*The Elementary Basic Principles of the Unified Theory of Relativity. C₂: Applications I**

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* Prepared under joint contract with the Office of Naval Research and the Army Office of Ordnance Research.

INTRODUCTION

This paper is the first of a series of papers which deal with the physical applications of the unified theory of relativity as proposed by A. Einstein [1], [2]*. It is an organic continuation of the papers [3], [4], [5], and [6] as far as results and notations are concerned.

The Einstein theory is based on the conditions

$$(0.1a) \quad D_{\omega} g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} \quad [6] \text{ II}(1.4a) \dagger$$

$$(0.1b) \quad S_{\lambda} \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha} = 0 \quad [6] \text{ II}(1.4b)$$

$$(0.2) \quad R_{\mu\lambda} = \partial_{[\mu} X_{\lambda]} \quad [6] \text{ II}(1.5)$$

where X_{λ} is an arbitrary gradient or an arbitrary vector. These two possibilities characterize *the special* and the *general case* (cf. [6] II, Section 1).

In this paper we deal with a very special case characterized by the condition

$$(0.3) \quad U^{\nu}_{\lambda\mu} = 0 \quad [6] \text{ I}(1.10).$$

This equation is a structural restriction imposed on the tensor $g_{\lambda\mu}$. On the other hand we shall see that for weak electromagnetic fields this condition is *automatically satisfied*. This holds in particular for experimental tests of the unified theory by celestial mechanics. This is also the reason why we started the physical interpretation with this very simple case.‡

The paper is divided in two chapters. In the first chapter we deal with physical applications in the absence of moving particles; the second chapter deals with the same problem in the presence of moving particles. The main idea may be roughly described by the following steps.

a) There is a skew symmetric tensor $m_{\lambda\nu}$ (cf. [6] I(3.8)) such that (0.1b) is equivalent to

$$(0.4a) \quad \partial_{[\mu} m_{\lambda\nu]} = 0.$$

Moreover, the tensors $k_{\lambda\nu} = g_{(\lambda\nu)}$, $h_{\lambda\mu} = g_{(\lambda\mu)}$ define a vector density v^{ν} for which we have the identity

$$(0.5) \quad v^{\nu} \equiv \partial_{\lambda} m^{\lambda\nu} \quad (m^{\lambda\nu} = \sqrt{|\mathfrak{h}|} m^{\lambda\nu}).$$

In order to convert this identity into an equation we substitute into $R_{[\mu\lambda]}$ the solution

$$(0.1c) \quad \Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}_{\lambda\mu}$$

* Numbers in square brackets refer to the bibliography at the end of the paper.

† The Roman numbers *I*, *II* refer to Chapters. The remaining numbers in round brackets refer here to equations.

‡ The next paper will deal with cases where (0.3) is not satisfied.

of (0.1a) and keep in mind (0.3). Then $R_{[\mu\lambda]}$ appears as a function of v^ν , $m_{\lambda\mu}$ and $h_{\lambda\mu}$

$$R_{[\mu\lambda]} \stackrel{\text{def}}{=} E_{\mu\lambda}(v^\cdot, m_{\cdot\cdot}, h_{\cdot\cdot})$$

so that the skew symmetric part of (0.2) is equivalent to

$$(0.5a) \quad E_{\mu\lambda}(v^\cdot, m_{\cdot\cdot}, h_{\cdot\cdot}) = \partial_{[\mu} X_{\lambda]}.$$

This system represents six differential equations for eight unknowns v^ν and X_λ expressed by means of auxiliary variables $m_{\lambda\mu}$, $h_{\lambda\mu}$. Substituting its solution

$$v^\nu = w^\nu(X_\cdot, m_{\cdot\cdot}, h_{\cdot\cdot})$$

into (0.5) we obtain a set of four *equations*

$$(0.4b) \quad w^\nu = \partial_\lambda m^{\lambda\nu}$$

for $m_{\lambda\mu}$ (in terms of X_λ and $h_{\lambda\mu}$). The equations (0.4) are the Maxwell equations for the electromagnetic field $m_{\lambda\mu}$.

b) Substituting from (0.1c) into the symmetric part of (0.2) one obtains by virtue of (0.3)

$$(0.6) \quad H_{\mu\lambda} - \frac{1}{2} H h_{\lambda\mu} = T_{\lambda\mu}.$$

Here $H_{\mu\lambda}$ is the contracted curvature tensor of the $h_{\lambda\mu} \equiv g_{(\lambda\mu)}$ and $H = H_{\lambda\mu} h^{\lambda\mu}$. The tensor $T_{\lambda\mu}$ is a well defined function of all arguments which enter into the total momentum energy tensor of the gravitational theory in the presence of an electromagnetic field. The equations (0.6) have the form of field equations of the gravitational theory. Hence, we identify $h_{\lambda\mu}$ with the gravitational potentials. Then (0.6) represent 10 differential equations for ten gravitational potentials $h_{\lambda\mu}$ expressed in terms of electromagnetic field $m_{\lambda\mu}$.

The tensor $T_{\lambda\mu}$ involved in (0.6) satisfies the equations

$$(0.7) \quad D_\lambda T^{\lambda\nu} = 0$$

which lead to the equations of motion

$$(0.8) \quad w^\lambda D_\lambda w^\nu = P^\nu.$$

Here $w^\lambda = w^\lambda / \sqrt{|\mathfrak{h}|}$ and P^ν is a well defined function of w^ν and $m_{\lambda\mu}$. In the equations (0.8) the symbol D_λ may be replaced by ∇_λ (*i.e.* by the symbol of covariant derivative with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$).

c) Let us replace the skew symmetric part of (0.2) by another set of conditions

$$(0.9) \quad D_\lambda \mathfrak{E}^{\alpha\mu\gamma\lambda} S_{\alpha\beta\gamma} S_{[\xi\eta\zeta]} = -\sigma \sqrt{|\mathfrak{g}|} S_{[\xi\eta\zeta]}^*.$$

* For $\mathfrak{E}^{\omega\mu\lambda\nu}$ see [6] I Section 2. The function of position σ is an arbitrary one.

Using this condition in the same way as we did the skew symmetric part of (0.2) we obtain again the Maxwell equations (0.4), the gravitational field equations (0.6) as well as (0.7). The latter equation *together with* (0.9) yields (instead of (0.8)) the equations of motion in the form

$$(0.10) \quad w^\lambda D_\lambda w^\nu = \sigma w^\nu$$

which by virtue of (0.3) is equivalent to

$$(0.11) \quad w^\lambda \nabla_\lambda w^\nu = \sigma w^\nu.$$

Hence, a particle moving in the $g_{\lambda\mu}$ -field without being subject to external forces describes the autoparallel lines (0.10) of $g_{\lambda\mu}$ which are identical with the geodesics (0.11) of $h_{\lambda\mu}$ (*the law of inertia*).

d) Assume that the electromagnetic field of the sun is weak outside of the sun so that quadratic products of its components may be neglected. This is the supposition one makes in the celestial mechanics of gravitational theory in the presence of electromagnetic field. In our case this assumption reduces (0.6) to

$$(0.12) \quad H_{\lambda\mu} = 0,$$

i.e. to the field equations of gravitational theory outside of the matter. Denote by $h_{\lambda\mu}$ its Schwarzschild solution. Then (0.11) describes its geodesics. In the gravitational theory these have been successfully identified with planetary orbits as well as with paths of photons. Since this identification is based solely on the Schwarzschild solution $h_{\lambda\mu}$ of (0.12) and of its geodesics (0.11) *we have to accept it for the unified theory in so far as this theory yields the equations* (0.11) *and* (0.12). On the other hand we know that the red shift of spectral lines, the deflection of light as well as the advance of perihelion of Mercury have been predicted on basis of (0.11) and (0.12) alone. *Hence, in the unified theory we obtain the same predictions concerning these three phenomena.*

The experimental confirmation of these three predictions has been looked upon as a *final* test in favor of the gravitational theory—at least as far as these three phenomena are concerned. The situation is more complicated when dealing with the *unified* theory. Here the equations (0.11) of motion are obtained as a consequence of (0.7) and of (0.9). Hence the experimental confirmation of the prediction concerning the red shift, the deflection of light and the advance of perihelion of Mercury cannot be regarded as a *final* test in favor of the unified theory, for here also equation (0.9) has to be tested. Hence the result may be formulated as follows:

The unified theory yields the same red shift of spectral lines, the same deflection of light and the same advance of the perihelion of Mercury as does the gravitational theory. These phenomena are caused by the gravitational field of the sun satisfying the gravitational field equations (0.6) as well as by the electromagnetic field of the sun, which has to satisfy the electromagnetic field equations (0.9). Hence the final test would consist in checking whether equations (0.9) are satisfied by the electromagnetic field of the sun. Here the last word belongs to the astrophysicists.*

* Based on (0.1a,b) on the symmetric part of (0.2) and on (0.9).

CHAPTER I

Empty space. Special space

1. The constants g , k , K .

Agreement (1.1a). Throughout this chapter we shall confine ourselves to the case

$$(1.1) \quad U^\nu_{\lambda\mu} = 0 \quad ([6] \text{ I}(1.10))^*$$

without mentioning it explicitly.

Remark I. The meaning of (1.1) is very simple: This equation is a necessary and sufficient condition that the extremals of the variational problem

$$\delta \int \sqrt{h_{\lambda\mu} dx^\lambda dx^\mu} = 0$$

be autoparallel (paths) of the connection $\Gamma^\nu_{\lambda\mu}$ (cf. [4] p. 30).

Remark II. We know also that the set of all tensors satisfying (0.1), (0.2), and (1.1) is not an empty set (Theorem [6] II(7.2)).

Theorem (1.1). All three scalars g ([6] I(1.4)), k ([6] I(1.4)) and

$$(1.2) \quad 2K = k_{\alpha\beta} k^{\alpha\beta}$$

are constant.

Proof. The statement about g follows at once from (1.1) and [6] (1.13b). On the other hand we have from [6] II(7.2a)

$$h^{\rho\omega} k_{(\rho}{}^\alpha K_{\omega)\mu\alpha} = 0.$$

Remembering the definition of $K_{\omega\mu\alpha}$ ([6] I(1.12)) we see that the last equation yields

$$\begin{aligned} 0 &= k^{\omega\alpha} K_{\omega\mu\alpha} = k^{\omega\alpha} (\nabla_\omega k_{\alpha\mu} + \nabla_\mu k_{\omega\alpha} + \nabla_\alpha k_{\omega\mu}) \\ &= k^{\omega\alpha} \nabla_\mu k_{\omega\alpha} = \nabla_\mu K = \partial_\mu K, \end{aligned}$$

and this equation proves the statement about K . The scalars g and K being constant we see from

$$(1.3) \quad g = 1 + K + k \ (\neq 0) \quad ([6] \text{ I}(1.5))$$

that k is also a constant.

Theorem (1.2). The conditions (0.1a, b) yield

$$(1.4) \quad \nabla_\alpha k_\omega{}^\alpha = 0.$$

* Cf. the note following Agreement (1.1b).

The proof follows from [6] II(7.6a) and (1.1).

Theorem (1.3). *The set of all tensors for which (0.1), (0.2), (1.1), and*

$$(1.5) \quad S_{[\omega\mu\nu]} = 0$$

hold is not an empty set.

Proof. We know (Theorem (7.4) in [4]*) that

$$(1.6a) \quad -\nabla_{[\omega} k_{\mu\lambda]} = 2S_{[\omega\mu\lambda]}.$$

On the other hand we have for the field dealt with in the proof of Theorem ([6] II(7.2))

$$(1.6b) \quad \nabla_{[\omega} k_{\mu\lambda]} = 0.$$

The equations (1.6) together with the proof of the above mentioned theorem yield the proof of our statement.

Agreement (1.1b). *Throughout this chapter we shall confine ourselves to the fields $k_{\lambda\mu} \neq 0$ † which do not belong to the first class (cf. [6] I p. 5) and for which (1.1) and (1.5) hold.*

Note. The physical meaning of this agreement will be dealt with in the last section of this chapter.

2. Electromagnetic field $m_{\lambda\nu}$.

Theorem (2.1). *The fields $m_{\lambda\nu}$, $m^{\lambda\nu}$, $k_{\lambda\mu}$ (cf. [6] I(3.8) and [6] I(4.5)) are related by*

$$(2.1) \quad \begin{aligned} \text{a)} \quad m_{\lambda\nu} &= -\frac{1}{2} \frac{\sqrt{|\mathfrak{h}|}}{\sqrt{g}} \epsilon_{\lambda\nu\alpha\beta} k^{\alpha\beta}, \\ \text{b)} \quad m^{\lambda\nu} &= \frac{1}{2} \frac{1}{\sqrt{g}} \mathfrak{E}^{\lambda\nu\alpha\beta} k_{\alpha\beta}, \\ \text{c)} \quad k_{\lambda\nu} &= \frac{1}{2} \sqrt{g|\mathfrak{h}|} \epsilon_{\lambda\nu\alpha\beta} m^{\alpha\beta}, \\ \text{d)} \quad k^{\lambda\nu} &= -\frac{1}{2} \frac{\sqrt{g}}{\sqrt{|\mathfrak{h}|}} \mathfrak{E}^{\lambda\nu\alpha\beta} m_{\alpha\beta}. \end{aligned}$$

Proof. According to the agreement (1.1b) we have

$$(2.2) \quad k = 0$$

so that the equations [6] I(3.8), [6] I(4.5), and [6] I(4.2) yield (2.1).

* There is an obvious misprint in the corresponding equation.

† The case $k_{\lambda\nu} = 0$ leading to

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$$

has to be excluded in a *unified theory*.

Theorem (2.2). *The fields $m_{\lambda\nu}$, $m^{\lambda\nu}$ satisfy the following conditions*

$$(2.3) \quad \begin{aligned} \text{a)} \quad & \nabla_{[\omega} m_{\mu\lambda]} \equiv \partial_{[\omega} m_{\mu\lambda]} = 0 \\ \text{b)} \quad & \nabla_{\lambda} m^{\lambda\nu} \equiv \partial_{\lambda} m^{\lambda\nu} = 0. \end{aligned}$$

Proof. First we obtain from (2.1b) and (1.6)

$$\nabla_{\lambda} m^{\lambda\nu} = \frac{1}{2\sqrt{g}} \mathfrak{E}^{\lambda\nu\alpha\beta} \nabla_{\lambda} k_{\alpha\beta} = -\frac{1}{2\sqrt{g}} \mathfrak{E}^{\nu\lambda\alpha\beta} \nabla_{[\lambda} k_{\alpha\beta]} = 0.$$

On the other hand the equations (1.4) and (2.1d) yield

$$0 = \nabla_{\lambda} k^{\lambda\nu} = -\frac{1}{2} \frac{\sqrt{g}}{\sqrt{|\mathfrak{h}|}} \mathfrak{E}^{\lambda\nu\alpha\beta} \nabla_{\lambda} m_{\alpha\beta} = \frac{1}{2} \frac{\sqrt{g}}{\sqrt{|\mathfrak{h}|}} \mathfrak{E}^{\nu\lambda\alpha\beta} \nabla_{[\lambda} m_{\alpha\beta]}.$$

The remaining statements of our theorem are almost obvious.

Definition (2.1). *The equations (2.3) will be looked upon as Maxwell equations for the electromagnetic field $m_{\lambda\nu}$ in an empty space.*

Remark. By means of this definition we have introduced the Maxwell equations. The exact meaning of the words “empty space” will be given later.

Theorem (2.3). *There is a vector m_{λ} (“the world vector”) such that*

$$(2.4) \quad m_{\lambda\nu} = 2\partial_{[\lambda} m_{\nu]} = 2\nabla_{[\lambda} m_{\nu]}.$$

Proof. The integrability conditions of (2.4) are (2.3a).

Remark. This theorem enables us to refer to the six components $m_{\lambda\nu}$ as representing only four unknowns m_{λ} .

Theorem (2.4). *The electromagnetic field $m_{\lambda\nu}$ is a solution of*

$$(2.5a) \quad 2\partial_{[\mu} X_{\lambda]} = e_{\mu\lambda\nu\omega} \Delta m^{\nu\omega} \sqrt{g}.$$

Proof. The equations (1.5), (1.6), and ([6] (1.12)) yield

$$(2.6) \quad K_{\omega\mu\nu} = 2\nabla_{\nu} k_{\omega\mu}.$$

On the other hand the skew symmetric part of (0.2) is equivalent to [6] II(7.2c) and this latter equation reduces by virtue of (2.6) to

$$(2.7) \quad \partial_{[\mu} X_{\lambda]} = \Delta k_{\mu\lambda}.$$

Substituting into this equation for $k_{\mu\lambda}$ from (2.1) we obtain (2.5a).

Remark. In the general case (*i.e.* in the case of X_{λ} being not a gradient) the system (2.5a) of six equations contains eight unknowns, namely four unknowns X_{λ} and four unknowns (m_{λ} represented by six components) $m_{\lambda\nu}$.^{*} Therefore we

^{*} The equation (2.5a) is a consequence of the skew symmetric part of (0.2) and of (2.3b). Hence, this latter equation is “incorporated” in (2.5a), where the $h_{\lambda\mu}$ are looked upon as auxiliary variables.

could just as well have prescribed some other condition for these unknowns. However, we want to prescribe a condition for the *general* case, which is automatically satisfied in the *special* case:

Agreement (2.1). *Throughout this chapter we shall confine ourselves to the case where*

$$(2.5b) \quad \mathfrak{E}^{\omega\mu\lambda\nu}(\partial_\omega X_\mu)m_{\lambda\nu} = 0.$$

Illustration (2.1). Let $m^{\lambda\nu}$ be a solution of

$$\Delta m^{\lambda\nu} = \rho m^{\lambda\nu}, \quad \rho \neq 0$$

belonging to the third class

$$m^{\lambda\nu} m_{\lambda\nu} = 0.$$

Then according to (2.5a) we have

$$\begin{aligned} 2\mathfrak{E}^{\mu\lambda\alpha\beta}(\partial_\mu X_\lambda)m_{\alpha\beta} &= \mathfrak{E}^{\mu\lambda\alpha\beta} \epsilon_{\mu\lambda\nu\omega} \sqrt{g} m_{\alpha\beta} \Delta m^{\nu\omega} \\ &= 4 \sqrt{g}(m_{\alpha\beta} \Delta m^{\alpha\beta}) \sqrt{|\mathfrak{h}|} \\ &= 4 \sqrt{g|\mathfrak{h}|} \rho m_{\alpha\beta} m^{\alpha\beta} = 0. \end{aligned}$$

Remark I. Later on we shall deal with the condition (2.5b) in more detail.

Remark II. In the special case the left hand term of (2.5a) is equal to zero. It is possible even in the general case to get rid of X_λ as the following theorem shows.

Theorem (2.5). *The solution $m^{\lambda\nu}$ of (2.5) satisfies also the conditions*

$$(2.8) \quad 0 = 2(\nabla^\alpha H_{\alpha[\omega\mu}{}^\beta] \epsilon_{\lambda]\beta\xi\eta} m^{\xi\eta} + (\nabla_\alpha m^{\xi\eta}) \{ \tfrac{1}{2} H_{[\omega}{}^\alpha \epsilon_{\mu\lambda]\xi\eta} - 2\epsilon_{\xi\eta\beta[\mu} H_{\omega}{}^\alpha{}_{\lambda]}{}^\beta \},$$

where $H_{\omega\mu\lambda}{}^\nu$ is defined by ([6] I(1.2)) and

$$(2.9) \quad H_{\mu\lambda} = H_{\alpha\mu\lambda}{}^\alpha.$$

Proof. The equations (2.5a) and (2.7) are equivalent, so that we may operate on (2.7) instead of dealing with (2.5a). The solution $k_{\mu\lambda}$ of (2.7) must obviously satisfy also the equation

$$(2.10) \quad \nabla_{[\omega} \Delta k_{\mu\lambda]} = 0.$$

This equation may be transformed. First of all we have

$$\begin{aligned} (2.11) \quad \text{a)} \quad \nabla_{[\omega} \Delta k_{\mu\lambda]} &= \nabla_{[\omega} \nabla^\alpha \nabla_{|\alpha|} k_{\mu\lambda]} \\ &= \nabla^\alpha \nabla_{[\omega} \nabla_{|\alpha|} k_{\mu\lambda]} + H_{[\omega}{}^\alpha{}_{|\alpha|}{}^\beta \nabla_{|\beta|} k_{\mu\lambda]} \\ &\quad + H_{[\omega}{}^\alpha{}_\mu{}^\beta \nabla_{|\alpha} k_{\beta|\lambda]} + H_{[\omega}{}^\alpha{}_\lambda{}^\beta \nabla_{|\alpha|} k_{\mu\beta]} \\ \text{b)} \quad \nabla^\alpha \nabla_{[\omega} \nabla_{|\alpha|} k_{\mu\lambda]} &= \nabla_\alpha \{ \nabla^\alpha \nabla_{[\omega} k_{\mu\lambda]} + 2H_{[\omega}{}^\alpha{}_\mu{}^\beta k_{|\beta|\lambda]} \}. \end{aligned}$$

Substituting from (2.11b) into (2.11a) and replacing in the so obtained equation $k_{\alpha\mu}$ by $m_{\lambda\mu}$ according to (2.1) and taking in account (1.6b) one obtains (2.8).

3. Gravitational field. In the previous section we identified the electromagnetic field with $m_{\lambda\mu}$ on account of (2.3). In this chapter we shall look for the gravitational field. To this effect we shall prove first the following

Theorem (3.1). *Put*

$$(3.1) \quad \begin{aligned} \text{a)} \quad T_{\mu\lambda} &= T_{\lambda\mu} \equiv -\frac{1}{4}g\{\mathfrak{E}^{\alpha\nu\xi\eta}\epsilon_{\beta\lambda\gamma\delta}(\nabla^\beta m_{\xi\eta})(\nabla_\alpha m^{\gamma\delta})h_{\nu\mu} \\ &\quad - h_{\mu\lambda}(\nabla_\alpha m_{\beta\gamma})(\nabla^\alpha m^{\beta\gamma})\} \\ \text{b)} \quad H &= H_{\mu\lambda}h^{\mu\lambda}. \end{aligned}$$

Then the symmetric part of (0.2) is equivalent to

$$(3.2) \quad H_{\mu\lambda} - \frac{1}{2}Hh_{\lambda\mu} = T_{\lambda\mu}$$

and the tensor $T_{\lambda\mu}$ satisfies the conditions

$$(3.3) \quad \nabla^\lambda T_{\lambda\mu} = 0.$$

Proof. The symmetric part of (0.2) is equivalent in our case to [6] II(7.2b) i.e.,

$$(3.4a) \quad H_{\mu\lambda} = \frac{1}{4}K_{\alpha\lambda}{}^\beta K_{\beta\lambda}{}^\alpha$$

so that

$$(3.4b) \quad H = \frac{1}{4}K_{\alpha\gamma\beta}K^{\beta\gamma\alpha}.$$

Substituting into (3.4a, b) from (2.6) we obtain

$$(3.5) \quad H_{\mu\lambda} - \frac{1}{2}Hh_{\mu\lambda} = (\nabla^\beta k_{\alpha\mu})(\nabla^\alpha k_{\beta\lambda}) - \frac{1}{2}h_{\mu\lambda}(\nabla_\beta k_{\alpha\gamma})\nabla^\alpha k^{\beta\gamma}.$$

Hence if we put

$$(3.6) \quad T_{\lambda\mu} = T_{\mu\lambda} \equiv (\nabla^\beta k_{\alpha\mu})(\nabla^\alpha k_{\beta\lambda}) - \frac{1}{2}h_{\mu\lambda}(\nabla_\beta k_{\alpha\gamma})\nabla^\alpha k^{\beta\gamma},$$

we obtain from (3.5) the equation (3.2), where $T_{\mu\lambda}$ is given by (3.6). Replacing in (3.6) the tensor $k_{\lambda\mu}$ by $m_{\lambda\mu}$ according to (2.1) and remembering (2.3) one obtains (3.1a). The equation (3.3) is a consequence of (3.2) and of the known relation

$$(3.7) \quad \nabla^\lambda (H_{\lambda\mu} - \frac{1}{2}Hh_{\lambda\mu}) = 0.$$

Remark. Equation (3.2) has the same form as in the gravitational theory (where of course the tensor $T_{\lambda\mu}$ is defined in another way). We use this analogy for the identification of the gravitational field.

Definition (3.1). *The field $h_{\lambda\mu}$ satisfying (3.2) will be looked upon as the gravitational field.*

Remark. The tensor $T_{\lambda\mu}$ as defined by (3.1) is a special case of a tensor dealt with in the next chapter. This latter tensor (denoted also by $T_{\lambda\mu}$) will be referred

to as the *total momentum energy tensor*, while the corresponding equations (3.3) will describe (the principle of conservation of energy as well as) *the motion*.

After having identified the electromagnetic field and the gravitational field we are able to prove the following

Theorem (3.2). *The connection $\Gamma_{\lambda\mu}^\nu$ is expressed by means of gravitation and electromagnetism as follows*

$$(3.8) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{\sqrt{g}}{2} \epsilon_{\lambda\mu\alpha\beta} \nabla^\nu m^{\alpha\beta}.$$

Proof. First we have from [6] I(1.9) and (1.1)

$$(3.9a) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}^\nu.$$

On the other hand the equations [6] I(1.11b), (1.1), (2.6), and (2.4) yield

$$(3.9b) \quad S_{\lambda\mu}^\nu = \frac{1}{2} K_{\lambda\mu}^\nu = \nabla^\nu k_{\lambda\mu} = \frac{\sqrt{g}}{2} \epsilon_{\lambda\mu\alpha\beta} \nabla^\nu m^{\alpha\beta}.$$

Substituting from (3.9b) into (3.9a) we obtain (3.8).

4. Another form of (2.5a). The system (2.5a) defining the electromagnetic field (in terms of the gravitational field) is a differential one. It is possible to convert it into an algebraic system, so that the *tensor $m_{\lambda\mu}$ may be obtained in a purely algebraic way*. In order to perform this conversion we shall prove first the auxiliary

Theorem (4.1). *The equation (2.7) is equivalent to*

$$(4.1) \quad (H_{\mu\lambda\alpha\beta} - 2H_{\beta[\mu} h_{\lambda]\alpha}) k^{\beta\alpha} = \partial_{[\mu} X_{\lambda]}.$$

Proof. Equation (1.6b) is equivalent to

$$(4.2) \quad 2\nabla_{[\omega} k_{\mu]\alpha} + \nabla_\alpha k_{\omega\mu} = 0$$

and this equation yields

$$(4.3) \quad 2\nabla^\alpha \nabla_{[\omega} k_{\mu]\alpha} + \Delta k_{\omega\mu} = 0.$$

On the other hand we have

$$(4.4a) \quad 2\nabla^\alpha \nabla_{[\omega} k_{\mu]\alpha} = 2\nabla_{[\omega} \nabla^\alpha k_{\mu]\alpha} + 2H_{[\omega\mu]}^\beta k_{\beta\alpha} + 2H_{[\omega|\alpha]}^\beta k_{\mu]\beta}.$$

This equation may be simplified because of (1.4) and

$$2H_{\alpha[\omega\mu]\beta} = H_{\alpha\omega\mu\beta} - H_{\alpha\mu\omega\beta} = H_{\alpha\omega\mu\beta} + H_{\mu\alpha\omega\beta} = -H_{\omega\mu\alpha\beta},$$

so that

$$(4.4b) \quad \begin{aligned} 2\nabla^\alpha \nabla_{[\omega} k_{\mu]\alpha} &= H_{\omega\mu\alpha\beta} k^{\alpha\beta} - 2H_{\beta[\omega} k_{\mu]}^\beta \\ &= (H_{\omega\mu\alpha\beta} - 2H_{\beta[\omega} h_{\mu]\alpha}) k^{\alpha\beta}. \end{aligned}$$

Substituting this result into (4.3) we obtain

$$(4.5) \quad \Delta k_{\mu\lambda} = (H_{\mu\lambda\alpha\beta} - 2H_{\beta[\mu} h_{\lambda]\alpha}) k^{\beta\alpha}$$

and comparing this equation with (2.7) one obtains (4.1).

Theorem (4.2). *The equation (2.5a) is equivalent to*

$$(4.6) \quad G^{\mu\lambda\gamma\delta} m_{\gamma\delta} = \frac{1}{\sqrt{|\mathfrak{h}|}} \mathfrak{G}^{\mu\lambda\gamma\delta} \partial_\gamma X_\delta$$

where

$$(4.7) \quad G^{\mu\lambda\gamma\delta} = -\frac{1}{2} \frac{\sqrt{g}}{\mathfrak{h}} \mathfrak{G}^{\beta\alpha\gamma\delta} \mathfrak{G}^{\omega\nu\mu\lambda} (H_{\omega\nu\alpha\beta} - 2H_{\beta[\omega} h_{\nu]\alpha}).$$

The proof follows at once from (4.1) and (2.1).

Theorem (4.3). *The condition (2.5b) is equivalent to*

$$(4.8) \quad G^{\omega\mu\lambda\nu} m_{\omega\mu} m_{\lambda\nu} = 0.$$

The proof follows from (4.7) and (2.5b).

Remark. A simple inspection shows that in the case of constant curvature H

$$H_{\omega\mu\lambda\nu} = -\frac{H}{12} (h_{\omega\lambda} h_{\mu\nu} - h_{\mu\lambda} h_{\omega\nu})$$

every field of the third class satisfies the condition (4.8) for an arbitrary vector X_λ .

5. Empty space, special space. Thanks to the conditions (0.2) (and (1.5)) we have been able to identify the electromagnetic field as well as the gravitational field.* The situation is less favorable if we try to identify the mass density by means of our conditions only. There we have only one hint how to proceed: In the gravitational theory the (rest) mass density $\mathfrak{M}$ is defined by

$$\mathfrak{M} = |H\sqrt{|\mathfrak{h}|}|.$$

On the other hand if we assume $m_{\lambda\nu} = 0$ then we have $k_{\lambda\nu} = 0$ (cf. Theorem I(4.2) in [6] I), and the unified theory reduces to the gravitational theory for “external” problems

$$H_{\lambda\mu} = 0^\dagger$$

so that we have $\mathfrak{M} = 0$. Hence, if we denote by $\mathfrak{M}$ the mass density in unified theory we know that we must have $\mathfrak{M} = 0$ for $H = 0$. This is the only hint we have for the identification of the mass density.

* Of course we cannot claim that this identification is the only possible one!

† This is the reason why we are excluding the case $k_{\lambda\mu} = 0$ (cf. agreement (1.1b)). Here we use this case only for heuristic purposes!

Definition (5.1). By the mass density $\mathfrak{M}$ we understand a nonnegative (continuous) function of H

$$(5.1a) \quad \mathfrak{M} = \mathfrak{M}(H)$$

such that

$$(5.1b) \quad \mathfrak{M} = 0 \text{ only for } H = 0.$$

Furthermore, whenever (5.1b) is satisfied we shall refer to the space as an empty space.*

Theorem (5.1). The space dealt with in this chapter is an empty space.

Proof. According to definition (5.1) it is sufficient to prove that we have in this chapter

$$H = 0.$$

In order to do so let us observe first that we obtain from (3.4b) and (2.6)

$$(5.2) \quad H = (\nabla_\beta k_{\alpha\gamma})(\nabla^\alpha k^{\beta\gamma}).$$

On the other hand the equation (1.6b) is equivalent to

$$\nabla_\beta k_{\alpha\gamma} = -2\nabla_{[\alpha} k_{\gamma]\beta}.$$

Hence, we may write for (5.2) by virtue of (2.1) and Theorem (1.1)

$$(5.3) \quad \begin{aligned} H &= -(\nabla_{[\beta} k_{\gamma]\alpha})\nabla^\alpha k^{\beta\gamma} = \tfrac{1}{2}(\nabla_\alpha k_{\beta\gamma})\nabla^\alpha k^{\beta\gamma} \\ &= -\tfrac{1}{2}k^{\alpha\beta}\Delta k_{\alpha\beta} = \tfrac{1}{2}gm_{\gamma\delta}\Delta m^{\gamma\delta}. \end{aligned}$$

On the other hand we obtain from (2.5)

$$(5.4) \quad \begin{aligned} 0 &= 2\mathfrak{E}^{\omega\mu\lambda\nu}(\partial_\omega X_\mu)m_{\lambda\nu} = \mathfrak{E}^{\omega\mu\lambda\nu}\epsilon_{\omega\mu\xi\eta}\sqrt{g}(\Delta m^{\xi\eta})m_{\lambda\nu} \\ &= 4\sqrt{g}m_{\lambda\nu}\Delta m^{\lambda\nu} = 4\sqrt{g|\mathfrak{h}|}m_{\gamma\delta}\Delta m^{\gamma\delta}. \end{aligned}$$

The equations (5.3) and (5.4) prove our assertion.

In order to derive some consequence of the previous theorem, we need another

Definition (5.2). A space without gravitation, i.e. the space with a coordinate system for which

$$(5.5) \quad h_{\lambda\mu} = \left(\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \right)$$

will be referred to as a special space.

* In the next chapter we shall be able to gather more information about $\mathfrak{M}$ and its connection with electric charge density.

Remark. In the gravitational theory whenever the electromagnetic field is absent one obtains from $H = 0$ the condition $H_{\lambda\mu} = 0$. The corresponding space is often called a free space. In the unified theory one has to distinguish between an *empty* space, where $H = 0$ but in general $H_{\lambda\mu} \neq 0$, and a *special* space, as defined by definition (5.2) (where not only $H \equiv 0$, $H_{\lambda\mu} \equiv 0$ but also $H_{\omega\mu\lambda}{}^\nu \equiv 0$). Let us stress again that according to our definition the special space is necessarily an empty space but not vice versa. Moreover, there are some more characteristic features of the unified theory (in our restricted case) which will be described in the subsequent theorems.

Theorem (5.2). *The following properties hold in the special case (i.e. in the case of X_λ being a gradient):*

- 1) The *electromagnetic* field may be described in terms of the *gravitational* field.
- 2) There are electromagnetic fields which produce a special space. This is true in particular for the electromagnetic field of the plane wave in the electromagnetic theory of light.
- 3) In all these cases the space is an empty space in the sense of definition (4.1).

Proof. Theorem (5.1) is valid for the special as well as the general case. Therefore the statement 3 holds. If we consider the components $m_{\lambda\nu}$ involved in (3.2) as auxiliary variables, then the solution $h_{\lambda\mu}$ of this system is expressed in terms of the electromagnetic field. Substituting this solution into (4.6) we obtain a system of differential equations for the unknowns $m_{\lambda\nu}$. Moreover, one could also proceed the other way round starting with (4.6) and considering $h_{\lambda\mu}$ involved in these equations as auxiliary variables so that the solution $m_{\lambda\nu}$ of this system is expressible in terms of the gravitational field $h_{\lambda\mu}$. Now substituting this solution into the right hand term of (3.2) one obtains a set of differential equations for the unknowns $h_{\lambda\mu}$. This proves the first statement of our theorem. In order to prove the second statement we shall consider the field used in the proof of Theorem ([6] II(7.2)). This field satisfies the equations ([6] II(7.2)). The last two of these equations are equivalent to (3.2) and (4.6).^{*} According to the aforementioned proof these equations are satisfied by [6] I(8.6) and [6] I(8.8). An easy inspection of these equations shows that for a suitable choice of the functions a, b the tensor $m_{\lambda\nu}$ as defined by (2.1) may be looked upon as describing the electromagnetic field of the plane wave of the electromagnetic theory of light. Hence, this field produces a special space.

Remark. The second part of the second statement might be looked upon as an experimental confirmation of the unified theory in the special case applied to this particular field. The corresponding connection (3.8) is

$$(5.6) \quad \Gamma_{\lambda\mu}^\nu = 2 E^\nu E_{[\lambda} u'_{\mu]}.^\dagger$$

^{*} While the first one is equivalent to (1.1).

[†] One has to bear in mind that in this particular case $|\mathfrak{h}| = 1$ is a scalar!

A theorem similar to Theorem (5.2) holds for the general case (*i.e.* for the case where X_λ is not a gradient).

Theorem (5.3). *The following properties hold in the general case:*

- 1) The *electromagnetic* field may be described in terms of the *gravitational* field.
- 2) Both fields are essential in the sense that the absence of either one* of them leads to contradiction.
- 3) The space is an empty space.

Proof. The first and third statements may be proved in the same way as the corresponding statements in the previous theorem. The second statement follows from the fact that in the general case we have

$$(5.7) \quad \partial_{[\lambda} X_{\mu]} \neq 0$$

while in the absence of either one* of the two fields $h_{\lambda\mu}$, $m_{\lambda\nu}$ we obtain according to (4.6)

$$\partial_{[\lambda} X_{\mu]} = 0$$

which contradicts (5.7).

6. Final remarks. All considerations of this chapter have been based on the following restrictions

$$(6.1) \quad \begin{array}{ll} \text{a)} & U^\nu_{\lambda\nu} = 0 \\ \text{b)} & k = 0 \\ \text{c)} & S_{[\omega\mu\lambda]} = 0 \end{array}$$

following from Agreement (1.1b).

Let us consider first the conditions (6.1a, b). In general they are of a *structural* nature. On the other hand there are cases in which these conditions are of a *quantitative nature only*: Assume that the field $k_{\lambda\nu}$ is of the type $\epsilon K_{\lambda\nu}$, where $K_{\lambda\nu}$ are bounded components of a tensor and ϵ is a constant so small that ϵ^2 may be neglected. Then the necessary and sufficient condition

$$k_{(\rho}{}^\alpha K_{\omega)\mu\alpha} = 0^\dagger$$

for (6.1a) is *automatically satisfied* and consequently (6.1a) holds. Moreover, we have

$$k = \mathfrak{k}/\mathfrak{h}$$

and $\mathfrak{k}$ is a square of a sum of quadratic products $k_{\alpha\beta} k_{\gamma\delta}$ so that (6.1b) is *also automatically satisfied*.

* The case $m_{\lambda\nu} = 0$ has been already excluded by Agreement (1.1b) according to which $k_{\lambda\nu} \neq 0$.

† Cf. Theorems [4] (9.1) and [4] (9.2a).

The condition (6.1c) is however a *structural restriction* for the field in question. The equation (1.6a) shows that (6.1c) is not automatically satisfied for the field $\epsilon K_{\lambda\nu}$. The physical significance of this condition follows from its equivalent form (2.3b). In the *unified theory* the condition (6.1c) refers to the “external” problem, where moving particles are not present. Hence, the unified theory as developed in this chapter may be looked upon as an *external theory for a weak field* ($\epsilon^2 \rightarrow 0$). In the next chapter, where the condition (6.1c) will be dropped, we shall deal with the theory in presence of moving particles.

Remark. The quantitative nature of the restrictions (6.1a, b) is very important. It shows in particular that the field $k_{\lambda\nu}$ (or $m_{\lambda\nu}$) of the first class *cannot be approximated** by neglecting the quadratic products of its components. In anticipating our results in the next paper we may say that *it is not possible to find a set of equations for the field of the first class whose approximation will have the form of Maxwell equations*. Any such approximation reduces the generality of the field and replaces it by another *field not belonging to the first class*. That is the reason why we looked even in this very special case (dealt with in this chapter) for *exact* Maxwell equations and not for a set of equations whose approximation† yields Maxwell equations. Looking for such a system does not yield an algorithm valid for *all* classes of $k_{\lambda\nu}$ and therefore has to be omitted (even in special cases). In the following chapter we shall see that the algorithm used here will gratify us with exact Maxwell equations.

* By a field of the same class.

† In the sense mentioned above.

CHAPTER II

General case

1. Fundamental relations for $m_{\lambda\mu}$.

Agreement (1.1). *Throughout this chapter we shall deal with the fields $k_{\lambda\mu} \neq 0$ satisfying the following requirements:*

1) *First requirement*

$$(1.1) \quad U_{\lambda\mu}^\nu = 0.$$

2) *Second requirement*

$$(1.2) \quad S_{[\omega\mu\lambda]} \neq 0.$$

3) *Third requirement*

$$(1.3) \quad S_{[\omega\mu\lambda]} S^{[\omega\mu\lambda]} \leq 0.$$

Remark. The only condition common to Agreement $I(1.1)^*$ and this one is the requirement (1.1).

Theorem (1.1). *Theorems $I(1.1)$ and $I(1.2)$ hold also under Agreement (1.1).*

The proof of these theorems depends only on (1.1) and not on (1.2) and (1.3).

In the Agreement $I(1.1b)$ we assumed that $k_{\lambda\nu}$ does not belong to the first class. The same statement may be proved here.

Theorem (1.2). *By virtue of (0.1b) and (1.2) the field $k_{\lambda\mu}$ does not belong to the first class, so that*

$$(1.4) \quad k = 0.$$

Moreover, the field $m_{\lambda\mu}$ satisfies the equations $I(2.1)$ and

$$(1.5) \quad \nabla_{[\omega} m_{\mu\lambda]} \equiv \partial_{[\omega} m_{\mu\lambda]} = 0.$$

Proof. According to Corollary [4] (7.5) the requirement (0.1) is equivalent to [6] $I(5.2)$ so that we have by virtue of Theorems (1.1), [6] $I(5.2)$ and according to [6] $I(3.8a)$

$$(1.6) \quad \partial_{[\mu} m_{\lambda\nu]} = \kappa \frac{\sqrt{|k|}}{\sqrt{g}} \nabla_{[\mu} k_{\lambda\nu]} = 0.$$

The last equation may be written by virtue of $I(1.6a)$

$$(1.7) \quad \frac{\sqrt{|k|}}{\sqrt{g}} S_{[\mu\lambda\nu]} = 0.$$

* I refers to the first chapter.

Comparing this equation with (1.2), we obtain (1.4). The remaining part of the theorem has been proved together with Theorem I(2.1).

Definition (1.1). Let the vector density v^r be defined by

$$(1.8) \quad \sqrt{g} v^r \stackrel{\text{def}}{=} \mathfrak{S}^{r\omega\mu\lambda} S_{\omega\mu\lambda}.$$

Theorem (1.3). The vector density v^r satisfies the following identities.*

$$(1.9) \quad \begin{aligned} \text{a)} \quad & \sqrt{g} v^r \equiv \frac{1}{2} \mathfrak{S}^{\omega\mu\lambda r} \partial_\omega k_{\mu\lambda} \\ \text{b)} \quad & \sqrt{g} v^r \equiv \frac{1}{2} \mathfrak{S}^{r\omega\mu\lambda} K_{\omega\mu\lambda}^\dagger \end{aligned}$$

and

$$(1.10) \quad v^r \equiv \partial_\lambda m^{\lambda r} \equiv \nabla_\lambda m^{\lambda r}.$$

Moreover, we have

$$(1.11) \quad v^r \neq 0 \quad v^2 \stackrel{\text{def}}{=} v^r v_r \geq 0.$$

Proof. The identity (1.9a) follows at once from (1.8) and I(1.6a). On the other hand we obtain from [6] I(1.12)

$$K_{[\omega\mu\nu]} = -\nabla_{[\omega} k_{\mu\nu]} + \nabla_{[\mu} k_{\omega\nu]} + \nabla_{[\nu} k_{\omega\mu]} = -\nabla_{[\omega} k_{\mu\nu]}$$

and (1.9b) follows easily from (1.9a). The identity (1.10) follows from (1.9a) and I(2.1b). The conditions (1.11) follow from (1.2) and (1.3) by virtue of [6] I(2.2c) and [6] I(2.4).

In the third section we shall convert the identity (1.10) into an equation by means of the skew symmetric part of (0.2).

2. The skew symmetric part of (0.2). We prove first the following auxiliary

Theorem (2.1a). The skew symmetric part of (0.2) is equivalent to

$$(2.1a) \quad 2\partial_{[\mu} X_{\lambda]} = -\sqrt{g} \mathfrak{e}_{\mu\lambda\omega\nu} \nabla^\omega v^\nu + 2\Delta k_{\mu\lambda}.$$

Proof. First of all we obtain from (1.9a)

$$(2.2) \quad \sqrt{g} \mathfrak{e}_{\omega\mu\lambda\nu} v^\nu \equiv 3\nabla_{[\omega} k_{\mu\lambda]}$$

and consequently according to [6] I(1.12)

$$(2.3) \quad \begin{aligned} K_{\mu\lambda\alpha} &\equiv 3\nabla_{[\mu} k_{\alpha\lambda]} + 2\nabla_\alpha k_{\mu\lambda} \\ &\equiv \sqrt{g} \mathfrak{e}_{\mu\alpha\lambda\nu} v^\nu + 2\nabla_\alpha k_{\mu\lambda}. \end{aligned}$$

* i.e. identities with respect to $k_{\lambda\nu}$.

† Cf. [6] I (1.12).

On the other hand the skew symmetric part of (0.2) is equivalent to [6] II(7.2c) i.e.,

$$2\partial_{[\mu} X_{\lambda]} = \nabla^\alpha K_{\mu\lambda\alpha}.$$

Substituting in this equation from (2.3) one obtains easily (2.1a).

Theorem (2.1b). *The equation (2.1a) is equivalent to*

$$(2.1b) \quad \sqrt{g} \epsilon_{\mu\lambda\omega\nu} \nabla^\omega v^\nu = 2\partial_{[\mu} X_{\lambda]} + 2(H_{\mu\lambda\alpha\beta} - 2H_{\beta[\mu} h_{\lambda]\alpha}) k^{\alpha\beta}.$$

Proof. First we obtain from (2.2)

$$\sqrt{g} \epsilon_{\mu\lambda\omega\nu} \nabla^\omega v^\nu \equiv \Delta k_{\mu\lambda} + \nabla^\omega 2\nabla_{[\mu} k_{\lambda]\omega},$$

or according to I(4.4b)

$$(2.4) \quad 2\sqrt{g} \epsilon_{\mu\lambda\omega\nu} \nabla^\omega v^\nu \equiv 2\Delta k_{\mu\lambda} + 2(H_{\mu\lambda\alpha\beta} - 2H_{\beta[\mu} h_{\lambda]\alpha}) k^{\alpha\beta}.$$

Eliminating $2\Delta k_{\mu\lambda}$ from (2.1a) and (2.4) one obtains (2.1b).

Theorem (2.2). *The equations (2.1) are equivalent to the mutually equivalent equations**

$$(2.5) \quad \begin{aligned} \text{a)} \quad & 2\sqrt{g} [\Delta m^{\omega\nu} - \nabla^{[\omega} v^{\nu]}] = \mathfrak{G}^{\mu\lambda\omega\nu} \partial_\mu X_\lambda \\ \text{b)} \quad & \frac{2\sqrt{g}}{\sqrt{|\mathfrak{h}|}} \nabla^{[\omega} v^{\nu]} = \frac{1}{\sqrt{|\mathfrak{h}|}} \mathfrak{G}^{\mu\lambda\omega\nu} \partial_\mu X_\lambda - G^{\lambda\omega\nu} m_{\mu\lambda} \end{aligned}$$

where $G^{\mu\lambda\omega\nu}$ is defined by I(4.7).

Proof. Substituting into (2.1) for $k_{\lambda\nu}$ and $k^{\lambda\nu}$ from I(2.1c, d) one obtains (2.5).

Note. The elimination of $\partial_{[\mu} X_{\lambda]}$ from (2.5a, b) leads to an equation which is a consequence of (1.5).

Theorem (2.3). *Every vector density v^ν satisfying (2.5b) satisfies also the equations*

$$(2.6) \quad \frac{\sqrt{g}}{\sqrt{|\mathfrak{h}|}} (\Delta v^\nu + H_\lambda{}^\nu v^\lambda) = -\nabla_\omega G^{u\lambda\omega\nu} m_{\mu\lambda}.$$

Proof. First of all we have

$$(2.7) \quad \begin{aligned} 2\nabla_\omega \nabla^{[\omega} v^{\nu]} &= \nabla_\omega (\nabla^\omega v^\nu - \nabla^\nu v^\omega) \\ &= \Delta v^\nu - \nabla_\omega \nabla^\nu v^\omega. \end{aligned}$$

On the other hand

$$(2.8) \quad -\nabla_\omega \nabla^\nu v^\omega = -\nabla^\nu \nabla_\omega v^\omega + H_\omega{}^\nu{}_\lambda{}^\omega v^\lambda.$$

This equation may be simplified. First we have from (1.10)

$$\nabla_\omega v^\omega \equiv 0$$

* Using Theorem [6] I (4.3) one could express g in terms of $m_{\lambda\mu}$. However, g is constant in our case so that it is irrelevant whether we express it by means of $k_{\lambda\mu}$ or $m_{\lambda\mu}$.

and moreover

$$H_{\omega}{}^{\nu}{}_{\lambda} v^{\lambda} = H_{\lambda}{}^{\nu} v^{\lambda}$$

so that we obtain from (2.7) and (2.8)

$$(2.9) \quad 2\nabla_{\omega} \nabla^{[\omega} v^{\nu]} = \Delta v^{\nu} + H_{\lambda}{}^{\nu} v^{\lambda}.$$

Applying the operator ∇_{ω} on (2.5b) and substituting from (2.9) into the derived equation (2.5b) one obtains (2.6).*

Note. The equation (2.6) may be very useful whenever we are dealing with a harmonic vector density v^{ν}

$$\Delta v^{\nu} = 0.$$

In this case the equation (2.6) reduces to an *algebraic* equation for v^{λ} .

Theorem (2.4). *A necessary and sufficient condition for v_{λ} to be a gradient is*

$$(2.10) \quad 2\sqrt{g}\Delta m^{\omega\nu} = G^{\mu\lambda\omega\nu} m_{\mu\lambda}.$$

Proof. Eliminating $\mathfrak{E}^{\mu\lambda\omega\nu} \partial_{\mu} X_{\lambda}$ from (2.5a, b) and

$$\nabla^{[\omega} v^{\nu]} = 0$$

one obtains (2.10).

Note. From (2.10) we see that whenever $H_{\omega\mu\lambda\nu} = 0$ the vector density v_{λ} is a gradient provided that $m^{\omega\nu}$ is harmonic.

Agreement (2.1). *We shall confine ourselves to the cases where (2.10) is not satisfied, i.e. we shall exclude the case when v_{λ} is a gradient.*

3. The electromagnetic field $m_{\lambda\mu}$ and its Maxwell equations. Let us first stress that Theorem I(2.3) holds also in our present considerations, so that the *six* components $m_{\lambda\mu}$ represent *four* unknowns m_{λ} only.

Consider now the $h_{\lambda\mu}$ involved in (2.5) as auxiliary variables. Then either one of the two sets (2.5) may be looked upon as a set of *six* differential equations for twelve unknowns v^{ν} , m_{μ} and X_{λ} .† Denote by

$$(3.1) \quad v^{\nu} \stackrel{\text{def}}{=} w^{\nu}(h_{..}, m_{..}, X_{..})$$

a solution of (2.5a) (or (2.5b)), where w^{ν} is a function of the $h_{\lambda\mu}$, $m_{\lambda\mu}$ and X_{λ} ††.

* One has for any vector X_{λ}

$$\mathfrak{E}^{\mu\lambda\omega\nu} \nabla_{\omega} \partial_{\mu} X_{\lambda} = \mathfrak{E}^{\mu\lambda\omega\nu} \nabla_{[\omega} \nabla_{\mu} X_{\lambda]} = 0.$$

† In the special case only 8 unknowns v^{ν} , m_{μ} are involved in (2.5).

†† And possibly also a function of the coordinates. The set (2.5) defines v^{ν} up to a gradient. Hence even in the special case (when X_{λ} is a gradient and therefore does not appear in (2.5)) we have to regard X_{λ} as an argument in the function w^{ν} .

Theorem (3.1). Upon substituting from (3.1) into the identity (1.10) one obtains a set of four differential equations

$$(3.2) \quad w^\nu = \partial_\lambda m^{\lambda\nu}$$

for four unknowns m_λ expressed by means of X_λ and $h_{\lambda\mu}$.

Proof. Assume that (3.1) is a solution of (2.5b). Then the left term of (3.2) contains $H_{\omega\mu\lambda\nu}$ while the right term contains $\sqrt{|\mathfrak{h}|}$ (and its derivative) and no other terms involving $H_{\omega\mu\lambda\nu}$. Hence (3.2) cannot be an identity. The remaining part of the theorem is obvious.

Agreement (3.1). In the equations (1.5) and (3.2), i.e.

$$(3.3) \quad \begin{array}{ll} \text{a)} & \partial_{[\omega} m_{\mu\lambda]} = 0 \\ \text{b)} & w^\nu = \partial_\lambda m^{\lambda\nu} \end{array}$$

which have the form of Maxwell equations,* we identify

- a) $m_{\lambda\mu}$ with the electromagnetic field,
 - b) w^ν with the world current density vector, and
 - c) $w^2 \stackrel{\text{def}}{=} w^\nu w_\nu$ with the square of the electric charge density whenever $w^2 \neq 0$.
- d) Moreover, we define the vector w^ν by means of

$$(3.4) \quad w^\nu \stackrel{\text{def}}{=} w^\nu / \sqrt{|\mathfrak{h}|}$$

the scalar w^2 by

$$(3.5) \quad w^2 \stackrel{\text{def}}{=} w^\nu w_\nu / |\mathfrak{h}|$$

and whenever $w^2 \neq 0$ we introduce “the four dimensional velocity unit vector”

$$(3.6) \quad u^\nu \stackrel{\text{def}}{=} w^\nu / \sqrt{w^2}.$$

Note that according to Theorem (3.1) the electromagnetic field is expressed in terms of auxiliary variables $h_{\lambda\mu}$ which for the time being are not fixed.

The following two theorems will be proved simultaneously.

Theorem (3.2). The square of the electric charge density is non-negative. The world current density is different from zero.

Theorem (3.3). The conservation law for the charge density is

$$(3.7) \quad \partial_\lambda w^\lambda = 0.$$

Proof. The statement of the first theorem is an obvious consequence of (1.11) and (3.1). The second theorem follows at once from (3.3b).

4. Continuation. In this section we shall derive some consequences from the previous results.

We obtained the field $m_{\lambda\mu}$ by means of ten equations (2.5) and (3.2) for twelve

* If we neglect dielectric constant and magnetic permeability and use a suitably chosen system of units.

unknowns v^ν , m_λ , X_λ (where in the special case the gradient vector X_λ has been involved only in (3.2)). Hence we may prescribe at least one new condition.

Agreement (4.1a). *We shall confine ourselves in the general case to electromagnetic fields satisfying the condition*

$$(4.1a) \quad 2\sqrt{g}m_{\omega\nu}\nabla^{[\omega}v^{\nu]} + \mathfrak{E}^{\mu\lambda\omega\nu}m_{\omega\nu}\partial_\mu X_\lambda = 0.$$

Note. In the general case this condition *does not restrict the generality* of the field $m_{\lambda\mu}$, for (4.1a) may be looked upon as a condition for the *arbitrary* non-gradient vector. On the other hand in the special case the second term on the left hand side is equal to zero and the arbitrary gradient X_λ is not involved.

Agreement (4.1b). *We shall restrict the generality of the problem in the special case assuming the validity of the condition*

$$(4.1b) \quad m_{\omega\nu}\nabla^{[\omega}v^{\nu]} = 0.$$

Note. We shall use the condition (4.1a) also in the special case for in this case it reduces to (4.1b).

Theorem (4.1). *The condition (4.1a) is equivalent to*

$$(4.2) \quad 2\sqrt{g}\mathfrak{E}^{\mu\lambda\omega\nu}m_{\omega\nu}\partial_\mu X_\lambda = \sqrt{|\mathfrak{h}|}G^{\mu\lambda\omega\nu}m_{\omega\nu}m_{\mu\lambda}.$$

Proof. One obtains (4.2) by eliminating $\nabla^{[\omega}v^{\nu]}$ from (4.1a) and (2.5b).

Note. According to this theorem, the condition (4.1a) may be converted into an equivalent one which does not contain v^ν and is of purely algebraic nature as far as $m_{\omega\nu}$ is concerned.

In the next section we shall use the following

Theorem (4.2). *The connection $\Gamma_{\lambda\mu}^\nu$ may be expressed by means of the electromagnetic field and the world current density vector in the following way*

$$(4.3) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2}\sqrt{g}e_{\lambda\mu\alpha\beta}(\nabla^\nu m^{\alpha\beta} - h^{\nu\alpha}w^\beta).$$

Proof. First we have by virtue of (1.1)

$$S_{\lambda\mu\alpha} = \frac{1}{2}K_{\lambda\mu\alpha} = \frac{1}{2}[3\nabla_{[\lambda}k_{\alpha\mu]} + 2\nabla_\alpha k_{\lambda\mu}]$$

and according to (2.2) and I(2.1c)

$$\begin{aligned} \frac{3}{2}\nabla_{[\lambda}k_{\alpha\mu]} &= \frac{1}{2}\sqrt{g}e_{\lambda\alpha\mu\nu}w^\nu \\ \nabla_\alpha k_{\lambda\mu} &= \frac{1}{2}\sqrt{g}\nabla_\alpha m^{\gamma\delta}e_{\lambda\mu\gamma\delta}. \end{aligned}$$

Hence

$$(4.4) \quad S_{\lambda\mu\alpha} = \frac{1}{2}K_{\lambda\mu\alpha} = \frac{1}{2}\sqrt{g}e_{\lambda\mu\gamma\delta}(\nabla_\alpha m^{\gamma\delta} - \delta_\alpha^\gamma w^\delta).$$

5. Symmetric part of (0.2). We prove first the following auxiliary *Theorem (5.1)*. Put

$$(5.1) \quad {}^*T'_{\lambda\mu} \stackrel{\text{def}}{=} \frac{1}{4} K_{\alpha\mu}{}^{\beta} K_{\beta\lambda}{}^{\alpha}.$$

Then

$$(5.2) \quad \begin{aligned} {}^*T'_{\lambda\mu} = {}^*T'_{\mu\lambda} = & -\frac{1}{2}g[-w_{\lambda} w_{\mu} + w^2 h_{\lambda\mu} - 2w^{\alpha} \nabla_{(\lambda} m_{\mu)\alpha} \\ & + \frac{1}{2}h_{\gamma\lambda} e_{\mu\alpha\xi\eta} \mathfrak{E}^{\gamma\beta\rho\nu} (\nabla_{\beta} m^{\xi\eta}) \nabla^{\alpha} m_{\rho\nu}]. \end{aligned}$$

Proof. We have by virtue of (4.4), (5.1) and [6] I(2.4b)

$$(5.3a) \quad \begin{aligned} {}^*T'_{\lambda\mu} &= \frac{1}{4} K_{\alpha\mu\beta} K^{\beta\gamma\alpha} h_{\lambda\gamma} = \frac{1}{4} K_{\mu\alpha\beta} K^{\gamma\beta\alpha} h_{\lambda\gamma} \\ &= \frac{g\mathfrak{h}^{-1}}{4} e_{\mu\alpha\xi\eta} \mathfrak{E}^{\gamma\beta\rho\nu} h_{\gamma\lambda} (\nabla_{\beta} m^{\xi\eta} - \delta_{\beta}^{\xi} w^{\eta}) (\nabla^{\alpha} m_{\rho\nu} - \delta_{\rho}^{\alpha} w_{\nu}). \end{aligned}$$

Moreover,

$$(5.3b) \quad {}^*T'_{\mu\lambda} = \frac{1}{4} K_{\lambda\alpha\beta} K^{\gamma\beta\alpha} h_{\mu\gamma} = \frac{1}{4} K_{\mu\beta\alpha} K^{\gamma\alpha\beta} h_{\lambda\gamma} = {}^*T'_{\lambda\mu},$$

so that the tensor ${}^*T'_{\mu\lambda}$ is a symmetric one. Using now the equations [6] I(2.2) as well as (3.3b), (3.4) and (3.5) we obtain (5.2) from (5.3).

Another auxiliary theorem needed later on is

Theorem (5.2). We have

$$(5.4) \quad {}^*T'_{\lambda\mu} h^{\lambda\mu} = +\frac{1}{2}gw^2.$$

Proof. First of all we obtain from (5.2)

$$(5.5) \quad {}^*T'_{\lambda\mu} h^{\lambda\mu} = -\frac{1}{2}g[-w^2 + 4w^2 - 2w^{\alpha} \nabla^{\lambda} m_{\lambda\alpha} + \frac{1}{2}e_{\mu\alpha\xi\eta} \mathfrak{E}^{\mu\beta\rho\nu} (\nabla_{\beta} m^{\xi\eta}) \nabla_{\alpha} m_{\rho\nu}].$$

This equation may be simplified. First of all we have

$$(5.6) \quad -2w^{\alpha} \nabla^{\lambda} m_{\lambda\alpha} = -2w^{\alpha} w_{\alpha} = -2w^2.$$

Moreover, by virtue of [6] I(2.2a)

$$(5.7a) \quad \begin{aligned} \frac{1}{2}e_{\mu\alpha\xi\eta} \mathfrak{E}^{\mu\beta\rho\nu} (\nabla_{\beta} m^{\xi\eta}) \nabla^{\alpha} m_{\rho\nu} &= 3\delta_{[\alpha\xi\eta]}^{\beta\rho\nu} (\nabla_{\beta} m^{\xi\eta}) \nabla^{\alpha} m_{\rho\nu} \\ &= (\nabla^{\alpha} m^{\beta\gamma}) \nabla_{\alpha} m_{\beta\gamma} - 2w^2. \end{aligned}$$

On the other hand we have from I(2.1)

$$\begin{aligned} k_{\lambda\nu} k^{\lambda\nu} &= -\frac{1}{4}g e_{\lambda\nu\alpha\beta} \mathfrak{E}^{\lambda\nu\gamma\delta} m^{\alpha\beta} m_{\gamma\delta} \\ &= -gm^{\alpha\beta} m_{\alpha\beta}. \end{aligned}$$

We know that the left term, as well as g , are constant (Theorem (1.1)). Hence $m^{\alpha\beta} m_{\alpha\beta}$ is constant and consequently

$$(5.7b) \quad (\nabla^{\alpha} m^{\beta\gamma}) \nabla_{\alpha} m_{\beta\gamma} = -m^{\beta\gamma} \Delta m_{\beta\gamma}.$$

Applying now the condition (4.1) we see that the right term in (5.7b) vanishes. Hence the right term of (5.7a) is equal to $-2w^2$. Substituting this expression in (5.5) and remembering (5.6) we obtain (5.4).

Theorem (5.3). *The symmetric part of (0.2) is equivalent to*

$$(5.8a) \quad H_{\lambda\mu} - \frac{1}{2}Hh_{\lambda\mu} = T_{\lambda\mu}$$

where

$$(5.8b) \quad \begin{aligned} T_{\mu\lambda} = T_{\lambda\mu} = {}^*T_{\lambda\mu} - \frac{1}{2}h_{\lambda\mu} {}^*T_{\alpha\beta} h^{\alpha\beta} \\ = -\frac{1}{2}g\{-w_\lambda w_\mu + \frac{3}{2}w^2 h_{\lambda\mu} - 2w^\alpha \nabla_\alpha m_{\mu}\}_\alpha \\ + \frac{1}{2}h_{\gamma\lambda} e_{\mu\alpha\xi\eta} \mathfrak{E}^{\gamma\beta\rho\nu} (\nabla_\beta m^{\xi\eta})(\nabla^\alpha m_{\rho\nu})\}. \end{aligned}$$

Proof. The symmetric part of (0.2) is equivalent to [6] II(7.2) and this equation is by virtue of (5.1) equivalent to

$$(5.9a) \quad H_{\lambda\mu} = {}^*T_{\lambda\mu}$$

where ${}^*T_{\lambda\mu}$ is given by (5.2). On the other hand we obtain from (5.9a) and (5.4)

$$(5.9b) \quad H = \frac{w^2 g}{2} = {}^*T_{\lambda\mu} h^{\lambda\mu}$$

and the proof follows easily.

6. Metric and gravitation.

Agreement (6.1). *Since the equation (5.8a) has a form similar to the field equations of the pure gravitational theory, in the unified theory we identify*

- a) $h_{\lambda\mu}$ with the gravitational potentials
- b) $T_{\lambda\mu}$ with the total momentum energy tensor.

Note. In the gravitational theory the gravitational field $h_{\lambda\mu}$ plays also the role of the metric tensor of the space. In order to investigate the geometrical role of the gravitational field in the unified theory we will first make precise what we understand by a metric tensor in a space endowed with a connection $\Gamma_{\lambda\mu}^\nu$. In the following definition we denote by D_μ the symbol of the covariant derivative with respect to this connection.

Definition (6.1). *Let $t_{\lambda\mu}$ be a tensor with the following properties:*

- a) *it is symmetric*

$$t_{\lambda\mu} = t_{\mu\lambda}$$

- b) *it is of rank n^**
- c) *any system of differential equations*

$$(6.1) \quad V^\lambda D_\lambda V^\nu = 0$$

* In our case $n = 4$.

admits a first quadratic integral

$$(6.2) \quad V^\lambda V^\mu t_{\lambda\mu} = \text{const. along } V^\nu.$$

Whenever the requirements a), b), c) are satisfied then we say that $t_{\lambda\mu}$ is a metric tensor* in the space endowed with the connection $\Gamma_{\lambda\mu}^\nu$.

Theorem (6.1). The tensor $h_{\lambda\mu}$ is a metric tensor in the space endowed with the connection $\Gamma_{\lambda\mu}^\nu$ which is defined in the unified theory by

$$(6.3a) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha}.$$

Proof. We know (cf. Theorem (2.1a) in [4]) that equation (6.3a) is equivalent to

$$(6.3b) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

so that

$$(6.4) \quad D_\omega h_{\lambda\mu} = 2S_{\omega(\mu}{}^\alpha g_{\lambda)\alpha}.$$

Consider now the scalar

$$h_{\lambda\mu} V^\lambda V^\mu = V^2$$

where the vector V^λ satisfies (6.1). Then we have

$$\begin{aligned} V^\nu \partial_\nu V^2 &= V^\nu D_\nu V^2 \\ &= V^\nu (D_\nu h_{\lambda\mu}) V^\lambda V^\mu + 2h_{\lambda\mu} V^\nu (D_\nu V^\lambda) V^\mu \end{aligned}$$

or by virtue of (6.4)

$$V^\nu \partial_\nu V^2 = (D_\nu h_{\lambda\mu}) V^\nu V^\lambda V^\mu = 2S_{\nu(\mu}{}^\alpha g_{\lambda)\alpha} V^\nu V^\mu V^\lambda = 0.$$

Hence (6.2) is satisfied along V^ν .

Note. According to this theorem the tensor $h_{\lambda\mu}$ is a metric tensor in the space of pure gravitational theory as well as in the space of the unified theory.

Theorem (6.2a). The equations (5.8a) may be looked upon

a) either as a set of ten differential equations for ten unknowns $h_{\lambda\mu}$ expressed by means of the auxiliary variables $m_{\lambda\mu}$ alone

b) or as a set of ten differential equations for ten unknowns $h_{\lambda\mu}$ expressed in terms of X_λ alone.†

* It is obvious according to this definition that $h_{\lambda\mu}$ is a metric tensor in the space endowed with the connection $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$.

† We do not exclude the case that the $h_{\lambda\mu}$ contain also the coordinates besides the argument already mentioned in text. We mean by this formulation that out of two possible arguments $m_{\lambda\mu}$, X_λ we have only the one mentioned in the text.

Proof. Substituting in (5.8a) for w^λ from (3.3b) and (3.4) we obtain a set of ten equations for ten unknown $h_{\lambda\mu}$ expressed in terms of the $m_{\lambda\mu}$. On the other hand $m_{\lambda\mu}$ as a solution of (2.5) and (3.2) is expressed in terms of $h_{\lambda\mu}$ and X_λ .

Theorem (6.2)b. *The electromagnetic field is a function of the gravitational field.*

Both statements have been already proved.

Note. According to Agreement (1.1) we exclude the case $k_{\lambda\mu} = 0$ so that the electromagnetic field is always present, $m_{\lambda\mu} \neq 0$. Hence this field may be looked upon as a basic field, so that the statement of the previous theorem about the gravitational field may be reworded by saying that the electromagnetic field creates the gravitational field. Then the question arises whether there are electromagnetic fields which produce a special space (*i.e.* space without gravitation, *cf.* Definition I(5.2)) as this was the case dealt with in the previous chapter (Theorem I(5.1)). If such an electromagnetic field exists then we must have by virtue of (4.1)

$$\mathfrak{E}^{\mu\lambda\omega\nu} m_{\omega\nu} \partial_\mu X_\lambda = 0$$

and according to (2.5b)

$$2\sqrt{g}\nabla^{(\omega} w^{\nu)} = \mathfrak{E}^{\mu\lambda\omega\nu} \partial_\mu X_\lambda.$$

Moreover we must have by virtue of (5.9a)

$$w_\lambda w_\mu - \frac{3}{2}w^2 h_{\lambda\mu} + 2w^\alpha \nabla_{(\lambda} m_{\mu)\alpha} = \frac{1}{2}h_{\gamma\lambda} e_{\mu\alpha\xi\eta} \mathfrak{E}^{\gamma\beta\rho\nu} (\nabla_\beta m^{\xi\eta}) \nabla^\alpha m_{\rho\nu}^*$$

* This equation yields $w = 0$.

where we have to substitute for w_λ from (3.3) and (3.4). Despite a considerable effort I have been not able to exhibit such a field (in contrast to the case treated in the previous chapter).

7. Mass density. Motion.

Theorem (7.1). *The mass density as defined by Definition I(5.1) satisfies the condition*

$$(7.1) \quad \mathfrak{M} = \mathfrak{M}(gw^2).$$

Moreover, we have

$$(7.2) \quad \mathfrak{M} = 0$$

if and only if the particle moves with the velocity of light.

The proof follows at once from Definition I(5.1) and from (5.9b).

Agreement (7.1). *We impose another condition on $\mathfrak{M} \neq 0$, namely that it satisfies the continuity equation.*

Note. Whenever $\mathfrak{M} \neq 0$ we have also $w^2 \neq 0$ so that the defining equation (3.6) has a sense. Then the continuity equation is

$$(7.3) \quad \partial_\lambda \mathfrak{M} u^\lambda = 0.$$

Theorem (7.2). *The mass*

$$M \stackrel{\text{def}}{=} \mathfrak{M} / \sqrt{\mathfrak{h}}$$

and the electric charge w are related by

$$(7.4a) \quad M = |w/\mu|$$

where $\mu \neq 0$ is an arbitrary scalar function of position (to be found experimentally) which is constant along the trajectory of the particle:

$$(7.4b) \quad u^\lambda \partial_\lambda \mu = 0.$$

Proof. The continuity equation (7.3) is equivalent to

$$u^\lambda \partial_\lambda \ln \mathfrak{M} + \partial_\lambda u^\lambda = 0$$

while the conservation law (3.7) is equivalent to

$$\frac{1}{2} u^\lambda \partial_\lambda \ln w^2 + \partial_\lambda u^\lambda = 0.$$

Comparing these two equations we obtain

$$u^\lambda \partial_\lambda \ln \frac{\mathfrak{M}}{\sqrt{w^2}} = u^\lambda \partial_\lambda \ln \frac{M}{w} = 0$$

and the theorem follows easily.

Theorem (7.3). *Whenever $\mathfrak{M} \neq 0$ the momentum energy tensor $T_{\lambda\mu}$ may be written*

$$(7.5a) \quad T_{\lambda\mu} = -\frac{1}{2}g\{(M\mu)^2[-u_\lambda u_\mu + \frac{3}{2}h_{\lambda\mu}] - 2M\mu^* u^\alpha \nabla_\alpha m_\mu\}_\mu \\ + \frac{1}{2}h_{\gamma\lambda} \epsilon_{\mu\alpha\epsilon\eta} \mathfrak{E}^{\gamma\beta\rho\nu} (\nabla_\beta m^{\epsilon\eta}) \nabla^\alpha m_{\rho\nu}\}$$

where

$$(7.5b) \quad \mu^* = |\mu| \operatorname{sgn} w.$$

The proof is obvious.

Note. According to this theorem the momentum energy tensor in the unified theory has the same elements

$$M, u_\lambda, m_{\lambda\mu}, h_{\lambda\mu}$$

as in the gravitational theory in the presence of an electromagnetic field. However, in the unified theory we have one element more, namely the function μ , which has to be found experimentally.

In the subsequent considerations we shall need the following auxiliary

Theorem (7.4). *Put*

$$(7.6) \quad P^{\lambda\mu} = P^{\mu\lambda} \stackrel{\text{def}}{=} \frac{3}{2} w^2 h^{\lambda\mu} - 2w_\alpha \nabla^{(\lambda} m^{\mu)\alpha} + \frac{1}{2} \mathfrak{E}^{\mu\alpha\xi\eta} \mathfrak{E}^{\lambda\beta\rho\nu} \nabla_\beta m_{\xi\eta} \nabla_\alpha m_{\rho\nu} \mathfrak{h}^{-1}.$$

Then we have

$$(7.7) \quad P^\mu \stackrel{\text{def}}{=} \nabla_\lambda P^{\lambda\mu} = D_\lambda P^{\lambda\mu}.$$

Proof. We have for any symmetric tensor $Q^{\lambda\mu}$ whatsoever by virtue of (1.1)

$$(7.8) \quad D_\lambda Q^{\lambda\nu} = \nabla_\lambda Q^{\lambda\nu} + S_{\alpha\lambda}{}^\lambda Q^{\alpha\nu} + S_{\alpha\lambda}{}^\nu Q^{\lambda\alpha} = \nabla_\lambda Q^{\lambda\nu}.$$

Theorem (7.5). *We have*

$$(7.9a) \quad \nabla_\lambda T^{\lambda\nu} = 0$$

(conservation of momentum and energy) and this equation is equivalent to

$$(7.9b) \quad D_\lambda T^{\lambda\nu} = 0.$$

Proof. The left hand term of (5.8a) satisfies the condition

$$(7.9c) \quad \nabla^\lambda (H_{\lambda\mu} - \frac{1}{2} H h_{\lambda\mu}) = 0$$

which is a consequence of the Bianchi identity. This equation (7.9c) is obviously equivalent to (7.9a) and this latter is equivalent to (7.9b) as it follows from (7.8).

Theorem (7.6). *The equations of motion are*

$$(7.10a) \quad w^\lambda \nabla_\lambda w^\mu = P^\mu$$

where P is defined by (7.7). They are equivalent to

$$(7.10b) \quad w^\lambda D_\lambda w^\mu = P^\mu.$$

Proof. Substituting into (7.9a) from (5.8) one obtains

$$(7.11) \quad \nabla_\lambda w^\lambda w^\mu = P^\mu.$$

On the other hand we have from (3.4) and (3.7)

$$\partial_\lambda w^\lambda = \nabla_\lambda \sqrt{|\mathfrak{h}|} w^\lambda = \sqrt{|\mathfrak{h}|} \nabla_\lambda w^\lambda = 0$$

so that

$$\nabla_\lambda w^\lambda w^\mu = w^\lambda \nabla_\lambda w^\mu.$$

On the other hand by virtue of (1.1)

$$w^\lambda D_\lambda w^\mu = w^\lambda \nabla_\lambda w^\mu + S_{\alpha\lambda}{}^\mu w^\lambda w^\alpha = w^\lambda \nabla_\lambda w^\mu.$$

Note. Whenever $M \neq 0$ the equation (7.10a) or (7.9b) may be written

$$(7.10c) \quad (u^\lambda D_\lambda u^\nu =) u^\lambda \nabla_\lambda u^\nu = \frac{P^\nu}{(M_\mu)^2} - u^\nu u^\lambda \partial_\lambda \ln M_\mu.$$

8. Discussion of test devices. If we want to test the unified theory *as developed in this chapter* we have to look for an electromagnetic field *for which the restriction (1.1) is a quantitative one rather than a structural one*. Let us consider the tensor $k_{\lambda\mu}$ which has been dealt with in the section 6 of the first chapter:

$$k_{\lambda\mu} = \epsilon K_{\lambda\mu}, \quad \epsilon^2 \rightarrow 0.$$

Then the equations I(2.1) yield

$$(8.1a) \quad m_{\lambda\mu} = \epsilon M_{\lambda\mu}, \quad \epsilon^2 \rightarrow 0$$

where

$$(8.1b) \quad M_{\lambda\nu} = -\frac{1}{2} \frac{\sqrt{|g|}}{\sqrt{g}} c_{\lambda\nu\alpha\beta} K^{\alpha\beta}.$$

We shall discuss a possibility for tests, based on (8.1), analysing each step.

First step. Assume that the sun's electromagnetic field is so weak that in the region exterior to the sun it satisfies (8.1). Then the equations (5.8a) reduce to

$$(8.2) \quad H_{\lambda\mu} = 0.$$

This is the same condition as in the gravitational theory. We take for $h_{\lambda\mu}$ the Schwarzschild solution of (8.2). Hence, the first step (where we did not assume anything about the mass density of the moving particle) led us to the gravitational field.

On the other hand we see from (2.5a) that the condition (4.1) is equivalent to

$$(8.3) \quad m_{\alpha\beta} \Delta m^{\alpha\beta} = 0,$$

and this condition is automatically satisfied (regardless of the choice of the vector X_λ) by virtue of our assumption about the weakness of the field.

This assumption does not require a deep knowledge of the electromagnetic field of the sun. This situation changes with the assumption of the

Second step. Assume that we actually know the electromagnetic field $m_{\lambda\nu}$ of the sun.* Then the equations of motion (7.10) contain only four unknowns, namely w^ν expressed in terms of the known fields $h_{\lambda\mu}$ and $m_{\lambda\mu}$. On the other hand they reduce to the identity $0 = 0$ on the basis of our assumption in the first step. Therefore if we wish some more information about the motion we have to adopt the method of successive approximations as used in the celestial mechanics of the gravitational theory: In the first approximation (*i.e.* in neglecting terms of

* Expressed in terms of the unknowns v^ν .

order ϵ^2) we obtained the field equations (8.2). In order to get any information from (7.10) we have to neglect all terms of order ϵ^3 but retain terms of order ϵ^2 . In doing so we see that both terms in (7.10) are of the order of magnitude of ϵ^2 and therefore they cannot be simplified further.

Third step. The equations of motion yield trajectories which for $w = 0$ might be identified with the trajectories of photons, for $w \neq 0$ with the planetary orbits. Now two possibilities have to be considered. Either this identification leads to the experimentally tested deflection of light and the advance of the perihelion of Mercury, or it does not. The latter case would indicate that our interpretation of Einstein's unified theory is wrong. In contrast to the situation in the gravitational theory the first possibility does not in itself confirm our interpretation.

Fourth step. Assume that the above mentioned identification leads to the deflection of light and the advance of the perihelion of Mercury. Then we have to examine whether or not in both these cases there exists a vector X_λ such that the equations (2.5a) (or (2.5b)) together with (3.3b) are satisfied for the corresponding solution w^ν of (7.10). If the answer turns out to be positive then our interpretation describes at least the two physical phenomena, mentioned in the second step.

Note I. The trajectories of photons as well as the planetary orbits are described in the gravitational theory by the geodesics of the Schwarzschild solution $h_{\lambda\mu}$ of (8.2). On the other hand if we put

$$w^\nu = \rho \frac{dx^\nu}{dp}, \quad \rho \neq 0$$

where p is a suitably chosen parameter (which for $w \neq 0$ might be identified with the four dimensional line element) then we have

$$w^\nu \nabla_\lambda w^\nu = \rho^2 \left\{ \left[\frac{d^2 x^\nu}{dp^2} + \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} \right] + \frac{d \ln \rho}{dq} \frac{dx^\nu}{dp} \right\}.$$

Hence, if there exists such a function ρ for which

$$\frac{1}{\rho^2} [P^\nu - w^\nu w^\lambda \partial_\lambda \ln \rho]$$

is proportional to w^ν , then we may be sure that the third step will lead to the experimentally tested deflection of light as well as to the advance of the perihelion of Mercury. We shall deal with this problem in the next section.

9. The law of inertia. The skew symmetric part of (0.2) is equivalent to the four original Einstein equations

$$(9.1) \quad \partial_{[\omega} R_{\mu\lambda]} = 0.$$

In this section we shall replace these conditions by another set of four conditions. In order to do so we start again with Definition (1.1), which leads to the identity (1.10). Moreover we introduce the expressions

$$(9.2) \quad v^\nu \stackrel{\text{def}}{=} v^\nu / \sqrt{|\mathfrak{h}|} \quad v^2 \stackrel{\text{def}}{=} v^\nu v_\nu.$$

Agreement (9.1a). In this section we replace the conditions (9.1) by another set of four differential equations

$$(9.3a) \quad D_\lambda \mathfrak{E}^{\alpha\beta\gamma\lambda} S_{\alpha\beta\gamma} S_{[\xi\eta\zeta]} = -\sigma \sqrt{|\mathfrak{g}|} S_{[\xi\eta\zeta]},$$

where the scalar σ is for the time being an arbitrary function of position.

Theorem (9.1). The condition (9.3a) is equivalent to

$$(9.3b) \quad P^\nu = \sigma v^\nu$$

where the vector P^ν is defined by (7.7).

Proof. Put

$$\mathfrak{a}^{\lambda\nu} \stackrel{\text{def}}{=} \mathfrak{E}^{\alpha\beta\gamma\lambda} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\alpha\beta\gamma} S_{\xi\eta\zeta}.$$

Then $\mathfrak{a}^{\lambda\nu}$ is a symmetric density of weight -2 and (9.3a) is equivalent to

$$D_\lambda \mathfrak{a}^{\lambda\nu} = -\sigma \sqrt{|\mathfrak{g}|} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\xi\eta\zeta}.$$

On the other hand we have by virtue of (1.1)

$$\begin{aligned} D_\lambda \mathfrak{a}^{\lambda\nu} &= \nabla_\lambda \mathfrak{a}^{\lambda\nu} + S_{\alpha\lambda}{}^\nu \mathfrak{a}^{\lambda\alpha} + S_{\alpha\lambda}{}^\lambda \mathfrak{a}^{\alpha\nu} - 2S_{\alpha\lambda}{}^\alpha \mathfrak{a}^{\lambda\nu} \\ &= \nabla_\lambda \mathfrak{a}^{\lambda\nu} \end{aligned}$$

so that (9.3a) is equivalent to

$$(9.3c) \quad \nabla_\lambda \mathfrak{E}^{\alpha\beta\gamma\lambda} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\alpha\beta\gamma} S_{\xi\eta\zeta} = -\sigma \sqrt{|\mathfrak{g}|} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\xi\eta\zeta}.$$

Moreover, we obtain by virtue of (5.8), (7.6), (9.2) and (1.8)

$$\begin{aligned} H^{\lambda\nu} - \tfrac{1}{2} H h^{\lambda\nu} - T^{\lambda\nu} + \tfrac{1}{2} g v^\lambda v^\nu &= H^{\lambda\nu} - \tfrac{1}{2} H h^{\lambda\nu} + \tfrac{1}{2} g P^{\lambda\nu} \\ &= \tfrac{g}{2} v^\lambda v^\nu = \tfrac{1}{2|\mathfrak{h}|} \mathfrak{E}^{\alpha\beta\gamma\lambda} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\alpha\beta\gamma} S_{\xi\eta\zeta}, \end{aligned}$$

and consequently according to (9.3c) and (7.9c)

$$\tfrac{g}{2} \nabla_\lambda P^{\lambda\nu} = \tfrac{1}{2|\mathfrak{h}|} \sigma \sqrt{|\mathfrak{g}|} \mathfrak{E}^{\xi\eta\zeta\nu} S_{\xi\eta\zeta}.$$

Applying (1.8) and (9.2) to this equation we obtain (9.3b).

Agreement (9.1b). We choose the arbitrary function σ involved in (9.3) in such a way that

$$(9.4) \quad m_{\alpha\beta} \Delta m^{\alpha\beta} = 0.$$

Consider now the $h_{\lambda\mu}$ involved in (9.3) as auxiliary variables. Then (9.3) may be looked upon as a set of four differential equations for *nine* unknowns v^ν , m_λ , σ . Denote by

$$(9.5a) \quad v^\nu \stackrel{\text{def}}{=} w^\nu(h_{..}, m_{..}, \sigma)$$

a solution of this system.

Theorem (9.1b). *Upon substituting from (9.5a) into the identity (1.10) one obtains a set of four differential equations (3.2) (where w^ν is defined by (3.4)) for the unknowns*

$$(9.5b) \quad m_\lambda = m_\lambda(h_{..}, \sigma).$$

Substituting the solution m_λ of these equations into (9.4) one obtains

$$(9.6) \quad \sigma = \sigma(h_{..}).$$

Proof. The identity (1.10) does not contain σ . If we substitute into it from (9.5a) then the left term (but not the right one) contains σ . Hence, (3.2) cannot be an identity. The remaining part of the theorem is obvious.

According to this theorem we can now use Agreement (3.1), as well as Theorems (3.2), (3.3), (5.1), (5.2), and (5.3). However, Theorem (6.2a) has to be changed.

Theorem (9.2). *The equations (5.8a) may be looked upon as a set of ten differential equations for ten unknowns $h_{\lambda\mu}$ expressed in terms of auxiliary variables $m_{\lambda\mu}$.*

The proof follows at once from (3.2) and (5.8).

Theorem (6.2b) holds too as well as does Theorem (7.1). (The condition (9.4) imposed on σ has been introduced only for the sake of this theorem and does not influence the main features of this theory.) However, as soon as we adopt the condition (9.4) we see that we may apply also Agreement (7.1) as well as Theorems (7.2) and (7.3). Moreover, Theorem (7.5) holds independently of the condition (9.4). However, Theorem (7.6) has to be changed.

Theorem (9.3). *The equations of motion are expressed by either one of the equivalent sets*

$$(9.7) \quad w^\mu \nabla_\mu w^\lambda = \sigma w^\lambda$$

$$(9.8) \quad w^\mu D_\mu w^\lambda = \sigma w^\lambda.$$

Proof. Comparing (9.3b) with (7.10a, b), we see that the latter equations are equivalent to (9.7) as well as to (9.8).

If we substitute into

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

from I(2.1c) we obtain

$$(9.9) \quad g_{\lambda\mu} = h_{\lambda\mu} + \frac{1}{2} \sqrt{g} \epsilon_{\lambda\nu\alpha\beta} m_{\gamma\delta} h^{\gamma\alpha} h^{\delta\beta}.$$

Hence $g_{\lambda\mu}$ is described by this equation in terms of the gravitational as well as electromagnetic field. The tensor field $g_{\lambda\mu}$ defined by (9.9) will be referred to as *the unified field*. By an *autoparallel line of the unified field* we understand any autoparallel line with respect to the connection (4.3). By a *geodesic of the gravitational field* we understand any autoparallel line with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$.

Theorem (9.4). *Whenever a particle moves freely in the unified field it describes an autoparallel line of this field. This autoparallel line is at the same time a geodesic of the gravitational field. (The law of inertia.)*

Proof. Put

$$w^\nu = \rho \frac{dx^\nu}{dp},$$

where ρ is a solution of

$$\frac{d\rho}{dp} = \sigma.$$

Then we obtain from (9.7) and (4.3)

$$(9.10) \quad \frac{d^2 x^\nu}{dp^2} + \Gamma_{\lambda\mu}^\nu \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = \frac{d^2 x^\nu}{dp^2} + \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0.$$

These equations prove our statement.

Note. The first part of Theorem (9.4) describes physical phenomena *in terms of $g_{\lambda\mu}$ and $\Gamma_{\lambda\mu}^\nu$ without splitting these objects with regard to gravitation and electromagnetism.*

Note. We know that $I(4.8)$ is equivalent to $I(5.4)$ i.e. to (9.4). Moreover, in the case treated in Chapter I, $v^\nu = 0$, the equations (9.3) are identically satisfied and therefore do not represent new conditions. The desired four conditions for $m_{\lambda\mu}$ in the absence of $I(2.5a)$ are given by $I(2.3b)$ (which have been incorporated in $I(2.5a)$ following from the restriction $I(1.5)^*$). Hence, the theory as depicted in this section holds also in the case of $I(1.5)$ dealt with in the first chapter.

10. Discussion of test devices. In this section we shall consider the test discussed in Section 8, this time from the point of view of the theory developed in the previous section.

The first step is the same as in Section 8 and yields the Schwarzschild solution $h_{\lambda\mu}$ of

$$(10.1) \quad H_{\lambda\mu} = 0.$$

On the other hand we see that the equation

$$(10.2) \quad m_{\alpha\beta} \Delta m^{\alpha\beta} = 0$$

is automatically satisfied by virtue of (8.1) and regardless the choice of σ .

* We do not need more than four conditions, since the eight unknowns $m_{\lambda\mu}$, v^ν reduce here to only four unknowns $m_{\lambda\lambda}$.

Second step. At this stage we do not need the knowledge of the electromagnetic field of the sun, for the equations of motion (9.10) depend only on $h_{\lambda\mu}$. Moreover, the identification mentioned in the third step of Section 8 yields for the paths of photons the equation (9.10) with

$$(10.3a) \quad h_{\lambda\mu} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0$$

while the planetary orbits are described by (9.10) with

$$(10.3b) \quad h_{\lambda\mu} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} \neq 0.$$

The equations (10.1), (9.10) and (10.3) allow us to predict the *red shift of spectral lines, the bending of light as well as the advance of the perihelion of Mercury*. The same equations being used in the gravitational theory for explaining the above mentioned three phenomena, we see that the *new theory predicts the same shift of spectral lines, the same deflection of light as well as the same advance of the perihelion of Mercury as obtained by the gravitational theory*. We know that the experimental evidence seems to favor these predictions obtained originally by the gravitational theory.

Third step. The equations (9.10) of motion appear in the unified theory as a consequence of (9.3a), equivalent to (9.3b):

$$(10.4) \quad \nabla_\lambda [\tfrac{3}{2} w^2 h^{\lambda\nu} - 2 w_\alpha \nabla^\alpha m^{(\nu)\alpha} + \tfrac{1}{2} \mathfrak{E}^{\mu\alpha\xi\eta} \mathfrak{E}^{\lambda\beta\rho\nu} (\nabla_\beta m_{\xi\eta}) \nabla_\alpha m_{\rho\nu} \mathfrak{h}^{-1}] = \sigma w^\nu.*$$

Hence, if we want to test this theory we have to know the electromagnetic field $m_{\lambda\mu}$ of the sun. Assume that we master this knowledge (as certainly we shall at some future time). Hence, in order to test the theory *we have to check whether the equations (10.4)† are satisfied* by the electromagnetic field of the sun. Here the last word belongs to the astrophysicists.

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* The vector w^ν describes the geodesic (9.10) in consideration and satisfies the equations (3.2). In deriving (10.4) we again used the method of successive approximations, keeping in mind that σ is either equal to zero or of the order of magnitude of ϵ , as is seen from (9.3a).

† The vector w^ν describes the geodesic (9.10) in consideration and satisfies the equations (3.2).