The Stability of Viscous Flow between Rotating Cylinders in the Presence of a Radial Temperature Gradient

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CONTENTS

1. Introduction	182
2. The equations of the problem	183
3. The case $\Omega = \text{constant}$; the equations governing marginal stability. . .	186
4. A variational procedure for solving the characteristic value problem in case $\beta(r) = \beta r$ (case I)	187
5. The variational solution of the characteristic value problem for case I. One confining cylinder	190
6. The variational solution of the characteristic value problem for case I. Two confining cylinders	192
7. The solution of the characteristic value problem for case II. One con- fining cylinder	197
8. The solution of the characteristic value problem for case II. Two con- fining cylinders	202
9. An illustrative example	204
10. Acknowledgements	206
References	207

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1. Introduction. Recent experimental studies by D. Fultz [1] and R. Hide [2] on the patterns of fluid motion which occur in the space between two rotating co-axial cylinders in the presence of a radial temperature gradient have disclosed remarkable analogies with the phenomenon of the jet stream in the upper atmosphere. It is the opinion of Fultz and Hide that under the conditions of their experiments the effect of gravity was at least as important as that of the Coriolis force. Nevertheless, in this paper we shall ignore the effect of gravity and consider the simpler problem of the stability of the two-dimensional viscous flow between rotating cylinders when a radial temperature gradient is present. This simpler problem is of some interest in itself: it discloses a type of rotationally induced thermal instability which appears to be novel; and as we shall see this new type of thermal instability has several features in common with the more familiar Bénard type of gravitationally induced thermal instability.

It may be stated here that the present paper is one of a series which is devoted to a systematic study of the various problems of stability in hydrodynamics and hydromagnetics (Chandrasekhar [3]–[9]; also Chandrasekhar & Fermi [10]).

2. The equations of the problem. As we have stated we shall consider the problem as two-dimensional. The equations of motion and heat conduction appropriate to the problem on hand are in cylindrical polar co-ordinates (r, θ) ,

$$(1) \quad \rho \left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} \right) u_r - \rho \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + \rho \nu \left(\nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right),$$

$$(2) \quad \rho \left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} \right) u_\theta + \rho \frac{u_\theta u_r}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \rho \nu \left(\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right)$$

and

$$(3) \quad \left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} \right) T = \kappa \nabla^2 T,$$

where ρ , p and T denote the density, pressure and temperature, u_r and u_θ the components of the velocity in the radial and the transverse (θ) directions and ν and κ are the coefficients of kinematic viscosity and thermometric conductivity, respectively. Also, we have the “equation of state”

$$(4) \quad \rho = \rho_0(1 - \alpha \Delta T), \quad \Delta T = T - T_0,$$

where α denotes the coefficient of volume expansion, ρ_0 the density corresponding to a mean temperature T_0 and ΔT is the deviation of the local temperature from T_0 .

It may be readily verified that equations (1) and (2) admit the stationary solution

$$(5) \quad u_r = 0, \quad u_\theta = V(r) = Ar + \frac{B}{r} \quad \text{and} \quad p = p_0(r),$$

where A and B are constants (related to the angular velocities of rotation, Ω_1 and Ω_2 of the inner and the outer cylinders of radii R_1 and R_2 , $R_2 > R_1$) and $p_0(r)$ is determined by

$$(6) \quad \frac{\partial p_0}{\partial r} = \rho_0 \{1 - \alpha \Delta T_0(r)\} \frac{V^2}{r}.$$

In equation (6), $\Delta T_0(r)$ represents a stationary solution of equation (3). Regarding this latter, we shall make two assumptions:

$$(7) \quad \begin{array}{ll} \text{case I:} & \kappa \nabla^2 \{\Delta T_0(r)\} = \epsilon, \\ \text{and} & \\ \text{case II:} & \kappa \nabla^2 \{\Delta T_0(r)\} = 0. \end{array}$$

Case I will be appropriate to a situation when there is a uniform distribution of heat sources such that in the absence of conduction the temperature at each point will rise at the rate ϵ ; while case II will be appropriate to a situation when the two cylinders confining the liquid are maintained at constant, different, temperatures. The stationary temperature gradients maintained in the two cases will be

$$(8) \quad \text{case I:} \quad \frac{d}{dr} \Delta T_0(r) = \beta r \quad \left(\beta = \frac{\epsilon}{2\kappa} = \text{constant} \right)$$

and

$$(9) \quad \text{case II:} \quad \frac{d}{dr} \Delta T_0(r) = \frac{\beta}{r} \quad (\beta = \text{constant}).$$

In general we shall write

$$(10) \quad \frac{d}{dr} \Delta T_0(r) = \beta(r).$$

We shall now suppose that the solution represented by equations (5) and (6) is slightly perturbed. Let this perturbed motion be characterized by

$$(11) \quad u_r = u, \quad u_\theta = V + v, \quad \Delta T = \Delta T_0(r) + \tau \quad \text{and} \quad p = p_0 + \delta p,$$

where u and v are small compared to V while τ and δp are small compared to ΔT_0 and p_0 , respectively. With these assumptions the linearized form of equations (1) and (2) are

$$(12) \quad \rho_0 \left(\frac{\partial u}{\partial t} + \frac{V}{r} \frac{\partial u}{\partial \theta} \right) + \rho_0 \alpha \tau \frac{V^2}{r} - \rho_0 \frac{2Vv}{r} = -\frac{\partial}{\partial r} \delta p + \rho_0 \nu \left(\nabla^2 u - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right)$$

and

$$(13) \quad \rho_0 \left(\frac{\partial v}{\partial t} + u \frac{\partial V}{\partial r} + \frac{V}{r} \frac{\partial v}{\partial \theta} \right) + \rho_0 \frac{Vu}{r} = -\frac{1}{r} \frac{\partial}{\partial \theta} \delta p + \rho_0 \nu \left(\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} \right),$$

where it should be noted that in obtaining these equations the further assumption has been made that α , except when multiplied by V^2 , leads to a quantity of the second order of smallness. The assumption in particular that $\alpha\tau V^2$ is a quantity of the first order of smallness is a necessary one: on it depends the onset of instability when the temperature gradient which is maintained exceeds a certain critical value. It may be recalled in this connection that a similar assumption is made in the theory of the gravitationally induced thermal instability (Rayleigh [11], Jeffreys [12]) when the effect of the variation in density (due to thermal expansion) is ignored in all terms in the equations of motion except the one in gravity.

In the framework of the approximations leading to equations (12) and (13) the equation of continuity reduces simply to the statement that the velocity is a solenoidal vector. Thus,

$$(14) \quad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0.$$

On making use of the particular form of V (equation (5)) we find that equations (12) and (13) can be reduced to the forms

$$(15) \quad \frac{\partial \varpi}{\partial r} = -\frac{\partial u}{\partial t} - \frac{V}{r} \frac{\partial u}{\partial \theta} - \alpha\tau \frac{V^2}{r} + 2 \frac{Vv}{r} + \nu \left(\nabla^2 u - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right)$$

and

$$(16) \quad \frac{1}{r} \frac{\partial \varpi}{\partial \theta} = -\frac{\partial v}{\partial t} - 2Au - \frac{V}{r} \frac{\partial v}{\partial \theta} + \nu \left(\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} \right),$$

where for the sake of brevity we have written

$$(17) \quad \varpi = \delta p / \rho_0.$$

Eliminating ϖ between equations (15) and (16) we obtain

$$(18) \quad \begin{aligned} \frac{\partial}{\partial \theta} \left\{ -\frac{\partial u}{\partial t} - \frac{V}{r} \frac{\partial u}{\partial \theta} - \alpha\tau \frac{V^2}{r} + 2 \frac{Vv}{r} + \nu \left(\nabla^2 u - \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{u}{r^2} \right) \right\} \\ = \frac{\partial}{\partial r} r \left\{ -\frac{\partial v}{\partial t} - 2Au - \frac{V}{r} \frac{\partial v}{\partial \theta} + \nu \left(\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} \right) \right\}. \end{aligned}$$

Returning to the equation of heat conduction (3), we find that the corresponding linearized form of this equation is

$$(19) \quad \frac{\partial \tau}{\partial t} + \beta(r)u + \frac{V}{r} \frac{\partial \tau}{\partial \theta} = \kappa \nabla^2 \tau$$

We shall now seek solutions of equations (14), (18) and (19) which are of the forms

$$(20) \quad u = u(r)e^{i(n\theta + \sigma t)}, \quad v = v(r)e^{i(n\theta + \sigma t)}, \quad \tau = \tau(r)e^{i(n\theta + \sigma t)},$$

where n is an integer and σ is a constant unspecified for the present. For solutions of the form (20) equations (14), (18) and (19) become

$$(21) \quad rD^* u = D(ru) = -inv,$$

$$(22) \quad \begin{aligned} in \left\{ -i\sigma u - in \frac{Vu}{r} - \alpha\tau \frac{V^2}{r} + 2 \frac{Vv}{r} + v \left[\left(DD^* - \frac{n^2}{r^2} \right) u - 2in \frac{v}{r^2} \right] \right\} \\ = rD^* \left\{ -i\sigma v - 2Au - in \frac{Vv}{r} + v \left[\left(DD^* - \frac{n^2}{r^2} \right) v + 2in \frac{u}{r^2} \right] \right\} \end{aligned}$$

and

$$(23) \quad i\sigma\tau + \beta(r)u + in \frac{V\tau}{r} = \kappa \left(D^* D - \frac{n^2}{r^2} \right) \tau,$$

where

$$(24) \quad D = \frac{d}{dr} \quad \text{and} \quad D^* = D + \frac{1}{r}.$$

The operators D and D^* satisfy the commutation relation

$$(25) \quad DD^* = D^* D - \frac{1}{r^2} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2};$$

this relation is useful in the subsequent reductions.

Equation (22) can be rearranged in the manner

$$(26) \quad \begin{aligned} rD^* \left[v \left(DD^* - \frac{n^2}{r^2} \right) - in\Omega - i\sigma \right] v - in \left[v \left(DD^* - \frac{n^2}{r^2} \right) - in\Omega - i\sigma \right] u \\ - 2 \left[rD^* \left(A - in \frac{v}{r^2} \right) u + in \left(\frac{V}{r} - in \frac{v}{r^2} \right) v \right] = -in\alpha \frac{V^2}{r} \tau, \end{aligned}$$

where

$$(27) \quad \Omega = \frac{V}{r} = A + \frac{B}{r^2},$$

denotes the angular velocity. We can now eliminate v from equation (26) by making use of equation (21). We find

$$(28) \quad \begin{aligned} \left[rD^* \left\{ v \left(DD^* - \frac{n^2}{r^2} \right) - in\Omega - i\sigma \right\} rD^* - n^2 \left\{ v \left(DD^* - \frac{n^2}{r^2} \right) - in\Omega - i\sigma \right\} \right. \\ \left. - 2in \frac{B}{r} D^* - \frac{4n^2 v}{r^2} \right] u = -(n^2 \Omega^2 \alpha) r\tau; \end{aligned}$$

while equation (23) can be written in the form

$$(29) \quad \left\{ \kappa \left(D^* D - \frac{n^2}{r^2} \right) - in\Omega - i\sigma \right\} \tau = \beta(r)u.$$

Equations (28) and (29) must be solved together with the boundary conditions which state that at $r = R_1$ and $r = R_2$ the fluctuations in the velocity and in the temperature must vanish, *i.e.*,

$$(30) \quad u = v = 0 \quad \text{and} \quad \tau = 0 \quad \text{at} \quad r = R_1 \quad \text{and} \quad r = R_2.$$

3. The case $\Omega = \text{constant}$; the equations governing marginal stability. In the remaining part of this paper the assumption will be made that the two cylinders are rotated at the same constant rate. As may be expected on general grounds and as is also apparent from equations (28) and (29) this assumption will lead to considerable simplification.

When

$$(31) \quad \Omega = A = \text{constant and } B = 0,$$

it is permissible to introduce a *phase angular velocity*

$$(32) \quad \omega = n\Omega + \sigma.$$

In terms of ω equations (28) and (29) take the simpler forms

$$(33) \quad \left[rD^* \left\{ \nu \left(DD^* - \frac{n^2}{r^2} \right) - i\omega \right\} rD^* \right. \\ \left. - n^2 \left\{ \nu \left(DD^* - \frac{n^2}{r^2} \right) - i\omega \right\} - \frac{4n^2 \nu}{r^2} \right] u = -(n^2 \Omega^2 \alpha) r\tau$$

and

$$(34) \quad \left[\kappa \left(D^* D - \frac{n^2}{r^2} \right) - i\omega \right] \tau = \beta(r)u,$$

while the form of the solutions sought is (*cf.* equations (20))

$$(35) \quad u = u(r)e^{in(\theta - \Omega t) + i\omega t} \quad \text{and} \quad \tau = \tau(r)e^{in(\theta - \Omega t) + i\omega t}.$$

Accordingly, in a frame of reference rotating with the angular velocity Ω , what distinguishes stability from instability of a pattern of fluid motion with $2n$ vortices, is the real part, $\Re(\omega)$, of ω . Now, it is known that in several related problems (Meksyn [13]; also Pellew & Southwell [14]) the situation in marginal stability (*i.e.*, one on the verge of stability) is characterized by $\omega = 0$ (and not merely by $\Re(\omega) = 0$); in other words, the principle of the exchange of stabilities is valid. In this paper we shall assume that this principle is valid also for

the problem on hand. In a later paper we shall return to a justification of the principle.

On the assumption then that the principle of the exchange of stabilities is valid, the equations governing marginal stability are

$$(36) \quad \left\{ r D^* \left(D D^* - \frac{n^2}{r^2} \right) r D^* - n^2 \left(D D^* - \frac{n^2}{r^2} \right) - \frac{4n^2}{r^2} \right\} u = - \frac{n^2 \Omega^2 \alpha}{\nu} r \tau$$

and

$$(37) \quad \kappa \left(D^* D - \frac{n^2}{r^2} \right) \tau = \beta(r) u.$$

On expanding the differential operator on the left-hand side of equation (36), we find that the equation is in fact equivalent to

$$(38) \quad \left(D^* D - \frac{n^2}{r^2} \right)^2 (ru) = - \frac{n^2 \Omega^2 \alpha}{\nu} \tau.$$

From equations (37) and (38) it now follows that

$$(39) \quad \left(D^* D - \frac{n^2}{r^2} \right)^3 (ru) = - \frac{n^2 \Omega^2 \alpha}{\kappa \nu} \beta(r) u,$$

where it may be noted that (*cf.* equation (25))

$$(40) \quad D^* D - \frac{n^2}{r^2} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2}.$$

According to equations (21) and (38) the boundary conditions with respect to which we must solve equation (39) are

$$(41) \quad \begin{aligned} u = 0, \quad D(ru) = 0 \quad \text{and} \quad \left(D^* D - \frac{n^2}{r^2} \right)^2 (ru) = 0 \\ \text{at} \quad r = R_1 \quad \text{and} \quad r = R_2. \end{aligned}$$

It is evident that solving equation (39) together with the six boundary conditions (41), three at each of the two boundaries, is equivalent to a characteristic value problem. In the following sections it will be shown how we can solve this problem for the two cases (8) and (9). It is the manner of solving this characteristic value problem that constitutes the essential mathematical content of this paper.

4. A variational procedure for solving the characteristic value problem in case $\beta(r) = \beta r$ (case I). In this case, letting

$$(42) \quad W = ru$$

and measuring r in units of the radius, R_2 , of the outer cylinder, we may first observe that the characteristic value problem we have to solve is one of determining

$$(43) \quad S_n = \frac{\Omega^2 \alpha \beta}{\kappa \nu} R_2^6,$$

such that the equation

$$(44) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right)^3 W = -n^2 S_n W,$$

may have a non-trivial solution which satisfies the boundary conditions

$$(45) \quad W = 0, \quad DW = 0 \quad \text{and} \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right)^2 W = 0$$

at $r = 1$ and $r = R_1/R_2 = \eta$ (say).

Reduced in this manner, the problem is seen to be very similar to the one encountered in the theory of the thermal instability of fluid spheres and spherical shells (*cf.* Chandrasekhar [4], [5] and [6]). And as in this latter theory the solution for the lowest characteristic number, S_n , can be effected by the application of a variational principle which we shall now proceed to formulate.

Letting

$$(46) \quad \begin{aligned} G &= \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) W = \frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) - \frac{n^2}{r^2} W, \\ F &= \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right)^2 W = \frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) - \frac{n^2}{r^2} G, \end{aligned}$$

we can rewrite the differential equation governing W in the form

$$(47) \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{dF}{dr} \right) - \frac{n^2}{r^2} F = -n^2 S_n W.$$

The boundary conditions (45) now require that

$$(48) \quad F = W = DW = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad r = \eta.$$

Multiply equation (47) by rF and integrate over the range of r . The left-hand side of the equation gives

$$(49) \quad \int_{\eta}^1 F \frac{d}{dr} \left(r \frac{dF}{dr} \right) r dr - n^2 \int_{\eta}^1 F^2 \frac{dr}{r}.$$

By integrating by parts the first of the two integrals in (49) and remembering that F vanishes at both limits, we obtain

$$(50) \quad -\int_{\eta}^1 \left\{ r \left(\frac{dF}{dr} \right)^2 + \frac{n^2 F^2}{r} \right\} dr.$$

Turning next to the right-hand side of equation (47) we have (cf. equation (46))

$$(51) \quad \int_{\eta}^1 r W F dr = \int_{\eta}^1 W \frac{d}{dr} \left(r \frac{dG}{dr} \right) dr - n^2 \int_{\eta}^1 W G \frac{dr}{r}.$$

After two integrations by parts the foregoing becomes

$$(52) \quad \int_{\eta}^1 r W F dr = \left(r W \frac{dG}{dr} - r G \frac{dW}{dr} \right)_{\eta}^1 + \int_{\eta}^1 r G \left\{ \frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) - \frac{n^2}{r^2} W \right\} dr.$$

The integrated parts vanish on account of the boundary conditions (cf. equation (48)) and we are left with

$$(53) \quad \int_{\eta}^1 r W F dr = \int_{\eta}^1 r G^2 dr.$$

The result of multiplying equation (47) by rF and integrating is, therefore,

$$(54) \quad n^2 S_n = \frac{\int_{\eta}^1 r \{ (dF/dr)^2 + n^2 F^2/r^2 \} dr}{\int_{\eta}^1 r G^2 dr}.$$

This formula expresses S_n as the ratio of two positive definite integrals.

If we now consider the effect on S_n of a variation δW in W compatible with the boundary conditions, we readily find that

$$(55) \quad n^2 \delta S_n = -\frac{2}{\int_{\eta}^1 r G^2 dr} \int_{\eta}^1 r \left\{ \frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} - \frac{n^2}{r^2} F + n^2 S_n W \right\} \delta F dr.$$

Hence to the first order, $\delta S_n \equiv 0$ for all small arbitrary variations δF . Further, it follows from (55) that the true solution of the problem gives a minimal value for S_n . This last fact enables us to formulate the following variational procedure of solving equation (47) and satisfying the boundary conditions of the problem:

Assume for F an expression involving one or more parameters A_j which vanishes at $r = 1$ and $r = \eta$. With the chosen form of F determine W as a solution of the equation

$$(56) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) W = F,$$

which satisfies the boundary conditions

$$(57) \quad W = 0 \quad \text{and} \quad \frac{dW}{dr} = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad r = \eta.$$

Then evaluate S_n according to formula (54) and minimize it with respect to the parameters A_j . In this way we shall obtain the "best" value of S_n for the chosen form of F .

5. The variational solution of the characteristic value problem for case I. One confining cylinder. We shall first consider the case $\eta = 0$ *i.e.*, when there is only one confining cylinder. In this case the continuity of the solutions at $r = 0$ and the form of the equation to be solved for W suggest that we assume for F the trial function

$$(58) \quad F = \sum_j A_j J_n(\alpha_j r),$$

where J_n denotes the Bessel function of order n , the α_j 's ($j = 1, 2, \dots$) are its zeros and the A_j 's are the variational parameters. With this choice, F vanishes at $r = 1$ as required. It may be recalled here that the functions $J_n(\alpha_j r)$ (for a given n) satisfy the orthogonality relations

$$(59) \quad \int_0^1 r J_n(\alpha_j r) J_n(\alpha_k r) dr = \frac{1}{2} [J'_n(\alpha_j)]^2 \delta_{jk},$$

where primes denote differentiation with respect to the argument of the Bessel function and δ_{jk} is the usual Kronecker symbol.

With F given by equation (58) the equation to be solved for W is

$$(60) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) W = \sum_j A_j J_n(\alpha_j r).$$

The general solution of this equation which has no singularity at the origin is

$$(61) \quad W = \sum_j (A_j / \alpha_j^4) J_n(\alpha_j r) + Br^n + Cr^{n+2},$$

where B and C are constants of integration. The condition $W = 0$ at $r = 1$ requires $B = -C$ and we have

$$(62) \quad W = \sum_j (A_j / \alpha_j^4) J_n(\alpha_j r) + B(r^n - r^{n+2}).$$

The constant B is determined by the remaining condition at $r = 1$, namely that here dW/dr must vanish. This leads to

$$(63) \quad B = \frac{1}{2} \sum_j (A_j / \alpha_j^3) J'_n(\alpha_j).$$

Turning next to the evaluation of S_n according to formula (54), we find that for F and W given by equations (58) and (62)

$$\begin{aligned}
 \int_0^1 r \left\{ \left(\frac{dF}{dr} \right)^2 + \frac{n^2 F^2}{r^2} \right\} dr \\
 &= - \int_0^1 r F \left\{ \frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} - \frac{n^2 F}{r^2} \right\} dr \\
 (64) \quad &= \int_0^1 r \Sigma_j A_j J_n(\alpha_j r) \Sigma_k A_k \alpha_k^2 J_n(\alpha_k r) dr \\
 &= \frac{1}{2} \Sigma_j A_j^2 \alpha_j^2 [J'_n(\alpha_j)]^2
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^1 r G^2 dr &= \int_0^1 r W F dr \\
 (65)^* \quad &= \frac{1}{2} \Sigma_j (A_j^2 / \alpha_j^4) [J'_n(\alpha_j)]^2 + B \Sigma_j A_j \int_0^1 (r^{n+1} - r^{n+3}) J_n(\alpha_j r) dr \\
 &= \frac{1}{2} \Sigma_j (A_j^2 / \alpha_j^4) [J'_n(\alpha_j)]^2 + 2B \Sigma_j (A_j / \alpha_j^2) J_{n+2}(\alpha_j).
 \end{aligned}$$

The resulting expression for S_n is, therefore

$$(66) \quad n^2 S_n = \frac{\Sigma_j A_j^2 \alpha_j^2 [J'_n(\alpha_j)]^2}{\Sigma_j (A_j^2 / \alpha_j^4) [J'_n(\alpha_j)]^2 + 2 \Sigma_j (A_j / \alpha_j^2) J_{n+2}(\alpha_j) \Sigma_k (A_k / \alpha_k^3) J'_n(\alpha_k)},$$

where we have substituted for B from equation (63). By minimizing this last expression with respect to the A_j 's we shall obtain the "best" value for the lowest characteristic number S_n for the chosen form of F .

The simplest trial function for F of the chosen form is

$$(67) \quad F = J_n(\alpha_1 r),$$

where α_1 is the first zero of J_n . For this choice of F there is no variational parameter with respect to which we have to minimize and equation (66) directly gives

$$(68) \quad n^2 S_n = \frac{\alpha_1^2 J'_n(\alpha_1)}{\alpha_1 J'_n(\alpha_1) + 2 J_{n+2}(\alpha_1)}.$$

Values of S_n obtained with the aid of this formula are listed in Table 1.

The values given by (68) can be improved by including a second term in F . Thus, with the assumption

$$(69) \quad F = J_n(\alpha_1 r) + A J_n(\alpha_2 r),$$

* The transformations used in going from (49) to (50) and similarly from (51) to (53) can be used in the reverse fashion since these depend (apart from definitions) only on the boundary conditions of the problem.

TABLE 1
The Lowest Characteristic Numbers S_n for Case I and One Confining Cylinder

n	FIRST APPROXIMATION	A	SECOND APPROXIMATION
1	6954	0.0811	6873
2	8416	0.0944	8297
3	1.235×10^4	0.1017	1.216×10^4
4	1.828×10^4	0.1059	1.800×10^4
5	2.648×10^4	0.1083	2.607×10^4
6	3.731×10^4	0.1097	3.675×10^4
7	5.123×10^4	0.1104	5.048×10^4
8	6.871×10^4	0.1107	6.773×10^4
9	9.027×10^4	0.1107	8.902×10^4
10	1.165×10^5	0.1105	1.149×10^5
11	1.478×10^5	0.1101	1.459×10^5
12	1.851×10^5	0.1097	1.828×10^5
13	2.287×10^5	0.1092	2.260×10^5
14	2.795×10^5	0.1086	2.762×10^5
15	3.381×10^5	0.1080	3.343×10^5

where α_2 denotes the second zero of J_n and A is a variational parameter, equation (66) gives

$$(70) \quad n^2 S_n = \alpha_1^7 \{ [J'_n(\alpha_1)]^2 + A^2 (\alpha_2/\alpha_1)^2 [J'_n(\alpha_2)]^2 \} \times [J'_n(\alpha_1) \{ \alpha_1 J'_n(\alpha_1) + 2J_{n+2}(\alpha_1) \} \\ + 4A(\alpha_1/\alpha_2)^3 J_{n+2}(\alpha_1) J'_n(\alpha_2) + A^2 (\alpha_1/\alpha_2)^5 J'_n(\alpha_2) \{ \alpha_2 J'_n(\alpha_2) + 2J_{n+2}(\alpha_2) \}]^{-1}.$$

The values of S_n obtained after minimizing the foregoing expression with respect to A are listed in Table 1 together with the values of A which give the minimum values.

From a comparison of the results obtained in the first and the second approximations it would appear that the second approximation gives values which are probably correct to one part in 10^4 .

6. The variational solution of the characteristic value problem for case I. Two confining cylinders. Turning next to the solution of the characteristic value problem for case I when there are two confining cylinders, we shall assume as a trial function for F a linear combination of the Bessel functions, $J_n(\alpha r)$ and $Y_n(\alpha r)$, of the two kinds which vanishes at $r = 1$ and $r = \eta$. For this latter purpose we first define the cylinder function (of order ν)

$$(71) \quad \mathcal{C}_{n,\nu}(z) = Y_n(\alpha\eta)J_\nu(z) - J_n(\alpha\eta)Y_\nu(z),$$

where α is a constant which we shall leave unspecified for the present. Then,

$$(72) \quad \mathcal{C}_{n,n}(\alpha r) = Y_n(\alpha\eta)J_n(\alpha r) - J_n(\alpha\eta)Y_n(\alpha r),$$

clearly vanishes for $r = \eta$; it will also vanish for $r = 1$ provided

$$(73) \quad Y_n(\alpha\eta)J_n(\alpha) - J_n(\alpha\eta)Y_n(\alpha) = 0.$$

It is known (*cf.* Gray & Mathews [15] p. 82, theorem X) that equation (73) admits an infinite number of roots all of which are real and simple; and that if $\alpha_j (j = 1, 2, \dots)$ are the distinct roots of the equation, the functions $\mathcal{C}_{n,n}(\alpha_j r)$ ($j = 1, 2, \dots$) form an orthogonal set with the integral property

$$(74) \quad \int_{\eta}^1 r \mathcal{C}_{n,n}(\alpha_j r) \mathcal{C}_{n,n}(\alpha_k r) dr = N_{j,n} \delta_{jk},$$

where

$$(75) \quad N_{j,n} = \frac{2}{\pi^2 \alpha_j^2} \left\{ \frac{J_n^2(\alpha_j \eta)}{J_n^2(\alpha_j)} - 1 \right\}.$$

For later use we may note here that the derivatives of $\mathcal{C}_{n,n}(\alpha_j r)$ at $r = 1$ and $r = \eta$ (which we shall denote by $\mathcal{C}'_n(\alpha_j)$ and $\mathcal{C}'_n(\alpha_j \eta)$, respectively) are given by

$$(76) \quad \mathcal{C}'_n(\alpha_j) = \left[\frac{d}{dr} \mathcal{C}_{n,n}(\alpha_j r) \right]_{r=1} = -\frac{2}{\pi} \frac{J_n(\alpha_j \eta)}{J_n(\alpha_j)}$$

and

$$(77) \quad \mathcal{C}'_n(\alpha_j \eta) = \left[\frac{d}{dr} \mathcal{C}_{n,n}(\alpha_j r) \right]_{r=\eta} = -\frac{2}{\eta\pi}.$$

Also, since $\mathcal{C}_{n,\nu}(z)$ is a cylinder function of ν it satisfies (with respect to ν) the same recurrence relations as the Bessel functions J_ν and Y_ν .

Returning to the variational solution of the characteristic value problem we assume for F the trial function

$$(78) \quad F(r) = \sum_j A_j \mathcal{C}_{n,n}(\alpha_j r),$$

where the A_j 's are the variational parameters. With this choice, F vanishes at $r = 1$ and $r = \eta$. With F given by (78) the equation governing W (equation (56)) can be explicitly solved and for $n > 1$ we have (*cf.* equation (61))

$$(79) \quad W = \sum_j (A_j / \alpha_j^4) \mathcal{C}_{n,n}(\alpha_j r) + B_1 r^n + B_2 r^{n+2} + B_3 r^{-n} + B_4 r^{-n+2},$$

where B_1, B_2, B_3 and B_4 are constants of integration to be determined by the boundary conditions (57). (It may be explicitly noted here that the solution

given by (79) does not apply for $n = 1$; we shall return to this case presently.) The boundary conditions (57) lead to the equations

$$(80) \quad B_1 + B_2 + B_3 + B_4 = 0,$$

$$(81) \quad B_1 \eta^n + B_2 \eta^{n+2} + B_3 \eta^{-n} + B_4 \eta^{-n+2} = 0,$$

$$(82) \quad nB_1 + (n+2)B_2 - nB_3 - (n-2)B_4 = -\Sigma_j (A_j/\alpha_j^4) \mathcal{C}'_n(\alpha_j),$$

$$(83) \quad nB_1 \eta^{n-1} + (n+2)B_2 \eta^{n+1} - nB_3 \eta^{-n-1} - (n-2)B_4 \eta^{-n+1} \\ = -\Sigma_j (A_j/\alpha_j^4) \mathcal{C}'_n(\alpha_j \eta),$$

where $\mathcal{C}'_n(\alpha_j)$ and $\mathcal{C}'_n(\alpha_j \eta)$ are defined as in equations (76) and (77). On solving the foregoing equations we find that the solution can be expressed in the form

$$(84) \quad B_i = \Sigma_j (A_j/\alpha_j^4) b_{ij} \quad (i = 1, 2, 3 \text{ and } 4),$$

where the constants b_{1j} , b_{2j} , b_{3j} and b_{4j} depend only on α_j . Thus

$$(85) \quad b_{2j} = K_n \{ (n-1)(1-\eta^2)\Delta_n(\alpha_j) + (1-\eta^{-2n+2})\delta_n(\alpha_j) \}$$

and

$$(86) \quad b_{4j} = K_n \{ (1-\eta^{2n+2})\Delta_n(\alpha_j) - (n+1)(1-\eta^2)\delta_n(\alpha_j) \},$$

where

$$(87) \quad \Delta_n(\alpha_j) = -\frac{1}{2} \{ \mathcal{C}'_n(\alpha_j) - \eta^{-n+1} \mathcal{C}'_n(\alpha_j \eta) \},$$

$$(88) \quad \delta_n(\alpha_j) = -\frac{1}{2} \{ \mathcal{C}'_n(\alpha_j) - \eta^{n+1} \mathcal{C}'_n(\alpha_j \eta) \}$$

and

$$(89) \quad K_n = [(n^2-1)(1-\eta^2)^2 + (1-\eta^{2n+2})(1-\eta^{-2n+2})]^{-1}.$$

With b_{2j} and b_{4j} given by equations (85) and (86), b_{1j} and b_{3j} follow from the equations:

$$(90) \quad b_{1j} = -\frac{1}{n} \{ \frac{1}{2} \mathcal{C}'_n(\alpha_j) + (n+1)b_{2j} + b_{4j} \}$$

and

$$(91) \quad b_{3j} = +\frac{1}{n} \{ \frac{1}{2} \mathcal{C}'_n(\alpha_j) + b_{2j} - (n-1)b_{4j} \}.$$

Turning next to the evaluation of S_n we find that for F and W given by equations (78) and (79) the integrals occurring in (54) can be reduced by steps similar to those indicated in (64) and (65). Thus, we now find

$$(92) \quad \int_{\eta}^1 r \left\{ \left(\frac{dF}{dr} \right)^2 + \frac{n^2 F^2}{r^2} \right\} dr = \Sigma_j A_j^2 \alpha_j^2 N_{j,n}$$

and

$$(93) \quad \int_{\eta}^1 r G^2 dr = \Sigma_j (A_j^2 / \alpha_j^4) N_{j,n} \\ + \Sigma_j A_j \int_{\eta}^1 \mathfrak{C}_{n,n}(\alpha_j r) [B_1 r^{n+1} + B_2 r^{n+3} + B_3 r^{-n+1} + B_4 r^{-n+3}] dr.$$

The remaining integrals on the right-hand side of (93) can be evaluated if proper use is made of the various recurrence relations satisfied by the cylinder functions $\mathfrak{C}_{n,\nu}$ and also of equations (80) and (81); and we find after some lengthy reductions that

$$(94) \quad \int_{\eta}^1 r G^2 dr = \Sigma_j (A_j^2 / \alpha_j^4) N_{j,n} - 2B_2 \Sigma_j (A_j / \alpha_j^2) \mathfrak{M}_{n,n+2}^{(j)} \\ - 2B_4 \Sigma_j (A_j / \alpha_j^2) \mathfrak{M}_{n,n-2}^{(j)},$$

where

$$(95) \quad \mathfrak{M}_{n,\nu}^{(j)} = \mathfrak{C}_{n,\nu}(\alpha_j) - \eta^{\nu} \mathfrak{C}_{n,\nu}(\alpha_j \eta)$$

and

$$(96) \quad \mathfrak{N}_{n,\nu}^{(j)} = \mathfrak{G}_{n,\nu}(\alpha_j) - \eta^{-\nu} \mathfrak{C}_{n,\nu}(\alpha_j \eta).$$

The resulting expression for S_n is, therefore,

$$(97) \quad n^2 S_n = \frac{\Sigma_j A_j^2 \alpha_j^2 N_{j,n}}{\Sigma_j \{ (A_j^2 / \alpha_j^4) N_{j,n} - 2B_2 (A_j / \alpha_j^2) \mathfrak{M}_{n,n+2}^{(j)} - 2B_4 (A_j / \alpha_j^2) \mathfrak{M}_{n,n-2}^{(j)} \}}$$

where B_2 and B_4 are given by equations (84) to (89).

As we have already stated the foregoing solution does not apply for the case $n = 1$. The reason why we must distinguish this case is that the complementary function in the solution (79) is no longer the general one as the terms in B_1 and B_4 become identical when $n = 1$. However, it is readily found that in the case $n = 1$ the general solution for W is

$$(98) \quad W = \Sigma_j (A_j / \alpha_j^4) \mathfrak{C}_{1,1}(\alpha_j r) + B_1 r + B_2 r^3 + B_3 r^{-1} + B_4 r \log r,$$

and the equations determining the constants of integration are:

$$(99) \quad B_1 + B_2 + B_3 = 0, \\ B_1 \eta + B_2 \eta^3 + B_3 \eta^{-1} + B_4 \eta \log \eta = 0, \\ B_1 + 3B_2 - B_3 + B_4 = -\Sigma_j (A_j / \alpha_j^4) \mathfrak{C}'_1(\alpha_j), \\ B_1 + 3B_2 \eta^2 - B_3 \eta^{-2} + B_4 (1 + \log \eta) = -\Sigma_j (A_j / \alpha_j^4) \mathfrak{C}'_1(\alpha_j \eta).$$

On solving these equations we find that the solutions for B_1 , B_2 , B_3 and B_4 can be expressed in the same way as before (*i.e.* by equations of the form (84)); but equations (85), (86), (89), (90) and (91) are now replaced by

$$\begin{aligned}
 b_{2j} &= K_1 \{ \tfrac{1}{2}(1 - \eta^2) \Delta_1(\alpha_j) + (\log \eta) \delta_1(\alpha_j) \}, \\
 b_{4j} &= K_1 \{ - (1 - \eta^4) \Delta_1(\alpha_j) + 2 (1 - \eta^2) \delta_1(\alpha_j) \}, \\
 (100) \quad K_1 &= [(1 - \eta^2)^2 + (1 - \eta^4) \log \eta]^{-1}, \\
 b_{1j} &= -[\tfrac{1}{2} \mathcal{C}'_1(\alpha_j) + 2b_{2j} + \tfrac{1}{2}b_{4j}] \\
 \text{and} \\
 b_{3j} &= +[\tfrac{1}{2} \mathcal{C}'_1(\alpha_j) + b_{2j} + \tfrac{1}{2}b_{4j}].
 \end{aligned}$$

The reduction of the formula for S_1 proceeds somewhat differently but essentially along the same lines as for $n > 1$. We finally obtain

$$(101) \quad S_1 = \frac{\sum_j A_j^2 \alpha_j^2 N_{j,1}}{\sum_j \{ (A_j/\alpha_j^4) N_{j,1} - 2B_2(A_j/\alpha_j^2) \mathfrak{M}_{1,3}^{(j)} + 2B_4(A_j/\alpha_j^3) \mathfrak{M}_{1,0}^{(j)} \}}.$$

(In equations (100) and (101) the general definitions of $\Delta_n(\alpha_j)$, $\delta_n(\alpha_j)$, $\mathfrak{M}_{n,\nu}^{(j)}$ and $\mathfrak{N}_{n,\nu}^{(j)}$ are retained without modifications.)

The simplest trial function for F of the chosen form is

$$(102) \quad F = \mathcal{C}_{n,n}(\alpha_1 r),$$

where α_1 is the first zero of the expression on the left-hand side of equation (73). For this choice of F there is no variational parameter with respect to which we have to minimize and equations (97) and (101) directly give (after substituting for B_2 and B_4):

$$\begin{aligned}
 (103) \quad S_n &= \frac{\alpha_1^8}{n^2} N_{1,n} \{ \alpha_1^2 N_{1,n} \\
 &\quad - 2K_n[(n-1)(1 - \eta^2) \Delta_n(\alpha_1) + (1 - \eta^{-2n+2}) \delta_n(\alpha_1)] \mathfrak{M}_{n,n+2}^{(1)} \\
 &\quad - 2K_n[(1 - \eta^{2n+2}) \Delta_n(\alpha_1) - (n+1)(1 - \eta^2) \delta_n(\alpha_1)] \mathfrak{N}_{n,n-2}^{(1)} \}^{-1} \\
 &\quad (n > 1)
 \end{aligned}$$

and

$$\begin{aligned}
 (104) \quad S_1 &= \alpha_1^8 N_{1,1} \{ \alpha_1^2 N_{1,1} - 2K_1[\tfrac{1}{2}(1 - \eta^2) \Delta_1(\alpha_1) + (\log \eta) \delta_1(\alpha_1)] \mathfrak{M}_{1,3}^{(1)} \\
 &\quad + 2K_1[-(1 - \eta^4) \Delta_1(\alpha_1) + 2(1 - \eta^2) \delta_1(\alpha_1)] \mathfrak{M}_{1,0}^{(1)} / \alpha_1 \}^{-1} \quad (n = 1).
 \end{aligned}$$

In Table 2 are listed the values of S_n obtained with the aid of the foregoing formulae and the recent tabulation of Chandrasekhar & Donna Elbert [16]

TABLE 2

The Lowest Characteristic Numbers S_n for Case I and Two Confining Cylinders: η denotes the Ratio of the Radius of the Inner to that of the Outer Cylinder

n	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$	$\eta = 0.5$	$\eta = 0.6$	$\eta = 0.8$
1	2.399×10^4	4.945×10^4	1.184×10^5	3.418×10^5	1.275×10^6	7.974×10^7
2	1.225×10^4	2.006×10^4	4.057×10^4	1.038×10^5	3.553×10^5	2.038×10^7
3	1.343×10^4	1.736×10^4	2.856×10^4	6.204×10^4	1.886×10^5	9.395×10^6
4	1.855×10^4	2.052×10^4	2.786×10^4	5.065×10^4	1.332×10^5	5.557×10^6
5	2.649×10^4	2.725×10^4	3.223×10^4	4.960×10^4	1.118×10^5	3.789×10^6
6		3.765×10^4	4.069×10^4	5.436×10^4	1.053×10^5	2.837×10^6
7					1.075×10^5	2.274×10^6
8					1.160×10^5	1.918×10^6
9						1.686×10^6
10						1.532×10^6
11						1.445×10^6
12						1.368×10^6

of the roots of equation (73). The dependence of S_n on n is further illustrated in Figure 1. From Table 2 and Figure 1 it is apparent how the pattern of convection which first appears at marginal stability shifts progressively to systems with a large number of vortices as η approaches unity.

The values given by (103) and (104) could be improved by including a second term in F . However, in view of the fact that for $\eta = 0$ the first approximation already gives values accurate to a fraction of a per cent (*cf.* Table 1) it was not considered necessary to carry out this improvement.

7. The solution of the characteristic value problem for case II. One confining cylinder. In case II the confining cylinders are maintained at constant, but different, temperatures. From a laboratory standpoint this is therefore the more important of the two cases considered.

In case II (*cf.* equation (9))

$$(105) \quad \beta(r) = \beta/r \quad (\beta = \text{a constant}),$$

and equation (39) becomes

$$(106) \quad \left(D^* D - \frac{n^2}{r^2}\right)^3 W = - \frac{n^2 \Omega^2 \alpha \beta}{\kappa \nu} \frac{W}{r^2},$$

where, as before, $W = ru$. Again, if we measure r in units of the radius R_2 of the outer cylinder the problem reduces to one of determining

$$(107) \quad S_n = \frac{\Omega^2 \alpha \beta}{\kappa \nu} R_2^4,$$

FIGURE 1. The criterion (S_n) for the onset of instability with a convection pattern with $2n$ vortices in the viscous flow between two rotating cylinders in the presence of a radial temperature gradient, for various values of the ratio (η) of the radii of the two cylinders and for case I. The curves labelled 1, 2, ..., 7 refer to $\eta = 0, 0.2, 0.3, 0.4, 0.5, 0.6$ and 0.8 , respectively.

such that the differential equation

$$(108) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right)^3 W = -n^2 S_n \frac{W}{r^2},$$

allows a non-trivial solution which satisfies the boundary conditions (45).

Now, it can be readily verified that the occurrence of $1/r^2$ on the right-hand side of equation (108) prevents the formulation of a variational procedure for solving the underlying characteristic value problem. We shall therefore describe a different method of solution which, it will appear, is quite satisfactory for the

purposes of determining the lowest characteristic number S_n . We shall illustrate the method by considering first the special case when there is only one confining cylinder. Strictly, the consideration of this special case under the conditions postulated is improper: For, the admission of a temperature gradient which varies as $1/r$ implies a logarithmic singularity in the temperature distribution at the origin if r should take the value zero. However, in spite of this singularity at the origin, the characteristic value problem itself has a meaning and the consideration of this case as a limit of permissible situations may be allowed.

As in §5 we shall let

$$(109) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right)^2 W = F,$$

and rewrite equation (108) in the form

$$(110) \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) F = -n^2 S_n \frac{W}{r^2}.$$

We shall now assume that in the interval $(0, 1)$, F can be expanded as a series in the form

$$(111) \quad F(r) = \sum_j A_j J_n(\alpha_j r),$$

where (*cf.* equation (59))

$$(112) \quad A_j = 2[J'_n(\alpha_j)]^{-2} \int_0^1 r F(r) J_n(\alpha_j r) dr.$$

With F given by (111) the solution of equation (109) which is continuous at the origin and satisfies the boundary conditions $W = dW/dr = 0$ at $r = 1$, is the same as that given in §5. We have (*cf.* equations (62) and (63))

$$(113) \quad W = \sum_j (A_j / \alpha_j^4) J_n(\alpha_j r) + B(r^n - r^{n+2}),$$

where

$$(114) \quad B = \frac{1}{2} \sum_j (A_j / \alpha_j^3) J'_n(\alpha_j).$$

Substituting (111) and (113) in equation (110), we are left with

$$(115) \quad \sum_j A_j \alpha_j^2 J_n(\alpha_j r) = \frac{n^2 S_n}{r^2} \{ \sum_j (A_j / \alpha_j^4) J_n(\alpha_j r) + B(r^n - r^{n+2}) \}.$$

Now multiplying equation (115) by $r J_n(\alpha_k r)$ and integrating over the range r , we obtain

$$(116) \quad \frac{1}{2} A_k \alpha_k^2 [J'_n(\alpha_k)]^2 = n^2 S_n \int_0^1 \frac{1}{r} \left\{ \sum_j \frac{A_j}{\alpha_j^4} J_n(\alpha_j r) + B(r^n - r^{n+2}) \right\} J_n(\alpha_k r) dr.$$

Letting

$$(117) \quad \left(j \left| \frac{1}{r^2} \right| k \right) = \int_0^1 J_n(\alpha_j r) \frac{1}{r} J_n(\alpha_k r) dr$$

and

$$(118) \quad D_k = \int_0^1 (r^{n-1} - r^{n+1}) J_n(\alpha_k r) dr,$$

we can rewrite equation (116) in the form

$$(119) \quad \frac{1}{2} A_k \alpha_k^2 [J'_n(\alpha_k)]^2 Q = \Sigma_j (A_j / \alpha_j^4) (j | r^{-2} | k) + B D_k,$$

where

$$(120) \quad Q = \frac{1}{n^2} S_n.$$

Finally, substituting for B from equation (114) we obtain the following infinite system of homogeneous equations for the A_j 's:

$$(121) \quad \Sigma_j \frac{A_j}{\alpha_j^4} \left\{ \left(j \left| \frac{1}{r^2} \right| k \right) + \frac{1}{2} \alpha_j J'_n(\alpha_j) D_k - \frac{1}{2} \alpha_j^6 [J'_n(\alpha_j)]^2 Q \delta_{jk} \right\} = 0.$$

Hence Q is to be determined as a root of the infinite determinantal equation

$$(122) \quad \left\| \left(j \left| \frac{1}{r^2} \right| k \right) + \frac{1}{2} \alpha_j J'_n(\alpha_j) D_k - \frac{1}{2} \alpha_j^6 [J'_n(\alpha_j)]^2 \delta_{jk} \right\| = 0.$$

By using the known properties of the Bessel functions, we can show that

$$(123) \quad \begin{aligned} D_k = \frac{1}{\alpha_k^n} \{ & 2^{n-1} (n-1)! - 2(n-1) \alpha_k^{n-2} J_{n-2}(\alpha_k) \\ & - 2^2 (n-1)(n-2) \alpha_k^{n-3} J_{n-3}(\alpha_k) \\ & - 2^3 (n-1)(n-2)(n-3) \alpha_k^{n-4} J_{n-4}(\alpha_k) - \dots - 2^{n-1} (n-1)! J_0(\alpha_k) \}. \end{aligned}$$

Also the diagonal elements of the matrix $(j | r^{-2} | k)$ can be explicitly evaluated. We find (*cf.* Watson [17] p. 137) that

$$(124) \quad \left(j \left| \frac{1}{r^2} \right| j \right) = \frac{1}{2n} \left\{ 1 - 2 \sum_{m=1}^{n-1} J_m^2(\alpha_j) - J_0^2(\alpha_j) \right\}.$$

But it does not appear that the non-diagonal elements can be similarly evaluated.

Now a method of solving equation (122) of infinite order would be to set the determinant formed by the first j rows and columns equal to zero and let j take increasingly larger values. In practise the success of this method will depend on how rapidly the lowest root of the equation of order j tends to its limit

as $j \rightarrow \infty$. It will appear that for the problem on hand the process converges sufficiently rapidly.

On the method of solution outlined in the preceding paragraph the *first approximation* for the lowest characteristic number S_n will be given by setting the $(1, 1)$ element of the matrix equal to zero. Thus

$$(125) \quad \frac{1}{2} \alpha_1^2 [J'_n(\alpha_1)]^2 Q = \frac{1}{\alpha_1^4} \left(1 \left| \frac{1}{r^2} \right| 1 \right) + \frac{J'_n(\alpha_1)}{2\alpha_1^3} D_1,$$

where α_1 denotes the first zero of $J_n(z)$. Substituting for $(1|r^{-2}|1)$ and D_1 in accordance with equations (123) and (124), we obtain

$$(126) \quad \begin{aligned} n^2 S_n = & \alpha_1^6 [J'_n(\alpha_1)]^2 \left[\frac{1}{n} \left\{ 1 - 2 \sum_{m=1}^{n-1} J_m^2(\alpha_1) - J_0^2(\alpha_1) \right\} \right. \\ & + \frac{J'_n(\alpha_1)}{\alpha_1^{n-1}} \{ 2^{n-1}(n-1)! - 2(n-1)\alpha_k^{n-2} J_{n-2}(\alpha_1) \\ & \left. - 2^2(n-1)(n-2)\alpha_1^{n-3} J_{n-3}(\alpha_1) - \dots - 2^{n-1}(n-1)! J_0(\alpha_1) \} \right]^{-1}. \end{aligned}$$

The values of S_n obtained with the aid of the foregoing formula for $n = 1, 2, \dots, 6$ are given in Table 3. For these same values of n a second approximation was also carried out by setting the determinant formed by the first two rows and columns of (122) equal to zero; and for $n = 1$ a third approximation was carried out as well. In carrying out these higher approximations the non-diagonal elements $(j|r^{-2}|k)(j \neq k)$ had to be evaluated numerically. This part of the work was enormously lightened by tables of the functions

$$(127) \quad J_n(\alpha_j) \text{ for } j = 1(1)5, \quad n = 1(1)6 \text{ and } r = 0(0.01)1.00,$$

calculated for the purpose by the Electronic Computer Project of the Institute for Advanced Study at Princeton. (A fuller acknowledgement is made at the end of the paper.)

A comparison of the values of S_n given in Table 3 shows that on going from the first to the second approximation we reduce the value of S_n by a nearly

TABLE 3
The Lowest Characteristic Numbers S_n for Case II and One Confining Cylinder

n	FIRST APPROXIMATION	SECOND APPROXIMATION	THIRD APPROXIMATION
1	1180	1075	1077
2	2340	2128	
3	4390	4000	
4	7560	6910	
5	12180	11170	
6	18570	17100	

constant factor: in fact, the factor varies only between 1.100 and 1.086 in the tabulated range of n . Also, in the one case for which the result of a third approximation is available, the value of S_n was changed from that given by the second approximation by only one part in 500. From these facts we may conclude that already the second approximation provides the required characteristic numbers with errors not probably exceeding one per cent.

8. The solution of the characteristic value problem for case II. Two confining cylinders. Turning finally to the solution for case II when there are two confining cylinders, we shall assume that F can be expanded as a series in the form (cf. equations (74) and (75))

$$(128) \quad F = \sum_j A_j \mathcal{C}_{n,n}(\alpha_j r),$$

$$(129) \quad \text{where} \quad A_j = \frac{1}{N_{j,n}} \int_{\eta}^1 r F(r) \mathcal{C}_{n,n}(\alpha_j r) dr.$$

With F given by (128) the solution of equation (109) which satisfies the boundary conditions (57) is evidently the same as that given in §6. Combining equations (79) and (84) (and similarly (98) and (84)) we can write the required solution for W in the form

$$(130) \quad W = \sum_j (A_j / \alpha_j^4) \{ \mathcal{C}_{n,n}(\alpha_j r) + b_{1j} r^n + b_{2j} r^{n+2} + b_{3j} r^{-n} + b_{4j} r^{-n+2} \text{ (or } r \log r \text{ in case } n = 1) \},$$

where the coefficients b_{1j} , etc., have the same meanings as in §6 (equations (85) to (91) and (100)).

Now substituting for F and W according to equations (128) and (130) in equation (109) we obtain

$$(131) \quad \sum_j A_j \alpha_j^2 \mathcal{C}_{n,n}(\alpha_j r) = \frac{n^2 S_n}{r^2} \sum_j (A_j / \alpha_j^4) \{ \mathcal{C}_{n,n}(\alpha_j r) + b_{1j} r^n + b_{2j} r^{n+2} + b_{3j} r^{-n} + b_{4j} r^{-n+2} \text{ (or } r \log r \text{ in case } n = 1) \}.$$

Next multiplying this equation by $r \mathcal{C}_{n,n}(\alpha_k r)$ and integrating over the range of r we obtain (cf. equations (74) and (75))

$$(132) \quad A_k \alpha_k^2 N_{k,n} Q = \sum_j (A_j / \alpha_j^4) \times \{ (j | r^{-2} | k) + b_{1j} \langle r^{n-1} \rangle_k + b_{2j} \langle r^{n+1} \rangle_k + b_{3j} \langle r^{-n-1} \rangle_k + b_{4j} \langle r^{-n+1} \rangle_k \text{ (or } \langle \log r \rangle_k \text{ in case } n = 1) \},$$

where $Q = 1/n^2 S_n$,

$$(133) \quad \left(j \left| \frac{1}{r^2} \right| k \right) = \int_{\eta}^1 \mathcal{C}_{n,n}(\alpha_j r) \frac{1}{r} \mathcal{C}_{n,n}(\alpha_k r) dr$$

and

$$(134) \quad \langle X \rangle_k = \int_{\eta}^1 X \mathfrak{C}_{n,n}(\alpha_k r) dr.$$

Equation (132) represents the following infinite system of homogeneous equations for the constants A_j :

$$(135) \quad \Sigma_j (A_j / \alpha_j^4) \{ (j | r^{-2} | k) + b_{1j} \langle r^{n-1} \rangle_k + b_{2j} \langle r^{n+1} \rangle_k + b_{3j} \langle r^{-n-1} \rangle_k \\ + b_{4j} \langle r^{-n+1} \rangle_k \text{ (or } \langle \log r \rangle_k \text{ in case } n = 1) - Q \alpha_j^6 N_{j,n} \delta_{jk} \} = 0.$$

Hence Q is to be determined as a root of the infinite determinantal equation

$$(136) \quad \| (j | r^{-2} | k) + b_{1j} \langle r^{n-1} \rangle_k + b_{2j} \langle r^{n+1} \rangle_k + b_{3j} \langle r^{-n-1} \rangle_k \\ + b_{4j} \langle r^{-n+1} \rangle_k \text{ (or } \langle \log r \rangle_k \text{ in case } n = 1) - Q \alpha_j^6 N_{j,n} \delta_{jk} \| = 0.$$

By using the known properties of cylinder functions, we can show that

$$(137) \quad \langle r^{n-1} \rangle_k = -\frac{1}{\alpha_k^n} \{ \alpha_k^{n-1} \mathfrak{M}_{n,n-1}^{(k)} + 2(n-1) \alpha_k^{n-2} \mathfrak{M}_{n,n-2}^{(k)} \\ + 2^2(n-1)(n-2) \alpha_k^{n-3} \mathfrak{M}_{n,n-3}^{(k)} + \dots + 2^{n-1}(n-1)! \mathfrak{M}_{n,0}^{(k)} \},$$

$$(138) \quad \langle r^{n+1} \rangle_k = \mathfrak{M}_{n,n+1}^{(k)} / \alpha_k,$$

$$(139) \quad \langle r^{-n+1} \rangle_k = -\mathfrak{N}_{n,n-1}^{(k)} / \alpha_k$$

and

$$(140) \quad \left(k \left| \frac{1}{r^2} \right| k \right) = -\frac{1}{2n} \left\{ [\mathfrak{C}_{n,0}^2(\alpha_k) - \mathfrak{C}_{n,0}^2(\alpha_k \eta)] \right. \\ \left. + 2 \sum_{m=1}^{n-1} [\mathfrak{C}_{n,m}^2(\alpha_k) - \mathfrak{C}_{n,m}^2(\alpha_k \eta)] \right\}.$$

But it does not appear that

$$(141) \quad \langle r^{-n-1} \rangle_k = \int_{\eta}^1 \frac{dr}{r^{n+1}} \mathfrak{C}_{n,n}(\alpha_k r), \\ \langle \log r \rangle_k = \int_{\eta}^1 \log r \mathfrak{C}_{1,1}(\alpha_k r) dr \quad (n = 1),$$

or the non-diagonal elements of the matrix (133) can be expressed similarly in terms of known quantities; in these cases numerical integration would seem unavoidable.

As in §7 a first approximation to the value of the lowest characteristic number S_n can be obtained by setting the (1, 1) element of the matrix on the left-hand

TABLE 4

The Lowest Characteristic Numbers S_n for Case II and Two Confining Cylinders: η denotes the Ratio of the Radius of the Inner to the Outer Cylinder

n	FIRST APPROXIMATION				SECOND APPROXIMATION
	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$	$\eta = 0.5$	$\eta = 0.5$
1	7478			1.855×10^5	1.897×10^5
2	4109	7936		5.652×10^4	5.813×10^4
3	5013	7188	1.356×10^4	3.399×10^4	3.505×10^4
4	7750	9025	1.299×10^4	2.799×10^4	2.893×10^4
5			1.620×10^4	2.772×10^4	2.868×10^4
6				3.081×10^4	3.171×10^4

side of equation (136) equal to zero. The values of S_n obtained in such a first approximation are given in Table 4 for various values of n and η .

We already know (§7) that for $\eta = 0$, the error of the first approximation is to give values of S_n which are too *high* by ten per cent. To estimate the errors in the present calculations, a second approximation was carried out for $\eta = 0.5$. The values obtained in the second approximation (by setting the determinant formed by the first two rows and columns of the secular matrix equal to zero) are also listed in Table 4. And a comparison of the values of S_n ($n = 1, 2, \dots, 6$) obtained in the first and in the second approximations indicates that now the error of the first approximation is to give values of S_n which are too *low* by three per cent. From these facts we may conclude that no entry in Table 4 is likely to be in error by more than ten per cent; indeed, for $\eta = 0.3, 0.4$ and 0.5 the errors may be very much less.

The results of the calculations of this as well as of the preceding section are illustrated in Figure 2. The remarkable similarity of these results with those obtained for case I (compare particularly Figures 1 and 2) is especially noteworthy.

9. An illustrative example. As an example illustrative of the theory developed in this paper we shall consider the case of water (at room temperatures) enclosed between two long cylinders maintained at constant different temperatures and of radii 5.0 cm and 2.5 cm, respectively. Let the entire system be rotated at a constant angular velocity of 1 radian per second. Then according to the results of Table 4, the lowest mode of instability is one with ten vortices ($n = 5$) and occurs when

$$(142) \quad S = \frac{\Omega^2 \alpha \beta}{\kappa \nu} R_2^4 = 2.87 \times 10^4.$$

(Actually, the values of S at which convection patterns with eight or twelve vortices can occur are not substantially higher.) Inserting the numerical values

$$\Omega = 1 \text{ sec}^{-1}; \quad R_2 = 5 \text{ cm}; \quad \alpha = 2.1 \times 10^{-4} \text{ degree}^{-1},$$

FIGURE 2. Same as Figure 1 but for case II. The curves labelled 1, 2, 3, 4 and 5 refer to $\eta = 0, 0.2, 0.3, 0.4$ and 0.5 , respectively; of the two curves labelled 1 the dotted one refers to the results of the second approximation while the other to the results of the first approximation. For $\eta = 0.5$ (curve 5) the results of the second approximation are plotted; the remaining curves refer to the results of the first approximation.

(143)

$$\kappa = 1.4 \times 10^{-3} \text{ cm}^2/\text{sec} \quad \text{and} \quad \nu = 1.0 \times 10^{-2} \text{ cm}^2/\text{sec},$$

we find from (142) that the critical value of β at which instability will set in is

$$(144) \quad \beta_{\text{critical}} = 3.1 \text{ degree/cm.}$$

Under the circumstances considered (namely that of case II)

$$(145) \quad \frac{dT}{dr} = \frac{\beta}{r};$$

accordingly, β is related to the difference in temperature of the two cylinders by

$$(146) \quad \Delta T = T_2 - T_1 = \beta \log (R_2/R_1) = -\beta \log \eta.$$

Hence (144) implies a difference in temperature of

$$(147) \quad \Delta T = 3.1 \log_e 2 = 2.1^\circ.$$

In other words, for water enclosed between two cylinders of radii 5.0 and 2.5 cm and rotated at the rate 1 radian/sec, instability of the type considered will arise when the difference in temperature between the two cylinders increases above 2.1° . On account of the occurrence of R_2^4 in the definition of S , this difference in temperature predicted for $R_2 = 5.0$ cm and $R_2 = 2.5$ cm will be reduced by a factor 16 if the scale of the apparatus is increased by a factor 2 and the rate of rotation is unaltered *i.e.*, $\Delta T = 0.13^\circ$ under these latter circumstances. Now the experiments of Fultz and Hide seem to have been performed with cylinders of radii 10 cm or more and rates of rotation much larger than 1 radian/sec; moreover, the patterns of motion they have studied are for differences of temperature generally in excess of one degree. Consequently, in their experiments factors not included in this theory must have been operative. Nevertheless, it would be of interest to repeat their experiments with cylinders of radii 5 cm or less (or with smaller rates of rotation) to see if the effects predicted here can be observed.

10. Acknowledgements. As has already been acknowledged the calculations for case II when there is only one confining cylinder (§7) were greatly facilitated by the Electronic Computer Project of the Institute for Advanced Study at Princeton having provided tables of the functions

$$(148) \quad J_n(\alpha_j r) \text{ for } n = 1(1)6, \quad j = 1(1)5 \text{ and } r = 0(0.01)1.00.$$

Similarly the calculations presented in §8 for two confining cylinders were facilitated by the Project having provided tables of the functions

$$(149) \quad \mathcal{C}_{n,n}(\alpha_j r) \text{ for } \eta = 0.2(0.1)0.6 \text{ and } 0.8, \quad n = 1(1)6 \text{ and } j = 1$$

and, also

$$(150) \quad \mathcal{C}_{n,n}(\alpha_j r) \text{ for } \eta = 0.5, \quad n = 1(1)6 \text{ and } j = 2 \text{ and } 3.$$

All these functions have been tabulated at intervals of 0.01 to eight significant figures of which at least seven are believed to be reliable.

It must be evident that without the tables provided by the Electronic Computer Project of the Institute for Advanced Study the task of evaluating numerically the various integrals needed in the calculation of S_n (for case II) would have been too formidable to have undertaken. It is therefore a particular pleasure to acknowledge here my indebtedness to Professor J. von Neumann and Dr. Herman H. Goldstine of the Institute for Advanced Study for their most generous co-operation. My thanks are also due to Miss Donna Elbert who gave valuable assistance in all the remaining calculations.

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