*On Bodies Achieving Extreme Values of the Critical Mach Number, I**

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1. Introduction. One of the challenging problems in the theory of subsonic flows is that of determination of the range of incident Mach numbers for which the flow past a body remains everywhere subsonic. While recent results [1, 2] have established the existence of subsonic flows past (essentially) arbitrary plane profiles for a range of incident Mach numbers so large that the corresponding maximum local Mach numbers of the flows cover all values between zero and one, the existence proofs yield no information concerning that value of the incident Mach number for which sonic velocity is first achieved on a given body. The determination of this *critical Mach number* for a given profile is obviously an involved computational problem, and so it is appropriate from the standpoint of general methods to seek only upper and lower bounds on this quantity. We are thus led naturally to consider the extremal problem of determining among all bodies of a given class the one achieving the maximum or minimum value of the critical Mach number.

This extremal problem, while of theoretical interest, is perhaps of greater importance in its relation to airfoil design. Since observations show the occurrence of local supersonic regions and shock waves to be connected with an increase in

* Prepared under Navy Contracts N6onr-180, Task Order V, with Indiana University, and Nonr-225(11) (NR-041-086) with Stanford University.

drag and other unwanted effects, it is evidently of interest to the aircraft designer to forestall these occurrences by shaping the wing profile to achieve the maximum critical Mach number consistent with design limitations. From this point of view our extremal problems are related to the design of optimal airfoils.

In this paper we present an exact theory of the following extremal problems for symmetric plane flows:¹

1. Among all profiles with given thickness ratio² to determine the shape achieving the maximum critical Mach number.
2. Among all profiles having given width and given maximum curvature,² to determine the shape achieving the minimum critical Mach number.

Although these two problems and certain extensions will be our principal concern, we shall consider also several related problems of practical interest.

The solution of the above problems demonstrates the strength and simplicity of the comparison methods which have previously been applied to other problems in the theory of subsonic and incompressible flows.³ The arguments rest squarely on the following comparison theorem: In symmetric subsonic flow about two bodies B_1 and B_2 let $B_1 \subset B_2$ and let the two profiles have a common point of contact at P off the axis of symmetry. If now the respective free stream Mach numbers obey the inequality $M_1 \geq M_2$, then the two local Mach numbers at P are unequal in the same order, $M_1(P) \geq M_2(P)$, and the equality holds if and only if the two flows and the two profiles are identical. (This statement does not mention the regularity conditions necessary for proof.)

In the problems considered here the comparison theorem must be applied in situations where P may be a sonic point. This makes the proof considerably more difficult than in the purely subsonic case, for which the analogous theorem has already been proved in [3]. This extension of the comparison theorem to include sonic boundary points is the only analytical detail of any complexity in the paper.

The extremal profile of Problem 1 is characterized as the solution of a free boundary problem. The figure consists of two equal vertical segments whose endpoints on each side of the axis of symmetry are joined by a convex arc, the resulting profile having the prescribed thickness ratio and the essential property that there is a subsonic flow past it for which the speed is sonic at all points on the arc.⁴ The incident Mach number in this flow is the maximum critical Mach number possessed by the competing profiles. The actual construction of the

¹ Throughout this paper the x -axis, which is parallel to the free stream flow, is taken to be the axis of symmetry.

² The *thickness ratio* is the ratio of width to length of a profile, the width being the absolute difference of the extreme ordinates on the body, and the length the absolute difference between the extreme abscissae. Curvature at a point on a closed profile will be measured positive when the curvature is directed toward the interior of the body.

³ Cf. [3] and the references cited therein.

⁴ The analogous free surface flow for incompressible fluids is the well-known Riabouchinsky cavity.

extremal body and the determination of the corresponding critical velocity is an interesting calculational problem which we postpone to Part II of this work.

The solution of Problem 2 is (above the axis of symmetry) simply the circular arc with the prescribed width and with its curvature equal to the prescribed maximum curvature. If the width exceeds the radius, then the profile consists of a semicircle mounted on two vertical segments.

Once the comparison theorem is available, the proof that the above profiles are indeed the required extremal bodies is both elementary and simple. Consider Problem 1, for example, and let us show that the profile described above has a larger critical Mach number than any other of the same thickness ratio. We shall show more, namely, that any profile B with critical Mach number equal to or greater than that of the extremal body E , and having the same length as E , must lie entirely within the latter, or be identical with it. For suppose the contrary; then by a similarity contraction of B , which does not change its critical Mach number, we can obtain a body lying within E and having contact with it, say at P , along the sonic arc of the latter (Figure 1). Since, by assumption, the critical Mach number of B is at least as great as that of E , the conditions of the comparison theorem are now fulfilled. It follows that the flow speed at P is larger on B than on E , contradicting the fact that P is a sonic point of the flow past E . This establishes the extremal character of E . The proof that the circular arc is a solution of Problem 2 is based on a similar idea and is just as simple.

In Section 6 we see that essentially the same considerations apply to axially symmetric flows, and that the two extremal problems for this class of flows have as solutions precisely the bodies of revolution corresponding to the extremals of the plane problems.

A problem for further investigation is the extension of the present results to asymmetric flows with and without circulation. This will require an appropriate generalization of the comparison theorem to include the case of asymmetric flows past closed profiles.

2. Preliminary results on critical flows. The flows considered here will be irrotational, subsonic and parallel at infinity, and will not exceed sonic speed in the interior. We shall distinguish between *critical* flows, in which the maximum flow speed is sonic, and *subcritical* flows, in which the maximum flow speed is subsonic; the term "subsonic flow" will include both the critical and subcritical flows, and will be used when no effort is made to distinguish the two classes of flows. We limit ourselves to plane flows in all but Section 6, where the flows under consideration will be axially symmetric.

For fluids obeying an equation of state, $p = p(\rho)$, where p is the pressure and ρ is the density, the motion is described by the equation for the potential ϕ ,

$$(1) \quad \left(1 - \frac{\phi_x^2}{c^2}\right)\phi_{xx} - \frac{2\phi_x\phi_y}{c^2}\phi_{xy} + \left(1 - \frac{\phi_y^2}{c^2}\right)\phi_{yy} = 0$$

and by the equation for the stream function ψ ,

$$(2) \quad \left(1 - \frac{\psi_y^2}{(\rho c)^2}\right) \psi_{xx} + \frac{2\psi_x \psi_y}{(\rho c)^2} \psi_{xy} + \left(1 - \frac{\psi_x^2}{(\rho c)^2}\right) \psi_{yy} = 0.$$

In the first of these equations the sound speed $c = [p'(\rho)]^{\frac{1}{2}}$ is a function of the flow speed $q = (\phi_x^2 + \phi_y^2)^{\frac{1}{2}}$, ρ and c being defined implicitly in terms of q by means of Bernoulli's equation.

$$(3) \quad \int_0^\rho c^2 \frac{d\rho}{\rho} = \frac{1}{2}(q_M^2 - q^2);$$

the constant q_M is the limit speed of the flow. For a polytropic flow, in which $p = A\rho^\gamma$, we have

$$c^2 = \frac{\gamma - 1}{2} (q_M^2 - q^2).$$

In equation (2) the quantities ρ and c and hence ρc are functions of the specific mass flow $Q = (\psi_x^2 + \psi_y^2)^{\frac{1}{2}}$, these functions being defined implicitly by the relation $Q = \rho q$, which gives q and hence both ρ and c as functions of Q . From the fact that

$$(4) \quad \frac{dQ}{dq} = \rho \left(1 - \frac{q^2}{c^2}\right) > 0 \quad \text{if} \quad q/c < 1$$

we have that q is a single-valued monotonically increasing function of Q in the subsonic range, and therefore in (2) ρc is a single-valued function of ψ_x and ψ_y for any subsonic flow.

It is convenient to write the equations (1) and (2) with the flow quantities in non-dimensional form so that the constants of the motion, such as q_M and the free stream values, do not appear in the coefficients. To this end we convert all velocities (including the sound speed) to non-dimensional form by dividing by q_M , while as reference quantities for the pressure and density we choose the rest values. Thus, in non-dimensional form $\rho = 1$ and $p = 1$ when the fluid is at rest, while $q = 1$ when $\rho = 0$ (we assume the limit speed achieved at zero density). In a polytropic flow the relation between non-dimensional pressure and density becomes $p = \rho^\gamma$, and we have

$$\rho = (1 - q^2)^{1/\gamma-1}, \quad c^2 = \frac{\gamma - 1}{2} (1 - q^2).$$

Equations (1) and (2), without the constants of the motion present in the coefficients, now describe all flows of the same fluid simultaneously. Henceforth the values of all flow quantities under consideration will be assumed non-dimensional.

It is well known that a non-trivial subcritical flow cannot achieve its maximum

speed in its interior, and that in such a flow past a finite body, the maximum is achieved on the body itself; (cf. [1, 2, 3]). However, for the problem at hand in which critical flows are admitted, the flow is subsonic at infinity, but is not assumed to be subsonic everywhere in its interior. The question arises as to whether such a flow may contain sonic points in its interior. We therefore prove, without restriction to symmetric flows, the following *maximum principle for the flow speed*:

If a uniform flow past a closed profile is subsonic at infinity and nowhere supersonic, then the maximum flow speed is achieved on the profile and not in the interior of the flow.

It follows that all flows considered in this paper are subsonic in their interior.

To prove the stated theorem we observe from the preceding remarks that it suffices to discuss flows in which the maximum speed is sonic and to show that $q < c_*$ in the interior of the flow, where c_* is the sonic flow speed. To establish this result let us first differentiate (1) with respect to x , and then insert in the coefficients of the resulting equation for ϕ_x the values of $\phi(x, y)$ for the flow under consideration. Thus the x -component of velocity, $u = \phi_x$, satisfies in the flow region R a linear differential equation with continuous coefficients

$$(5) \quad L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0,$$

where a, b, c are the coefficients of $\phi_{xx}, \phi_{xy}, \phi_{yy}$ in (1), considered now as known functions of x and y . The coefficients a, b, c , determine a positive definite or semidefinite quadratic form at every point in R ,

$$a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 \geq 0, \quad ac - b^2 \geq 0;$$

furthermore, at any point where $ac - b^2 = 0$ the speed is sonic and $a + c = 1$, so that a and c are not both zero simultaneously, and in particular, if $u = c_*$, we have $a = 0, b = 0, c = 1$.

The required result rests on the following lemma: If $u = c_*$ at a point P on the circumference of a circle in whose interior $u < c_*$, then the radius from P to the center of the circle is parallel to the x -axis. The proof of this lemma is patterned very closely after E. Hopf's proof of the maximum principle for elliptic differential equations [4]. Evidently we may assume that the circle C of the lemma is such that $u = c_*$ only at P on the circumference. For convenience let us choose the origin of coordinates at the center of C , and consider the function (used by Hopf for the same purpose),

$$v = e^{-\alpha r^2} - e^{-\alpha R^2},$$

where α is a positive constant to be specified, $r^2 = x^2 + y^2$, and R is the radius of C . Let us form $L(v)$

$$L(v) = e^{-\alpha r^2}[4\alpha^2(ax^2 + 2bxy + cy^2) - 2\alpha(a + c + dx + ey)].$$

At P , $a = b = 0$ and $c = 1$. Hence if, contrary to the statement of the lemma, the radius from P to the center of C (the origin) is not on the line $y = 0$, we have that $ax^2 + 2bxy + cy^2 > 0$ in a small circle C' about P , and consequently, for suitably large α the bracketed expression, and also $L(v)$, is positive in C' . Now, on the arc of C' interior to and on the circumference of C we have, by hypothesis, $u < c_* = u(P)$, while outside C , $v < 0$, so that for an appropriate $\epsilon > 0$ the function $w = u + \epsilon v$ is less than $u(P)$ on the circumference of C' . Therefore w has a maximum in the interior of C' , say at Q . At Q , we have that $w_x = w_y = 0$ and hence, by virtue of the inequality $L(w) > 0$ in C' ,

$$L[w(Q)] = (aw_{xx} + 2bw_{xy} + cw_{yy})(Q) > 0.$$

Since a, b, c are the coefficients of a positive definite or semidefinite quadratic form at every point in R , while $w_{xx}(Q), w_{xy}(Q), w_{yy}(Q)$ are coefficients of a negative definite or semidefinite form, it follows that $L(w) \leq 0$ at Q , in contradiction to the inequality obtained above. This proves the lemma.

Suppose now that $u = c_*$ at some point P_1 . Let L be the longest vertical segment containing P_1 and lying wholly in the flow region R . We shall establish that $u = c_*$ on all of L .⁵ For suppose the contrary. Then there would be points on L in whose neighborhood $u \neq c_*$. Let Q be such a point and let P_2 be the closest point (to Q) on L for which $u = c_*$; we may choose Q closer to P_2 than to the boundary of R . At Q lay out a horizontal segment M bisected by L and with the property that $u \neq c_*$ on it. Consider now the family of ellipses, each having M as a principal axis, the other axis varying in length l from zero (in which case the ellipse degenerates to the segment M) to twice the length QP_2 . By continuity, the ellipses for which l is close to zero do not contain points at which $u = c_*$ either in their interior or on their boundary. At the other extreme $u = c_*$ on the ellipse whose semi-axis is QP_2 . Hence there is a first ellipse on the boundary of which $u = c_*$, say at P_3 , while $u < c_*$ in its interior. Since the interior normal at P_3 is not horizontal, the lemma proved above is obviously contradicted, and hence we conclude $u = c_*$ on all of L .

We turn now to complete the proof of the maximum principle for flow speed. Let us assume, contrary to the theorem, the existence of a sonic point P in the interior of the flow region. We may choose our coordinates so that the velocity at P is in the x -direction. Now consider the vertical line through P . If either the upper or lower half line through P does not intersect the profile, then from the above it follows that $u = c_*$ on the entire half line; but this contradicts the requirement that the flow be subsonic at infinity. Consider then the other possibility that the straight line through P is intercepted by the profile both above and below P . On the segment L thus cut off by the body we have $u = c_*$. Since the body is assumed to be a simple closed contour, L divides it into two

⁵ The following argument is based on that by L. NIRENBERG, "A Strong Maximum Principle for Parabolic Equations," *Comm. on Pure and Appl. Math.* **6**, 167-177 (1953); the result itself was obtained independently.

arcs, one of which bounds with L a finite simply connected portion of the flow region. This arc is a streamline of the flow, while L is orthogonal to the flow velocity at every point. This evidently contradicts the equation of continuity in the region bounded by L and the streamline arc. Hence interior sonic points do not exist, and the maximum principle for flow speed is proved.

We conjecture but lack the proof of the following stronger result: A flow which is nowhere supersonic in an open connected region is either strictly subsonic throughout the region or is the uniform sonic flow. This proposition can obviously be formulated as a general theorem for a class of partial differential equations similar to (1) and (2).

Consider now a symmetric flow past a profile B . Let ψ be the stream function of the flow, taken equal to zero on B , and let Q_0 be the mass flow at infinity; thus $\psi_y \rightarrow Q_0$, $\psi_x \rightarrow 0$ at infinity. Since the flow is assumed to be subsonic at infinity, we have from Theorem 5 of [3]⁶ that

$$(6) \quad \psi(x, y) - Q_0 y \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow \infty.$$

Suppose now that we have a second symmetric body $\bar{B}$ and a flow past it described by the stream function $\bar{\psi}$ with mass flow $\bar{Q}_0$ at infinity. Suppose, further, that $\bar{B} \subset B$ and $\bar{Q} \geq Q_0$. Let us now consider only the upper half of the two flows, in which $\psi > 0$ and $\bar{\psi} > 0$. Since $\bar{B} \subset B$, we have $\bar{\psi} \geq \psi = 0$ on B , and $\bar{\psi} = \psi = 0$ on the axis of symmetry. Thus $\bar{\psi} - \psi \geq 0$ on the boundary of the open flow region D bounded by B and the axis of symmetry. From the maximum principle for the flow speed, both flows are everywhere subsonic in D . It follows from well known results concerning equations (such as (2)) with coefficients not containing the dependent variable explicitly, that the difference $\bar{\psi} - \psi \equiv \omega$ of two subsonic solutions obeys a linear elliptic differential equation of the form (5), with continuous coefficients in D . The function ω therefore obeys the maximum principle: it cannot take on its maximum or minimum value in a region unless it is constant in the region. We have seen above that $\omega \geq 0$ on the boundary of D , while from

$$\psi - Q_0 y \rightarrow 0, \quad \bar{\psi} - \bar{Q}_0 y \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow \infty$$

it follows

$$\omega - (\bar{Q}_0 - Q_0)y \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow \infty.$$

We conclude from the maximum principle for ω that $\omega \geq 0$ throughout D ; furthermore, the maximum principle asserts that either $\omega \equiv 0$ or $\omega > 0$ everywhere in D . Since the mass flow Q is a monotonically increasing function of speed, we can summarize this result in the following form:

Let ψ and $\bar{\psi}$ be stream functions describing symmetric subsonic flows past each of two profiles B and $\bar{B}$, with $\psi = 0$ on B and $\bar{\psi} = 0$ on $\bar{B}$. If $\bar{B} \subset B$ and

⁶ Although the flow is assumed subcritical in [3], the proof of this theorem given in [3] requires only that the flow be uniform and subsonic at infinity.

the respective free stream velocities q_0 , $\bar{q}_0$ obey the inequality, $\bar{q}_0 \geq q_0$, then either

$$(7) \quad \bar{\psi} - \psi > 0$$

throughout the flow region D bounded by B and the axis of symmetry, or $\psi \equiv \bar{\psi}$, in which case the two flows are identical and $B = \bar{B}$.

This result will be applied to situations in which not only is $\bar{B}$ contained in B , but also the two profiles have contact at a point P off the axis of symmetry. In this case we have $\omega(P) = \bar{\psi}(P) - \psi(P) = 0$ as well as $\omega \geq 0$ in D . It is obvious that

$$\bar{Q}(P) - Q(P) = \frac{\partial \bar{\psi}}{\partial n}(P) - \frac{\partial \psi}{\partial n}(P) = \frac{\partial \omega}{\partial n}(P) \geq 0,$$

where n is the normal directed into D , and hence that the flow speeds satisfy the inequality

$$\bar{q}(P) \geq q(P).$$

However, a much stronger statement can be made under certain general circumstances, namely, that *the strict inequality holds unless the two flows are identical and $B = \bar{B}$* . When we have proved this boundary point lemma we shall have established the comparison theorem on which the solution of our extremal problems is based.

It has been shown in [3] for subcritical flows that if P is a regular boundary point⁷ of D and if the second derivatives of ψ or $\bar{\psi}$ remain bounded near P , then the strict inequality $\bar{q}(P) > q(P)$ prevails unless the two flows are identical. To establish this result for *critical* flows as well seems to require a subtler argument than in the subcritical case. We have succeeded in proving it under the additional restriction that the second derivatives of ψ and $\bar{\psi}$ be continuous up to and including P . The details of proof of this boundary point lemma are given in Section 7, where proofs under alternative assumptions are also presented. Assuming this result, we are now able to assert the fundamental comparison theorem.

Comparison Theorem: *Let B and $\bar{B}$ be symmetric profiles such that $\bar{B} \subset B$, and let B and $\bar{B}$ have a common point of contact at a regular point P . If two symmetric subsonic flows past B and $\bar{B}$ (whether subcritical or critical) have free stream velocities q_0 and $\bar{q}_0$, with $\bar{q}_0 \geq q_0$, then at P the respective flow speeds satisfy the inequality*

$$\bar{q}(P) \geq q(P);$$

furthermore, the equality holds if and only if the two flows are identical. In particular, the speed $q(P)$ can be sonic if and only if the two flows are identical.

⁷ A point is a regular boundary point of a region if it is on the circumference of a circle whose interior lies entirely within the region.

In the following, wherever the comparison theorem is used, we assume that the profiles and flows under consideration satisfy the regularity conditions imposed in the proof of the theorem.

There are several immediate consequences of the comparison theorem for critical flows.⁸ The uniqueness of critical flows, and therefore of the critical Mach number, for a given body is clear. For if there are two critical flows past the same body, and q_0 , $\bar{q}_0$ are their respective (critical) free stream velocities with $\bar{q}_0 \geq q_0$, then the comparison theorem, applied to the common flow region, asserts that the flow speed can be sonic for the flow with smaller free stream velocity if and only if the two flows are identical. Another result states that the convex hull of a body B has a critical Mach number at least as great as that of B , and that the two values are equal if and only if B is already convex.

3. Bodies achieving maximum critical velocity. We now consider the following extremal problem for plane symmetric flows:

Problem 1. *Among all profiles with given ratio of width to length, to determine the shape achieving the maximum critical velocity.*

A second problem, apparently distinct from the preceding, but, as we shall see, actually possessed of the same solution, is the following:

Problem 1'. *Among all profiles with given length and enclosing a given area, to determine the shape achieving the maximum critical velocity.*

We assert that *the solution of these problems is a body consisting (above the axis of symmetry) of two equal vertical segments joined by a convex arc C , such that the resulting profile has the prescribed dimensions and has the property that there is a subsonic flow past it which is sonic everywhere on C .*

For the sake of definiteness, we take as admissible profiles those which satisfy, in the open interval between the two ends, the regularity conditions under which the boundary point lemma is proved (cf. Section 7). This allows the extremal body, which turns out to have infinite curvature at the end points of C , but finite curvature elsewhere, to be included among the admissible profiles.

Let us assume for the present the existence of a profile E with the stated properties. We shall return to the problem of construction of such a body in the following section. Then it is easy to prove that this profile does indeed solve the extremal problem. As outlined in the Introduction, the proof is based on the fact that any admissible profile $B \neq E$ having the same length as the extremal body E and a critical velocity at least as great, $q_{cr}(B) \geq q_{cr}(E)$, must lie entirely below the arc C in the open interval between the two extreme abscissae. Suppose, contrary to this assertion, that B either intersects or touches the open arc C . In particular, this will occur if B and E have the same width or enclose the same area. Then a suitable contraction (possibly the identity), centered at any interior point of E on the axis of symmetry, takes B into a

⁸ The actual existence of critical flows past finite profiles of bounded curvature has been proved by M. SHIFFMAN (unpublished).

profile $\bar{B}$ contained in E and touching it at some point P on C (Figure 1); this contraction takes the critical flow past B into a similar critical flow past $\bar{B}$ with the same free stream velocity, $q_{cr}(B) = q_{cr}(\bar{B})$. It is clear that the conditions of the comparison theorem are now satisfied by the flows past $\bar{B}$ and E , and therefore we have the inequality for the respective flow speeds $q_{\bar{B}}, q_E$ at P ,

$$q_{\bar{B}}(P) > q_E(P) = c_*$$

This is of course impossible, and from the contradiction we conclude that the extremal profile E solves both Problems 1 and 1'.

FIGURE 1a

FIGURE 1b

It is evident the problems have the same solution if the class of admissible profiles is widened in Problem 1 to include bodies of larger thickness ratio than the given value, and in Problem 1' to include bodies of larger area than the one specified (but with the same given length).

While the solutions of Problems 1 and 1' serve to provide upper bounds on the critical Mach number of bodies whose gross dimensions alone are prescribed, they are obviously unsuitable for aircraft and are of little assistance in the design of optimum airfoils. However, if additional restrictions are imposed on the class of admissible profiles, it is possible to modify Problem 1 (and 1') in such a way that the resultant extremal bodies describe more realistic wing forms. Suppose, for example, that in Problem 1 the maximum negative slope of the profile is prescribed, as well as the thickness ratio.^{8a} Among this more limited class of profiles again we seek the shape achieving the maximum critical velocity.

In close analogy with the result of Problem 1 we find the extremal profile to be an arc E of the prescribed dimensions composed of two straight segments S_1 and S_2 joined by a convex arc on which the speed is sonic for the critical flow past E ; the forward segment S_1 is vertical, while S_2 has the prescribed maximum negative slope and forms the trailing edge of the profile E (Figure 2). The profiles thus obtained are obviously closer to the needs of the aeronautical engineer than the previous solutions. Both the construction of such a body and the proof that it solves the modified extremal problem proceed in much the same fashion as for Problem 1. After we make the observation that the trailing edge of any admissible profile must lie in the angle formed by S_2 and the axis of sym-

^{8a} The thickness ratio in this problem cannot be arbitrarily large, but has the given maximum slope as an upper bound.

FIGURE 2

metry without crossing S_2 , the details of proof are identical with those in the solution of Problem 1.

4. Theoretical construction and properties of the extremal profile. We turn now to a more detailed study of the extremal body, the existence of which has been postulated in the preceding section, and which until now has been characterized only by the few properties required in the course of proof. We shall take up several of the theoretical questions related to construction of the profile, but leave the details of calculation to Part II of this work.

It is convenient to approach the problem of construction in reverse direction from that suggested by the extremal problem, namely, to begin by fixing the free stream velocity rather than the dimensions of the body and then seek a profile consisting of two vertical segments joined by an (unknown) free streamline on which the flow speed is sonic.⁹

As is customary in free boundary problems, the construction of the solution is based on the hodograph method. Let (q, θ) be polar coordinates in the hodograph plane, θ being the inclination of the velocity vector. The image in the hodograph plane of the streamline $IOAA'O'I$ bounding the (upper half) flow region (Figure 3a) is a closed curve consisting of the semicircular arc AA'^{10} on which $q = c_*$, the two vertical segments OA and $O'A'$ ($\theta = \pm \frac{1}{2}\pi$) corresponding to the vertical segments of the extremal body, and the two coincident segments OI , $O'I$ which form edges of the slit $\theta = 0$ extending from the origin to the point I ($q_0, 0$), where q_0 is the preassigned free stream speed (Figure 3b). To determine the unknown shape of the physical flow region and thereby the extremal body from the known shape of the image region R in the hodograph

⁹ Except for the fact that we are here dealing with compressible fluids, this problem is identical with that of the Riabouchinsky finite cavity in the classical theory of incompressible fluids. In the latter theory this model of cavity flow was first suggested by RIABOUCHINSKY (*Proc. Second Int. Cong. Appl. Mech. Zürich*, 1926, pp. 512–518) in an effort to simulate the observed finite cavities of cavitation flow past bodies. Curiously, although this model is physically unrealistic in its original setting as a solution of the finite cavitation problem, here it proves to be the solution of a genuine physical problem in the theory of compressible fluids.

¹⁰ The lettering in the hodograph plane corresponds to that in the physical plane.

FIGURE 3a

FIGURE 3b

plane we are led to seek either a mapping $(q, \theta) \rightarrow (\theta, \psi)$, defined by the equations,

$$(8) \quad \begin{aligned} \phi_\theta &= \frac{q}{\rho} \psi_q \\ \phi_q &= -\frac{1}{\rho q} \left(1 - \frac{q^2}{c^2} \right) \psi_\theta, \end{aligned}$$

which takes the open region R in the hodograph plane into the half plane $\psi > 0$; or we may seek a positive solution of the hodograph equation

$$(9) \quad \psi_{qq} + \frac{1}{q} \left(1 + \frac{q^2}{c^2} \right) \psi_q + \frac{1}{q^2} \left(1 - \frac{q^2}{c^2} \right) \psi_{\theta\theta} = 0,$$

in which the solution $\psi(q, \theta)$ is required to vanish on the boundary of R and to have a singularity at $I(q_0, 0)$ of the type¹¹

$$(10) \quad \text{Im} \left(\frac{1}{\sqrt{qe^{-i\theta} - q_0}} \right).$$

The mapping procedure is the natural one for incompressible fluids because of the availability of conformal mapping techniques. However, in the present problem, for lack of effective methods of treating the quasi-conformal mappings defined by (8), we have found it simpler to take the second approach suggested above, and to solve the boundary value problem directly. The existence of a positive solution $\psi(q, \theta)$ of (9) that vanishes on the boundary of R and has a

¹¹ This is the singularity for the stream function of uniform incompressible potential flows without circulation. The nature of the singularity is the same for subsonic flows.

singularity of the required type at $(q_0, 0)$ can be demonstrated without great difficulty by the method of integral equations. We leave the details for Part II of this work.

Having once obtained the required solution $\psi(q, \theta)$, we can construct the extremal body itself from the following formulas connecting the physical and hodograph planes,

$$x(q, \theta) = \int x_q dq + x_\theta d\theta, \quad y(q, \theta) = \int y_q dq + y_\theta d\theta,$$

where

$$\begin{aligned} x_q &= -\frac{1}{\rho q} \left[\frac{\cos \theta}{q} \left(1 - \frac{q^2}{c^2} \right) \psi_\theta + \sin \theta \psi_q \right], & x_\theta &= \frac{1}{\rho q} (q \cos \theta \psi_q - \sin \theta \psi_\theta) \\ y_q &= \frac{1}{\rho q} \left[-\frac{\sin \theta}{q} \left(1 - \frac{q^2}{c^2} \right) \psi_\theta + \cos \theta \psi_q \right], & y_\theta &= \frac{1}{\rho q} (q \sin \theta \psi_q + \cos \theta \psi_\theta). \end{aligned}$$

To determine the extremal profile, we use the fact that $\psi_q = 0$ and $\theta = \pm\pi/2$ on its straight sections, while $\psi_\theta = 0$ on the curved portion, so that the latter is given in parametric form by the formulas,

$$\begin{aligned} (11) \quad x(\theta) &= \frac{1}{\rho_*} \int_{\pi/2}^{\theta} \cos \theta \psi_q(c_*, \theta) d\theta + \text{const.} & (\rho_* = \rho(q)_{q=c_*}); \\ y(\theta) &= - \int_0^{c_*} \frac{1}{\rho q^2} \left(1 - \frac{q^2}{c^2} \right) \psi_\theta \left(q, \frac{\pi}{2} \right) dq + \frac{1}{\rho_*} \int_{\pi/2}^{\theta} \sin \theta \psi_q(c_*, \theta) d\theta. \end{aligned}$$

The shape of the extremal profile, but not its size, is fixed by this process, since ψ is determined only up to a constant factor by the boundary value problem.

Let us now examine some properties of the flow yielded by the hodograph solution $\psi(q, \theta)$. It is clear from the construction that the solution describes a uniform subsonic flow past a profile consisting of two vertical segments joined by a sonic streamline arc. That the flow region is the entire simply covered exterior of the extremal body follows from the fact that any subsonic solution $\psi(q, \theta)$ defines a quasi-conformal mapping into the physical plane, and these mappings have the same covering properties as the conformal mappings. Among the properties of the extremal profile we observe that the sonic arc is strictly convex. This is a consequence of the fact, to be observed during proof of the boundary point lemma (in Section 7) that the curvature of a streamline at a sonic boundary point is positive.¹² At the end points of the sonic arc the curvature becomes infinite, for the curvature on the sonic arc, which by (11) is equal to

$$(12) \quad - \frac{\rho}{\psi_q} (c_*, \theta),^{13}$$

¹² This can also be proved in the hodograph plane by showing that $\psi_q < 0$ on the sonic arc $q = c_*$. The curvature, $-\rho_*/\psi_q$, of the sonic arc is therefore positive.

¹³ The minus sign results from the definition of positive curvature in Footnote 2.

becomes infinite at the endpoints of the arc, where evidently $\psi_q = 0$. It is a simple consequence of the comparison theorem that the thickness ratio of the extremal body is a monotonically decreasing function of the critical Mach number. Furthermore, in the course of proof of existence of the extremal profile (Part II) it is shown that the profile varies continuously with the critical velocity; hence the thickness ratio and critical Mach number are continuous and strictly monotonic functions of one another. In the same process it becomes clear, too, that the thickness ratio approaches zero as the free stream Mach number approaches unity.

5. Bodies achieving minimum critical velocity. To obtain lower bounds on the critical velocity it no longer suffices to seek extremal bodies among profiles whose gross dimensions alone are specified. For it is obvious that by making the competing profiles sufficiently rough, arbitrarily small critical velocities can be achieved. This suggests that a well posed extremal problem for determination of a body achieving minimum critical velocity should impose limitations on the curvature of the admissible profiles. Such a problem is the following.

Problem 2. Among all profiles of given width and given maximum curvature, to determine the shape achieving the minimum critical velocity.

This problem can be stated non-dimensionally and in slightly more general form by defining the admissible class to consist of profiles for which the product of width and maximum curvature is less than or equal to some fixed quantity.

The resulting solution of the problem is not an unexpected one. If the width of the admissible profiles is less than or equal to the prescribed minimum radius of curvature, the extremal profile is simply the circular arc of the given dimensions. If, on the other hand, the width exceeds the allowed minimum radius of curvature, the solution is a semicircle of that radius mounted on two equal vertical segments which are so chosen that the complete figure has the required width.

The proof that the circular arc is the extremal solution follows from an appropriate application of the comparison theorem. Let B be any profile of the same width as a circular arc C and having its maximum curvature equal to the curvature of C . It is clear that all values of the slope on B are realized also on C , and that at points where B and C have the same slope the ordinate is greater or at least as great on C , for otherwise the width of B would exceed that of C . Suppose now that P is a sonic point in the critical flow past B . Draw the normal to B at P and extend it until it meets the axis of symmetry, say at O . It is possible by a suitable translation of C to bring it in such a position that the extension of OP intersects C at a point P' of the same slope as B at P . From our previous remark we know that either $P = P'$ or that P' lies outside B . A contraction centered at O (possibly the identity) takes C into a circular arc $\tilde{C}$ tangent to B at P and of non-decreased curvature; hence $\tilde{C}$ is contained in B .

If now the critical flow past C (and therefore past $\tilde{C}$) has its free stream velocity greater than or equal to that in the critical flow past B , or equivalently,

$$q_{cr}(\tilde{C}) = q_{cr}(C) \geq q_{cr}(B),$$

then the conditions of the comparison theorem are fulfilled, and we have a contradiction to the assumption that P is sonic, unless $B = \tilde{C} = C$. This proves that the circular arc is extremal.

In case the width of the admissible profiles exceeds their minimum radius of curvature, it is necessary to assume of the profiles that vertical lines intersect them in at most a single point or a segment. The argument to establish the extremal character of the asserted solution then proceeds as above without essential change.

There are several modifications of Problem 2 for which the solutions can be easily established by the same methods. For example, if the admissible profiles have prescribed area and maximum curvature, it is clear that the extremal body achieving the minimum critical velocity is again the circular arc of the appropriate dimensions. Or, to formulate a problem more suitable for airfoil design, let the admissible profiles have prescribed width (or area) and also prescribed maximum curvatures of different value at the points of positive and negative slope. In this case the extremal profile consists of two circular arcs meeting at the point of maximum ordinate, the leading and trailing arcs having the prescribed maximum curvatures (Figure 4). Although the proof that this

FIGURE 4

profile solves the extremal problem is simple as before, we must observe that the body itself is one for which the comparison theorem, in the generality proved here, does not immediately apply, for the profile has discontinuous curvature at the point where the two circular arcs meet; furthermore, the existence of a critical flow past this body has not been established. However, despite these omissions, which will probably be filled in time, it is reasonable from a practical point of view to accept this profile as solution of the extremal problem.

The numerical determination of the minimum critical velocities achieved by the extremal bodies of this section is a difficult computational problem for which an adequate procedure has not yet been formulated. However, if only a lower bound on the critical velocity is desired, the results of Section 3 can be used to good effect. Let us assume that the maximum critical speed as a function of thickness ratio has been computed for Problem 1 and is known. Consider now the circular arc solution C of Problem 2, and the extremal body E of Problem 1 having at its midpoint the same thickness and curvature as C . E has the property

that the minimum curvature of its sonic arc occurs at the midpoint. We now observe that E has a smaller critical velocity than C . For if sonic speed is achieved on C at a point P , a suitable horizontal translation and contraction of E will bring it to a position where it is interior to C and tangent to it at P . It is clear that the critical speed of E must be less than that of C , since otherwise the comparison theorem would apply and yield a contradiction at P .

In light of the extremal property of the circular arc we can therefore state: The extremal profile E of Problem 1 has a smaller critical velocity than all other profiles of the same thickness possessing maximum curvature equal or less than the curvature of E at its midpoint. Thus, if the critical velocities associated with the extremal profiles of Problem 1 are completely known, we have simultaneously upper and lower bounds on the critical velocities of arbitrary bodies of known shape.

6. Axially symmetric flows. While the results of the preceding sections have been developed for plane flows, the same results, as we shall see in this section, obtain for axially symmetric flows as well. In the latter flows equations (1) and (2) are replaced by

$$(13) \quad \left(1 - \frac{\phi_x^2}{c^2}\right)\phi_{xx} - \frac{2\phi_x\phi_y}{c^2}\phi_{xy} + \left(1 - \frac{\phi_y^2}{c^2}\right)\phi_{yy} + \frac{\phi_y}{y} = 0$$

and

$$(14) \quad \left(1 - \frac{\psi_x^2}{(\rho cy)^2}\right)\psi_{xx} + \frac{2\psi_x\psi_y}{(\rho cy)^2}\psi_{xy} + \left(1 - \frac{\psi_y^2}{(\rho cy)^2}\right)\psi_{yy} - \frac{\psi_y}{y} = 0,$$

where x, y are now rectangular coordinates in the meridian plane, x being the axial, and y the radial, variable. The maximum principle for the flow speed holds for these flows as well, although it is simpler to obtain this result directly from the equation for the potential in three dimensions than from either (13) or (14). By following the argument in the plane case and making the obvious modifications required for three-dimensional flows, we obtain the result that in any (not necessarily symmetric) spatial flow past a body in which the free stream velocity is subsonic and the flow is nowhere supersonic, the maximum flow speed is not achieved in the interior of the flow. For axially symmetric flows we know, furthermore, that the maximum speed is taken on the body.¹⁴

We see from the case of plane flows that the proof of the comparison theorem rests on the result that $\psi(x, y) - Q_0 y \rightarrow 0$ as $(x, y) \rightarrow \infty$. The analogue of this theorem for axially symmetric flows states that

$$\psi(x, y) - \frac{Q_0}{2} y^2 \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow \infty,$$

¹⁴ Cf. [3], pp. 249-250, where this result is established for subcritical flows.

a result which as yet is unproved, although undoubtedly true. The fundamental inequality (7)

$$\omega \equiv \bar{\psi} - \psi > 0,$$

required for the proof of the comparison theorem, can be inferred in a different way if we make the reasonable hypothesis that the critical flow described by the stream function ψ is the limit of subcritical flows ψ_M past the same body, these flows depending continuously on the free stream Mach number M . For if M_0 is the free stream Mach number of the flow described by $\bar{\psi}$ and M is that corresponding to ψ_M , $M < M_0$, we have (as proved in [3]) the inequality

$$\bar{\psi} - \psi_M > 0,$$

and from this follows in the limit

$$\bar{\psi} - \psi \geq 0.$$

From the maximum principle either the strict inequality holds everywhere in the common region or the two stream functions are identical.

The boundary point lemma can be proved for equation (14) just as for plane flows, the modifications in proof due to the difference between equations (2) and (14) for plane and axisymmetric flows being quite inessential, as the proof in Section 7 shows. However, the argument breaks down if the subject boundary point lies on the axis of symmetry ($y = 0$), a situation that does not occur in this paper.

The arguments in Sections 3 and 5 show that the earlier extremal problems, now formulated for axially symmetric bodies, have as solutions the bodies of revolution corresponding exactly to the extremal profiles of the plane problems. Namely, the solution to Problem 1 is now a body of revolution bounded by two circular discs spanned by a curved surface on which the flow is everywhere sonic for the critical flow, while the solution of Problem 2 is simply the surface obtained by rotating a circular arc of the appropriate dimensions. However, for axially symmetric problems we still lack a constructive procedure for calculating either the extremal body of Problem 1 or the critical Mach number associated with the extremal solution of Problem 2. The subsidiary extremal problems of Sections 3 and 5, and their solutions, also have their axisymmetric analogues.

7. Boundary point lemma. In Section 2 we have shown that under the conditions of the comparison theorem the inequality $\bar{\psi} - \psi \geq 0$ prevails in the flow region D bounded by the profile B and the axis of symmetry, and that the strict inequality holds in D if $\psi \equiv \bar{\psi}$. To complete the proof of the comparison theorem we have still to show that at any regular point of contact P between B and $\bar{B}$ the strict inequality $\bar{q}(P) > q(P)$ holds between the respective speeds when the

two flows are not identical. Since this result has already been proved for subcritical flows (*cf.* [3]) it remains only to prove the inequality in case $\bar{q}(P) = c_*$.

We offer first a relatively simple proof¹⁵ based on the assumption that the flow past $\bar{B}$ is the limit of subcritical flows past $\bar{B}$ depending continuously on the free stream velocity. This assumption is quite reasonable, and is supported by the fact that the critical flows for which existence has been proved by Shiffman (*cf.* Footnote 8) are indeed such limits of subcritical flows. To proceed with the proof, let Γ be an arc lying in D with its endpoints Q_1, Q_2 on the axis of symmetry on opposite sides of B . Let $\bar{\psi}'$ represent the stream functions of the family having $\bar{\psi}$ as its limit; then a $\bar{\psi}'$ can be found which is arbitrarily close to $\bar{\psi}$ over the entire length of Γ and such that simultaneously $\partial\bar{\psi}'/\partial n$ is arbitrarily close to $\partial\bar{\psi}/\partial n$ at Q_1 and Q_2 . Since $\bar{\psi} - \psi > 0$ in D and Q_1, Q_2 are by choice regular subsonic boundary points of D , we have

$$\frac{\partial\bar{\psi}}{\partial n}(Q_1) > \frac{\partial\psi}{\partial n}(Q_1), \quad \frac{\partial\bar{\psi}}{\partial n}(Q_2) > \frac{\partial\psi}{\partial n}(Q_2).$$

It is therefore possible to determine a $\bar{\psi}'$ such that $\bar{\psi}' > \psi$ on the (open) arc Γ . Since at the same time $\bar{\psi}' \geq \psi$ on the portion of the boundary of D intercepted by Γ , it follows that $\bar{\psi}' > \psi$ in the finite subregion of D cut off by Γ . Thus $\bar{\psi}' - \psi > 0$ in the neighborhood of P , while $\bar{\psi}'(P) = \psi(P) = 0$. This gives the inequality

$$\frac{\partial\bar{\psi}'}{\partial n}(P) \geq \frac{\partial\psi}{\partial n}(P),$$

or equivalently,

$$\bar{q}'(P) \geq q(P).$$

On the other hand, since $\bar{\psi}'$ by assumption describes a subcritical flow, we have also

$$\bar{q}(P) = c_* > \bar{q}'(P),$$

and finally the desired result,

$$c_* = \bar{q}(P) > q(P).$$

The principal objection to a proof based on the continuous dependence assumption, other than objection to use of a deep existence theory, is that it makes it necessary to establish the uniqueness of the critical flow by a separate argument, at present lacking. Without uniqueness, the extremal character of our solutions would then be proved only with respect to those competing flows to which the continuous dependence argument applies.

We now prove the boundary point lemma in the following local form.

¹⁵ This proof uses suggestions made to one of the authors by J. SERRIN.

Boundary point lemma. *Let γ and $\bar{\gamma}$ be continuously curved arcs which touch each other at P , but do not cross. Let D be an open neighborhood of P on one side of γ in which a subsonic stream function ψ is defined such that $\psi > 0$ in D , $\psi = 0$ on γ , and such that the second derivatives of ψ are continuous in $D + P$; let $\bar{\psi}$ be an analogous solution in a one-sided neighborhood $\bar{D}$ of $\bar{\gamma}$, with $D \subset \bar{D}$. In D we suppose $\bar{\psi} - \psi > 0$. Then*

$$\bar{q}(P) > q(P),$$

and, in particular, $q(P) < c_*$.

We treat first the relatively simple case in which the respective curvatures $\kappa, \bar{\kappa}$ are unequal at P . Since the lemma is proved in the subcritical case, it remains only to show that $q(P) < c_*$. Suppose, to the contrary, that $c_* = \bar{q}(P) = q(P)$. From the irrotationality of the flows we have at P ,

$$\frac{\partial q}{\partial n}(P) + \kappa q(P) = 0, \quad \frac{\partial \bar{q}}{\partial n}(P) + \bar{\kappa} \bar{q}(P) = 0,$$

while from the assumed inequality of the curvatures we have $\bar{\kappa} - \kappa > 0$. Hence

$$\frac{\partial(\bar{q} - q)}{\partial n}(P) = \kappa q(P) - \bar{\kappa} \bar{q}(P) = (\kappa - \bar{\kappa})c_* < 0.$$

Thus, along a small arc N normal at P , it follows that $\bar{q} < q$, and accordingly $|\text{grad } \bar{\psi}| < |\text{grad } \psi|$. In particular, if we take N to be a segment of the equipotential $\varphi = \text{const.}$ orthogonal to the streamlines $\psi = \text{const.}$, and let $\partial/\partial n$ designate differentiation along N , we obtain (on N)

$$\bar{\psi} - \psi = \int_N \left(\frac{\partial \bar{\psi}}{\partial n} - \frac{\partial \psi}{\partial n} \right) dn \leq \int_N (|\text{grad } \bar{\psi}| - |\text{grad } \psi|) dn < 0,$$

which contradicts the given inequality $\bar{\psi} - \psi > 0$.

It remains now to prove the lemma in case $\kappa = \bar{\kappa}$. As a preliminary step we show first that if $q(P) = c_*$ the curvature κ is not only non-negative, as is evident, but is actually positive. For convenience choose the origin of coordinates at P , with the x -axis tangential to γ at P , and the positive y -axis directed into the flow region D . Since the irrotationality condition

$$(15) \quad \frac{\partial q}{\partial n} + \kappa q = 0$$

holds at P , it suffices to prove that $\frac{\partial q}{\partial n}(P) < 0$, or equivalently, that $\varphi_{xy}(P) = u_y(P) < 0$. As noted in Section 2, u obeys equation (5), which is elliptic in the neighborhood of P , while $a(P) = b(P) = 0$ and $c(P) = 1$. In some circle C lying in D and tangent to γ at P let us consider the comparison function v previously defined in Section 2,

$$v = e^{-\alpha r^2} - e^{-\alpha R^2} \quad (r^2 = x^2 + (R - y)^2).$$

If C' is any concentric circle in C , we can make $L(v) > 0$ in the annulus between C and C' by choosing α sufficiently large. For suitable $\epsilon > 0$ the inequality

$$(16) \quad w \equiv c_* - u - \epsilon v \geq 0$$

is satisfied on C' , and trivially on C . Since $L(w) \leq 0$ in the annulus, we are assured that w has its minimum on the boundary (cf. [4]), and hence the inequality (16) holds throughout the annulus. Observing now that $w(P) = 0$, we conclude $w_y(P) \geq 0$, and therefore

$$u_y(P) \leq -\epsilon v_y(0, 0) < 0.$$

This establishes the result that streamline curvature is positive at a sonic boundary point.

We now complete the proof of the lemma by examining in detail the equation obeyed by $\omega = \bar{\psi} - \psi$ near P where $q(P) = \bar{q}(P) = c_*$. If the equation for ψ is subtracted from that for $\bar{\psi}$ we obtain

$$0 = a\omega_{xx} + 2b\omega_{xy} + c\omega_{yy} + \Delta,$$

where a, b, c , are the functions of x and y which result from inserting $\bar{\psi}(x, y)$ in the coefficients of the equation for $\bar{\psi}$, and

$$(17) \quad \Delta = \left(\frac{\psi_y^2}{(\rho c)^2} - \frac{\bar{\psi}_y^2}{(\bar{\rho} \bar{c})^2} \right) \psi_{xx} + 2 \left(\frac{\bar{\psi}_x \bar{\psi}_y}{(\bar{\rho} \bar{c})^2} - \frac{\psi_x \psi_y}{(\rho c)^2} \right) \psi_{xy} + \left(\frac{\psi_x^2}{(\rho c)^2} - \frac{\bar{\psi}_x^2}{(\bar{\rho} \bar{c})^2} \right) \psi_{yy}.$$

Application of the mean value theorem shows that Δ can be written in the form

$$\Delta = d\omega_x + e\omega_y,$$

where $d(x, y)$ and $e(x, y)$ are appropriate mean value quantities. Thus ω is a solution of the elliptic equation with continuous coefficients in D ,

$$(18) \quad L(\omega) = a\omega_{xx} + 2b\omega_{xy} + c\omega_{yy} + d\omega_x + e\omega_y = 0.$$

In the case of subcritical flows the coefficients d and e are bounded in the neighborhood of P , and this fact makes quite simple the proof that $\frac{\partial \omega}{\partial n}(P) > 0$.¹⁶

However, in the problem at hand d and e become infinite at P , necessitating a more refined analysis. To study these coefficients more closely let us consider, for example, the first term in the expression (17) for Δ . We have

$$(19) \quad \left(\frac{\psi_y^2}{(\rho c)^2} - \frac{\bar{\psi}_y^2}{(\bar{\rho} \bar{c})^2} \right) \psi_{xx} = \omega_x \left\{ \frac{2\psi_y^2 \psi_x}{(\rho c)^3 Q} \frac{d(\rho c)}{dQ} \right\} \psi_{xx} + \omega_y \left\{ \frac{2\psi_y^3}{(\rho c)^3 Q} \frac{d(\rho c)}{dQ} - \frac{2\psi_y}{(\rho c)^2} \right\} \psi_{xx},$$

¹⁶ Cf. [3] and E. HOPF, *Proc. Amer. Math. Soc.* **3**, 791-793 (1952).

where the quantities in braces are to be evaluated at certain intermediate values between ψ_x, ψ_y and $\bar{\psi}_x, \bar{\psi}_y$ for each point in D . We shall be interested only in the coefficient of ω_y . The singularity in the coefficient is contributed by the term

$$\frac{d(\rho c)}{dQ} = \frac{\frac{dc}{dq} - \frac{q}{c}}{1 - \frac{q^2}{c^2}}.$$

For $q = c = c_*$ the numerator becomes $c'(c_*) - 1$, and

$$\frac{d}{dq} \left(1 - \frac{q^2}{c^2} \right)_{q=c_*} = \frac{2}{c_*} (c'(c_*) - 1).$$

Therefore, in the neighborhood of $P(0, 0)$,

$$\frac{d(\rho c)}{dQ} = \frac{c_*}{2} \frac{(1 + \epsilon_1)}{\delta q}, \quad \epsilon_1 \rightarrow 0 \quad \text{as} \quad r \rightarrow 0,$$

where δq is the (intermediate) value of $q - c_*$ appearing (implicitly) in (19). We estimate δq by observing that the flow speed near P can be written in the form

$$q(x, y) = c_* - \mu(x) + q_y(x, ty)y, \quad 0 < t < 1,$$

where $\mu(x) \geq 0$, $\mu(x) \rightarrow 0$ as $x \rightarrow 0$, and

$$q_y(x, ty) = -c_* \kappa y + \epsilon_2, \quad \epsilon_2 \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

The latter estimate is based on (15) and the assumed continuity of the velocity derivatives at P . Since an analogous result holds for $\bar{q}(x, y)$, and recalling that $\kappa = \bar{\kappa}$, we have

$$\delta q = -c_* \kappa y + \bar{\epsilon}_2 y - \bar{\mu}(x) \leq -c_* \kappa y + \bar{\epsilon}_2 y$$

where $\bar{\epsilon}_2 \rightarrow 0$ as $r \rightarrow 0$, and $\bar{\mu}(x) \geq 0$, $\bar{\mu}(x) \rightarrow 0$ as $x \rightarrow 0$. Also, the quantities

$$\frac{2\psi_y^3}{(\rho c)^3 Q} - \frac{2}{\rho_* c_*}, \quad \psi_{xx} - \kappa \rho_* c_*, \quad \text{and} \quad \frac{2\bar{\psi}_y}{(\rho c)^2} - \frac{2}{\rho_* c_*},$$

vanish as $r \rightarrow 0$. Combining these results, we obtain the estimate for $y > 0$,

$$\left\{ \frac{2\psi_y^3}{(\rho c)^3 Q} \frac{d(\rho c)}{dQ} - \frac{2\psi_y}{(\rho c)^2} \right\} \psi_{xx} \geq - \frac{(1 + \epsilon_3)}{y}, \quad \epsilon_3 \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

From the vanishing of ψ_x, ψ_{xy} , etc., at P it is evident that the remaining expressions in (17) contribute terms of order $\epsilon(r)/y$ to e . Consequently we may assert

$$e(x, y) \geq \frac{-(1 + \epsilon(x, y))}{y} \quad \text{where} \quad \epsilon \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

Let us now choose a semicircle $S: r \leq r_0, y \geq 0$, in which

$$\frac{1 + \epsilon}{c} - 1 \leq \sigma < 1,$$

and such that the end points $x = \pm r_0$ do not lie on γ . Since $\kappa > 0$, and $\epsilon \rightarrow 0$, $c \rightarrow 1$ at P , such a semicircle can surely be found in D . In the interior of S we have

$$L(y^{2+\sigma}) \geq y^\sigma(2 + \sigma)[(1 + \sigma)c - (1 + \epsilon)] \geq 0,$$

while $\omega > 0$ on the arc $r = r_0$. It follows that for suitable $\alpha > 0$

$$(20) \quad \omega \geq \alpha y^{2+\sigma}$$

on the boundary of S , and since

$$L(\omega - \alpha y^{2+\sigma}) \leq 0$$

in S , the inequality (20) prevails in the interior of S as well as on the boundary (maximum principle). We now arrive at a contradiction, for the first two members of equation (2) for ψ and $\bar{\psi}$ vanish at P at least to the first order in r , from which it follows that ψ_{yy} and $\bar{\psi}_{yy}$ are both $O(r)$ as $r \rightarrow 0$, and hence $\omega_{yy} = O(r)$. But this implies $\omega \leq ky^3$ on $x = 0$ near P , contradicting (20). This completes proof of the boundary point lemma.

REFERENCES

- [1] M. SHIFFMAN, On the existence of subsonic flows of a compressible fluid, *J. Rational Mech. Anal.* **1**, 605-652 (1952).
- [2] L. BERS, Existence and uniqueness of a subsonic gas flow past a given profile (abstract), *Bull. Amer. Math. Soc.* **59**, 360 (1953).
- [3] D. GILBARG, Comparison methods in the theory of subsonic flows, *J. Rational Mech. Anal.* **2**, 233-251 (1953).
- [4] E. HOPF, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitzungsberichte Akad. Berlin* **1927**, 147-152; also BATEMAN, *Partial Differential Equations*, pp. 135-137.