Surface Waves of Finite Energy

J. KAMPÉ DE FÉRIET & J. KOTIK¹

The University, Lille and Harvard University, Cambridge, Massachusetts

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1. During the last twenty years, a good deal of work has been done with the linearized gravity wave theory. In the most popular situation, one considers an infinite or semi-infinite ocean, of two or three dimensions, containing some sort of an obstacle, and proves the existence and uniqueness of motions which are regular except perhaps at the obstacle, and which behave like simple harmonic waves at infinity. The obstacles considered have been such things as beaches, submerged cylinders, vertical and horizontal plane barriers, docks, lumpy seabeds and the like. Besides proving the existence and uniqueness of the solution, one computes physical quantities like reflection coefficients in terms of the characteristic parameters of the wave and the geometry of the obstacle.

We will take up a different problem which seems to have attracted little attention, *the initial-value problem: if at time $t = 0$ the free surface has a prescribed form and if no forces except gravity act, what is the subsequent motion?*

The answer was given (in the case of infinite depth) by Cauchy [2] and Poisson [3] in case the initial elevation is the Dirac δ -function. The solution is in the form of a definite integral from which valuable information is obtained by using the method of stationary phase [1]. Thinking of the initial elevation as the limit of a sequence of step-functions, one could try to solve the general initial value problem by using a weighted superposition of these classical solutions; this was done by J. Kotik in [4]. On the other hand, one can think of the initial elevation as a superposition of sines and cosines, or Fourier Integral, in which case the solution is obtained by associating with each Fourier component of the initial elevation the appropriate simply harmonic wave and adding up. The formal expressions obtained in both cases are iterated integrals, and they differ

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only in the order of integration. The second method, which we use here, is the less powerful, in that it works for a smaller class of initial data; on the other hand, the result is obtained more easily and the class of initial data has real physical interest: *all the motions considered here have finite energy.*

2. We consider two-dimensional motions in water of infinite depth; if the depth were h we would have only to replace $e^{-\lambda y}$ by $\cosh \lambda(y - h)/\cosh \lambda h$ throughout, and use similar arguments.

We suppose that the physical units have been so chosen that the density of the fluid is $\rho = 1$ and the acceleration of gravity is $g = 1$.

The fluid occupies the half-plane

$$\{x, y: -\infty < x < \infty, y \geq 0\};$$

we define the sets

$$D = \{x, y, t: -\infty < x < \infty, y > 0, -\infty < t < \infty\},$$

$$B = \{x, y, t: -\infty < x < \infty, y = 0, -\infty < t < \infty\},$$

$$\bar{D} = B \cup D.$$

Usually one does not consider negative time in initial value problems, but in our problem the solutions are automatically defined for $-\infty < t < \infty$.

The salient function is the velocity potential $\phi(x, y, t)$ which has the following properties [1]:

$$(1) \quad \phi_{xx} + \phi_{yy} = 0 \quad \text{in } D,$$

$$(2) \quad \phi_{tt} = \phi_y \quad \text{in } B.$$

Under weak additional assumptions, we shall automatically have the additional condition

$$(2') \quad \phi_{tt} = \phi_y \quad \text{in } D,$$

and our solutions will have this additional property. From $\phi(x, y, t)$ we get the velocities u, v , in the x, y directions using

$$u = -\phi_x, \quad v = -\phi_y;$$

the elevation $\eta(x, t)$ of the surface above the mean level $y = 0$ is given by

$$(3) \quad \eta(x, t) = -\phi_t(x, 0, t) \quad [1].$$

Our purpose is to find the velocity potential $\phi(x, y, t)$ satisfying the general conditions (1), (2), (3) and the initial conditions

$$(4) \quad \phi(x, 0, 0) = 0$$

$$(5) \quad \phi_t(x, 0, 0) = F(x),$$

under the hypotheses

$$\begin{aligned} H_1 & F(x) \in L^2, \\ H_2 & F(x) \text{ absolutely continuous,} \\ H_3 & F'(x) \in L^2, \\ H_4 & F(x) = F(-x). \end{aligned}$$

Equation (4) means that the initial impulse applied to the surface is zero; (5) means that the surface elevation is initially $-F(x)$. H_1 states that the potential energy,

$$(6) \quad \frac{1}{2} \int_{-\infty}^{\infty} [\eta(x, 0)]^2 dx,$$

due to the initial elevation is finite. H_3 implies that $F(x)$ is bounded and vanishes at infinity. H_4 is an inessential simplification.

We will define a velocity potential $\phi(x, y, t)$ explicitly and show that

a) $\phi(x, y, t)$ is defined and continuous in $\bar{D}$, satisfies (1) and (2') in D , is analytic in all variables in D , and satisfies (4).

b) $\phi_t(x, y, t)$ exists and is continuous in $\bar{D}$, and satisfies (5). Furthermore, the surface elevation $\eta(x, t)$ is L^2 in x for each t and L^2 in t for each x ; also $\eta(x, t)$ is bounded. Finally

$$\begin{aligned} \lim_{|t| \rightarrow \infty} \eta(x, t) &= 0, & \lim_{|x| \rightarrow \infty} \eta(x, t) &= 0, \\ \lim_{y \rightarrow \infty} \phi(x, y, t) &= 0, & \lim_{y \rightarrow \infty} \phi_t(x, y, t) &= 0. \end{aligned}$$

c) $\phi_y(x, 0, t)$ and $\phi_{tt}(x, 0, t)$ exist (for all x) for almost all t , and are equal, satisfying (2). Also $\phi_y(x, 0, t) \in L^2$ with respect to x for each t , which expresses the finiteness of the kinetic energy. A slight strengthening of the hypotheses H_1, H_2, H_3 will ensure that (2) is satisfied everywhere.

d) The solution $\phi(x, y, t)$ is essentially unique.

3. Certain facts which will be used often are stated below and will henceforth be referred to by letter.

$\mathfrak{A}$. If, whenever α, β, γ are in intervals A, B, C , respectively, we have $f(x, \alpha, \beta, \gamma)$ continuous in $\alpha, \beta, \gamma, f(x, \alpha, \beta, \gamma)$ measurable in x , and $|f(x, \alpha, \beta, \gamma)| \leq G(x) \in L$, then

$$J(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} f(x, \alpha, \beta, \gamma) dx$$

is continuous for α, β, γ in A, B, C , respectively.

℔. If, in addition, $|f_\alpha(x, \alpha, \beta, \gamma)| \leq H(x) \varepsilon L$, for α, β, γ in A, B, C , respectively, then

$$J_\alpha(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} f_\alpha(x, \alpha, \beta, \gamma) dx$$

for α, β, γ in A, B, C , respectively.

℥. If $F(x)$ is absolutely continuous, $F(x)$ and $F'(x) \varepsilon L^2$, then $f(\lambda)$, the Fourier-Plancherel transform of $F(x)$, is

$$a) \quad f(\lambda) = \lim_{A \rightarrow +\infty} \frac{2}{\pi} \int_0^A \cos \lambda x F(x) dx,$$

and the Fourier-Plancherel transform of $F'(x)$ is $-\lambda f(\lambda)$,

$$b) \quad -\lambda f(\lambda) = \lim_{A \rightarrow +\infty} \frac{2}{\pi} \int_0^A \sin \lambda x F'(x) dx.$$

Also,

$$c) \quad f(\lambda) \text{ and } \lambda f(\lambda) \varepsilon L^2,$$

$$d) \quad |\lambda|^\alpha f(\lambda) \varepsilon L \quad \text{for } 0 \leq \alpha < \frac{1}{2},$$

$$e) \quad F(x) = \int_0^{+\infty} \cos \lambda x f(\lambda) d\lambda \quad (\text{an ordinary Lebesgue integral,})$$

$$f) \quad -F'(x) = \lim_{A \rightarrow +\infty} \int_0^A \sin \lambda x \lambda f(\lambda) d\lambda.$$

℧. If $g(\xi) \varepsilon L$ and $\xi g(\xi) \varepsilon L^2$, $G(t)$ being the Fourier transform of $g(\xi)$,

$$G(t) = \int_0^{+\infty} \cos \xi t g(\xi) d\xi,$$

and $H(t)$ the Fourier-Plancherel transform of $\xi g(\xi)$:

$$H(t) = \lim_{A \rightarrow +\infty} \int_0^A \sin \xi t \xi g(\xi) d\xi,$$

then one has for almost all t :

$$H(t) = -G'(t).$$

℘. If $g(\xi) \varepsilon L^2$, the function

$$v(y, t) = \int_0^\infty e^{-\xi^2 y} \sin \xi t g(\xi) d\xi$$

is defined and continuous in the half-plane $y > 0$, is analytic in y and t for $y > 0$, and

$$v_{ti} = v_y, \quad y > 0.$$

Putting

$$v(t) = \text{l.i.m.}_{A \rightarrow \infty} \int_0^A \sin \xi t g(\xi) d\xi,$$

one has

$$\lim_{y \rightarrow 0+} v(y, t) = v(t), \quad \text{for almost all } t.$$

4. We shall define the velocity potential by

$$(7) \quad \phi(x, y, t) = \int_0^\infty e^{-\lambda y} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \cos \lambda x f(\lambda) d\lambda,$$

where $f(\lambda)$ is the Fourier-Plancherel transform of the given function $F(x)$ satisfying H_1, H_2, H_3, H_4 ; $f(\lambda)$ is defined by $\mathfrak{C}(a)$. Due to $\mathfrak{C}(a)$, the integrand is in L as a function of λ for $x, y, t \in \bar{D}$; hence $\phi(x, y, t)$ is defined in $\bar{D}$. Applying $\mathfrak{A}$ to the sets $\bar{D}_n = \bar{D} \cap T_n$, where T_n is $\{x, y, t: |t| \leq N\}$ we see that $\phi(x, y, t)$ is continuous in $\bar{D}$ since the integrand is absolutely less than $|Nf(\lambda)|$ in T_n . Furthermore, due to $\mathfrak{B}$, $\phi(x, y, t)$ is infinitely differentiable in D , and (1) and (2') hold. Actually, without $\mathfrak{B}$, $\phi(x, y, t)$ is analytic in D .

Setting $t = 0$, we see that the initial condition (4) is satisfied.

5. Now let us consider

$$(8) \quad J(x, y, t) = \int_0^\infty e^{-\lambda y} \cos \sqrt{\lambda} t \cos \lambda x f(\lambda) d\lambda.$$

Using $\mathfrak{A}$ we have $J(x, y, t)$ continuous in $\bar{D}$, since the integrand is absolutely less than $|f(\lambda)|$ in $\bar{D}$. $\mathfrak{B}$ implies that $J(x, y, t) = \phi_t(x, y, t)$ in $\bar{D}$.

Setting $y = 0$, we have the surface elevation

$$(9) \quad \eta(x, t) = - \int_0^\infty \cos \sqrt{\lambda} t \cos \lambda x f(\lambda) d\lambda.$$

The important fact to note is that, at a given time t_0 , $\eta(x, t_0)$ appears as the Fourier transform of $-\cos \sqrt{\lambda} t_0 f(\lambda)$; but, putting $\lambda = \xi^2$, we have

$$(10) \quad \eta(x, t) = - \int_0^\infty \cos \xi t \cos \xi^2 x \cdot 2\xi f(\xi^2) d\xi,$$

and at a given x_0 , $\eta(x_0, t)$ appears as the Fourier transform of $-2\xi f(\xi^2) \cos \xi^2 x_0$. Both integrals are ordinary Lebesgue integrals due to the fact that $f(\lambda)$, and hence $2\xi f(\xi^2) \in L$. [$\mathfrak{C}(d)$].

Setting $t = 0$, we have

$$\eta(x, 0) = - \int_0^\infty \cos \lambda x f(\lambda) d\lambda = -F(x) \quad [\mathfrak{E}(e)],$$

and thus the initial condition (5) is satisfied. From (9) and (10) we see that $\eta(x, t)$ is in L^2 in x and t separately. One would expect the first result on physical grounds, since the total energy is initially finite. The elevation is bounded:

$$|\eta(x, t)| \leq \int_0^\infty |f(\lambda)| d\lambda.$$

By the Riemann-Lebesgue theorem,

$$\lim_{|t| \rightarrow \infty} \eta(x, t) = 0, \quad \lim_{|x| \rightarrow \infty} \eta(x, t) = 0,$$

showing that the energy radiates out to infinity (from whence it came, at $t = -\infty$). Note also that ϕ and all its derivatives, and in particular the fluid velocities, vanish as y tends to $+\infty$.

6. Let us put

$$\phi(\xi, x) = -2\xi \cos \xi^2 x f(\xi^2);$$

we can write (10) in the form

$$\eta(x, t) = \int_0^{+\infty} \cos \xi t \phi(\xi, x) d\xi.$$

From $\mathfrak{E}(c)$ and (d) it is readily seen that for every x

$$\xi^\alpha \phi(\xi, x) \varepsilon L, \text{ if } 0 \leq \alpha < 1,$$

$$\xi^\alpha \phi(\xi, x) \varepsilon L^2, \text{ if } 0 \leq \alpha < \frac{3}{2}.$$

Thus let us put

$$(11) \quad \theta(x, t) = \text{l.i.m.}_{A \rightarrow +\infty} \int_0^A \sin \xi t \xi \phi(\xi, x) d\xi.$$

Applying $\mathfrak{D}$, we have immediately: for every given x , one has for almost all t :

$$-\phi_{tt}(x, 0, t) = \eta_t(x, t) = -\theta(x, t).$$

Now, we have in D :

$$\begin{aligned} \phi_y(x, y, t) &= - \int_0^{+\infty} e^{-\lambda y} \sin \sqrt{\lambda} t \cos \lambda x \sqrt{\lambda} f(\lambda) d\lambda \\ &= \int_0^{+\infty} e^{-\xi^2 y} \sin \xi t \xi \phi(\xi, x) d\xi. \end{aligned}$$

Thus, due to $\mathfrak{E}$: for every given x , one has for almost all t

$$\lim_{y \rightarrow +0} \phi_y(x, y, t) = \theta(x, t).$$

But if for given x and t , $\lim_{y \rightarrow +0} \phi_y(x, y, t)$ exists, then $\phi_y(x, 0, t)$ also exists and is equal to the former; in fact, for given x, t , $\phi(x, y, t)$ is a continuous function of y on the closed interval $y \geq 0$ and has a derivative $\phi_y(x, y, t)$ on the open interval $y > 0$; by the mean value theorem

$$\frac{\phi(x, y, t) - \phi(x, 0, t)}{y} = \phi_y(x, c, t), \quad 0 < c < y.$$

If $\lim_{y \rightarrow +0} \phi_y(x, y, t)$ exists, the limit of the left-hand member also exists, and, by definition, is equal to $\phi_y(x, 0, t)$; thus

$$\phi_y(x, 0, t) = \lim_{y \rightarrow +0} \phi_y(x, y, t).$$

Putting everything together, we have the final result: For every given x , one has for almost all t

$$\eta_t(x, t) = -\phi_y(x, 0, t).$$

Thus (2) is satisfied for almost all t .

7. If we make the extra assumption $\lambda^{\frac{1}{2}} f(\lambda) \in L$, then the arguments of Section 6 are unnecessary, since then

$$\phi_y(x, y, t) = - \int_0^{+\infty} e^{-\lambda y} \sin \sqrt{\lambda} t \cos \lambda x \sqrt{\lambda} f(\lambda) d\lambda$$

is continuous in $\bar{D}$, and (2) is trivial. But as pointed out in $\mathfrak{E}(d)$, hypotheses H_1, H_2, H_3 imply only that $|\lambda|^\alpha f(\lambda) \in L$ for $0 \leq \alpha < \frac{1}{2}$; that $\alpha < \frac{1}{2}$ is the best possible is shown by the following example:

$$\begin{aligned} f(\lambda) &= \frac{1}{\lambda^{\frac{1}{2}} \log \lambda}, & \text{for } \lambda \geq 2, \\ &= 0, & \text{for } \lambda < 2. \end{aligned}$$

Clearly, $f(\lambda), \lambda f(\lambda)$ are in L^2 , and so are their Fourier-Plancherel transforms $F(x)$ and $F'(x)$, but $\lambda^{\frac{1}{2}} f(\lambda)$ is not in L .

Since the condition $\lambda^{\frac{1}{2}} f(\lambda) \in L$ does not bear directly on the original functions $F(x), F'(x)$ we find a condition which does.

Lemma: Let $g(\lambda) \in L(0, \infty)$ and $\int_x^\infty g(\lambda) d\lambda = O(x^{-2\alpha})$. Then if $\delta < 2\alpha$, $\lambda^\delta g(\lambda) \in L(0, \infty)$.

$$\begin{aligned} \text{Proof: } \int_x^R \lambda^\delta g(\lambda) d\lambda &= \lambda^\delta \int_\infty^\lambda g(t) dt \Big|_x^R - \delta \int_x^R \lambda^{\delta-1} \int_\infty^\lambda g(t) dt \cdot d\lambda \\ &= R^\delta \int_\infty^R g(\lambda) d\lambda - x^\delta \int_\infty^x g(\lambda) d\lambda - \delta \int_x^R \lambda^{\delta-1} O(\lambda^{-2\alpha}) d\lambda. \end{aligned}$$

Now Theorem 85 of [8] says that if $F'(x) \in L^2$ has Fourier transform $\lambda f(\lambda)$ then the following conditions are equivalent:

$$\begin{aligned} \text{A)} \quad & \int_{-\infty}^\infty |F'(x+h) - F'(x-h)|^2 dx = O(|h|^{-2\alpha}), \\ \text{B)} \quad & \left(\int_{-\infty}^{-x} + \int_x^\infty \right) [\lambda f(\lambda)]^2 d\lambda = O(|x|^{-2\alpha}). \end{aligned}$$

Hence we may add A) to our basic assumptions H_1-H_4 , deduce B), and obtain $\lambda^{1+\delta/2} f(\lambda) \in L^2$ by applying the lemma to $g(\lambda) = [\lambda f(\lambda)]^2$. But then the Schwarz inequality gives

$$\int_R^\infty |\lambda^{1/2} f(\lambda)| d\lambda \leq \left(\int_R^\infty [\lambda^{1+\delta/2} f(\lambda)]^2 d\lambda \right)^{1/2} \left(\int_R^\infty \lambda^{-(1+\delta)} d\lambda \right)^{1/2},$$

so that $\lambda^{1/2} f(\lambda) \in L$.

The lemma has an almost exact converse: if $\int_x^\infty \lambda^\delta g(\lambda) d\lambda$ is a convergent Lebesgue integral, then $\int_x^\infty g(\lambda) d\lambda = O(x^{-\delta})$. The condition A) fails to be equivalent to $\lambda^{1/2} f(\lambda) \in L$ by the discrepancy of the partial converse and the fact that we have actually shown that $\lambda^{1+\epsilon} f(\lambda) \in L$ for small $\epsilon > 0$, which is too much.

8. If γ is a region in the fluid and Γ is its boundary, then the kinetic energy in γ is given by

$$T_\gamma = \frac{1}{2} \iint_\gamma (\text{grad } \phi)^2 dS = -\frac{1}{2} \int_\Gamma \phi \phi_n ds.$$

Taking γ to be a rectangle with one side on the free surface, and verifying that as the rectangle grows the contribution of the submerged sides vanishes, we have

$$T = -\frac{1}{2} \int_{-\infty}^\infty \phi \phi_y dx.$$

The potential energy is

$$V = \frac{1}{2} \int_{-\infty}^\infty [\phi_l]^2 dx.$$

Using the definition of ϕ and the Parseval formula, we have

$$\begin{aligned}
 -\frac{1}{2} \int_{-\infty}^{\infty} \phi \phi_y dx &= \frac{1}{2} \int_{-\infty}^{\infty} [f(\lambda)]^2 \sin^2 \lambda t d\lambda \\
 \frac{1}{2} \int_{-\infty}^{\infty} [\phi_t]^2 dx &= \frac{1}{2} \int_{-\infty}^{\infty} [f(\lambda)]^2 \cos^2 \lambda t d\lambda.
 \end{aligned}$$

The sum is

$$\frac{1}{2} \int_{-\infty}^{\infty} [f(\lambda)]^2 d\lambda = \frac{1}{2} \int_{-\infty}^{\infty} [F(x)]^2 dx,$$

which is the energy at $t = 0$.

9. We now show that the solution $\phi(x, y, t)$ is unique within the class of velocity potentials having the following properties:

- a) $\phi_{tt} = \phi_y$ for $y \geq 0$, $t > 0$
- b) ϕ, ϕ_t harmonic for $y > 0$, $t \geq 0$
- c) $\phi_t(x, y, 0)$ bounded, $\phi_y(x, y, 0) \rightarrow 0$ as $y \rightarrow \infty$.

Suppose, by linearity, that

$$-\eta(x, 0) = \phi_t(x, 0, 0) = 0; \quad \phi(x, 0, 0) = 0.$$

Then it follows from [6] that $\phi_t(x, y, 0) = ky$, $\phi(x, y, 0) = k'y$, and from c) above, that

$$(12) \quad \phi_t(x, y, 0) = \phi(x, y, 0) = 0.$$

Now, for fixed x , (12) constitutes *analytic Cauchy data* for the heat equation a). The relevant uniqueness theory [7] tells us that for each x the unique solution of the heat equation is

$$\phi(x, y, t) \equiv 0.$$

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