Abel and Riesz Transforms of Series Having Bounded Partial Sums

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1. Introduction. It is our purpose to obtain best estimates relating the Abel and Riesz transforms

$$(1.1) \quad A(t) = \sum_{k=0}^{\infty} t^k u_k, \quad R^{(r)}(n) = \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right)^r u_k$$

of series $\sum u_n$ of complex terms satisfying the condition

$$(1.2) \quad \limsup_{n \rightarrow \infty} |s_n| < \infty$$

where

$$(1.21) \quad s_n = \sum_{k=0}^n u_k \quad n = 0, 1, 2, \dots$$

is the sequence of partial sums of $\sum u_n$. Some of our work applies to Riesz transformations of arbitrary complex order, but we assume everywhere that r is real and $r \geq 0$; it is not assumed that r is an integer. Our problems are, roughly speaking, to determine relative values of t and n such that the quantity

$$(1.22) \quad |A(t) - R^{(r)}(n)|$$

will be small and to determine how small the quantity can be made by best choices of t and n .

Before formulating our problems, we need some terminology. If (1.2) holds, then the sequence s_n has a nonempty bounded set Z of limit points. There is a

unique circle of minimum radius that contains Z ; let ζ and R denote the center and radius of this circle, and let

$$(1.3) \quad x_n = s_n - \zeta.$$

Then

$$(1.4) \quad \limsup_{n \rightarrow \infty} |x_n| = R.$$

If $\sum u_n$ converges to s , then $\zeta = s$ and $R = 0$. If $\sum u_n$ diverges, then $R > 0$. Obviously $\limsup |s_n| \geq R$, with equality only when $\zeta = 0$. We note in particular that if s_n is a divergent sequence of zeros and ones, then $R = \frac{1}{2}$. Applications of our results to this special case are interesting.

Among the problems which we shall solve is the determination for each order r of the least constant H_r , which turns out to be finite and positive, with the following property. There exist a function $t(\alpha)$ for which $0 < t(\alpha) < 1$ when $\alpha > 0$ and $t(\alpha) \rightarrow 1$ as $\alpha \rightarrow \infty$, and a function $n(\alpha)$ for which $n(\alpha)$ is a positive integer for each $\alpha > 0$ and $n(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$, such that

$$(1.5) \quad \limsup_{\alpha \rightarrow \infty} |A(t) - R^{(r)}(n)| \leq H_r R$$

for each series $\sum u_n$ having bounded partial sums. Optimal functions $t(\alpha)$ and $n(\alpha)$ will be characterized and related results will be obtained. An upper bound for H_r follows from a recent paper [3] with which the present one has much in common. It was shown there that, for each $r > 0$, there is a least constant H_r^* such that

$$(1.6) \quad \limsup_{\alpha \rightarrow \infty} |A(t) - R^{(r)}(n)| \leq H_r^* \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \sum_{k=0}^n ku_k \right|.$$

Let $\sum u_n$ be a series for which $\limsup |s_n - \zeta| = R < \infty$, and let

$$(1.61) \quad M_n = R^{(1)}(n) = (s_0 + s_1 + \cdots + s_n)/(n+1), \quad n = 0, 1, 2, \dots$$

denote the sequence of arithmetic means. Then $\limsup |M_n - \zeta| \leq R$. From the Kronecker formula

$$(1.62) \quad \frac{1}{n+1} \sum_{k=0}^n ku_k = s_n - M_n = (s_n - \zeta) - (M_n - \zeta)$$

we obtain

$$(1.63) \quad \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \sum_{k=0}^n ku_k \right| \leq 2R$$

and hence, from (1.6),

$$(1.64) \quad \limsup_{\alpha \rightarrow \infty} |A(t) - R^{(r)}(n)| \leq 2H_r^* R.$$

It follows that $H_r \leq 2H_r^*$. We shall find that H_r is less than $2H_r^*$. Some of our results take the following form. Let $\sum u_n$ be a divergent series having bounded partial sums, and let $\epsilon > 0$. Let $R^{(r)}(n)$ denote the Riesz transform of large order r of $\sum u_n$. If we set, for each large n ,

$$(1.7) \quad t = 1 - \frac{r + .34856746}{n}$$

then the Abel transform will be so close to the Riesz transform that

$$(1.71) \quad |A(t) - R^{(r)}(n)| \leq \left[\frac{.45910551}{r} + O\left(\frac{1}{r^2}\right) \right] R.$$

More precise results and tables are given in Section 4 and following sections.

An exposition and full bibliography of the field in which our problems lie is given in [2].

2. Preliminary formulas. In order to clarify and exhibit the generality of our preliminary work, we define functions $b(k, t)$ and $\rho(k, n)$ by the formulas

$$(2.1) \quad b(k, t) = t^k \quad 0 < t < 1; k = 1, 2, 3, \dots$$

and

$$(2.11) \quad \rho(k, n) = \left(1 - \frac{k}{n+1}\right)^r \quad 0 \leq k \leq n$$

and $\rho(k, n) = 0$ when $k > n$ so that

$$(2.12) \quad A(t) = \sum_{k=0}^{\infty} b(k, t)u_k, \quad R^{(r)}(n) = \sum_{k=0}^{\infty} \rho(k, n)u_k.$$

We suppose now that $\sum u_n$ is a given series having bounded partial sums and use the notation of the introduction. Then

$$(2.2) \quad u_k = s_k - s_{k-1} = (s_k - \zeta) - (s_{k-1} - \zeta) = x_k - x_{k-1}$$

and $\limsup |x_k| = R < \infty$. Putting this in (2.12) gives

$$(2.3) \quad A(t) = \zeta + \sum_{k=0}^{\infty} [b(k, t) - b(k+1, t)]x_k$$

and

$$(2.31) \quad R^{(r)}(n) = \zeta + \sum_{k=0}^{\infty} [\rho(k, n) - \rho(k+1, n)]x_k.$$

Let functions $t(\alpha)$ and $n(\alpha)$ be such that $0 < t(\alpha) < 1$ when $\alpha > 0$, $t(\alpha) \rightarrow 1$ as $\alpha \rightarrow \infty$, $n(\alpha)$ is a positive integer when $\alpha > 0$, and $n(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$.

In what follows, we simplify formulas by writing n and t for $n(\alpha)$ and $t(\alpha)$. Let

$$(2.4) \quad y(\alpha) = A(t) - R^{(r)}(n).$$

Using (2.3) and (2.31), we see that

$$(2.41) \quad y(\alpha) = \sum_{k=0}^{\infty} a_k(\alpha) x_k$$

where

$$(2.42) \quad a_k(\alpha) = [b(k, t) - b(k+1, t)] - [\rho(k, n) - \rho(k+1, n)].$$

It is easily verified that

$$(2.43) \quad \lim_{\alpha \rightarrow \infty} a_k(\alpha) = 0 \quad k = 0, 1, 2, \dots$$

Let H be the constant defined by

$$(2.5) \quad \limsup_{\alpha \rightarrow \infty} \sum_{k=0}^{\infty} |a_k(\alpha)| = H;$$

this constant H , which depends upon the given functions $t(\alpha)$ and $n(\alpha)$, is finite in the case of the functions in (2.11) and (2.12) because

$$(2.51) \quad \begin{aligned} \sum_{k=1}^{\infty} |a_k(\alpha)| &\leq \sum_{k=0}^{\infty} |b(k, t) - b(k+1, t)| \\ &\quad + \sum_{k=0}^{\infty} |\rho(k, n) - \rho(k+1, n)| = 2. \end{aligned}$$

It follows from (2.43), (2.5), and a lemma of Agnew [1] that

$$(2.6) \quad \limsup_{\alpha \rightarrow \infty} |y(\alpha)| = H \limsup_{n \rightarrow \infty} |x_n| = HR.$$

Moreover the constant H in (2.5) and (2.6) is the best possible constant in the sense that there is a series $\sum u_n$ such that $0 < R < \infty$ and equality holds in (2.6). It can be expected that different sequences $t(\alpha)$ and $n(\alpha)$ will yield different numbers H . For each fixed r , the least constant H_r noted in connection with (1.5) is the minimum of the set of numbers H representable in the form (2.5).

3. Determination of H_r . From (2.42), (2.1), and (2.11) we obtain

$$(3.1) \quad a_k(\alpha) = t^k - t^{k+1} - \left[\left(1 - \frac{k}{n+1} \right)^r - \left(1 - \frac{k+1}{n+1} \right)^r \right], \quad 0 \leq k < n,$$

$$(3.11) \quad a_n(\alpha) = t^n - t^{n+1} - (n+1)^{-r}$$

and

$$(3.12) \quad a_k(\alpha) = t^k - t^{k+1} \quad k > n.$$

The case $r = 0$ is, as could easily have been seen from the beginning, quite trivial. In this case

$$(3.2) \quad \sum_{k=0}^{\infty} |a_k(\alpha)| = 2(1 - t^n - t^{n+1}) = o(1) + 2.$$

The conclusion to be drawn from this is weak in the sense that it is an obvious consequence of the following simple facts. If $\sum u_n$ is a series for which $\limsup |s_n - \zeta| \leq 1$, then also $\limsup |A(t) - \zeta| \leq 1$ as $t \rightarrow 1$; this shows that $H \leq 2$. On the other hand there exist real series, such as those for which $s_n = 2$ when n lies in a sequence of integers of density zero and $s_n = 0$ otherwise, for which $\liminf s_n = 0$, $\limsup s_n = 2$, $R = 1$, and $\lim A(t) = 0$; this shows that $H \geq 2$.

Henceforth, without further notice except in statements of theorems, we treat the nontrivial case in which $r > 0$. It is a convenience to introduce a parameter λ which is a function of n and t , and hence of α , defined by the equivalent formulas

$$(3.3) \quad \lambda = n \log t^{-1}, \quad t^n = e^{-\lambda}, \quad t = e^{-\frac{\lambda}{n}}$$

From (3.11) and (3.12) we obtain

$$(3.4) \quad \sum_{k=n}^{\infty} |a_k(\alpha)| = o(1) + t^n = o(1) + e^{-\lambda}$$

as $\alpha \rightarrow \infty$. When $0 \leq k < n$, we use (3.1) and (3.3) to put $|a_k(\alpha)|$ in the form

$$(3.5) \quad |a_k(\alpha)| = \left| \lambda e^{-\frac{\lambda \xi_k}{n}} - r \left(\frac{n}{n+1} \right) \left(1 - \frac{\xi_k}{n+1} \right)^{r-1} \right| \frac{1}{n}$$

where, for each k , ξ_k is a suitably chosen number between $k/(n+1)$ and $(k+1)/(n+1)$ obtained by use of the mean-value theorem. While the classic Riemann sums which approximate Riemann integrals are formed only for bounded functions, we may nevertheless interpret the sum over $k = 0, 1, 2, \dots, n-1$ of the numbers in (3.5) as being, for each $\alpha > 0$, a modified Riemann sum formed for a Lebesgue integrable function over the interval $0 < x < 1$ and see that

$$(3.51) \quad \sum_{k=1}^{n-1} |a_k(\alpha)| = o(1) + \int_0^1 |\lambda e^{-\lambda x} - r(1-x)^{r-1}| dx.$$

Combining (3.2) and (3.51), we obtain

$$(3.6) \quad \sum_{k=1}^{\infty} |a_k(\alpha)| = o(1) + G_r(\lambda)$$

where

$$(3.61) \quad G_r(\lambda) = e^{-\lambda} + \int_0^1 |\lambda e^{-\lambda x} - r(1-x)^{r-1}| dx.$$

For each fixed $r > 0$, $G_r(\lambda)$ is continuous and in fact differentiable over $0 \leq x < \infty$, and

$$(3.62) \quad \lim_{\lambda \rightarrow 0} G_r(\lambda) = 2, \quad \lim_{\lambda \rightarrow \infty} G_r(\lambda) = 2.$$

Since moreover

$$(3.63) \quad 0 < G_r(\lambda) < 2 \quad 0 < \lambda < \infty,$$

it follows that $G_r(\lambda)$ must have an absolute minimum $G_r(q_r)$ attained for at least one positive value q_r of λ and that

$$(3.64) \quad 0 < G_r(q_r) < 2.$$

We shall show later that there is a unique value of λ for which $G_r(\lambda)$ attains its minimum; meanwhile we assume this fact and let q_r be this unique value of λ . Using (3.6) and the inequality $G_r(\lambda) \geq G_r(q_r)$, we see that when H is defined by (2.5) we have $H \geq G_r(q_r)$. Therefore $H_r \geq G_r(q_r)$. Suppose now that q is any given positive number and we choose functions $t(\alpha)$ and $n(\alpha)$ such that $\lambda(\alpha) = n \log t^{-1} \rightarrow q$ as $\alpha \rightarrow \infty$. Then, since $G_r(\lambda)$ is a continuous function of λ , it follows from (3.6) that

$$(3.7) \quad \limsup_{\alpha \rightarrow \infty} \sum_{k=1}^{\infty} |a_k(\alpha)| = G_r(q).$$

If we set $q = q_r$ in (3.7), we find that $H_r \leq G_r(q_r)$. These results imply that

$$(3.8) \quad H_r = G_r(q_r).$$

Since $G_r(q_r)$ is continuous and attains its minimum only when $q = q_r$, (3.6) implies that we will have $H = H_r$ if and only if $t(\alpha)$ and $n(\alpha)$ are functions such that $\lambda(\alpha) = n \log t^{-1} \rightarrow q_r$ as $\alpha \rightarrow \infty$ or, what amounts to the same thing

$$(3.9) \quad \lim_{\alpha \rightarrow \infty} n(\alpha)[1 - t(\alpha)] = q_r.$$

4. Theorems on Abel and Riesz transforms. Before passing to calculations and approximations of q_r and $H_r = G_r(q_r)$, we collect some of the results we have obtained.

Theorem 4.1. *Let $r > 0$ and let $q > 0$. Let $n(t)$ be a function of t such that*

$$0 < t < 1$$

$$(4.11) \quad n(t) = \frac{q + o(1)}{1 - t}$$

where $o(1) \rightarrow 0$ as $t \rightarrow 1$. Then, with the notation of the introduction,

$$(4.12) \quad \limsup_{t \rightarrow 1} |A(t) - R^{(r)}(n(t))| \leq G_r(q)R$$

where the constant $G_r(q)$ is determined from (3.61). Moreover $G_r(q)$ is the best constant, and $G_r(q_r)$ is the best constant H_r in (1.5).

Theorem 4.3. Let $r > 0$ and let $q > 0$. Let t_n be a sequence such that

$$(4.21) \quad t_n = 1 - \frac{q + o(1)}{n}$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Then

$$(4.22) \quad \limsup_{n \rightarrow \infty} |A(t_n) - R^{(r)}(n)| \leq G_r(q)R,$$

and the remarks at the end of Theorem 4.1 are valid.

5. The case $r = 1$. On account of properties of the function $r(1 - x)^{r-1}$, it is convenient and perhaps necessary to consider separately the cases where $0 < r < 1$, where $r = 1$, and where $r > 1$. We take first the special and interesting case in which $r = 1$ and the Riesz transform $R^{(1)}(n)$ reduces to the arithmetic mean transform in (1.61). In this case (3.61) becomes

$$(5.1) \quad G_1(\lambda) = e^{-\lambda} + \int_0^1 |\lambda e^{-\lambda x} - 1| dx.$$

When $0 < \lambda \leq 1$ we have $G_1(\lambda) = 2e^{-\lambda}$ so $G_1(\lambda)$ is decreasing and the minimum occurs when $\lambda = q_1 > 1$.

In case $\lambda > 1$, let $z = z(\lambda)$ be defined by $z = \lambda^{-1} \log \lambda$. Since $\lambda e^{-\lambda x} > 1$ when $0 \leq x < z$ and $\lambda e^{-\lambda x} < 1$ when $x > z$, we see from (5.1) that

$$(5.11) \quad G_1(\lambda) = e^{-\lambda} + \int_0^z (\lambda e^{-\lambda x} - 1) dx - \int_z^1 (\lambda e^{-\lambda x} - 1) dx$$

and hence that

$$(5.12) \quad G_1(\lambda) = 2[1 + e^{-\lambda} - \lambda^{-1}(1 + \log \lambda)].$$

Since

$$(5.2) \quad G'_1(\lambda) = 2\lambda^{-2} [\log \lambda - \lambda^2 e^{-\lambda}],$$

we see that $G'_1(\lambda) = 0$ when and only when $\lambda = E$ where E is the unique positive number satisfying the equation

$$(5.3) \quad E^2 = e^E \log E.$$

Since $G_1'(\lambda) < 0$ when $\lambda < E$ and $G_1'(\lambda) > 0$ when $\lambda > E$, we see that $G_1(\lambda)$ attains its minimum value $G_1(q_1)$ when and only when $\lambda = q_1 = E$, being decreasing when $\lambda < E$ and increasing when $\lambda > E$.

To determine the numerical value of E , which is the geometric mean of its exponential and its logarithm, it seems desirable to put $x = \log E$ and to determine x as the zero of $f(x)$ where

$$(5.31) \quad f(x) = e^x + \log x - 2x.$$

This x is the arithmetic mean of its exponential and its logarithm. Using Newton's method to solve (5.31) gives the first three of the constants in the hierarchy

$$(5.32) \quad \log \log E = \log x = -.63970 \quad 63845 \quad 431322$$

$$(5.33) \quad \log E = x = .52744 \quad 72679 \quad 802111$$

$$(5.34) \quad E = e^x = 1.69460 \quad 09205 \quad 035543$$

$$(5.35) \quad e^E = e^{e^x} = 5.44447 \quad 27541 \quad 545132$$

Use of these constants shows that the minimum value of $G_1(\lambda)$ is

$$(5.4) \quad H_1 = G_1(q_1) = G_1(E) = .56462 \quad 29252 \quad 011374.$$

Some values of $G_1(\lambda)$ are

$$G_1(.25) = 1.557602 \quad G_1(1.75) = .565130$$

$$G_1(.50) = 1.213061 \quad G_1(2) = .577523$$

$$G_1(.75) = .944733 \quad G_1(3) = .700499$$

$$G_1(1) = .735759 \quad G_1(4) = .843484$$

$$G_1(1.25) = .615980 \quad G_1(5) = .969701$$

$$G_1(1.5) = .572307 \quad G_1(10) = 1.339492$$

$$G_1(E) = .564623 \quad G_1(20) = 1.600427$$

6. Notation. In order to simplify some of our formulas and calculations we define a function $\psi(x)$, which depends upon the parameters r and λ , by the formulas

$$(6.1) \quad \psi(x) = \lambda e^{-\lambda x} - r(1 - x)^{r-1} \quad 0 \leq x < 1$$

and

$$(6.11) \quad \psi(x) = \lambda e^{-\lambda x} \quad x \geq 1.$$

When $0 \leq x < 1$, let

$$(6.2) \quad \phi_1(x) = \lambda e^{-\lambda x}, \quad \phi_2(x) = r(1 - x)^{r-1}$$

and let

$$(6.21) \quad \phi_1(x) = \lambda e^{-\lambda x}, \quad \phi_2(x) = 0, \quad x \geq 1$$

so that

$$(6.22) \quad \psi(x) = \phi_1(x) - \phi_2(x)$$

when $x \geq 0$ and $x \neq 1$. If $r > 1$, then $\psi(x)$ is continuous and (6.22) holds when $x \geq 0$. From (3.61), (6.1), and (6.11) we obtain

$$(6.3) \quad G_r(\lambda) = \int_0^\infty |\psi(x)| dx,$$

and evaluation of elementary integrals shows that

$$(6.31) \quad \int_0^\infty \psi(x) dx = 0.$$

7. The case $0 < r < 1$. In this case the sign of $\psi(x)$ is easily determined by use of the functions $\phi_1(x)$ and $\phi_2(x)$ in (6.2). Note that $\phi_1(0) = \lambda$ and $\phi_1(x)$ is decreasing over $0 \leq x \leq 1$, and that $\phi_2(0) = r$ and $\phi_2(x)$ is increasing over $0 \leq x < 1$. In case $\lambda \leq r$, we have $\psi(x) < 0$ over $0 < x < 1$ and hence $G_r(\lambda) = 2e$. Thus $G_r(\lambda)$ is decreasing over $0 \leq \lambda \leq r$ and the minimum of $G_r(\lambda)$ is attained when $\lambda > r$.

When r is fixed and $\lambda > r$, properties of $\phi_1(x)$ and $\phi_2(x)$ show that there is a unique value z of x for which $\psi(x) = 0$ and that $\psi(x) > 0$ when $0 < x < z$ and $\psi(x) < 0$ when $z < x < 1$. This number z depends upon both r and λ and is determined by the conditions $0 < z < 1$ and

$$(7.1) \quad \lambda e^{-\lambda z} = r(1 - z)^{r-1}.$$

Properties of $\psi(x)$ and (6.3) show that

$$(7.2) \quad G_r(\lambda) = \int_0^z \psi(x) dx - \int_z^1 \psi(x) dx + \int_1^\infty \psi(x) dx$$

and hence because of (6.31) that

$$(7.21) \quad G_r(\lambda) = -2 \int_z^1 \psi(x) dx$$

and

$$(7.22) \quad G_r(\lambda) = 2 \int_z^1 [r(1 - x)^{r-1} - \lambda e^{-\lambda x}] dx.$$

Evaluating this integral gives

$$(7.23) \quad G_r(\lambda) = 2[(1 - z)^r + e^{-\lambda} - e^{-\lambda z}].$$

To compute $G_r(\lambda)$ for a given pair of values of r and λ , the procedure is to solve (7.1) for z and then use (7.23). It seems appropriate to solve (7.1) by finding the zero in the interval $0 < z < 1$ of the function

$$(7.24) \quad f(z) = \log(1 - z)^{-1} + [\lambda(1 - r)^{-1}]z - (1 - r)^{-1} \log(\lambda r^{-1}).$$

This function has a simple derivative, and Newton's method is easily applied. Some values of $G_{\frac{1}{3}}(\lambda)$ are

$$\begin{array}{ll} G_{\frac{1}{3}}(.25) = 1.557602 & G_{\frac{1}{3}}(1.75) = .928290 \\ G_{\frac{1}{3}}(.50) = 1.213061 & G_{\frac{1}{3}}(2) = .949502 \\ G_{\frac{1}{3}}(.75) = 1.021023 & G_{\frac{1}{3}}(3) = 1.068048 \\ G_{\frac{1}{3}}(1) = .944820 & G_{\frac{1}{3}}(5) = 1.292249 \\ G_{\frac{1}{3}}(1.25) = .917642 & G_{\frac{1}{3}}(10) = 1.575500 \\ G_{\frac{1}{3}}(1.50) = .915742 & G_{\frac{1}{3}}(20) = 1.756432 \end{array}$$

The formula (7.22) shows that, for each fixed r , the problem of minimizing $G_r(\lambda)$ is equivalent to the problem of minimizing the area of a region bounded by the line $x = 1$, a fixed curve determined by r , and a variable curve determined by λ . For investigation of the minimum of $G_r(\lambda)$, we seek the derivative $G'_r(\lambda)$ which can be obtained from (7.23) by taking into account the fact that z is a function of λ and r , or from (7.22) because the integrand vanishes when $x = z$. From (7.22) we obtain

$$(7.3) \quad G'_r(\lambda) = -2 \int_z^1 [1 - \lambda x] e^{-\lambda x} dx$$

and hence

$$(7.31) \quad G'_r(\lambda) = 2[ze^{-\lambda z} - e^{-\lambda}].$$

We now use these results to obtain information about $G_r(\lambda)$ and, in particular, to prove the following theorem.

Theorem 7.4. *If $0 < r < 1$, then the unique value q_r of λ for which $G_r(\lambda)$ attains its minimum value $H_r = G_r(q_r)$ is the first of the unique pair (λ, z) of numbers for which*

$$(7.41) \quad \lambda e^{-\lambda z} = r(1 - z)^{r-1}$$

and

$$(7.42) \quad ze^{-\lambda z} = e^{-\lambda}.$$

Moreover $G'_r(\lambda) < 0$ when $\lambda < q_r$ and $G'_r(\lambda) > 0$ when $\lambda > q_r$. Finally, $1 < q_r < z_0^{-1}$ where z_0 is defined by (7.52).

Since $0 < z < 1$, the formulas (7.1) and (7.41) defining z are equivalent to

$$(7.5) \quad \lambda ze^{-\lambda z} = rz(1 - z)^{r-1}.$$

Since $xe^{-x} \leq e^{-1}$ with equality only when $x = 1$, it follows from (7.5) that the inequality

$$(7.51) \quad r(1 - x)^{r-1} \leq e^{-1}x^{-1}$$

holds when $x = z$. Let z_0 be the unique number for which $0 < z_0 < 1$ and

$$(7.52) \quad r(1 - z_0)^{r-1} = e^{-1}z_0^{-1},$$

and note that (7.5) holds with $z = z_0$ if and only if $\lambda = z_0^{-1}$. The graph of $e^{-1}x^{-1}$ is the envelope of the family of graphs of the functions $\lambda e^{-\lambda x}$, and elementary facts concerning the graphs of $e^{-1}x^{-1}$, $\lambda e^{-\lambda x}$, and $r(1 - x)^{r-1}$ imply (i) that z increases from 1 to z_0 as λ increases from r to z_0^{-1} and that $dz/d\lambda > 0$ when $1 < \lambda < z_0^{-1}$ and (ii) that z decreases from z_0 to 0 as λ increases from z_0^{-1} and $\lambda \rightarrow \infty$ and that $dz/d\lambda < 0$ when $\lambda > z_0^{-1}$. Differentiating (7.41) gives

$$(7.53) \quad 1 - \lambda z = \left[\lambda \left(\lambda + \frac{1 - r}{1 - z} \right) \right] \frac{dz}{d\lambda}.$$

Since the factor in brackets is positive, our information about the sign of $dz/d\lambda$ shows that

$$(7.54) \quad 0 < \lambda z \leq 1 \qquad r < \lambda \leq z_0^{-1}$$

and

$$(7.55) \quad \lambda z \geq 1 \qquad \lambda \geq z_0^{-1}.$$

If $\lambda \leq 1$, then (7.3) shows that $G'_r(\lambda) < 0$ and hence (7.31) implies that (7.42) fails to hold. If $\lambda > z_0^{-1}$, then (7.55) shows that $\lambda z \geq 1$ and hence (7.3) implies that $G'_r(\lambda) > 0$ and that (7.42) fails to hold. This shows that $G_r(\lambda)$ can attain its minimum and that (7.42) can hold only for one or more values of λ in the interval $1 < \lambda < z_0^{-1}$.

We now put (7.31) in the form

$$(7.56) \quad G'_r(\lambda) = 2\lambda^{-1}[\lambda ze^{-\lambda z} - \lambda e^{-\lambda}].$$

Recalling that $G'_r(1) < 0$ and $G'_r(z_0^{-1}) > 0$, we confine attention to the interval $1 < \lambda < z_0^{-1}$. Since z is an increasing function of λ , so also are λz and $\lambda ze^{-\lambda z}$, the

latter being true because $0 < \lambda z < 1$. Moreover $\lambda e^{-\lambda}$ and $-\lambda e^{\lambda}$ are respectively decreasing and increasing functions of λ because $\lambda > 1$. Thus the quantity in brackets in (7.56) is increasing over $1 < \lambda < x_0^{-1}$ and there is a unique value of λ for which it is zero. This proves Theorem 7.4.

It seems unlikely that useful explicit formulas for q_r and $H_r = G_r(q_r)$ are obtainable, but we can obtain interesting estimates of them which serve as starting points for applications of methods of successive approximation. Starting with (7.22) and using the fact, demonstrated following (7.52), that $0 < z \leq z_0$ where z_0 is defined by (7.52), we obtain

$$(7.6) \quad G_r(\lambda) \geq 2 \int_{z_0}^1 [r(1-x)^{r-1} - \lambda e^{-\lambda x}] dx.$$

We now use the fact that, except for the points of tangency, all of the graphs of the functions $\lambda e^{-\lambda x}$ lie below their envelope which is the graph of $(ex)^{-1}$; this is equivalent to the inequality

$$(7.61) \quad \lambda e^{-\lambda x} = (\lambda x e^{-\lambda x}) x^{-1} \leq e^{-1} x^{-1}$$

in which equality holds only when $\lambda x = 1$. Therefore

$$(7.62) \quad G_r(\lambda) > 2 \int_{z_0}^1 [r(1-x)^{r-1} - e^{-1} x^{-1}] dx.$$

The right side being independent of λ , we have

$$(7.63) \quad H_r = G_r(q_r) > 2 \int_{z_0}^1 [r(1-x)^{r-1} - e^{-1} x^{-1}] dx$$

and hence

$$(7.64) \quad H_r = G_r(q_r) > 2[(1-z_0)^r - e^{-1} \log z_0^{-1}].$$

In case $r = \frac{1}{2}$, we find from (7.52) that

$$(7.65) \quad z_0 = .513296$$

and hence from (7.74) that

$$(7.651) \quad H_r = G_r(q_r) > .904604.$$

Because of (7.22) and elementary considerations involving the graphs of the functions $\lambda e^{-\lambda x}$ and their envelope, it can be expected that λ' and $G_r(\lambda')$ will be good approximations to q_r and $G_r(q_r)$ whenever λ' is chosen so that the point P at which the graph of $\lambda' e^{-\lambda' x}$ is tangent to the envelope lies about midway between the points on the envelope with x coordinates z_0 and 1. Accordingly we may set $\lambda' = [(1+z_0)/2]^{-1}$ and obtain an upper bound of $H_r = G_r(q_r)$ from the inequality

$$(7.66) \quad H_r = G_r(q_r) \leq G_r([(1+z_0)/2]^{-1})$$

where, as above, z_0 is determined from (7.52). For the case $r = \frac{1}{2}$, we find that

$$(7.661) \quad \lambda' = [(1 + z_0)/2]^{-1} = 1.321618,$$

the corresponding z determined from (7.42) is $z' = .484662$ and

$$(7.662) \quad H_r = G_r(q_r) \leq G_r(\lambda') = .915130.$$

While the matter is of secondary importance, graphs indicate that approximations are improved by taking the x coordinate of P to be a little less than the average of z_0 and z_1 and we quite arbitrarily take it to be $.6z_0 + .4$. Accordingly we set $\lambda'' = (.6z_0 + .4)^{-1}$ and obtain an upper bound of $H_r = G_r(q_r)$ from the inequality

$$(7.67) \quad H_r = G_r(q_r) \leq G_r((.6z_0 + .4)^{-1}).$$

In case $r = \frac{1}{2}$, (7.75) shows that

$$(7.671) \quad (.6z_0 + .4)^{-1} = 1.412474,$$

using this as a value of λ in (7.1) gives $z = .494049$, and (7.23) then shows that

$$(7.672) \quad H_r = G_r(q_r) \leq G(1.412474) = .914358.$$

In order to compute q_r and $H_r = G_r(q_r)$ by use of (7.23), it is necessary to solve the equations (7.41) and (7.42) for z and λ . When $0 < z < 1$, (7.42) can be put in the form

$$(7.7) \quad \lambda = \frac{\log z^{-1}}{1 - z}$$

which gives λ as an elementary function of z . For this reason, we seek z and λ by seeking the z for which $g(z) = 0$ where

$$(7.71) \quad g(z) = r(1 - z)^{r-1} - \lambda e^{-\lambda z}$$

where λ is the function of z defined by (7.7). The labor involved in finding z is greatly reduced by using $(z_0 + 1)/2$ or $(.6z_0 + .4)$ for the starting value of z . In case $r = \frac{1}{2}$, the solution of (7.41) and (7.42) is $z = .49250\ 2479$ and $\lambda = 1.39558\ 4718$. Thus

$$(7.72) \quad q_{\frac{1}{2}} = 1.39558\ 4718$$

and use of (7.23) gives

$$(7.73) \quad H_{\frac{1}{2}} = G_{\frac{1}{2}}(q_{\frac{1}{2}}) = .91431\ 8331.$$

8. The case $r > 1$. We suppose that r has a given value for which $r > 1$ and obtain information about $G_r(\lambda)$ when $\lambda > 0$. In order to use (6.3), we need in-

formation about the sign of $\psi(x)$ over the interval $0 < x < 1$ where, as in (6.1),

$$(8.01) \quad \psi(x) = \lambda e^{-\lambda x} - r(1-x)^{r-1}.$$

The facts are more complicated than in previous cases and are given in the following theorem.

Theorem 8.1. *Let $r > 1$. If $0 < \lambda \leq r$, then there is a number $z = z(\lambda)$ such that $0 < z < 1$ and*

$$(8.11) \quad \psi(x) < 0 \quad 0 < x < z$$

and

$$(8.12) \quad \psi(x) > 0 \quad z < x < 1.$$

If $\lambda > r$, then there are numbers $z^ = z^*(\lambda)$ and $z = z(\lambda)$ such that $0 < z^* < z < 1$ and*

$$(8.14) \quad \psi(x) > 0 \quad 0 < x < z^*$$

$$(8.15) \quad \psi(x) < 0 \quad z^* < x < z$$

and

$$(8.16) \quad \psi(x) > 0 \quad z < x < 1.$$

Moreover $z^* < r^{-1} < z$.

To prove this theorem, we note first that $\psi(r^{-1}) < 0$. This is true because

$$(8.17) \quad r^{-1}\psi(r^{-1}) = (\lambda/r)e^{-\lambda/r} - (1-r)^{r-1}$$

and

$$(8.171) \quad te^{-t} \leq e^{-1} < (1-r^{-1})^{r-1}$$

when $t > 0$ and $r > 1$. Since $\psi(r^{-1}) < 0$ and $\psi(1) = \lambda e^{-\lambda} > 0$, it follows that $\psi(x)$ must have at least one zero z for which $r^{-1} < z < 1$. The derivative of $\psi(x)$ being less informative, we use the function $f(x) = \psi(x)e^{\lambda x}$ which has the same zeros and obtain

$$(8.18) \quad f'(x) = r\lambda[x - 1 + (r-1)/\lambda](1-x)^{r-2}e^{\lambda x}.$$

The other factors being positive, the sign of $f'(x)$ is the same as the sign of the linear factor in brackets which is positive when $x = 1$. Therefore, depending upon whether $\lambda \leq r-1$ or $\lambda > r-1$, $f(x)$ is either increasing over the whole interval $0 < x < 1$ or is decreasing over a subinterval $0 < x < x_1$ and increasing over the complementary subinterval $x_1 < x < 1$ where $x_1 = 1 - (r-1)/\lambda$. In case $\lambda \leq r$, we have $f(0) = \lambda - r \leq 0$ and it follows that $f(x)$ and $\psi(x)$ have at most one zero in the interval $0 < x < 1$. The conclusion of the first part of Theorem 8.1 follows. Suppose now that $\lambda > r$. Then $f(0) = \lambda - r > 0$, $f(r^{-1}) < 0$,

and $f(1) = \lambda > 0$ so $f(x)$ must have at least one zero z^* for which $0 < z^* < r^{-1}$ and at least one zero z for which $r^{-1} < z < 1$. Since $f(x)$ is decreasing over $0 < x < x_1$ and increasing over $x_1 < x < 1$, the remaining conclusion of Theorem 8.1 follows.

We simplify some of our considerations by defining z^* to be zero when $\lambda \leq r$ so that z^* and z are now both defined over the entire interval $\lambda > 0$. The formulas (8.14), (8.15), and (8.16) are then valid when $\lambda > 0$, (8.14) being valid when $\lambda < r$ because there exist no values of x for which $0 < x < z^*$.

Using (6.3), (6.31), (6.11), (8.14), (8.15), (8.16) and finally (8.01), we obtain the formula

$$(8.2) \quad G_r(\lambda) = 2 \int_{z^*}^z [r(1-x)^{r-1} - \lambda e^{-\lambda x}] dx,$$

the integrand being $-\psi(x)$. Evaluating the integral in (8.2) gives

$$(8.21) \quad G_r(\lambda) = 2[e^{-\lambda z} - (1-z)^r + (1-z^*)^r - e^{-\lambda z^*}].$$

From (8.2) and (8.21) we see that the values and behavior of $G_r(\lambda)$ depend upon the values and behavior of z^* and z which are nonelementary functions of r and λ .

Information about z and z^* is obtainable from the formula

$$(8.3) \quad x\psi(x) = -x \left[\frac{1}{ex} - \lambda e^{-\lambda x} \right] - \left[rx(1-x)^{r-1} - \frac{1}{e} \right].$$

As (7.61) shows,

$$(8.31) \quad (ex)^{-1} - \lambda e^{-\lambda x} \geq 0$$

with equality only when $\lambda x = 1$. Let

$$(8.32) \quad g(x) = rx(1-x)^{r-1} - e^{-1}.$$

Then $g(0) = g(1) = -e^{-1} < 0$ and $g(r^{-1}) > 0$. Since

$$(8.33) \quad g'(x) = r(1-rx)(1-x)^{r-2},$$

we see that $g(x)$ is increasing over $0 < x < r^{-1}$ and decreasing over $r^{-1} < x < 1$. Therefore there exist numbers z_0^* and z_0 , depending upon r but not upon λ such that $0 < z_0^* < r^{-1} < z_0 < 1$, $g(z_0^*) = g(z_0) = 0$, and $g(x) > 0$ when $z_0^* < x < z_0$. Hence (8.3) and (8.31) show that $\psi(x) < 0$ when $z_0^* < x < z_0$. The numbers z^* and z , which depend upon both r and λ , therefore satisfy the condition

$$(8.34) \quad 0 \leq z^* \leq z_0^* < r^{-1} < z_0 \leq z < 1.$$

If $\lambda = z_0^{-1}$ then equality holds in (8.31) when $x = z_0$ and, since $g(z_0) = 0$, it follows from (8.3) that $\psi(z_0) = 0$ and hence $z = z_0$. If $\lambda = (z_0^*)^{-1}$, then the same argument shows that $z^* = z_0^*$. To determine the ways in which, for a fixed $r > 1$, the num-

bers z and z^* depend upon λ we let x denote one or the other of z and z^* so that $\psi(x) = 0$ and hence

$$(8.4) \quad \log \lambda - \lambda x = \log r + (r - 1) \log (1 - x).$$

Differentiation gives

$$(8.41) \quad \left(\frac{r-1}{1-x} - \lambda \right) \frac{dx}{d\lambda} = \frac{\lambda x - 1}{\lambda}.$$

The coefficient of $dx/d\lambda$ cannot be zero because if we equate it to zero and eliminate x from this equation and the equation $\psi(x) = 0$ we obtain

$$(8.42) \quad \lambda^r e^{-\lambda} = r[(r-1)/e]^{r-1}$$

and this is impossible because

$$(8.43) \quad \lambda^r e^{-\lambda} \leq r^r e^{-r} < r[(r-1)/e]^{r-1}$$

when $\lambda > 0$ and $r > 1$. In case $x = z$, this coefficient is positive when λ is near zero and hence is positive for each $\lambda > 0$. In case $x = z^*$, the coefficient is negative when λ is large and hence is negative for each $\lambda > 1$.

Theorem 8.5. *Let $r > 1$. Then the function λz is increasing over the interval $\lambda > 0$ and λz^* is increasing over the interval $\lambda > 1$.*

To prove this theorem let $w(\lambda)$ be the function of λ defined by $w(\lambda) = \lambda x$ where x is one or the other of z and z^* . With the aid of (8.41) we obtain

$$(8.61) \quad \left(\frac{r-1}{1-x} - \lambda \right) w'(\lambda) = \frac{rx - 1}{1-x}.$$

In case $x = z$, the coefficient of $w'(\lambda)$ is positive and, as (8.34) shows, $0 < x < 1$ and $rx > 1$ so $w'(\lambda) > 0$ and λz is increasing. In case $x = z^*$, the coefficient of $w'(\lambda)$ is negative and, as (8.34) shows, $0 < x < 1$ and $rx < 1$ so $w'(\lambda) > 0$ and λz^* is increasing when $\lambda > 1$. This proves Theorem 8.5. With the aid of this theorem and (8.41), we obtain the following theorem.

Theorem 8.6. *Let $r > 1$. When λ increases from 0, z decreases from 1 to attain its minimum value z_0 when $\lambda = \lambda_0 = z_0^{-1}$ and then increases to approach 1 as λ increases from λ_0 . When λ increases from 1, z^* increases from 0 to attain its maximum value z_0^* when $\lambda = \lambda_0^* = (z_0^*)^{-1}$ and then decreases to approach 0 as λ increases from λ_0^* .*

In case $r = 2$, z_0^* and z_0 are respectively the lesser and the greater of the values of x for which

$$(8.7) \quad x^2 - x + (2e^{-1}) = 0,$$

and hence

$$(8.71) \quad z_0^* = .24297806, \quad z_0 = .75702194$$

and

$$(8.72) \quad \lambda_0 = 1.32096567, \quad \lambda_0^* = 4.11559798.$$

Table 1, showing values of z^* , z , and $G_r(\lambda)$ when $r = 2$, contains sufficient entries to show all significant variations, in these functions of λ .

TABLE 1

λ	z^* when $r = 2$	z when $r = 2$	λz when $r = 2$	$G_2(\lambda)$
.1	.000000	.954552	.095455	1.813787
.25	.000000	.900190	.225048	1.577033
.5	.000000	.835356	.417678	1.262933
1.	.000000	.768039	.768039	.820232
1.25	.000000	.757544	.946929	.658290
λ_0	.000000	.757022	1.000000	.617682
1.5	.000000	.760212	1.140318	.524438
1.75	.000000	.774303	1.355030	.414001
2.	.000000	.796812	1.593624	.323805
2.25	.098371	.823692	1.853308	.274240
2.5	.158091	.851130	2.127825	.264435
2.75	.194414	.876576	2.410584	.275234
3.	.216471	.898840	2.696521	.297544
3.5	.237280	.933246	3.266362	.359181
4.	.242828	.956387	3.825548	.429256
λ_0^*	.242978	.960497	3.953020	.445678
4.25	.242800	.964797	4.100386	.464699
5.	.237485	.981525	4.907625	.566944
10.	.180899	.999772	9.997725	1.014305
20.	.121613	$1 - 2 \times 10^{-8}$	$20 - 4 \times 10^{-7}$	1.367451

9. $H_r = G_r(q_r)$ when $r > 1$. In this section we prove the following theorem, set forth a procedure for calculating q_r , and give a relevant table.

Theorem 9.1. *For each $r > 1$ there is a unique value q_r of λ for which $G_r(\lambda)$ attains its minimum value $H_r = G_r(q_r)$, the function $G_r(\lambda)$ being decreasing when $0 < \lambda < q_r$ and increasing when $\lambda > q_r$. Moreover $r < q_r < \lambda_0^* = (z_0^*)^{-1}$, and q_r is the first of the unique triplet (λ, z^*, z) for which $\lambda > r$, $0 < z^* < r^{-1} < z < 1$, and the three equations*

$$(9.11) \quad \lambda e^{-\lambda z^*} = r(1 - z^*)^{r-1}$$

$$(9.12) \quad \lambda e^{-\lambda z} = r(1 - z)^{r-1}$$

$$(9.13) \quad z^* e^{-\lambda z^*} = z e^{-\lambda z}$$

hold. The latter three equations are equivalent to

$$(9.14) \quad \begin{aligned} \lambda z^* e^{-\lambda z^*} &= \lambda z e^{-\lambda z} \\ &= r z^*(1 - z^*)^{r-1} = r z(1 - z)^{r-1}. \end{aligned}$$

Since the integrand of the integral in (8.2) is zero at the limits of integration, we can differentiate (8.2) to obtain

$$(9.2) \quad G'_r(\lambda) = -2 \int_{z^*}^z (1 - \lambda x) e^{-\lambda x} dx.$$

Evaluating this integral gives

$$(9.21) \quad G'_r(\lambda) = 2[z^* e^{-\lambda z^*} - z e^{-\lambda z}],$$

and we sometimes use the alternative formula

$$(9.22) \quad \frac{1}{2} \lambda G'_r(\lambda) = \lambda z^* e^{-\lambda z r^*} - \lambda z e^{-\lambda z}.$$

If $\lambda \leq r$, then $z^* = 0$ while $0 < z < 1$ and (9.21) shows that $G'_r(\lambda) < 0$ and $G_r(\lambda)$ is a decreasing function of λ . If $\lambda \geq \lambda_0^* = (z_0^*)^{-1}$, then Theorem 8.5 and (8.34) show that $1 \leq \lambda z^* < \lambda z$; hence, because the function $t e^{-t}$ is decreasing over the infinite interval $t \geq 1$, (9.22) shows that $G'_r(\lambda) > 0$ and $G_r(\lambda)$ is increasing. Hence $G'_r(\lambda)$ can be zero and $G_r(\lambda)$ can attain its minimum value only at one or more points of the interval $r < \lambda < \lambda_0^* = (z_0^*)^{-1}$. But for values of λ within this interval λz^* and λz are both increasing and $\lambda z^* < 1 < \lambda z$. Hence (9.22) and the fact that $t e^{-t}$ is increasing over $0 \leq t \leq 1$ and decreasing over $t > 1$ imply that $\lambda G'_r(\lambda)$ is increasing over $r < \lambda < \lambda_0^*$. Hence $\lambda G'_r(\lambda)$ and $G'_r(\lambda)$ have exactly one zero q_r and $r < q_r < \lambda_0^*$. The conclusion of Theorem 9.1 follows.

One way to compute the triplet (λ, z^*, z) of Theorem 9.1 may be described roughly as follows. For each λ in an adroitly chosen succession, compute z^* from (9.11) and z from (9.12), the goal being to determine λ so that the difference of the members of (9.13) is appropriately close to zero. At least for use with an ordinary calculating machine, it is simpler to use the following procedure in which z is chosen for the independent variable. The goal is to make $f(z)$ appropriately close to zero where

$$(9.3) \quad f(z) = \lambda e^{-\lambda z} - r(1 - z)^{r-1}.$$

The value of λ is obtained from

$$(9.31) \quad \lambda = (z - z^*)^{-1} \log (z/z^*)$$

after z^* has been obtained by making $g(z^*)$ appropriately close to zero where

$$(9.32) \quad g(z^*) = z^*(1 - z^*)^{r-1} - z(1 - z)^{r-1}$$

or, alternatively, by making $h(z^*)$ appropriately close to zero where

$$(9.33) \quad h(z^*) = \log z^* + (r - 1) \log (1 - z^*) \\ - \log z - (r - 1) \log (1 - z).$$

TABLE 2

r	z^* when $\lambda = q_r$	z when $\lambda = q_r$	q_r	$H_r = G_r(q_r)$
1.5	.2066496	.9635057	2.0341435	.3674497
2	.1521998	.8478002	2.4690195	.2642751
3	.0996672	.6464938	3.4192319	.1681308
4	.0741172	.5146989	4.3985812	.1231333
5	.0590011	.4259204	5.3872765	.0971137
6	.0490089	.3627496	6.3801421	.0801663
7	.0419121	.3156927	7.3752292	.0682529
8	.0366111	.2793481	8.3716397	.0594214
9	.0325008	.2504593	9.3689022	.0526131
10	.0292204	.2269586	10.3667455	.0472044
20	.0145431	.1169408	20.3573899	.0232750
50	.0058015	.0476026	50.3520356	.0092329

Even for relatively small values of r , the functions z^* , z , q_r , and H_r are closely approximated by simple functions of r . Appropriate forms for approximating functions can be surmised by inspection of the entries in the table and are established in Section 10. Let $f_1(r)$ be the function of the form $f_1(r) = Ar^{-1} + Br^{-2}$ which agrees with z^* when $r = 10$ and when $r = 20$. Then

$$(9.41) \quad f_1(r) = .2895209r^{-1} + .0268273r^{-2}.$$

The closeness by which $f_1(r)$ approximates z^* is indicated by the fact that if $r = 50$, then $f_1(r) = .0058011$ while $z^* = .0058015$. Let $f_2(r)$ be the function of the form $f_2(r) = Ar^{-1} + Br^{-2}$ which agrees with z when $r = 10$ and when $r = 20$. Then

$$(9.42) \quad f_2(r) = 2.4080483r^{-1} - 1.3846278r^{-2}.$$

When $r = 50$ we have $f_2(r) = .0476071$ while $z = .0476026$. Let $f_3(r)$ be the function of the form $f_3(r) = r + A + Br^{-1}$ which agrees with q_r when $r = 10$ and when $r = 20$. Then

$$(9.43) \quad f_3(r) = r + .3480342 + .1871136r^{-1}.$$

When $r = 50$ we have $f_3(r) = 50.3517764$ while $q_r = 50.3520356$. Finally, let $f_4(r)$ be the function of the form $f_4(r) = Ar^{-1} + Br^{-2}$ which agrees with H_r when $r = 10$ and when $r = 20$. Then

$$(9.44) \quad f_4(r) = .4589552r^{-1} + .1308857r^{-2}.$$

When $r = 50$ we have $f_4(r) = .0092315$ while $H_r = .0092329$. The constants in (9.41), (9.42), (9.43) and (9.44) are of interest in connection with approximations in Section 10.

10. Estimates when r is large. We begin with some estimates which are valid whenever $r > 0$ but will be used when r is large.

If $0 < h < 1$, then (3.61) shows that, where h may be a function of r and λ ,

$$(10.1) \quad \begin{aligned} G_r(\lambda) &= e^{-\lambda} + \left\{ \int_0^h + \int_h^1 \right\} | \lambda e^{-\lambda x} - r(1-x)^{r-1} | dx \\ &\geq e^{-\lambda} + \left| \left\{ \int_0^h - \int_h^1 \right\} [\lambda e^{-\lambda x} - r(1-x)^{r-1}] dx \right| \end{aligned}$$

and hence

$$(10.11) \quad G_r(\lambda) \geq e^{-\lambda} + | e^{-\lambda} + 2(1-h)^r - 2e^{-\lambda h} |.$$

Let λ be a function of r ; we simplify formulas by writing λ for $\lambda(r)$. Suppose that, as $r \rightarrow \infty$, $\limsup \lambda(r)/r = \infty$, and let r become infinite over a sequence such that $\lambda(r)/r \rightarrow \infty$. For each positive constant A we can put $h = A/r$ in (10.11) to obtain

$$(10.12) \quad G_r(\lambda) \geq e^{-\lambda} + \left| e^{-\lambda} + 2 \left(1 - \frac{A}{r} \right)^r - 2e^{-(\lambda/r)A} \right|,$$

and hence $G_r(\lambda) \geq o(1) + 2e^{-A}$. Thus in case $\limsup \lambda(r)/r = \infty$ we can obtain only the trivial estimate $\limsup G_r(\lambda) = 2$ for $G_r(\lambda)$. In case $\liminf \lambda(r)/r = 0$, we can put $h = A/\lambda$ in (10.12) to show that in this case also $G_r(\lambda) \geq o(1) + 2e^{-A}$ and hence that again $\limsup G_r(\lambda) = 2$.

We now show that nontrivial estimates of $G_r(\lambda)$ are obtainable in case

$$(10.2) \quad 0 < q_1 = \liminf_{r \rightarrow \infty} \frac{\lambda(r)}{r} \leq \limsup_{r \rightarrow \infty} \frac{\lambda(r)}{r} = q_2 < \infty.$$

Making a change of variable in (3.61) gives

$$(10.21) \quad G_r(\lambda) = e^{-\lambda} + \int_0^r \left| \frac{\lambda}{r} e^{-(\lambda/r)x} - \left(1 - \frac{x}{r} \right)^{r-1} \right| dx$$

and it follows from this that

$$(10.22) \quad G_r(\lambda) = o(1) + \int_0^\infty \left| \frac{\lambda}{r} e^{-(\lambda/r)x} - e^{-x} \right| dx$$

where $o(1) \rightarrow 0$ as $r \rightarrow \infty$. In case $\lambda = rq$ where q is a positive constant and more generally in case $\lambda(r)/r \rightarrow q > 0$ as $r \rightarrow \infty$, the members of (10.31) have limits as $r \rightarrow \infty$ and we obtain

$$(10.23) \quad G_r(\lambda) = o(1) + \int_0^\infty | qe^{-q} - e^{-x} | dx.$$

It follows from this that $G_r(\lambda) = o(1)$ if and only if $q_1 = q_2 = 1$, that is, if and only if $\lambda(r)/r \rightarrow 1$ as $r \rightarrow \infty$.

It is our next task to determine how rapidly $G_r(\lambda)$ converges to zero when $\lambda(r)/r \rightarrow 1$ and $r \rightarrow \infty$. For this purpose we put

$$(10.3) \quad \lambda(r) = r + \delta(r)$$

where $\delta(r)/r = o(1)$ as $r \rightarrow \infty$. To simplify formulas, let $\rho = r^{\frac{1}{2}}$ and $\rho' = \rho^{-1} = r^{-\frac{1}{2}}$. For use with (3.61), we put

$$(10.31) \quad I(r) = \int_{\rho'}^1 |\lambda e^{-\lambda x} - r(1-x)^{r-1}| dx$$

and find that

$$(10.32) \quad |I(r)| \leq \int_{\rho'}^1 \lambda e^{-\lambda x} dx + \int_{\rho'}^1 r(1-x)^{r-1} dx$$

and hence, as very crude estimates show, $I(r) = o(r^{-2})$. Using (3.61) gives

$$(10.33) \quad G_r(\lambda) = o(r^{-2}) + \int_0^{\rho'} |\lambda e^{-\lambda x} - r(1-x)^{r-1}| dx.$$

Changing the variable of integration by the substitution $x = y/r$ and then replacing y by x gives

$$(10.34) \quad G_r(\lambda) = o(r^{-2}) + \int_0^{\rho} \left| \frac{\lambda}{r} e^{-(\lambda/r)x} - \left(1 - \frac{x}{r}\right)^{r-1} \right| dx.$$

Using (10.3), we put this in the form

$$(10.35) \quad G_r(\lambda) = o(r^{-2}) + \int_0^{\rho} e^{-x} |F(x)| dx$$

where

$$(10.36) \quad \begin{aligned} F(x) &= \left(1 + \frac{\delta}{r}\right) e^{-(\delta/r)x} - e^x \left(1 - \frac{x}{r}\right)^{r-1} \\ &= F_1(x) + F_2(x). \end{aligned}$$

With the aid of

$$(10.37) \quad \begin{aligned} \log F_2(x) &= x - (r-1) \left(\frac{x}{r} + \frac{x^2}{2r^2} + \cdots \right) \\ &= \frac{x}{r} - \frac{x^2}{2r^2} + O\left(\frac{x^3}{r^3}\right) + O\left(\frac{x^3}{r^2}\right) \end{aligned}$$

and $F_2(x) = \exp [\log F_2(x)]$, we can show that

$$(10.38) \quad F_1(x) = 1 + \frac{\delta}{r} - \frac{\delta x}{r} + O\left(\frac{\delta^2 x^2}{r^2}\right)$$

and

$$(10.39) \quad F_2(x) = 1 + \frac{x}{r} - \frac{x^2}{2r} + O\left(\frac{x^3}{r^2}\right) + O\left(\frac{x^3}{r^2}\right).$$

The leading term in the expansion of $F(x)$ in powers of r^{-1} is therefore

$$(10.4) \quad \frac{1}{2r} (x^2 - 2x - 2\delta x + 2\delta).$$

Hence (10.35) gives

$$(10.41) \quad G_r(\lambda) = O\left(\frac{\delta^2}{r^2}\right) + O\left(\frac{1}{r^2}\right) + \frac{J(\delta)}{r}$$

where

$$(10.42) \quad J(\delta) = \frac{1}{2} \int_0^\infty e^{-x} |x^2 - 2(1 + \delta)x + 2\delta| dx;$$

the integral in (8.61) is first obtained with the upper limit $\rho = r^{\frac{1}{2}}$, but the difference is absorbed into the term $O(r^{-2})$ in (10.41). Investigation of (10.41) shows that when λ and δ are functions of r for which (10.3) holds, $G_r(\lambda)$ approaches zero most rapidly as r increases when $\delta(r)$ converges to the unique positive number δ_0 which minimizes the right member of (10.42).

The minimum of $J(\delta)$ occurs when $\delta \geq 0$. For such values of δ , the two zeros α and β of $x^2 - 2(1 + \delta)x + 2\delta$ are both positive and

$$(10.43) \quad \alpha = 1 + \delta - (1 - \delta^2)^{\frac{1}{2}}; \quad \beta = 1 + \delta + (1 - \delta^2)^{\frac{1}{2}}.$$

From (10.42) we find that

$$(10.44) \quad J(\delta) = - \int_\alpha^\beta e^{-x} [x^2 - (\alpha + \beta)x + \alpha\beta] dx$$

and

$$(10.45) \quad J'(\delta) = 2 \int_\alpha^\beta e^{-x} (x - 1) dx.$$

Therefore

$$(10.46) \quad J(\delta) = (\beta - \alpha + 2)e^{-\beta} + (\beta - \alpha - 2)e^{-\alpha}$$

and

$$(10.47) \quad J'(\delta) = 2[\alpha e^{-\alpha} - \beta e^{-\beta}]$$

where α and β are the functions of δ defined by (10.43). The value of δ for which $J'(\delta) = 0$ and $J(\delta)$ attains its minimum is that for which

$$(10.48) \quad \delta = \delta_0 = .34856746, \quad \alpha = .28955883 \quad \text{and} \quad \beta = 2.40757609.$$

The minimum value of $J(\delta)$ is

$$(10.49) \quad J(\delta_0) = .45910551.$$

It follows that if

$$(10.5) \quad \lambda_r = r + .34856746 + o(1)$$

then

$$(10.51) \quad G_r(\lambda_r) = \frac{.45910551}{r} + o\left(\frac{1}{r^2}\right).$$

This implies the result in (1.7) and (1.71).

If we replace (10.3) by

$$(10.6) \quad \lambda(r) = r + \delta + C_1/r + O(r^{-2})$$

where δ and C_1 are constants, then it is not difficult to modify the work following (10.3) to obtain a constant C_2 such that

$$(10.61) \quad G_r(\lambda) = J(\delta)/r + C_2/r^2 + O(r^{-3})$$

where $J(\delta)$ is defined by (10.42). Setting $\delta = 0$ and $\delta = 1$, we find that

$$(10.62) \quad J(0) = 4e^{-2} = .54134113$$

and

$$(10.63) \quad J(1) = .62002501$$

and hence

$$(10.64) \quad G_r(r) = .54134113r^{-1} + O(r^{-2})$$

and

$$(10.65) \quad G_r(r+1) = .62002501r^{-1} + O(r^{-2}).$$

Thus when r is large the numbers $G_r(r)$ and $G_r(r+1)$ exceed $G_r(q_r)$ by non-trivial factors.

From (10.41) and following work, we see that

$$(10.7) \quad q_r + r + \delta_0 + O(r^{-1})$$

where δ_0 is given by (10.48). This enables us to obtain asymptotic information about the triplets (λ, z^*, z) of numbers satisfying (9.11), (9.12), and (9.13). Putting $\lambda = r + \delta$ where $\delta = \delta_0 + O(r^{-1})$ in (9.12) gives

$$(10.71) \quad (1 + \delta r^{-1})e^{-(1+\delta)z} = [e^z(1 - z)]^{r^{-1}}.$$

Since $0 < z < 1$ and $\delta = \delta_0 + o(1)$, the left member of (10.71) has a positive inferior limit as $r \rightarrow \infty$; hence so also does the right member and this implies that $z \rightarrow 0$ as $r \rightarrow \infty$. Now let p be the function of r defined by $p = rz$. Then $p/r = z = o(1)$ as $r \rightarrow \infty$. Replacing z by p/r in (10.71) and taking logarithms gives

$$(10.72) \quad -\frac{\delta p}{r} + \log \left(1 + \frac{\delta}{r} \right) = p + (r - 1) \log \left(1 - \frac{p}{r} \right).$$

Expanding these logarithms by elementary formulas gives

$$(10.73) \quad -\frac{\delta p}{r} + \frac{\delta}{r} \left[1 - \frac{\delta}{2r} + \frac{\delta^2}{3r^2} - \frac{\delta^3}{4r^3} + \cdots \right] \\ = -\frac{p^2}{r} \left[\frac{1}{2} + \frac{p}{3r} + \frac{p^2}{4r^2} + \cdots \right] + \frac{p}{r} \left[1 + \frac{p}{2r} + \frac{p^2}{3r^2} + \frac{p^3}{4r^3} + \cdots \right].$$

Using the facts that $\delta = O(1)$ and $p/r = o(1)$. Using all this information with the equation obtained by multiplying (10.73) by p , we see that $p^3/r = o(1)$. Hence we can put (10.73) in the form

$$(10.74) \quad \frac{1}{2}p^2 - (1 + \delta)p + \delta = \frac{\delta^2}{r} \left[\frac{1}{2} - \frac{\delta}{3r} + \frac{\delta^2}{4r^2} + \cdots \right] \\ - \frac{p^3}{r} \left[\frac{1}{3} + \frac{p}{4r} + \frac{p^2}{5r^2} + \cdots \right] + \frac{p^2}{r} \left[\frac{1}{2} + \frac{p}{3r} + \frac{p^2}{4r^2} + \cdots \right]$$

and conclude that

$$(10.75) \quad p^2 - 2(1 + \delta)p + 2\delta = o(1)$$

as $r \rightarrow \infty$. Since $\delta = \delta_0 + O(r^{-1})$, this implies that $p = O(1)$ as $r \rightarrow \infty$. Returning to (10.75), we can now see that

$$(10.76) \quad p^2 - 2(1 + \delta)p + 2\delta = O(r^{-1})$$

and hence that

$$(10.77) \quad p^2 - 2(1 + \delta_0)p + 2\delta_0 = O(r^{-1}).$$

Therefore, since $p = rz$,

$$(10.78) \quad rz = (1 + \delta_0) \pm \sqrt{1 + \delta_0^2} + O(r^{-1}).$$

Using (9.11) in place of (9.12) replaces z by z^* in all of the work from (10.71) to (10.78) inclusive. Since $z^* < z$, the negative and positive signs in (10.78) belong respectively to z^* and z . Therefore, the constants being the numbers α and β following (10.48), we have

$$(10.8) \quad z^* = \frac{.28955883}{r} + O(r^{-2})$$

and

$$(10.81) \quad z = \frac{2.40757609}{r} + O(r^{-2}).$$

From (10.5), (10.51) and (10.7), we obtain

$$(10.82) \quad \lambda = q_r = r + .34856746 + O(r^{-1})$$

and

$$(10.83) \quad G_r(q_r) = \frac{.45910551}{r} + O(r^{-2}).$$

The constants in the four above formulas differ relatively little from the corresponding constants in the formulas (9.41), (9.42), (9.43) and (9.44). It seems certain that the estimates in (10.8), (10.81), (10.82) and (10.83) can be sharpened and that, for example, (10.83) can be replaced by

$$(10.84) \quad G_r(q_r) = .45910551r^{-1} + Br^{-2} + O(r^{-3})$$

where B is a constant having roughly the magnitude of the constant .1308857 in (9.44). That such estimates should be quite accurate even for small values of r is indicated by the following. Seeking a function $f(r)$ to approximate $G_r(q_r)$, we let $f(r)$ be the function of the form $f(r) = .45910551r^{-1} + Br^{-2}$ which agrees with $G_r(q_r)$ when $r = 50$ and find that

$$(10.9) \quad f(r) = .45910551r^{-1} + .1269694r^{-2}.$$

The function $f(r)$ in (10.9) is necessarily a good approximation to $G_r(q_r)$ when r is large and the second term is small in comparison to the first. It can be expected that $f(r)$ is a good approximation to $G_r(q_r)$ when $r = 20$ and we find that in this case $f(r) = .0232727$ while $G_r(q_r) = .0232750$. It is more surprising that $f(r)$ remains a good approximation to $G_r(q_r)$ clear down to and including $r = 1$. When $r = 2$ we have $f(r) = .26129$ while $G_r(q_r) = .26427$; when $r = 1.5$ we have $f(r) = .36250$ while $G_r(q_r) = .36744$; and when $r = 1$ we have $f(r) = .58607$

while $G_r(q_r) = .56462$. This indicates that $f(r)$ approximates $G_r(q_r)$ with an error less than 4 percent of $G_r(q_r)$ over the whole range $r \geq 1$.

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