0$.

Proof. Let δ_1 satisfy $0 < \delta_1 < \min(\delta_0, \tilde{\delta}_0, \tilde{\epsilon}_0)$ and let δ_1 be fixed throughout the proof. It follows readily, using the first equation of (4.1), H3 and H4, that $|x(t, \epsilon) - \phi(a)| \leq K(t - a)$ for some constant $K > 0$ as long as $|\beta| \leq \delta_1$, $|\xi| \leq \delta_1$, $a \leq t \leq b$, and $0 < \epsilon \leq \delta_1$. Simple continuity considerations then yield

$$(4.18) \quad \begin{aligned} |H| &\leq h(t - a) + \omega(\epsilon) = h(s\epsilon^r) + \omega(\epsilon), \\ \text{as long as } |\beta| &\leq \delta_1, \quad |\xi| \leq \delta_1, \quad a \leq t \leq b, \quad 0 < \epsilon \leq \delta_1. \end{aligned}$$

It is clear that there exists an $M > 0$ such that

$$(4.19) \quad \begin{aligned} |g(\phi(a), \psi(a) + \gamma + \beta, a, 0) - g(\phi(a), \psi(a) + \gamma, a, 0)| &\leq M |\beta| \\ \text{as long as } |\beta| &\leq \delta_1. \end{aligned}$$

Applying Lemma 2.2 to the inequality which results from (4.13) on applying (4.18) and (4.19), we readily obtain

$$(4.20) \quad \begin{aligned} |\beta| &\leq \int_0^s [h(\tau\epsilon^r) + \omega(\epsilon)] e^{M(s-\tau)} d\tau \\ \text{as long as } |\beta| &\leq \delta_1, \quad |\xi| \leq \delta_1, \quad a \leq t \leq b, \quad 0 < \epsilon \leq \delta_1. \end{aligned}$$

The conclusion is obvious from (4.20).

Loosely speaking, Lemma 4.1 states that

$$(4.21) \quad y(t, \epsilon) \approx \psi(a) + \gamma[(t - a)/\epsilon^r], \quad a \leq t \leq \tilde{t}.$$

We are now ready to state and prove Theorem 4.1.

Theorem 4.1. *H1, H2, H3, and H4 imply that for sufficiently small ϵ (4.1) has a unique solution $x(t, \epsilon)$, $y(t, \epsilon)$ satisfying the initial condition $x(a, \epsilon) = \phi(a)$, $y(a, \epsilon) = \Psi(a)$ (where $\Psi(a)$ is that of H4). Furthermore, $x(t, \epsilon)$, $y(t, \epsilon)$ exists over the entire interval $a \leq t \leq b$ and $|x(t, \epsilon) - \phi(t)| + |y(t, \epsilon) - \psi(t)| \rightarrow 0$ as $\epsilon \rightarrow 0+$ uniformly over $c \leq t \leq b$ for any $c > a$.*

Proof. The proof follows directly from Lemma 4.1 and Theorem 1.1. We simply take s_1 sufficiently large and ϵ sufficiently small to make, see (4.12), $|y(\tilde{t}, \epsilon) - \psi(\tilde{t})|$ and $|x(\tilde{t}, \epsilon) - \phi(\tilde{t})|$ sufficiently small so that the interval $\tilde{t} \leq t \leq b$ can be treated by Theorem 1.1. That $\tilde{t}$ is not constant causes no real difficulty.

5. Extension of results. We now generalize the preceding results to systems having more dependent variables and higher powers of ϵ . The method will be clear from the treatment of the system (5.1) and the associated degenerate system (5.2). These systems are

$$\begin{aligned} \frac{dx}{dt} &= f(x, y_1, y_2, t, \epsilon) \\ (5.1) \quad \epsilon^{r_1} \frac{dy_1}{dt} &= g_1(x, y_1, y_2, t, \epsilon) \end{aligned}$$

$$\epsilon^{r_2} \frac{dy_2}{dt} = g_2(x, y_1, y_2, t, \epsilon),$$

$$\begin{aligned} \frac{dx}{dt} &= f(x, y_1, y_2, t, 0) \\ (5.2) \quad 0 &= g_1(x, y_1, y_2, t, 0) \\ 0 &= g_2(x, y_1, y_2, t, 0), \end{aligned}$$

where x and f are real vectors of dimension $n_0 \geq 1$; y_i and g_i are real vectors of dimension $n_i \geq 1$, $i = 1, 2$; $\epsilon > 0$; and r_1 and r_2 are fixed constants satisfying $r_2 > r_1 > 0$.

It is well to stress the fact that y_1 and y_2 are separate vectors in their own right and should not be thought of as components of a vector y . This notation has been adopted because it also fits smoothly into more general cases.

The hypotheses that are now needed are very similar to H1, H2 and H3 and are denoted by H1', H2', and H3'. Under these hypotheses we shall prove Theorems 5.1 and 5.2 which are direct analogues of Theorems 1.1 and 1.2 respectively.

H1': *The degenerate system (5.2) is assumed to possess a solution $x = \phi(t)$, $y_1 = \psi_1(t)$, $y_2 = \psi_2(t)$, which we shall henceforth denote as $\phi(t)$, $\psi_1(t)$, $\psi_2(t)$, which is continuously differentiable on the closed finite interval $a \leq t \leq b$.*

A region, R' , is now defined which is analogous to R .

$$\begin{aligned} R': \quad & |x - \phi(t)| < \delta_0, \quad |y_1 - \psi_1(t)| < \delta_0, \quad |y_2 - \psi_2(t)| < \delta_0, \\ & a \leq t \leq b, \quad 0 \leq \epsilon < \delta_0, \quad \text{where } \delta_0 > 0. \end{aligned}$$

The notation used for Jacobian matrices is the same as that previously given. For example, by $\partial g_1 / \partial y_2$ we mean the n_1 by n_2 matrix

$$\frac{\partial g_1}{\partial y_2} = \left(\frac{\partial g_{1\alpha}(x, y_1, y_2, t, \epsilon)}{\partial y_{2\beta}} \right), \quad \alpha = 1, 2, \dots, n_1, \quad \beta = 1, 2, \dots, n_2,$$

where $g_{1\alpha}$ is the α component of g_1 and $y_{2\beta}$ is the β component of y_2 .

H2': This is a smoothness hypothesis on f , g_1 , and g_2 .

- (i) In R' : f , $\partial f/\partial x$, $\partial f/\partial y_1$, and $\partial f/\partial y_2 \in C$ with respect to $(x, y_1, y_2, t, \epsilon)$.
- (ii) Same as (i) with g_1 replacing f .
- (iii) Same as (i) with g_2 replacing f .

Remarks. It will be evident from the proof of Theorem 5.1 that some of the differentiability hypotheses on f , g_1 , and g_2 can be weakened to Lipschitz conditions without any change in the proof.

Before stating H3' we introduce the notation:

$$g_{i,j} = g_{i,j}(t) = \frac{\partial g_i}{\partial y_j}(\phi(t), \psi_1(t), \psi_2(t), t, 0), \quad i, j = 1, 2.$$

Thus, $g_{i,j}$ is an n_i by n_j matrix which by H1' and H2' is continuous on $a \leq t \leq b$.

H3': This is the crucial hypothesis. We assume that the real parts of all the characteristic roots of the two matrices $g_{2,2}$ and $g_{1,1} - g_{1,2} g_{2,2}^{-1} g_{2,1}$ are $\leq -\mu < 0$ on $a \leq t \leq b$.

Theorem 5.1. H1', H2', and H3' imply that if

$$\epsilon + |\Phi(a, \epsilon) - \phi(a)| + |\Psi_1(a, \epsilon) - \psi_1(a)| + |\Psi_2(a, \epsilon) - \psi_2(a)|$$

is sufficiently small, then (5.1) has a unique solution $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ satisfying the initial condition $x(a, \epsilon) = \Phi(a, \epsilon)$, $y_1(a, \epsilon) = \Psi_1(a, \epsilon)$, $y_2(a, \epsilon) = \Psi_2(a, \epsilon)$. Furthermore, $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ exists over the entire interval $a \leq t \leq b$ and $|x(t, \epsilon) - \phi(t)| + |y_1(t, \epsilon) - \psi_1(t)| + |y_2(t, \epsilon) - \psi_2(t)| \rightarrow 0$ uniformly over $a \leq t \leq b$ as $\epsilon + |\Phi(a, \epsilon) - \phi(a)| + |\Psi_1(a, \epsilon) - \psi_1(a)| + |\Psi_2(a, \epsilon) - \psi_2(a)| \rightarrow 0$.

Theorem 5.2. H1', H2', and H3' imply that there exists a $\delta > 0$, which is independent of ϵ , such that if $|\Psi_1(a) - \psi_1(a)| + |\Psi_2(a) - \psi_2(a)| \leq \delta$ (where $\Psi_1(a)$ and $\Psi_2(a)$ are constant vectors), then for sufficiently small ϵ (5.1) has a unique solution $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ satisfying the initial condition $x(a, \epsilon) = \phi(a)$, $y_1(a, \epsilon) = \Psi_1(a)$, $y_2(a, \epsilon) = \Psi_2(a)$. Furthermore, $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ exists over the entire interval $a \leq t \leq b$ and $|x(t, \epsilon) - \phi(t)| + |y_1(t, \epsilon) - \psi_1(t)| + |y_2(t, \epsilon) - \psi_2(t)| \rightarrow 0$ as $\epsilon \rightarrow 0$ uniformly over $c \leq t \leq b$ for any $c > a$.

Proof of Theorem 5.1. Let t_0 satisfy $a \leq t_0 < b$. We shall proceed as in the proof of Theorem 1.1 by proving the theorem for a particular closed subinterval, $I(t_0)$, of $a \leq t \leq b$ with t_0 as left-hand end point, whose length will be seen not to depend on t_0 or on ϵ . Let $(\Phi(t_0, \epsilon), \Psi_1(t_0, \epsilon), \Psi_2(t_0, \epsilon), t_0, \epsilon)$ belong to R' . Denote by $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ the unique solution of (5.1) existing on some closed subinterval of $a \leq t \leq b$ with t_0 as left-hand end point and satisfying $x(t_0, \epsilon) = \Phi(t_0, \epsilon)$, $y_1(t_0, \epsilon) = \Psi_1(t_0, \epsilon)$, $y_2(t_0, \epsilon) = \Psi_2(t_0, \epsilon)$.

Define $\xi = \xi(t, \epsilon)$, $\eta_1 = \eta_1(t, \epsilon)$, and $\eta_2 = \eta_2(t, \epsilon)$ by

$$\begin{aligned} x &= \phi + \xi \\ (5.3) \quad y_1 &= \psi_1 + \eta_1 \\ y_2 &= \psi_2 + \eta_2. \end{aligned}$$

Then ξ , η_1 , η_2 satisfy the system

$$\begin{aligned} \frac{d\xi}{dt} &= f(\phi + \xi, \psi_1 + \eta_1, \psi_2 + \eta_2, t, \epsilon) - f(\phi, \psi_1, \psi_2, t, 0) \\ (5.4) \quad \epsilon^{r_1} \frac{d\eta_1}{dt} &= g_1(\phi + \xi, \psi_1 + \eta_1, \psi_2 + \eta_2, t, \epsilon) - \epsilon^{r_1} \frac{d\psi_1}{dt}, \\ \epsilon^{r_2} \frac{d\eta_2}{dt} &= g_2(\phi + \xi, \psi_1 + \eta_1, \psi_2 + \eta_2, t, \epsilon) - \epsilon^{r_2} \frac{d\psi_2}{dt} \end{aligned}$$

Using the notation

$$\begin{aligned} A &= g_{1,1}(t_0) \\ B &= g_{1,2}(t_0) \\ (5.5) \quad C &= g_{2,1}(t_0) \\ D &= g_{2,2}(t_0), \end{aligned}$$

we may rewrite the last two equations of (5.4) as

$$\begin{aligned} \epsilon^{r_1} \frac{d\eta_1}{dt} &= A\eta_1 + B\eta_2 + G_1 \\ (5.6) \quad \epsilon^{r_2} \frac{d\eta_2}{dt} &= C\eta_1 + D\eta_2 + G_2, \end{aligned}$$

where

$$\begin{aligned} G_1 &= g_1(\phi + \xi, \psi_1 + \eta_1, \psi_2 + \eta_2, t, \epsilon) - \epsilon^{r_1} \frac{d\psi_1}{dt} - A\eta_1 - B\eta_2 \\ (5.7) \quad G_2 &= g_2(\phi + \xi, \psi_1 + \eta_1, \psi_2 + \eta_2, t, \epsilon) - \epsilon^{r_2} \frac{d\psi_2}{dt} - C\eta_1 - D\eta_2. \end{aligned}$$

Dividing the last equation of (5.6) by ϵ^{r_2} and applying the variation of constants formula to the resulting equation yields

$$(5.8) \quad \eta_2(t, \epsilon) = e^{D(t-t_0)/\epsilon r_2} \eta_2(t_0, \epsilon) + \frac{1}{\epsilon r_2} \int_{t_0}^t e^{D(t-\tau)/\epsilon r_2} C\eta_1(\tau, \epsilon) d\tau \\ + \frac{1}{\epsilon r_2} \int_{t_0}^t e^{D(t-\tau)/\epsilon r_2} G_2 d\tau.$$

Equation (5.8) is valid as long as $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ remains in R' .

Define $\tilde{\eta}_2 = \tilde{\eta}_2(t, \epsilon)$ by

$$(5.9) \quad \tilde{\eta}_2(t, \epsilon) = \eta_2(t, \epsilon) + D^{-1} C\eta_1(t, \epsilon).$$

It follows readily from H3' that

$$(5.10) \quad |\tilde{\eta}_2(t, \epsilon)| \leq |\eta_2(t, \epsilon)| + K |\eta_1(t, \epsilon)| \\ |\eta_2(t, \epsilon)| \leq |\tilde{\eta}_2(t, \epsilon)| + K |\eta_1(t, \epsilon)|.$$

The constant K and the function $\omega(\epsilon)$ will be used in the same manner as before. Let σ satisfy $0 < \sigma < \mu$; hold σ fixed for the remainder of the proof. Adding $D^{-1} C\eta_1(t, \epsilon)$ to both sides of (5.8) and then using H3' and Lemma 2.4, we obtain

$$(5.11) \quad |\tilde{\eta}_2(t, \epsilon)| \leq K e^{-\sigma(t-t_0)/\epsilon r_2} |\eta_2(t_0, \epsilon)| + |J| + \frac{K}{\epsilon r_2} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon r_2} |G_2| d\tau,$$

where

$$(5.12) \quad J = \frac{1}{\epsilon r_2} \int_{t_0}^t e^{D(t-\tau)/\epsilon r_2} C\eta_1(\tau, \epsilon) d\tau + D^{-1} C\eta_1(t, \epsilon).$$

Inequality (5.11) is valid for $t \geq t_0$ as long as $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ remains in R' .

Let $I(t_0)$ be a closed subinterval of $a \leq t \leq b$ with t_0 as left-hand end point and of, as yet undetermined, length $l(t_0)$. Let ϵ_1 satisfy $0 < \epsilon_1 \leq 1$. Let δ_1 satisfy $0 < \delta_1 < \delta_0$; hold δ_1 fixed throughout the proof. From the smoothness hypothesis, H2', it follows that there exists a $\delta_2 = \delta_2(\epsilon_1) > 0$, and a $\delta_3 = \delta_3(\epsilon_1)$ satisfying $0 < \delta_3 \leq \delta_1$, such that $|G_i| \leq K(|\xi| + \omega(\epsilon)) + \epsilon_1 |\eta_1| + \epsilon_1 |\eta_2|$, $i = 1, 2$, for $t \in I(t_0)$ if $l(t_0) \leq \delta_2$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_3$, $|\eta_2| \leq \delta_3$, $0 \leq \epsilon \leq \delta_1$; where K and $\omega(\epsilon)$ depend only on δ_1 (and, therefore, are well determined). It follows from (5.10) that

$$(5.13) \quad |G_i| \leq K(|\xi| + \omega(\epsilon) + \epsilon_1 |\eta_1|) + \epsilon_1 |\tilde{\eta}_2|, \quad i = 1, 2 \\ \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_1, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3,$$

$$0 \leq \epsilon \leq \delta_1.$$

From the first equation of (5.4) there exists a $K > 0$ such that

$$(5.14) \quad \left| \frac{d\xi}{dt} \right| \leq K$$

if $a \leq t \leq b$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_1$, $|\eta_2| \leq \delta_1$, $0 < \epsilon \leq \delta_1$.

This yields, in the same manner that we established (3.7),

$$(5.15) \quad \frac{1}{\epsilon^{r_i}} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^{r_i}} |\xi(\tau, \epsilon)| d\tau \leq \frac{1}{\sigma} |\xi(t, \epsilon)| + K\epsilon^{r_i}, \quad i = 1, 2$$

if $t_0 \leq t \leq b$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_1$, $|\eta_2| \leq \delta_1$, $0 < \epsilon \leq \delta_1$.

From the second equation of (5.4) there exists a $K > 0$ such that

$$(5.16) \quad \left| \frac{d\eta_1}{dt} \right| \leq \frac{K}{\epsilon^{r_1}}$$

if $a \leq t \leq b$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_1$, $|\eta_2| \leq \delta_1$, $0 < \epsilon \leq \delta_1$.

A simple application of the mean value theorem together with (5.16) yields

$$(5.17) \quad |\eta_1(t, \epsilon) - \eta_1(\tau, \epsilon)| \leq \frac{K}{\epsilon^{r_1}} |t - \tau|$$

if $a \leq t, \tau \leq b$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_1$, $|\eta_2| \leq \delta_1$, $0 < \epsilon \leq \delta_1$.

Using (5.17) together with an argument very similar to that used to establish (3.7) yields

$$(5.18) \quad \frac{1}{\epsilon^{r_2}} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^{r_2}} |\eta_1(\tau, \epsilon)| d\tau \leq \frac{1}{\sigma} |\eta_1(t, \epsilon)| + K\epsilon^{r_2-r_1}$$

if $t_0 \leq t \leq b$, $|\xi| \leq \delta_1$, $|\eta_1| \leq \delta_1$, $|\eta_2| \leq \delta_1$, $0 < \epsilon \leq \delta_1$.

The following relation follows immediately from (5.12):

$$J = -\frac{D^{-1}}{\epsilon^{r_2}} \int_{t_0}^t e^{D(t-\tau)/\epsilon^{r_2}} (-D)[C\eta_1(\tau, \epsilon) - C\eta_1(t, \epsilon)] d\tau + D^{-1} e^{D(t-t_0)/\epsilon^{r_2}} C\eta_1(t, \epsilon).$$

Hence, from H3' and Lemma 2.4, we have

$$(5.19) \quad |J| \leq \frac{K}{\epsilon^{r_2}} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^{r_2}} |\eta_1(\tau, \epsilon) - \eta_1(t, \epsilon)| d\tau + Ke^{-\sigma(t-t_0)/\epsilon^{r_2}} |\eta_1(t, \epsilon)|$$

for $t_0 \leq t \leq b$ as long as $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ remains in R' .

From (5.17) and (5.19) we obtain

$$(5.20) \quad \begin{aligned} |J| &\leq K e^{-\sigma(t-t_0)/\epsilon^{r_2}} |\eta_1(t, \epsilon)| + K \epsilon^{r_2-r_1} \\ \text{if } t_0 &\leq t \leq b, \quad |\xi| \leq \delta_1, \quad |\eta_1| \leq \delta_1, \quad |\eta_2| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1. \end{aligned}$$

Substituting (5.13), (5.15), (5.18), and (5.20) into (5.11) yields

$$(5.21) \quad \begin{aligned} |\tilde{\eta}_2(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/\epsilon^{r_2}} (|\eta_2(t_0, \epsilon)| + |\eta_1(t, \epsilon)|) \\ &\quad + K(|\xi(t, \epsilon)| + \omega(\epsilon) + \epsilon_1 |\eta_1(t, \epsilon)|) + \frac{K\epsilon_1}{\epsilon^{r_2}} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^{r_2}} |\tilde{\eta}_2(\tau, \epsilon)| d\tau \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\ &\quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

Applying Lemma 2.1 to (5.21) yields

$$(5.22) \quad \begin{aligned} |\tilde{\eta}_2(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/\epsilon^{r_2}} (|\eta_2(t_0, \epsilon)| + |\eta_1(t, \epsilon)|) + K(|\xi(t, \epsilon)| \\ &\quad + \omega(\epsilon) + \epsilon_1 |\eta_1(t, \epsilon)|) + \frac{K\epsilon_1}{\epsilon^{r_2}} \int_{t_0}^t e^{-(\sigma-K\epsilon_1)(t-\tau)/\epsilon^{r_2}} [e^{-\sigma(\tau-t_0)/\epsilon^{r_2}} (|\eta_2(t_0, \epsilon)| \\ &\quad + |\eta_1(\tau, \epsilon)|) + |\xi(\tau, \epsilon)| + \omega(\epsilon) + \epsilon_1 |\eta_1(\tau, \epsilon)|] d\tau \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\ &\quad 0 < \epsilon \leq \delta_3. \end{aligned}$$

We now choose ν , satisfying $0 < \nu \leq 1$, sufficiently small so that if $0 < \epsilon_1 \leq \nu$ then $\sigma - K\epsilon_1 \geq \frac{1}{2}\sigma$, where K here is the one appearing in (5.22). Thus ν is well determined. An elementary integration together with inequalities similar to (5.15) and (5.18) yields

$$(5.23) \quad \begin{aligned} |\tilde{\eta}_2(t, \epsilon)| &\leq K e^{-\sigma(t-t_0)/2\epsilon^{r_2}} (|\eta_2(t_0, \epsilon)| + |\eta_1(t, \epsilon)|) \\ &\quad + K(|\xi(t, \epsilon)| + \omega(\epsilon) + \epsilon_1 |\eta_1(t, \epsilon)|) \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\ &\quad 0 < \epsilon \leq \delta_3 \text{ and } 0 < \epsilon_1 \leq \nu. \end{aligned}$$

We now turn our attention to the first equation of (5.6) (which is, of course, the second equation of (5.4)). This equation may be rewritten as

$$(5.24) \quad \epsilon^{r_1} \frac{d\eta_1}{dt} = (A - BD^{-1}C)\eta_1 + B\tilde{\eta}_2 + G_1.$$

The matrix $A - BD^{-1}C$ will be denoted by F . Dividing both sides of (5.24) by ϵ^{r_1} and applying the variation of constants formula to the resulting equation yields

$$(5.25) \quad \eta_1(t, \epsilon) = e^{F(t-t_0)/\epsilon^{r_1}} \eta_1(t_0, \epsilon) + \frac{1}{\epsilon^{r_1}} \int_{t_0}^t e^{F(t-\tau)/\epsilon^{r_1}} (B\tilde{\eta}_2(\tau, \epsilon) + G_1) d\tau.$$

Equation (5.25) is valid as long as $x(t, \epsilon)$, $y_1(t, \epsilon)$, $y_2(t, \epsilon)$ remains in R' . Applying $H2'$, $H3'$, and Lemma 2.4 to (5.25), then substituting (5.13), (5.15), and (5.23) into the resulting inequality yields after an elementary integration and use of $|\eta_1| \leq \delta_1$

$$(5.26) \quad \begin{aligned} |\eta_1(t, \epsilon)| &\leq K[e^{-\sigma(t-t_0)/\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| + |\xi(t, \epsilon)| + \omega(\epsilon)] \\ &\quad + \frac{K\epsilon_1}{\epsilon^{r_1}} \int_{t_0}^t e^{-\sigma(t-\tau)/\epsilon^{r_1}} |\eta_1(\tau, \epsilon)| d\tau \end{aligned}$$

$$\text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3,$$

$$0 < \epsilon \leq \delta_3, \quad \text{and} \quad 0 < \epsilon_1 \leq \nu.$$

Lemma 2.1 is now applied to (5.26), yielding

$$(5.27) \quad \begin{aligned} |\eta_1(t, \epsilon)| &\leq K[e^{-\sigma(t-t_0)/\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| + |\xi(t, \epsilon)| + \omega(\epsilon)] \\ &\quad + \frac{K\epsilon_1}{\epsilon^{r_1}} \int_{t_0}^t e^{K\epsilon_1(t-\tau)/\epsilon^{r_1}} e^{-\sigma(t-t_0)/\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| d\tau \\ &\quad + \frac{K\epsilon_1}{\epsilon^{r_1}} \int_{t_0}^t e^{-(\sigma-K\epsilon_1)(t-\tau)/\epsilon^{r_1}} (|\xi(\tau, \epsilon)| + \omega(\epsilon)) d\tau \end{aligned}$$

$$\text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3,$$

$$0 < \epsilon \leq \delta_3, \quad \text{and} \quad 0 < \epsilon_1 \leq \nu.$$

We now choose ϵ_1 , satisfying $0 < \epsilon_1 \leq \nu$, sufficiently small so that $\sigma - K\epsilon_1 \geq \frac{1}{2}\sigma$ (K here stands for the one in (5.27) and is, of course, perfectly well determined); ϵ_1 is held fixed from now on at this value. This determines a δ_2 and a δ_3 , which are also held fixed from now on. An elementary integration together with an inequality similar to (5.15) (for $i = 1$) yields

$$(5.28) \quad \begin{aligned} |\eta_1(t, \epsilon)| &\leq Ke^{-\sigma(t-t_0)/2\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| + K(|\xi(t, \epsilon)| + \omega(\epsilon)) \\ \text{for } t \in I(t_0) \text{ if } l(t_0) &\leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \end{aligned}$$

$$0 < \epsilon \leq \delta_3.$$

From (5.10), (5.23) and (5.28) we obtain

$$\begin{aligned}
 (5.29) \quad & |\eta_2(t, \epsilon)| \leq K e^{-\sigma(t-t_0)/2\epsilon^{r_2}} |\eta_2(t_0, \epsilon)| \\
 & + K e^{-\sigma(t-t_0)/2\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| + K(|\xi(t, \epsilon)| + \omega(\epsilon)) \\
 & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\
 & 0 < \epsilon \leq \delta_3.
 \end{aligned}$$

The first equation of (5.4) implies the existence of a $K > 0$ and an $\omega(\epsilon)$ such that

$$\begin{aligned}
 (5.30) \quad & \left| \frac{d\xi}{dt} \right| \leq K(|\xi| + |\eta_1| + |\eta_2| + \omega(\epsilon)) \\
 & \text{for } a \leq t \leq b \text{ if } |\xi| \leq \delta_1, \quad |\eta_1| \leq \delta_1, \quad |\eta_2| \leq \delta_1, \quad 0 < \epsilon \leq \delta_1.
 \end{aligned}$$

Substituting (5.28) and (5.29) into (5.30) yields

$$\begin{aligned}
 (5.31) \quad & \left| \frac{d\xi}{dt} \right| \leq K[e^{-\sigma(t-t_0)/2\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| + e^{-\sigma(t-t_0)/2\epsilon^{r_2}} |\eta_2(t_0, \epsilon)| + \omega(\epsilon)] \\
 & + K|\xi(t, \epsilon)| \\
 & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\
 & 0 < \epsilon \leq \delta_3.
 \end{aligned}$$

Applying Lemma 2.2 to (5.31) yields

$$\begin{aligned}
 (5.32) \quad & |\xi(t, \epsilon)| \leq |\xi(t_0, \epsilon)| e^{K(t-t_0)} + K \int_{t_0}^t [e^{-\sigma(r-t_0)/2\epsilon^{r_1}} |\eta_1(t_0, \epsilon)| \\
 & + e^{-\sigma(r-t_0)/2\epsilon^{r_2}} |\eta_2(t_0, \epsilon)| + \omega(\epsilon)] e^{K(t-r)} dr \\
 & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\
 & 0 < \epsilon \leq \delta_3
 \end{aligned}$$

which easily yields

$$\begin{aligned}
 (5.33) \quad & |\xi(t, \epsilon)| \leq |\xi(t_0, \epsilon)| e^{K(t-t_0)} + K\omega(\epsilon) \\
 & \text{for } t \in I(t_0) \text{ if } l(t_0) \leq \delta_2, \quad |\xi| \leq \delta_3, \quad |\eta_1| \leq \delta_3, \quad |\eta_2| \leq \delta_3, \\
 & 0 < \epsilon \leq \delta_3.
 \end{aligned}$$

Combining (5.28), (5.29) and (5.33) we obtain

$$\begin{aligned}
 (5.34) \quad & | \xi(t, \epsilon) | \leq | \xi(t_0, \epsilon) | e^{K(t-t_0)} + K\omega(\epsilon) \\
 & | \eta_1(t, \epsilon) | \leq K e^{-\sigma(t-t_0)/2\epsilon^{r_1}} | \eta_1(t_0, \epsilon) | + K(| \xi(t_0, \epsilon) | + \omega(\epsilon)) \\
 & | \eta_2(t, \epsilon) | \leq K e^{-\sigma(t-t_0)/2\epsilon^{r_2}} | \eta_2(t_0, \epsilon) | \\
 & \quad + K e^{-\sigma(t-t_0)/2\epsilon^{r_1}} | \eta_1(t_0, \epsilon) | + K(| \xi(t_0, \epsilon) | + \omega(\epsilon)) \\
 & \text{for } t \in I(t_0) \quad \text{if } l(t_0) \leq \delta_2, \quad | \xi | \leq \delta_3, \quad | \eta_1 | \leq \delta_3, \quad | \eta_2 | \leq \delta_3, \\
 & \quad \quad \quad 0 < \epsilon \leq \delta_3.
 \end{aligned}$$

Recall that δ_2 and δ_3 have already been determined. Take $l(t_0) = \delta_2$. Thus $l(t_0)$ does not depend on t_0 or on ϵ . The remainder of the proof is evident from (5.34).

Theorem 5.2 follows directly from Theorem 5.1 and (5.34).

Added in proof: GRADSTEIN considers a problem similar to ours and also uses H3. I. S. GRADSTEIN, On a class of non-linear differential equations with small coefficients for certain derivatives. *Dokl. Akad. Nauk SSSR (N. S.)* **64**, 441–443 (1949) and later volumes of same journal.

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