*Nonexistence of Conformal Singularities in Solid Spaces**

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The most interesting regular spots of an analytic conformal mapping of a complex variable are its winding points or conformal singularities. The same is true for a conformal (also called isothermic) harmonic representation of a minimal surface. However, in mapping a region of classical riemannian space of dimension $n > 2$ conformally and analytically into another classical riemannian space it is customary to assume that the mapping, unless constant, be devoid of conformal singularities. This assumption is, as I shall show, superfluous: for it follows from the conformality and analyticity of the mapping together with the positive definite structure of classical riemannian space.

Conformal mapping. Let X be a region of real riemannian space of dimension $n > 2$ with real symmetric positive definite metric tensor $g_{\alpha\beta} = g_{\alpha\beta}(x)$ differentiable to all orders; and let Y be a real riemannian space with real symmetric positive definite metric which need not be specified. An analytic mapping $y = y(x)$ from X into Y is called conformal if there exists on X a scalar function $p = p(x)$ such that

$$y_{\alpha} y_{\beta} = p g_{\alpha\beta}$$

throughout X : covariant derivatives of y with respect to coordinates of x being denoted by subscripts and inner product in the space Y of a pair of derivatives of y by juxtaposition.

Conformal singularity. Since the metrics of X and Y are positive definite and

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$g^{\alpha\beta} g_{\alpha\beta} = n$, duplicate greek indices being summed out from 1 to n , the quadratic expression

$$\nabla_1(y, y) = g^{\alpha\beta} y_\alpha y_\beta$$

is positive definite and has, for conformal y , the value

$$\nabla_1(y, y) = np.$$

Therefore p is everywhere nonnegative in X , and vanishes at a point x_0 of X if and only if all first derivatives of y vanish at x_0 . Such a point x_0 at which p vanishes is called a conformal singularity of y .

Intention. We shall show by induction that at a conformal singularity x_0 of a conformal analytic mapping y from a region X of $n > 2$ dimensional space into Y all derivatives of all orders of y vanish: wherefrom it follows by virtue of analyticity of the mapping y over the region X that y is constant throughout X , that is, maps all of X into a single point of Y .

Outline of proof. To this end let x_0 be a conformal singularity of the conformal analytic mapping y . Thus $p = 0$ at x_0 , so all first covariant derivatives of y vanish at x_0 . Suppose that for some fixed $m \geq 2$ all covariant derivatives of y of order $< m$ also vanish at x_0 , whereupon we call x_0 a singularity of y of order $\geq m$. Then all partial derivatives of y of order $< m$ vanish at x_0 and all covariant derivatives of y of order m coincide with the corresponding partial derivatives so are symmetric in their indices. The inductive step of the proof consists in showing that all these covariant derivatives of y of order m also vanish at x_0 , that is, that the positive definite expression

$$\nabla_m(y, y) = g^{\alpha_1\beta_1} \dots g^{\alpha_m\beta_m} y_{\alpha_1 \dots \alpha_m} y_{\beta_1 \dots \beta_m}$$

vanishes at x_0 . This will be accomplished with the aid of a differential operator S , which when quadratically acting on conformal y vanishes throughout X , so all its k fold iterated laplacians

$$\Delta^k S = g^{\alpha_1\beta_1} \dots g^{\alpha_k\beta_k} S_{\alpha_1\beta_1 \dots \alpha_k\beta_k}$$

also vanish throughout X . Hence in particular

$$\Delta^{m-2} S(y, y) = 0$$

at x_0 by virtue of conformality of y . On the other hand we show from the $\geq m^{\text{th}}$ order singular nature of y at x_0 that

$$\Delta^{m-2} S(y, y) = (n-2)2^{m-2}\nabla_m(y, y) + \text{nonnegative terms.}$$

It therefore follows since $n > 2$ that $\nabla_m(y, y)$, being nonnegative, vanishes at x_0 . This summarizes the inductive step of the argument.

Conformal calculation of $\Delta^{m-2} S$. Define the contravariant four index tensor s as follows:

$$2s^{\alpha\beta\mu\nu} = ng^{\alpha\mu} g^{\beta\nu} + ng^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu}.$$

Thus s , being composed of g 's, is covariantly constant, and, as is easily verified,

$$s^{\alpha\beta\mu\nu} g_{\alpha\beta} = 0.$$

The differential operator S is defined with the aid of this tensor s as

$$S(y, y) = (s^{\alpha\beta\mu\nu} y_\alpha y_\beta)_{\mu\nu}.$$

For conformal y satisfying

$$y_\alpha y_\beta = p g_{\alpha\beta}$$

we evidently have¹

$$S(y, y) = 0$$

throughout X . Therefore all iterated laplacians of S also vanish throughout X . In particular

$$\Delta^{m-2} S(y, y) = 0$$

at x_0 , as was to be shown.

Singular calculation of $\Delta^{m-2} S$. In order to calculate $\Delta^{m-2} S$ at a singularity x_0 of y where all covariant derivatives of y of order $< m$ vanish we inductively build a formula for $\Delta^{k-2} S$ from $k = 2$ to $k = m$ as follows. Let $\mathfrak{s}_k (k = 2, \dots, m)$ denote the proposition that

$$\Delta^{k-2} S(y, y) = \sum_{h+i+j=k} \sigma_{hij} \nabla_h (\Delta^i y, \Delta^j y)$$

with nonnegative integral coefficients σ_{hij} symmetric in i and j such that

$$\sigma_{k00} = (n - 2)2^{k-2}.$$

This expression for $\Delta^{k-2} S$ is to be understood as an incomplete symbol which will later be operated on by Δ^{m-k} to produce $\Delta^{m-2} S$ and thereupon evaluated at x_0 . Thus all derivatives of y appearing in the $\mathfrak{s}_k$ formula may be supposed symmetric in their indices: this assumption of symmetry being eventually validated by evaluation at x_0 . Note that this is true even for derivatives of order $> m$, for in the final validating evaluation at x_0 such derivatives will be coupled with derivatives of order $< m$ which vanish at x_0 .

The initial proposition $\mathfrak{s}_2$ is easily verified by direct calculation using symmetry of differentiation. Application of the differential operator Δ to a single term of

¹ The cartesian version of this formula was used by BECKENBACH & BING in their paper *Conformal minimal varieties*, Duke Mathematical Journal, vol. **10** (1943) pp. 637-640, to show that a harmonic conformal mapping $y(x)$ from a region X of euclidean space of dimension $n > 2$ into a euclidean space Y is necessarily linear.

the $\mathfrak{s}_k$ formula for $k < m$, again using symmetry of differentiation, yields the following symbolic result² with operand y omitted:

$$\Delta \nabla_h(\Delta^i, \Delta^j) = 2\nabla_{h+1}(\Delta^i, \Delta^j) + \nabla_h(\Delta^{i+1}, \Delta^j) + \nabla_h(\Delta^i, \Delta^{j+1}).$$

Thus we see that $\mathfrak{s}_k$ implies $\mathfrak{s}_{k+1}$ for $k < m$. Consequently, by finite induction from $k = 2$ to $k = m$, we may assert $\mathfrak{s}_m$.

Consider the value of $\Delta^{m-2}S$ as determined by the $\mathfrak{s}_m$ formula evaluated at x_0 . All terms in this formula for which $i \neq j$ involve derivatives of order $< m$ hence vanish when evaluated at x_0 . And all terms for which $i = j$ are positive definite expressions with nonnegative coefficients so are nonnegative. Therefore

$$\Delta^{m-2}S(y, y) = (n - 2)2^{m-2}\nabla_m(y, y) + \text{nonnegative terms}$$

at x_0 , as was to be shown.

We have already demonstrated how the proof of nonexistence of conformal singularities for dimension $n > 2$ follows from this singular calculation together with the preceding conformal calculation of $\Delta^{m-2}S$.

Remark. Formula $\mathfrak{s}_m$ is of course valid for dimension $n = 2$ at a conformal singularity x_0 of order m where all derivatives of y of order $< m$ vanish but not all of order m . Such a conformal singularity of (exact) order m is not in this case of dimension $n = 2$ contradictory, for the coefficient of $\nabla_m(y, y)$ in the $\mathfrak{s}_m$ formula vanishes. However, the coefficient of $\nabla_{m-2}(\Delta y, \Delta y)$ is positive (see Footnote 2), so all derivatives of order $m - 2$ of Δy vanish at x_0 . From this fact together with the defining equations of conformality differentiated $2m - 2$ times at x_0 it can be shown by local analysis alone without intervention of complex function theory that differential distances at x_0 are reduced by the mapping in proportion to the m^{th} power and angles are magnified m times.

Indefinite spaces. The above results concerning conformal singularities of a conformal analytic mapping depend in essence upon the positive definite nature of the spaces X and Y . For it turns out, as I shall show in a later paper, that conformal singularities of any order $m \geq 2$ can occur in a definite or indefinite space X of dimension $n \geq 2$ if the embedding space Y of the mapping image is allowed to be indefinite. Furthermore, although distances at such a conformal singularity of order m are necessarily reduced in proportion to the m^{th} power, angles need not be magnified m times if $m \geq 4$, though they must be if $m = 2$ or 3 .

² Although for our present purpose (however see Remark) no more information than divulged by $\mathfrak{s}_m$ is needed about the σ coefficients, this result shows that they may be determined recursively by the formula

$$\sigma_{hij} = 2\sigma_{h-1ij} + \sigma_{hi-1j} + \sigma_{hij-1},$$

and thereupon explicitly by the formula

$$\sigma_{hij} = (n - 2)2^{h-2}c_{h-2ij} + (n - 1)2^{h-1}c_{h-1i-1j} + (n - 1)2^{h-1}c_{h-1ij-1} + n2^h c_{hi-1j-1}$$

where $c_{hij} = (h + i + j)!/h!i!j!$ is the symmetric trinomial coefficient.