Extensions in the Applicability of Riemann's Formula

G. S. S. LUDFORD

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md.

Communicated by HILDA GEIRINGER

§1. PRELIMINARIES

Introduction. The object of this paper is to investigate the Cauchy initial values problem for the linear hyperbolic differential equation

$$L(u) \equiv u_{xy} + au_x + bu_y + cu = f$$

when the initial data has a finite number of discontinuities, when the tangent to the initial curve has a finite number of discontinuous changes in direction, and when some characteristics meet the initial curve in more than one point.* It will be assumed that a, b, c together with a_x, b_y are continuous functions of x, y in any particular characteristic rectangle R considered; further, for convenience, it will be supposed that $f \equiv 0$.

With these conditions on a, b, c , it is well-known that there exists a function $v(x, y; \xi, \eta)$, the Riemann function, which satisfies the adjoint equation

$$(1) \quad M(v) \equiv v_{xy} - (av)_x - (bv)_y + cv = 0,$$

as well as the conditions

$$(2) \quad \begin{cases} v_y = av & \text{on } x = \xi, \\ v_x = bv & \text{on } y = \eta, \\ v = 1 & \text{at } x = \xi, y = \eta. \end{cases}$$

* The classical literature states that there is no solution, in general, of this problem, see [2].

In addition it is known that $u = v(\xi, \eta; x, y)$ satisfies

$$(3) \quad L(u) = 0.$$

This follows from the symmetry property of the Riemann function.

Certain other properties of the Riemann function will be needed subsequently, and although these are used implicitly by Rademacher [2] for a similar purpose, they do not seem to have been proved before.

Continuity Properties of the Riemann Function. If a sequence of continuous functions $v_n(x, y; \xi, \eta)$ is defined by the recurrence relation

$$(4) \quad \begin{aligned} v_n(x, y; \xi, \eta) = & \int_x^\xi \int_y^\eta \left[a(\alpha, \beta) \frac{\partial v_{n-1}}{\partial \alpha}(x, y; \alpha, \beta) + b(\alpha, \beta) \frac{\partial v_{n-1}}{\partial \beta}(x, y; \alpha, \beta) \right. \\ & \left. + c(\alpha, \beta) v_{n-1}(x, y; \alpha, \beta) \right] d\alpha d\beta + \int_x^\xi b(\alpha, y) v_{n-1}(x, y; \alpha, y) d\alpha \\ & + \int_y^\eta a(x, \beta) v_{n-1}(x, y; x, \beta) d\beta, \end{aligned}$$

with $v_0(x, y; \xi, \eta) \equiv 1$, then, see [1], $\sum_{n=0}^\infty v_n(x, y; \xi, \eta)$ converges uniformly, for (x, y) and (ξ, η) in R , to $v(x, y; \xi, \eta)$. The latter is therefore a continuous function of all its variables* in R . The fact that $v_x, v_y, v_\xi, v_\eta; v_{\xi x}, v_{\xi y}, v_{\eta x}, v_{\eta y}, v_{\xi \eta}; v_{\xi \eta x}, v_{\xi \eta y}$ are also continuous functions† in R , must now be demonstrated.

Lemma 1. *If v_{n-1} ($n \geq 4$) is continuous in R and bounded there by $Hd^{n-1}/(n-1)!$, where H is some positive constant and $d = |x - \xi| + |y - \eta|$; if $\partial v_{n-1}/\partial x, \partial v_{n-1}/\partial y, \partial v_{n-1}/\partial \xi, \partial v_{n-1}/\partial \eta$ are continuous in R and bounded there by $Hd^{n-2}/(n-2)!$; if $\partial^2 v_{n-1}/\partial \xi \partial x, \partial^2 v_{n-1}/\partial \xi \partial y, \partial^2 v_{n-1}/\partial \eta \partial x, \partial^2 v_{n-1}/\partial \eta \partial y, \partial^2 v_{n-1}/\partial \xi \partial \eta$ are continuous in R and bounded there by $Hd^{n-3}/(n-3)!$; and if $\partial^3 v_{n-1}/\partial \xi \partial \eta \partial x, \partial^3 v_{n-1}/\partial \xi \partial \eta \partial y$ are continuous in R and bounded there by $Hd^{n-4}/(n-4)!$, then the corresponding derivatives of v_n exist and are continuous in R , and are bounded there by*

$$HKd^n/n!, \quad HKd^{n-1}/(n-1)!, \quad HKd^{n-2}/(n-2)!, \quad HKd^{n-3}/(n-3)!$$

respectively, where K is a constant which depends only on the size of R and the bound M of the functions a, b, c, a_x, b_y , which are continuous in R .

* This is the sense in which "continuous" will be understood, unless otherwise stated.

† It will be clear from the proof that it is not necessary to specify the order of differentiation of these derivatives.

For from the continuity properties of v_{n-1} and its assigned derivatives it is clear that (4) may be differentiated to give, for example*

$$\begin{aligned}
 \frac{\partial v_n}{\partial x} &= \int_x^\xi \int_y^\eta \left[a(\alpha, \beta) \frac{\partial^2 v_{n-1}}{\partial x \partial \alpha} (x, y; \alpha, \beta) + b(\alpha, \beta) \frac{\partial^2 v_{n-1}}{\partial x \partial \beta} (x, y; \alpha, \beta) \right. \\
 &\quad \left. + c(\alpha, \beta) \frac{\partial v_{n-1}}{\partial x} (x, y; \alpha, \beta) \right] d\alpha d\beta \\
 &\quad + \int_y^\eta \left[a(x, \beta) \frac{\partial v_{n-1}}{\partial x} (x, y; x, \beta) - b(x, \beta) \frac{\partial v_{n-1}}{\partial \beta} (x, y; x, \beta) \right. \\
 &\quad \left. + \{a_x(x, \beta) - c(x, \beta)\} v_{n-1}(x, y; x, \beta) \right] d\beta \\
 &\quad + \int_x^\xi b(\alpha, y) \frac{\partial v_{n-1}}{\partial x} (x, y; \alpha, y) d\alpha - b(x, y) v_{n-1}(x, y; x, y), \\
 \frac{\partial v_n}{\partial \xi} &= \int_y^\eta \left[a(\xi, \beta) \frac{\partial v_{n-1}}{\partial \xi} (x, y; \xi, \beta) + b(\xi, \beta) \frac{\partial v_{n-1}}{\partial \beta} (x, y; \xi, \beta) \right. \\
 (5) \quad &\quad \left. + c(\xi, \beta) v_{n-1}(x, y; \xi, \beta) \right] d\beta + b(\xi, y) v_{n-1}(x, y; \xi, y), \\
 \frac{\partial^2 v_n}{\partial \xi \partial x} &= \int_y^\eta \left[a(\xi, \beta) \frac{\partial^2 v_{n-1}}{\partial \xi \partial x} (x, y; \xi, \beta) + b(\xi, \beta) \frac{\partial^2 v_{n-1}}{\partial x \partial \beta} (x, y; \xi, \beta) \right. \\
 &\quad \left. + c(\xi, \beta) \frac{\partial v_{n-1}}{\partial x} (x, y; \xi, \beta) \right] d\beta + b(\xi, y) \frac{\partial v_{n-1}}{\partial x} (x, y; \xi, y), \\
 \frac{\partial^2 v_n}{\partial \eta \partial x} &= \int_x^\xi \left[a(\alpha, \eta) \frac{\partial^2 v_{n-1}}{\partial x \partial \alpha} (x, y; \alpha, \eta) + b(\alpha, \eta) \frac{\partial^2 v_{n-1}}{\partial x \partial \eta} (x, y; \alpha, \eta) \right. \\
 &\quad \left. + c(\alpha, \eta) \frac{\partial v_{n-1}}{\partial x} (x, y; \alpha, \eta) \right] d\alpha + a(x, \eta) \frac{\partial v_{n-1}}{\partial x} (x, y; x, \eta) \\
 &\quad - b(x, \eta) \frac{\partial v_{n-1}}{\partial \eta} (x, y; x, \eta) + [a_x(x, \eta) - c(x, \eta)] v_{n-1}(x, y; x, \eta), \\
 \frac{\partial^2 v_n}{\partial \xi \partial \eta} &= a(\xi, \eta) \frac{\partial v_{n-1}}{\partial \xi} (x, y; \xi, \eta) + b(\xi, \eta) \frac{\partial v_{n-1}}{\partial \eta} (x, y; \xi, \eta) \\
 &\quad + c(\xi, \eta) v_{n-1}(x, y; \xi, \eta), \\
 \frac{\partial^3 v_n}{\partial \xi \partial \eta \partial x} &= a(\xi, \eta) \frac{\partial^2 v_{n-1}}{\partial \xi \partial x} (x, y; \xi, \eta) + b(\xi, \eta) \frac{\partial^2 v_{n-1}}{\partial \eta \partial x} (x, y; \xi, \eta) \\
 &\quad + c(\xi, \eta) \frac{\partial v_{n-1}}{\partial x} (x, y; \xi, \eta).
 \end{aligned}$$

* In the first of the equations (5) $\partial v_{n-1}(x, y; x, \beta)/\partial x$ signifies $[\partial v_{n-1}(x, y; \alpha, \beta)/\partial \alpha]_{\alpha=x}$.

The other assigned derivatives may be computed in the same way, and it is clear from (4) and (5) that v_n and all the derivatives are continuous in R . Further from the given bounds on v_{n-1} and its assigned derivatives, it may be deduced, by methods similar to those given in [1], that, for example,

$$\begin{aligned}
|v_n| &\leq HM \left[4 + \frac{d}{n+1} \right] \frac{d^n}{n!}, \\
\left| \frac{\partial v_n}{\partial x} \right| &\leq HM \left[6 + \frac{3d}{n} \right] \frac{d^{n-1}}{(n-1)!}, \\
\left| \frac{\partial v_n}{\partial \xi} \right| &\leq HM \left[3 + \frac{d}{n} \right] \frac{d^{n-1}}{(n-1)!}, \\
\left| \frac{\partial^2 v_n}{\partial \xi \partial x} \right| &\leq HM \left[3 + \frac{d}{n-1} \right] \frac{d^{n-2}}{(n-2)!}, \\
\left| \frac{\partial^2 v_n}{\partial \eta \partial x} \right| &\leq HM \left[4 + \frac{3d}{n-1} \right] \frac{d^{n-2}}{(n-2)!}, \\
\left| \frac{\partial^2 v_n}{\partial \xi \partial \eta} \right| &\leq HM \left[2 + \frac{d}{n-1} \right] \frac{d^{n-2}}{(n-2)!}, \\
\left| \frac{\partial^3 v_n}{\partial \xi \partial \eta \partial x} \right| &\leq HM \left[2 + \frac{d}{n-2} \right] \frac{d^{n-3}}{(n-3)!},
\end{aligned}$$

where $n \geq 4$. Since d is bounded in R by the sum, s , of adjacent sides of R , it is clear that the lemma is proved with $K = M [6 + s]$.

Lemma 2. v_3 and its assigned first, second and third derivatives are continuous in R and bounded by $Ld^3/3!$, $Ld^2/2!$, Ld , and L respectively, where $L = 3M^3(1+s)(4+5s)(2+s)$.

For since $v_0 \equiv 1$, it and its assigned first, second and third derivatives are bounded by 1, 0, 0, 0, respectively. Therefore, from (4) and (5), it is clear that v_1 and its assigned first, second, and third derivatives are continuous in R and bounded by $L_1 d$, L_1 , L_1 , 0 with $L_1 = 2M(1+s)$. Hence v_2 and its assigned first, second and third derivatives are continuous in R and bounded by $L_2 d^2/2!$, $L_2 d$, L_2 , L_2 with $L_2 = ML_1(4+5s) = 2M^2(1+s)(4+5s)$. Thus v_3 and its assigned first, second, and third derivatives are continuous in R and bounded by $Ld^3/3!$, $Ld^2/2!$, Ld , L with $L = ML_2(6 + \frac{3}{2}s) = 2M^3(1+s)(4+5s)(6 + \frac{3}{2}s) = 3M^3(1+s)(4+5s)(2+s)$.

Theorem 1. The Riemann function $v(x, y; \xi, \eta)$ and the particular derivatives $v_x, v_y, v_\xi, v_\eta; v_{\xi x}, v_{\xi y}, v_{\eta x}, v_{\eta y}, v_{\xi \eta}, v_{\xi \eta}; v_{\xi \eta x}, v_{\xi \eta y}$ are continuous functions in R .

It has been shown in the lemmas that v_n and the assigned derivatives of v_n are continuous in R , for $n = 0, 1, 2, \dots$. All that remains is to show that the cor-

responding infinite sums $\sum_{n=0}^{\infty} v_n, \sum_{n=0}^{\infty} \partial v_n / \partial x, \dots$, etc. are uniformly convergent in all variables. But from the lemmas it is clear that these series are dominated by $\frac{L}{K^3} \sum_{n=3}^{\infty} \frac{(Ks)^n}{n!}, \frac{L}{K^2} \sum_{n=2}^{\infty} \frac{(Ks)^n}{n!}, \frac{L}{K} \sum_{n=1}^{\infty} \frac{(Ks)^n}{n!}, L \sum_{n=0}^{\infty} \frac{(Ks)^n}{n!}$ respectively, when the first three terms in each of the former are neglected. Hence they are uniformly convergent in all their variables.

§2. AN IMMEDIATE GENERALISATION

First Extension of the Cauchy Initial Values Problem. It is of interest to investigate whether Riemann's solution of the Cauchy initial values problem for the equation (3) is still valid when the initial data has discontinuities, even infinite ones, and when the initial curve has discontinuities in tangent direction.

Definition. A function $u(x, y)$ will be called a regular solution of $L(u) = 0$ in R if it is continuous and has continuous derivatives of the first order at every interior point of R , except possibly at the points of a finite number of segments of characteristics, across which the normal derivative of u alone may have a (finite or infinite) discontinuity, and satisfies $L(u) = 0$ at all other interior points.

Theorem 2. Let C be an initial curve which is intersected in at most one point by any characteristic of either family and which has a tangent which turns continuously except at a finite number of points of C . Let the initial values $f(\sigma), g(\sigma), h(\sigma)$ of u, u_x, u_y be continuous functions of arc length σ , except at a finite number of points of C , where g and h may have finite or infinite discontinuities. Then there exists a unique regular solution of $L(u) = 0$ in R , the rectangle having the end points of C for opposite vertices, which takes the prescribed initial values f, g, h on C , provided certain integrals* of these values converge uniformly.

It will be sufficient to consider the case in which there is a single discontinuity point O on the curve, which does not lie at either end, and to suppose that both g and h as well as the tangent direction have discontinuities there, see Figure 1.

First, if such a regular solution u exists, the discontinuities of u_x can occur only on KOM and the discontinuities of u_y only on NOL , see [3]. Everywhere else† u, u_x, u_y and therefore u_{xy} , see (3), are continuous. Further v, v_x, v_y , and therefore v_{xy} , see (1), are continuous everywhere in R . Applying Green's theorem‡ successively to the regions $FJP_1, HPIE, P_2GD$, one therefore obtains

$$\begin{aligned}
 (6) \quad 0 &= \iint_{FJP_1} + \iint_{HPIE} + \iint_{P_2GD} [vL(u) - uM(v)] dS, \\
 &= \oint_{FJP_1} + \oint_{HPIE} + \oint_{P_2GD} [X \cos \widehat{nx} + Y \cos \widehat{ny}] db,
 \end{aligned}$$

* The integrals concerned are those appearing on the right hand sides of (8), (9) and (10). A fuller discussion of them is given at the end of this section.

† u , of course, is continuous at points of these segments also.

‡ When P lies in AOM or LOB the classical treatment suffices.

Fig. 1

where

$$X = \frac{1}{2}(vu_y - wv_y) + avu,$$

$$Y = \frac{1}{2}(vu_x - wv_x) + bvu,$$

$$v = v(x, y; \xi, \eta),$$

and n is the outward normal on the contour b considered. Now

$$\int_P^I [X \cos \hat{n}x + Y \cos \hat{n}y] db = \int_P^I X dy = \frac{1}{2}(vu)_I - \frac{1}{2}(vu)_P,$$

where use of (2) has been made. The integrals along JP_1 , P_2G , HP may be treated similarly, so that (6) becomes

$$\begin{aligned} u_P = & \frac{1}{2}[(vu)_{P_1} + (vu)_{P_2}] + \left(\int_{P_1}^F + \int_D^{P_2} \right) [X \cos \hat{n}x + Y \cos \hat{n}y] d\sigma \\ (7) \quad & + \frac{1}{2}[(vu)_I - (vu)_J] + \frac{1}{2}[(vu)_H - (vu)_G] \\ & + \left(\int_E^I - \int_F^J \right) Y dx + \left(\int_G^D - \int_H^E \right) X dy. \end{aligned}$$

Further,

$$\int_E^I Y dx = \int_E^I u(bv - v_x) dx + \frac{1}{2}(vu)_I - \frac{1}{2}(vu)_E,$$

and similarly for the integrals along FJ , GD , HE . Thus the second and third lines in (7) may be written

$$\begin{aligned} \mathfrak{R} = & [(vu)_I - (vu)_J] + [(vu)_H - (vu)_G] + \frac{1}{2}[(vu)_F + (vu)_D - 2(vu)_E] \\ & + \left(\int_E^I - \int_F^J \right) u(bv - v_x) dx + \left(\int_G^D - \int_H^E \right) u(av - v_y) dy. \end{aligned}$$

Now the last expression contains *only* a, b, u, v, v_x, v_y which are all continuous everywhere in R . Hence as $F, D, E \rightarrow O$, $\mathfrak{R} \rightarrow 0$, *whatever* discontinuities u_x and u_y have on KOM and NOL respectively. Thus

$$(8) \quad u_P = \frac{1}{2}[f(\sigma_1)v_{P_1} + f(\sigma_2)v_{P_2}] + \int_{P_1}^{P_2} [X \cos \hat{n}x + Y \cos \hat{n}y] d\sigma,$$

provided the integral converges. If the integral converges uniformly for (ξ, η) in R , then u is continuous in R , a necessary condition for u to be regular. In this case also u is uniquely determined by the Riemann formula (8).

In order to demonstrate existence, it is only necessary to show that (8) is a regular solution. The steps in Rademacher's proof [2] will be set down in order to show how it depends on the continuity properties of u and v . Let $P' = (\xi + \delta, \eta)$ and $v' = v(x, y; \xi + \delta, \eta)$, with P'_1 the corresponding point to P_1 . Then

$$\begin{aligned} \left(\frac{\partial u}{\partial \xi} \right)_P &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{1}{2}u_{P_1}[v'_{P_1} - v_{P_1}] + \frac{1}{2}[u_{P_1}' v'_{P_1} - u_{P_1} v'_{P_1}] + \frac{1}{2}u_{P_2}[v'_{P_2} - v_{P_2}] \right. \\ &\quad + \int_{P_1}^{P_2} [(X' - X) \cos \hat{n}x + (Y' - Y) \cos \hat{n}y] d\sigma \\ &\quad \left. + \int_{P'_1}^{P_1} [X' \cos \hat{n}x + Y' \cos \hat{n}y] d\sigma \right\}, \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{1}{2}u_{P_1}[v'_{P_1} - v_{P_1}] + \frac{1}{2}u_{P_2}[v'_{P_2} - v_{P_2}] \right. \\ &\quad + \int_{P_1}^{P_2} [(X' - X) \cos \hat{n}x + (Y' - Y) \cos \hat{n}y] d\sigma \\ &\quad \left. + \int_{P'_1}^{P_1} [u(av' - v'_y) \cos \hat{n}x + v'(bu + u_x) \cos \hat{n}y] d\sigma \right\}. \end{aligned}$$

Thus, since $v, v_x, v_\xi, v_{\xi x}, v_{\xi y}, v_{xy}$ are continuous* in R , and under the conditions

* The continuity of v_{xy} follows from that of $v_x, v_y, v, a, a_x, b, b_y$, from (1).

laid on f, g, h, a, a_x, b and the curve C , it is clear that $\partial u/\partial \xi$ exists at all points of R except those on KOM , and is given by

$$(9) \quad \left(\frac{\partial u}{\partial \xi} \right)_P = \frac{1}{2} \left[f(\sigma_1) \left(\frac{\partial v}{\partial \xi} \right)_{P_1} + f(\sigma_2) \left(\frac{\partial v}{\partial \xi} \right)_{P_2} \right] \\ + v_{P_1} [g(\sigma_1) + b_{P_1} f(\sigma_1)] + \int_{P_1}^{P_2} \left[\frac{\partial X}{\partial \xi} \cos \widehat{nx} + \frac{\partial Y}{\partial \xi} \cos \widehat{ny} \right] d\sigma,$$

provided the integral is a continuous function of ξ, η . This will be so if it converges uniformly for (ξ, η) in R , and then $\partial u/\partial \xi$ is continuous in R except on KOM , a necessary condition for u to be regular.

Similarly, since $v, v_u, v_\eta, v_{\eta x}, v_{\eta y}, v_{xy}$ are continuous in R , it may be shown that $\partial u/\partial \eta$ exists at all points of R except those on NOL , and is given by

$$(10) \quad \left(\frac{\partial u}{\partial \eta} \right)_P = \frac{1}{2} \left[f(\sigma_1) \left(\frac{\partial v}{\partial \eta} \right)_{P_1} + f(\sigma_2) \left(\frac{\partial v}{\partial \eta} \right)_{P_2} \right] \\ + v_{P_2} [h(\sigma_2) + a_{P_2} f(\sigma_2)] + \int_{P_1}^{P_2} \left[\frac{\partial X}{\partial \eta} \cos \widehat{nx} + \frac{\partial Y}{\partial \eta} \cos \widehat{ny} \right] d\sigma,$$

provided the integral is a continuous function of ξ, η . This will be so if it converges uniformly for (ξ, η) in R , and then $\partial u/\partial \eta$ is continuous in R except on NOL , a necessary condition for u to be regular.

Finally one has from (10), (or from (9)),

$$\left(\frac{\partial^2 u}{\partial \xi \partial \eta} \right)_P = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{1}{2} u_{P_1} \left[\left(\frac{\partial v'}{\partial \eta} \right)_{P_1} - \left(\frac{\partial v}{\partial \eta} \right)_{P_1} \right] + \frac{1}{2} \left[u_{P_1'} \left(\frac{\partial v'}{\partial \eta} \right)_{P_1'} - u_{P_1} \left(\frac{\partial v'}{\partial \eta} \right)_{P_1} \right] \right. \\ + \frac{1}{2} u_{P_2} \left[\left(\frac{\partial v'}{\partial \eta} \right)_{P_2} - \left(\frac{\partial v}{\partial \eta} \right)_{P_2} \right] + (v'_{P_2} - v_{P_2}) [(u_y)_{P_2} + a_{P_2} u_{P_2}] \\ + \int_{P_1}^{P_2} \left[\left(\frac{\partial X'}{\partial \eta} - \frac{\partial X}{\partial \eta} \right) \cos \widehat{nx} + \left(\frac{\partial Y'}{\partial \eta} - \frac{\partial Y}{\partial \eta} \right) \cos \widehat{ny} \right] d\sigma \\ \left. + \int_{P_1'}^{P_1} \left[\frac{\partial X'}{\partial \eta} \cos \widehat{nx} + \frac{\partial Y'}{\partial \eta} \cos \widehat{ny} \right] d\sigma \right\}, \\ = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{1}{2} u_{P_1} \left[\left(\frac{\partial v'}{\partial \eta} \right)_{P_1} - \left(\frac{\partial v}{\partial \eta} \right)_{P_1} \right] + (v'_{P_2} - v_{P_2}) [(u_y)_{P_2} + a_{P_2} u_{P_2}] \right. \\ + \frac{1}{2} u_{P_2} \left[\left(\frac{\partial v'}{\partial \eta} \right)_{P_2} - \left(\frac{\partial v}{\partial \eta} \right)_{P_2} \right] \\ + \int_{P_1}^{P_2} \left[\left(\frac{\partial X'}{\partial \eta} - \frac{\partial X}{\partial \eta} \right) \cos \widehat{nx} + \left(\frac{\partial Y'}{\partial \eta} - \frac{\partial Y}{\partial \eta} \right) \cos \widehat{ny} \right] d\sigma \\ \left. + \int_{P_1'}^{P_1} [u(av'_\eta - v'_{u\eta}) \cos \widehat{nx} + v'_\eta (u_x + bu) \cos \widehat{ny}] d\sigma \right\}.$$

Since* $v_\xi, v_\eta, v_{\xi\eta}, v_{\xi\eta x}, v_{\xi\eta y}$ are continuous in R , $\partial^2 u / \partial \xi \partial \eta$ exists at all points of R except those on KOM and NOL , and is given by

$$(11) \quad \left(\frac{\partial^2 u}{\partial \xi \partial \eta} \right)_P = \frac{1}{2} \left[f(\sigma_1) \left(\frac{\partial^2 v}{\partial \xi \partial \eta} \right)_{P_1} + f(\sigma_2) \left(\frac{\partial^2 v}{\partial \xi \partial \eta} \right)_{P_2} \right] \\ + \left(\frac{\partial v}{\partial \eta} \right)_{P_1} [g(\sigma_1) + b_{P_1} f(\sigma_1)] + \left(\frac{\partial v}{\partial \xi} \right)_{P_2} [h(\sigma_2) + a_{P_2} f(\sigma_2)] \\ + \int_{P_1}^{P_2} \left[\frac{\partial^2 X}{\partial \xi \partial \eta} \cos \widehat{nx} + \frac{\partial^2 Y}{\partial \xi \partial \eta} \cos \widehat{ny} \right] d\sigma,$$

provided the integral is a continuous function of ξ, η . This will be so if the preceding integrals converge uniformly.

In order to show that u is a regular solution of (3), there remains only the question of whether it satisfies (3). This follows immediately from (8), (9), (10), (11) since v, v_x, v_y and hence X, Y satisfy

$$\frac{\partial^2 v}{\partial \xi \partial \eta} + a(\xi, \eta) \frac{\partial v}{\partial \xi} + b(\xi, \eta) \frac{\partial v}{\partial \eta} + c(\xi, \eta)v = 0,$$

(see (3)) and in addition

$$(12) \quad v_\eta(\xi, y; \xi, \eta) + a(\xi, \eta)v(\xi, y; \xi, \eta) = 0, \\ v_\xi(x, \eta; \xi, \eta) + b(\xi, \eta)v(x, \eta; \xi, \eta) = 0,$$

which relations follow from (2). Lastly it is necessary to verify that u and its first derivatives take the prescribed initial values on C . As P tends to a point other than O , on C , P_1 and P_2 tend to the same point. Hence it is clear from (8), (9), (10) and (12) that u, u_ξ, u_η tend to f, g, h respectively, since $v \rightarrow 1$.

This completes the existence and uniqueness proofs for the case when the integrals in (8), (9), (10) are uniformly convergent. This latter restriction may be replaced by the one that $\int_{P_1}^{P_2} V(g \cos \widehat{nx} + h \cos \widehat{ny}) d\sigma$ shall be uniformly convergent for $V = v_\xi, v_\eta$ and all (ξ, η) in R , or for all (ξ, η) in $A'KOM$ and $B'MOL$. Since v_ξ, v_η are continuous in R and therefore bounded, this is certainly true if $\int_{C'} |g| d\sigma$ and $\int_{C'} |h| d\sigma$ are convergent for a segment C' of C containing O .

§3. RESOLUTION OF A CLASSICAL DIFFICULTY

Second Extension of the Cauchy Initial Values Problem. Suppose now that the initial curve is such that x, y as functions of σ ($\sigma_0 \leq \sigma \leq \sigma_{p+1}$) have a finite

* RADEMACHER uses the continuity of $v_{xy\eta}$. However the continuity of $v_{\xi\eta y}$ may be used instead.

number* p of extrema $\sigma = \sigma_1, \sigma_2, \dots, \sigma_p (\sigma_1 < \sigma_2 < \dots < \sigma_p)$, the arcs between these extrema having a piecewise continuous tangent direction and no segment coincident with a segment of a characteristic. The curve is then intersected by some characteristics in more than one point. It will be shown that Riemann's formula (8) gives the unique regular solution of (3), taking the prescribed initial values, on a sheeted surface† associated with the initial curve C . This is the case also when the first derivatives are prescribed to have discontinuities.

Definition. Let (x_n, y_n) be the coordinates of the point $\sigma = \sigma_n$ of the initial curve C , and $R_{mn} (m, n = 1, 2, \dots, p+1)$ the rectangle enclosed by the characteristics $x = x_{m-1}, x_m; y = y_{n-1}, y_n$. Then the sheeted surface R associated with C is that surface formed when R_{mn} is joined to $R_{m+1,n}$ along their common side $x = x_m$ and to $R_{m,n+1}$ along their common side $y = y_n$, the arc (σ_{n-1}, σ_n) of C being considered to lie in the same sheet as R_{nn} .

Definition. The sheeted surface R may be represented diagrammatically by an array of $(p+1)^2$ rectangles, such that the rectangle in the m^{th} column and n^{th} row is congruent coordinate-wise with R_{mn} and is joined to its neighbors in a similar manner. This diagram, which is simply-connected, will be called the unfolded diagram‡ associated with C .

The term "continuity" which occurs in the definition of a regular solution, is now defined with respect to position on the sheeted surface (or in the unfolded diagram). The values of $u_\xi, u_\eta, u_{\xi\eta}$ on a fold of the surface R are defined as the common limits of these functions respectively, as the fold is approached on the two sheets having the fold in common, provided these limits exist.

Theorem 3. Let C be an initial curve which has a finite number of extrema in the coordinates x, y , and which has a tangent which turns continuously except at a finite number of points of C . Let the initial values $f(\sigma), g(\sigma), h(\sigma)$ of u, u_x, u_y be continuous functions of arc length σ , except at a finite number of points of C , where g and h may have finite or infinite discontinuities. Then there exists a unique regular solution of $L(u) = 0$ in R , the sheeted surface defined above, which takes the prescribed initial values f, g, h on C , provided certain integrals§ of these initial values converge uniformly.

It will be sufficient to consider the case in which there is a single extremum O on the curve, which does not lie at either end, and to suppose that both g and h as well as the tangent direction have a discontinuity there, see Figure 2. The unfolded diagram is Figure 3, where the coordinates vary as indicated.

First, if such a regular solution u exists, an argument similar to that of the preceding theorem shows that discontinuities in u_x can take place only on KOM , and discontinuities in u_y only on NOL . Thus Green's theorem can be applied as

* The points may be extrema of x, y simultaneously.

† This concept has been used by KRISTIANOVICH [4] in a particular problem in elasticity, as well as by the author [5].

‡ It can be obtained by "unfolding" the sheeted surface along the sides of the rectangles R_{mn} .

§ These integrals are the analogues of those mentioned in Theorem 2.

Fig. 2

Fig. 3

before to the regions FJP_1 , $HP1E$, P_2GD , and the same argument shows that u must be given by the formula (8). This proves uniqueness. The proof that (8) gives a regular solution of (3) taking the correct initial values on C goes through as before, and the theorem is proved, provided the same integrals as before are uniformly convergent.

Remarks and Conclusions. It is clear that if the curve C is closed, it is necessary to specify which point is the beginning and end point of the curve. Otherwise the unfolded diagram is not unique, and likewise the solution is not unique. Further it is clear that the analysis extends to an infinite number of extrema and discontinuities provided they have no point of accumulation (the length of C now becoming infinite), and that if the sheeted surface R has unbounded coordinates x, y , then a, b, c, a_x, b_y are bounded. A case such as this occurs in [5] where C is the map of certain periodic conditions in another plane, and is thus a closed curve covered infinitely often.

It may be noted that the present definition of regularity is less restrictive than that of Goursat. It could have been used in the classical Riemann problem (no discontinuities in initial values of the first derivatives) in which case it would have followed immediately that such a regular solution, in the present meaning of the phrase, would be regular in the classical sense. Further, it seems reasonable to expect that the iterative method of proving uniqueness and existence, when in general, since a_x, b_y are not assumed continuous, there is no Riemann formula (8), could be adapted to the present extensions. However since the present approach was intended to extend the applicability of Riemann's formula, this will not be considered here.

Finally, it may be stated that there is no point in discussing the case in which the initial values of u itself are discontinuous; for then the solution u must have lines of discontinuity of u itself, which lines will be characteristics. The solution is thus not unique, unless the laws of propagation of these discontinuities along characteristics are prescribed.

REFERENCES

- [1] GOURSAT, E. "Cours d'Analyse Mathématique", Gauthier-Villars, Paris, Vol. III, 4th edition, 114–158, (1927).
- [2] FRANK P. & VON MISES R., "Die Differential- und Integralgleichungen der Mechanik und Physik", Vieweg, Braunschweig, Vol. I, 2nd edition, 801–809, (1930).
- [3] COURANT R. & HILBERT D., "Methoden der mathematischen Physik", Springer, Berlin, Vol. II, 2nd edition, 356–365, (1931).
- [4] KRISTIANOVICH, S., "The plane problem of the mathematical theory of plasticity in the case where the extremal forces are applied along a closed contour", *Matematicheskii Sbornik*, **1**, 511–534, (1936). Translation: Applied Mathematics group, Brown University, RMB-4, (1946).
- [5] LUDFORD, G. S. S., "On an extension of Riemann's method of integration, with applications to one-dimensional gas dynamics", *Proc. Camb. Phil. Soc.*, **48**, 499–510, (1952).