Conservation Laws in Compressible Fluid Flow and Associated Mappings¹

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Introduction. A steady irrotational compressible fluid flow in the x - y -plane is governed by the system of partial differential equations

$$(1) \quad u_y - v_x = 0, \quad (\rho u)_x + (\rho v)_y = 0.$$

The density $\rho = \rho(u, v)$ is to be thought of as function of the velocity components u and v obtained by use of Bernoulli's law and the equation of state adopted. If the differentiations indicated by the indices in (1) are performed, one obtains a system of partial differential equations of the form

$$(2) \quad a_1^i u_x + a_2^i v_x + b_1^i u_y + b_2^i v_y = 0 \quad (i = 1, 2).$$

The coefficients a_k^i and b_k^i are functions of the velocity components, *i.e.*, the two dependent variables of the system.

For the physical discussion of the contents of our differential equations, it is of importance to note, that, in their original form (1), they represent *conservation laws*. By this we mean here any differential equation for u and v as functions of x and y which can be put into the form

$$(3) \quad (\xi(u, v))_x + (\eta(u, v))_y = 0$$

with suitable continuously differentiable functions $\xi(u, v)$ and $\eta(u, v)$. Equations (1) are obtained by setting $\xi = -v$, $\eta = u$ and $\xi = \rho u$, $\eta = \rho v$ respectively.

The contents of (3) can also be expressed by saying that

$$(4) \quad d\chi = -\eta(u, v) dx + \xi(u, v) dy$$

is an exact differential and by integration of $d\chi$ over the boundary of domains

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relations for the boundary values of u and v are obtained. The importance of conservation laws lies in this fact. It is, therefore, of great interest *to determine all conservation laws which are satisfied by all solutions of system (1)*. We know, *e.g.*, that the conservation laws of momentum

$$(5) \quad (\rho u^2 + p)_x + (\rho uv)_y = 0, \quad (\rho uv)_x + (\rho v^2 + p)_y = 0$$

with pressure p expressed in terms of u and v are consequences of system (1) in the formal sense that the left sides of the equations (5) are linear combinations of the left sides of equations (1) with coefficients depending on u and v alone. We will take up this question in Section 1 for a general system (2). We shall show that the conservation laws which are formal consequences of system (2) in the above described sense and which we shall call conservation laws of system (2) may be characterized by a system of linear partial differential equations for ξ and η as functions of u and v . The relation between the new system of differential equations and the original system (2) will be formulated in form of a duality principle.

Given a conservation law of (2), one may coordinate with each solution of (2) a (not necessarily single valued) function $\chi = \int -\eta dx + \xi dy$. If two conservation laws are considered simultaneously, one may interpret the corresponding pair of functions

$$(6) \quad \begin{aligned} \chi_1 &= \int -\eta_1 dx + \xi_1 dy \\ \chi_2 &= \int -\eta_2 dx + \xi_2 dy \end{aligned}$$

as representation of a mapping from the x - y -plane into a χ_1 - χ_2 -plane. In the case of conservation laws (1) the functions χ_1 and χ_2 coincide with the potential function φ and stream function ψ . One knows that important features of a flow are reflected in the behavior of the mapping into the φ - ψ -plane. We may expect to discover new characteristics of flows by leaving greater freedom in the choice of χ_1 and χ_2 . In Section 2 some general characteristics of mappings (6) will be discussed.

In Section 3 a concept of *strong ellipticity* of systems (2) and of mappings will be introduced. A concept of strong ellipticity is due to M. Lavrentiev [I]. Ellipticity in the ordinary sense represents a local property of differential equations, *strong ellipticity* refers to their behavior in *the large*. Our concept of strong ellipticity is less restrictive than that of Lavrentiev. By utilizing the strong ellipticity of some mappings connected with subsonic flows we derive in Section 4 several *inequalities* concerning the behavior of a flow on the boundary of the flow region. They all give an estimate of a mean value of a quantity dependent on the speed, *e.g.*, pressure along the boundary.

1. Conservation Laws of Systems of Partial Differential Equations. We consider a system of partial differential equations of form (2) with coefficients a_k^i and b_k^i which depend only on u and v . We assume the latter to be defined in an open region Δ of the u - v -plane, continuous there, and we assume further that the matrix of the coefficients

$$(7) \quad \begin{pmatrix} a_1^1 & a_2^1 & b_1^1 & b_2^1 \\ a_1^2 & a_2^2 & b_1^2 & b_2^2 \end{pmatrix}$$

has rank 2 everywhere in Δ . Now let (3) be a conservation law with continuously differentiable functions $\xi(u, v)$ and $\eta(u, v)$ in Δ which is a formal consequence of the system (2). In explicit form the conservation law is given by

$$(3') \quad \xi_u u_x + \xi_v v_x + \eta_u u_y + \eta_v v_y = 0$$

The assumption that (3') follows formally from (2) is equivalent to the assertion that the 3-row matrix

$$(8) \quad \begin{pmatrix} a_1^1 & a_2^1 & b_1^1 & b_2^1 \\ a_1^2 & a_2^2 & b_1^2 & b_2^2 \\ \xi_u & \xi_v & \eta_u & \eta_v \end{pmatrix}$$

has rank 2 everywhere in Δ . This is completely expressed by setting two of the 3-row determinants containing a non-vanishing 2-row determinant from the first two rows equal to zero. The resulting equations are linear and homogeneous in the derivatives of ξ and η , and we write them in the form

$$(9') \quad c_1^i \eta_v - c_2^i \eta_u - d_1^i \xi_v + d_2^i \xi_u = 0 \quad (i = 1, 2).$$

The coefficients c_k^i and d_k^i are again continuous functions of u and v in Δ and the matrix

$$(10) \quad \begin{pmatrix} c_1^1 & c_2^1 & d_1^1 & d_2^1 \\ c_1^2 & c_2^2 & d_1^2 & d_2^2 \end{pmatrix}$$

has again everywhere in Δ rank 2. *The solutions of (9') give us the totality of all conservation laws of system (2).*

There is a certain discrepancy between the character of the original equations (2) and equations (9') governing their conservation laws. The latter are linear, the former non-linear. One may achieve symmetry by the following procedure: As long as one is interested only in the formal relations between the two systems one may restrict the considerations to solutions of system (2) whose Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0$. The role of dependent and independent variables can then, at

least in the small, be interchanged and equations (2) become

$$(2') \quad a_1^i y_v - a_2^i y_u - b_1^i x_v + b_2^i x_u = 0 \quad (i = 1, 2).$$

Systems (2') and (9') now have the same form. In a similar way symmetry can be achieved by admitting only solutions of (9') whose Jacobian $\begin{vmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{vmatrix} \neq 0$. Again converting the roles of the dependent and independent variables transforms (9') into system

$$(9) \quad c_1^i u_\xi + c_2^i v_\xi + d_1^i u_\eta + d_2^i v_\eta = 0 \quad (i = 1, 2)$$

which completely corresponds in form to (2). We will call the system (9) (or (9')) *dual to the system (2) (or (2')) respectively*.

By using the expression "dual" we anticipate that duality represents a *symmetric relation* between two systems of partial differential equations. We will prove it shortly. But let us first compute the dual system of system (1). As in all applications ρ is considered as a function of the speed $q = \sqrt{u^2 + v^2}$ alone, and we may write it as a function of $\frac{1}{2}q^2$: $\rho = \rho(\frac{1}{2}q^2)$. Its derivative with respect to $\frac{1}{2}q^2$ will be denoted by $\dot{\rho}$. Matrix (8) becomes here

$$(8') \quad \begin{pmatrix} 0 & -1 & 1 & 0 \\ \rho + \dot{\rho}u^2 & \dot{\rho}uv & \dot{\rho}uv & \rho + \dot{\rho}v^2 \\ \xi_u & \xi_v & \eta_v & \eta_u \end{pmatrix}.$$

One verifies easily that the system governing the conservation laws of (1) is given by

$$(10) \quad \begin{aligned} (\rho + \dot{\rho}v^2)\xi_u - (\rho + \dot{\rho}u^2)\eta_v &= 0 \\ (\rho + \dot{\rho}v^2)(\xi_v + \eta_u) - 2\dot{\rho}uv\eta_v &= 0. \end{aligned}$$

As a second example, let us consider axially symmetric incompressible steady flows in space. If u and v represent Cartesian coordinates in a meridian plane, the u -axis being the symmetry axis, the potential function $\varphi(u, v)$ and the corresponding stream function satisfy the equations

$$(11) \quad v\varphi_u = \psi_v, \quad v\varphi_v = -\psi_u.$$

The corresponding dual system is, as one computes easily,

$$(12) \quad v\xi_u = -\eta_v, \quad v\xi_v = \eta_u.$$

One obtains (12) from (11) by replacing φ and ψ by ξ and $(-\eta)$ respectively.

We now return to the general system and prove the already announced

Theorem 1. *Duality between the systems (2) and (9) (or, what is the same, between (2') and (9')) represents a symmetric relation.*

In order to put the proof into a simple form we first introduce a geometrical language suggested by the nature of the relations involved. It will prove useful also for other purposes. We will operate with 2-2-matrices

$$(13) \quad \gamma = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

which we may interpret as vectors in a 4-dimensional linear vector space M . If we do this we will call γ an M -vector. Into M we introduce an indefinite *Eucclidean metric* by defining a norm $N(\gamma)$ of γ according to the formula

$$(14) \quad N(\gamma) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

We define further the scalar product (γ^1, γ^2) of the two M -vectors

$$(15) \quad \gamma^1 = \begin{pmatrix} a_1^1 & a_2^1 \\ b_1^1 & b_2^1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} a_1^2 & a_2^2 \\ b_1^2 & b_2^2 \end{pmatrix}$$

by the corresponding polar form

$$(16) \quad (\gamma^1, \gamma^2) = \frac{1}{2} \left\{ \begin{vmatrix} a_1^1 & a_2^1 \\ b_1^2 & b_2^2 \end{vmatrix} + \begin{vmatrix} a_1^2 & a_2^2 \\ b_1^1 & b_2^1 \end{vmatrix} \right\}.$$

We have then $N(\gamma) = (\gamma, \gamma)$.

The M -vectors γ^1 and γ^2 will be called orthogonal if $(\gamma^1, \gamma^2) = 0$.

Quadratic form (14) defining the norm of γ has two positive and two negative dimensions. This is immediately seen if the new coordinates in M

$$(17) \quad c_1 = \frac{a_1 + b_2}{2}, \quad c_2 = \frac{a_2 + b_1}{2}, \quad c_3 = \frac{a_1 - b_2}{2}, \quad c_4 = \frac{a_2 - b_1}{2}$$

are introduced. We obtain then

$$(14') \quad N(\gamma) = c_1^2 + c_2^2 - c_3^2 - c_4^2.$$

We now rewrite system (2') in terms of our language. We form the M -vectors (15) from the coefficients of (2') and interpret also the Jacobian matrix

$$(18) \quad \omega = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

as an element of M . Equations (2') can then be written in the form

$$(19) \quad (\gamma^i, \omega) = 0 \quad (i = 1, 2)$$

expressing orthogonality of ω to γ^1 and γ^2 at each point (u, v) .

Correspondingly we introduce the M -vectors

$$(20) \quad \delta^i = \begin{pmatrix} c_1^i & c_2^i \\ d_1^i & d_2^i \end{pmatrix} \quad (i = 1, 2)$$

formed from the coefficients of the dual system (9') and the Jacobian matrix

$$(21) \quad \zeta = \begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix}.$$

Equations (9') are then representable in the form

$$(22) \quad (\delta^i, \zeta) = 0 \quad (i = 1, 2).$$

Finally, conservation law (3') reads after interchanging the role of the dependent and independent variables as follows:

$$(23) \quad (\omega, \zeta) = 0.$$

From equations (19), (22), and (23) one immediately derives the fact that the two-dimensional vector manifolds spanned by γ^1 and γ^2 or δ^1 and δ^2 are *orthogonal to each other in M* . This proves the duality theorem. It is reflected in the duality between orthogonal two-dimensional vector-manifolds in M .

We prove now a series of further simple theorems all showing how important features of the systems of partial differential equations considered here can be expressed in our geometrical language.

Let us start with the distinction between elliptic, hyperbolic, and parabolic systems. The system (2') is elliptic, hyperbolic, or parabolic at a point (u, v) according as whether the quadratic form

$$(24) \quad \begin{vmatrix} a_1^1 s_2 - a_2^1 s_1, & b_2^1 s_1 - b_1^1 s_2 \\ a_1^2 s_2 - a_2^2 s_1, & b_2^2 s_1 - b_1^2 s_2 \end{vmatrix} = A s_1^2 + 2B s_1 s_2 + C s_2^2$$

in s_1 and s_2 is definite, indefinite or semidefinite¹ there. A simple computation shows that the discriminant of (24)

$$(25) \quad AC - B^2 = \begin{vmatrix} (\gamma^1 \gamma^1) & (\gamma^1 \gamma^2) \\ (\gamma^2 \gamma^1) & (\gamma^2 \gamma^2) \end{vmatrix}.$$

The two dimensional manifold in M spanned by γ^1 and γ^2 consists of the linear combinations

$$(26) \quad \gamma = t_1 \gamma^1 + t_2 \gamma^2$$

¹ "Semidefinite" should always include here the case of an identically vanishing form.

and the norm of γ

$$(27) \quad N(\gamma) = (\gamma^1, \gamma^1)t_1^2 + 2(\gamma^1, \gamma^2)t_1t_2 + (\gamma^2, \gamma^2)t_2^2$$

represents a quadratic form in the coefficients t_1 and t_2 . We now introduce the

Definition 1: *The manifold (26) will be called definite or semidefinite according as form (27) is definite, indefinite, or semidefinite. In case of definiteness we shall speak of positive or negative definite manifolds according to the corresponding behavior of form (27).*

We now remark that the discriminant of (27) coincides with that of (24) and we have, therefore,

Theorem 2: *The system (2') is at a point (u, v) elliptic, hyperbolic, or parabolic according as the manifold (26) is definite, indefinite, or semidefinite.*

We will call two systems of partial differential equations of form (2') of the same type in a domain Δ of the u - v -plane if they are at each point (u, v) of Δ both either elliptic, hyperbolic, or parabolic. We have then

Theorem 3: *The dual systems (2') and (9') are of the same type.*

Indeed, by using Sylvester's inertia theorem for quadratic forms one sees easily that two two-dimensional *orthogonal* linear manifolds are both definite, indefinite, or semidefinite. In case of definiteness the signs of the corresponding quadratic forms are opposite.

It is often useful to transform equations (2') into a new form by subjecting the variables u and v or the variables x and y to transformations which do not destroy the character of the differential equations. The transformations in question are of one of the following types:

$$(28a) \quad \begin{aligned} \bar{u} &= \bar{u}(u, v) \\ \bar{v} &= \bar{v}(u, v) \end{aligned} \quad \left(\begin{vmatrix} \bar{u}_u & \bar{u}_v \\ \bar{v}_u & \bar{v}_v \end{vmatrix} \neq 0 \right),$$

i.e., an arbitrary continuously differentiable transformation of u and v with a non-vanishing Jacobian; or

$$(28b) \quad \begin{aligned} x &= b_{11} \bar{x} + b_{12} \bar{y} + b_1 \\ y &= b_{21} \bar{x} + b_{22} \bar{y} + b_2 \end{aligned} \quad \left(\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0 \right)$$

with a coefficient matrix depending upon u and v ,

$$(29) \quad \begin{pmatrix} b_{11} & b_{12} & b_1 \\ b_{21} & b_{22} & b_2 \end{pmatrix},$$

whose columns are solutions of equations (2').

In order to describe the transformations of the differential equations in an invariant way we need the knowledge of all "similitudes" of the geometry of M . By similitude we mean a linear vector transformation of M which does not

destroy orthogonality in M . Transformations of this type can easily be constructed. Take two elements α and β of M that considered as matrices have non-vanishing determinants, and a variable element σ . Then

$$(30') \quad \bar{\sigma} = \alpha\sigma\beta$$

is a similitude. Indeed

$$(31') \quad N(\bar{\sigma}) = N(\alpha)N(\beta)N(\sigma),$$

and hence, for any pair of elements σ^1 and σ^2

$$(31'') \quad (\bar{\sigma}^1, \bar{\sigma}^2) = N(\alpha)N(\beta)(\sigma^1, \sigma^2).$$

This shows that (30') represent similitudes. Consider further the transformations

$$(30'') \quad \bar{\sigma} = \alpha\sigma'\beta$$

where σ' is the transposed matrix of σ . Since $N(\sigma') = N(\alpha)N(\beta)N(\sigma)$, equations (31') and (31'') hold also for the transformations (30''). One can easily show that *formulas (30') and (30'') represent the totality of similitudes*. Formula (30') gives the proper, (30'') the improper transformations (transformations with positive or negative determinant in four dimensions).

We now subject system (2') to a transformation (28a). We have

$$(32') \quad \begin{pmatrix} x_{\bar{u}} & x_{\bar{v}} \\ y_{\bar{u}} & y_{\bar{v}} \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_{\bar{u}} & u_{\bar{v}} \\ v_{\bar{u}} & v_{\bar{v}} \end{pmatrix},$$

or if we set

$$(33a) \quad \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \omega, \quad \begin{pmatrix} x_{\bar{u}} & x_{\bar{v}} \\ y_{\bar{u}} & y_{\bar{v}} \end{pmatrix} = \bar{\omega}, \quad \begin{pmatrix} u_{\bar{u}} & u_{\bar{v}} \\ v_{\bar{u}} & v_{\bar{v}} \end{pmatrix} = \beta,$$

we may write (32') in the form

$$(32a) \quad \bar{\omega} = \omega\beta.$$

Transformation (32a) represents for each point (u, v) an associated similitude transforming ω into $\bar{\omega}$. Write now system (2') in the form (19) and subject γ^1 and γ^2 to the transformation (32a):

$$(34a) \quad \bar{\gamma}^i = \gamma^i\beta \quad (i = 1, 2).$$

We obtain then immediately

Theorem 4a: *If a transformation (28a) is applied to the system $(\gamma^i, \omega) = 0$ ($i = 1, 2$) the system $(\bar{\gamma}^i, \bar{\omega}) = 0$ ($i = 1, 2$) results, the $\bar{\gamma}^i$ being defined by (34a).*

We now apply a transformation (28b). One sees then immediately with the

abbreviations

$$(33b) \quad \bar{\omega} = \begin{pmatrix} \bar{x}_u & \bar{x}_v \\ \bar{y}_u & \bar{y}_v \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1}$$

that now modulo terms of lower order than one

$$(32b) \quad \bar{\omega} \equiv \alpha \omega$$

holds, which again represents for each (u, v) a similitude. If one sets now

$$(34b) \quad \bar{\gamma}^i = \alpha \gamma^i$$

one obtains in a similar way as in the previous case

Theorem 4b: *If a transformation (28b) is applied to the system $(\gamma^i, \omega) = 0$ ($i = 1, 2$) the system $(\bar{\gamma}^i \bar{\omega}) = 0$ ($i = 1, 2$) results, the $\bar{\gamma}^i$ being defined by (34b).*

The transformation properties of systems of partial differential equations discussed above and expressed in the last two theorems may be utilized to associate with each system of partial differential equation $(\gamma^i, \omega) = 0$ ($i = 1, 2$) a quadratic differential form, determined modulo a non-vanishing factor which may depend on u and v , which is invariantly connected with the system under the transformations (28a) and (28b).¹ One may say that with each such system a conformal geometry may be associated which is invariantly connected with the system under the transformation (28). We coordinate first with each line element (du, dv) the M -vectors

$$(33') \quad \kappa^1 = \begin{pmatrix} -dv & du \\ 0 & 0 \end{pmatrix}, \quad \kappa^2 = \begin{pmatrix} 0 & 0 \\ -dv & du \end{pmatrix}.$$

Under the transformations (28a) these vectors have, modulo a non-vanishing factor, the transformation character of ω as expressed as equation (32a). Indeed, one computes immediately that

$$(35) \quad \bar{\kappa}^1 = \begin{pmatrix} -d\bar{v} & d\bar{u} \\ 0 & 0 \end{pmatrix}, \quad \bar{\kappa}^2 = \begin{pmatrix} 0 & 0 \\ -d\bar{v} & d\bar{u} \end{pmatrix}$$

are obtained from κ^1 and κ^2 by

$$(36) \quad \bar{\kappa}^i = \kappa^i \cdot \beta \cdot \begin{vmatrix} \bar{u}_u & \bar{u}_v \\ \bar{v}_u & \bar{v}_v \end{vmatrix} \quad (i = 1, 2).$$

Now we form the quadratic differential form

$$(37) \quad 4F = \begin{vmatrix} (\gamma^1, \kappa^1) & (\gamma^1, \kappa^2) \\ (\gamma^2, \kappa^1) & (\gamma^2, \kappa^2) \end{vmatrix} = a(u, v) du^2 + 2b(u, v) du dv + c(u, v) dv^2.$$

¹ In the special case of equations (1) this form was introduced by BERS [II].

Its coefficients are

$$(38) \quad a = \begin{vmatrix} a_1^1 & b_1^1 \\ a_1^2 & b_1^2 \end{vmatrix}, \quad b = \frac{1}{2} \left\{ \begin{vmatrix} a_1^1 & b_1^1 \\ a_1^2 & b_2^2 \end{vmatrix} + \begin{vmatrix} a_2^1 & b_1^1 \\ a_2^2 & b_1^2 \end{vmatrix} \right\}, \quad c = \begin{vmatrix} a_2^1 & b_2^1 \\ a_2^2 & b_2^2 \end{vmatrix}.$$

F may also be written in the form of a four-row determinant

$$(37') \quad F = - \begin{vmatrix} a_1^1 & a_2^1 & b_1^1 & b_2^1 \\ a_1^2 & a_2^2 & b_1^2 & b_2^2 \\ -dv & du & 0 & 0 \\ 0 & 0 & -dv & du \end{vmatrix}.$$

In this form $-F$ may be geometrically interpreted as volume of the four-dimensional parallelepiped spanned by the four M -vectors $\gamma^1, \gamma^2, \kappa^1, \kappa^2$.

The invariant character of F under the transformations (28a) is an immediate consequence of (34a) and (36). If transformations (28b) are applied one obtains on account of the matrix relations

$$(38) \quad \alpha\kappa^1 = a_{11}\kappa^1 + a_{21}\kappa^2, \quad \alpha\kappa^2 = a_{12}\kappa^1 + a_{22}\kappa^2$$

$$(39) \quad \begin{vmatrix} (\alpha\gamma^1, \alpha\kappa^1) & (\alpha\gamma^1, \alpha\kappa^2) \\ (\alpha\gamma^2, \alpha\kappa^1) & (\alpha\gamma^2, \alpha\kappa^2) \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} (\gamma^1, \kappa^1) & (\gamma^1, \kappa^2) \\ (\gamma^2, \kappa^1) & (\gamma^2, \kappa^2) \end{vmatrix}.$$

On the other hand

$$(40) \quad (\alpha\gamma^i, \alpha\kappa^k) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} (\gamma^i, \kappa^k).$$

Equations (39) and (40) give

$$(41) \quad 4\bar{F} = \begin{vmatrix} (\bar{\gamma}^1, \kappa^1) & (\bar{\gamma}^1, \kappa^2) \\ (\bar{\gamma}^2, \kappa^1) & (\bar{\gamma}^2, \kappa^2) \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot 4F,$$

which proves the invariance under transformations (28b).

We remark that form $(-F)$ is obtained from form (24) by replacing in the latter the variables s_1 and s_2 by dv and $(-du)$ respectively. This leads to

Theorem 5: *The system (19) is a point (u, v) elliptic, hyperbolic, or parabolic according as the associated form F is at the point definite, indefinite, or semidefinite.*

We further prove

Theorem 6: *Two dual systems lead to the same differential form (modulo a non-vanishing factor).*

Indeed, from the equations expressing orthogonality of each δ^i with each γ^k one concludes that the coefficients of (37') and those of the corresponding form built from the dual system are proportional.

We need in the following section a formula relating the differential form as-

sociated with the system with a pair of known solutions of the system. In order to express it in a simple way we first remark that if $(\xi_1(u, v); \eta_1(u, v))$ and $(\xi_2(u, v); \eta_2(u, v))$ are two solutions of the dual system, the Jacobian matrices

$$(42) \quad \zeta^1 = \begin{pmatrix} (\xi_1)_u & (\xi_1)_v \\ (\eta_1)_u & (\eta_1)_v \end{pmatrix}, \quad \zeta^2 = \begin{pmatrix} (\xi_2)_u & (\xi_2)_v \\ (\eta_2)_u & (\eta_2)_v \end{pmatrix}$$

represent two M -vectors falling into the manifold spanned by γ^1 and γ^2 . Hence they can be written as linear combinations

$$(43) \quad \zeta^1 = l_1^1 \gamma^1 + l_2^1 \gamma^2, \quad \zeta^2 = l_1^2 \gamma^1 + l_2^2 \gamma^2.$$

The relation sought is expressed in

Theorem 7: *The determinant of the differentials*

$$(44) \quad \begin{vmatrix} d\xi_1 & d\eta_1 \\ d\xi_2 & d\eta_2 \end{vmatrix} = lF,$$

with F given by (37) and a proportionality factor

$$(45) \quad l = 4 \begin{vmatrix} l_1^1 & l_2^1 \\ l_1^2 & l_2^2 \end{vmatrix}.$$

Proof. One may write

$$(46) \quad d\xi_1 = (\zeta^1 \kappa^2), \quad d\eta_1 = -(\zeta^1 \kappa^1), \quad d\xi_2 = (\zeta^2 \kappa^2), \quad d\eta_2 = -(\zeta^2 \kappa^1).$$

Substitution of (43) into (46) and forming the determinant (44) verifies the identity.

Remark: The proportionality factor l vanishes where the Jacobian matrices ζ^1 and ζ^2 are linearly dependent.

2. Mappings Associated with a Pair of Conservation Laws. Let us consider a (not necessarily single valued) mapping

$$(47) \quad f = f(x, y), \quad g = g(x, y)$$

of an open domain D of the x - y -plane into the f - g -plane. We assume that f and g are twice continuously differentiable in D and their derivatives single-valued. We form the Jacobian matrix

$$(48) \quad \tau = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

and its differential

$$(49) \quad d\tau = \begin{pmatrix} df_x & df_y \\ dg_x & dg_y \end{pmatrix} = \begin{pmatrix} f_{xx} dx + f_{xy} dy, f_{yx} dx + f_{yy} dy \\ g_{xx} dx + g_{xy} dy, g_{yx} dx + g_{yy} dy \end{pmatrix}.$$

Equation (48) may be considered as an expression of a single-valued mapping of D into the four-dimensional vector-space M . In general, τ will describe a two-dimensional surface S in M . This is definitely the case if we restrict the consideration to a neighborhood of a point (x_0, y_0) where among the elements of the matrix $d\tau$ there are two linearly independent differentials. In this case S may be represented by a pair of equations

$$(50) \quad F(f_x, f_y, g_x, g_y) = 0, \quad G(f_x, f_y, g_x, g_y) = 0.$$

The functions F and G are continuously differentiable functions of their four arguments in the neighborhood of the point τ_0 of M corresponding to (x_0, y_0) and their differentials independent there. We may say that f and g satisfy a well defined system of partial differential equations of first order not containing explicitly the dependent and also the independent variables. These equations may be at $\tau = \tau_0$ elliptic, hyperbolic, or parabolic. The decision depends only on the tangent plane of S at $\tau = \tau_0$, *i.e.*, on the approximating system of linear equations. Using Theorem 2 of Section 1, the quadratic form representing the norm of a variable tangent element at $\tau = \tau_0$

$$(51) \quad N(d\tau) = \begin{vmatrix} df_x & df_y \\ dg_x & dg_y \end{vmatrix} = \begin{vmatrix} f_{xx}dx + f_{xy}dy, f_{yx}dx + f_{yy}dy \\ g_{xx}dx + g_{xy}dy, g_{yx}dx + g_{yy}dy \end{vmatrix}$$

gives the answer to our question: The system (50) is at $\tau = \tau_0$ *elliptic, hyperbolic, or parabolic according as the quadratic differential form (51) is at (x_0, y_0) definite, indefinite, or semidefinite.*

This result leads in a natural way to

Definition 2: Mapping (47) will be called *elliptic, hyperbolic, or parabolic at a point (x, y) of D according as the quadratic differential form (51) is there definite, indefinite, or semidefinite.*

We derive now several simple lemmas regarding the character of mappings obtained from given ones by some simple processes. In order to express them in a short form we introduce

Definition 3: A mapping σ_1 will be called of the same type at a point (x_1, y_1) as a mapping σ_2 at a point (x_2, y_2) if σ_1 and σ_2 are at these points both elliptic or hyperbolic or both parabolic.

Lemma 1: A mapping obtained from (47) by subjecting the x - y -plane or the f - g -plane to an affine transformation are in corresponding points of the same type.

A proof is immediately obtained from the fact that in the transition to the new mapping the form (51) is multiplied by a non-vanishing constant factor.

Lemma 2: Suppose that the Jacobian $\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \neq 0$ in D . Then the inverse mapping is in the image point of (x, y) of the same type as the original mapping at (x, y) .

Proof. Call the Jacobian matrix of the inverse mapping $\bar{\tau}$. Then

$$d\tau\bar{\tau} = \tau d\bar{\tau} + d\tau \cdot \bar{\tau} = 0,$$

or

$$d\bar{\tau} \cdot \bar{\tau}^{-1} = -\tau^{-1} d\tau,$$

and hence

$$(52') \quad \frac{\det(d\bar{\tau})}{\det(\bar{\tau})} = \frac{\det(d\tau)}{\det(\tau)}$$

or

$$(52) \quad \det(d\bar{\tau}) = \frac{\det(d\tau)}{(\det \tau)^2},$$

which proves the lemma.

Having fluid dynamical applications in mind we are in particular interested in mappings (6) originating from two conservation laws of a given system (2') of partial differential equations. We assume that we know two conservation laws

$$\frac{\partial \xi_i(u, v)}{\partial x} + \frac{\partial \eta_i(u, v)}{\partial y} = 0 \quad (i = 1, 2),$$

the functions ξ_i and η_i being defined in the same domain Δ of the u - v -plane. We shall speak of the conservation laws (ξ_i, η_i) . To each solution of (2) defined in a domain D of the x - y -plane leading to a mapping of D into Δ we can coordinate a mapping (6). We shall call it a mapping associated with the pair of conservation laws (ξ_i, η_i) ($i = 1, 2$). We shall study its character. Since

$$(\chi_1)_x = -\eta_1, \quad (\chi_1)_y = \xi_1, \quad (\chi_2)_x = -\eta_2, \quad (\chi_2)_y = \xi_2,$$

the determinant

$$\begin{vmatrix} (d\chi_1)_x & (d\chi_1)_y \\ (d\chi_2)_x & (d\chi_2)_y \end{vmatrix} = \begin{vmatrix} d\xi_1 & d\eta_1 \\ d\xi_2 & d\eta_2 \end{vmatrix},$$

and we obtain

Lemma 3: *The mapping (6) is a point (x, y) of D elliptic, hyperbolic, or parabolic according as the differential form*

$$(53) \quad \begin{vmatrix} d\xi_1 & d\eta_1 \\ d\xi_2 & d\eta_2 \end{vmatrix}$$

is at the corresponding point (u, v) of Δ definite, indefinite, or semidefinite. Only at points where the Jacobian of the mapping into the u - v -plane vanishes does (53) become semidefinite.

Combining this result with Theorem 7 we obtain

Theorem 8: *If we assume that the Jacobian matrices ζ^1 and ζ^2 of the pairs of functions (ξ_i, η_i) ($i = 1, 2$) are at each point of Δ linearly independent then our mapping is at a point (x, y) of D elliptic, hyperbolic, or parabolic according as the differential form F coordinated with the system (2) is definite, indefinite, or semidefinite at the corresponding point (u, v) . A similar degeneration into semidefiniteness may take place as in Lemma 3.*

We will apply the last theorem in the next section to the hydrodynamical system (1) and we will therefore investigate here under which conditions the assumption of the Theorem 8 is satisfied. We will consider only the case where one of the conservation laws is the conservation law of mass. Let us first compute the form F associated with system (1). According to (37')

$$(54) \quad -F = \begin{vmatrix} 0 & -1 & 1 & 0 \\ \rho + \dot{\rho}u^2 & \dot{\rho}uv & \dot{\rho}uv & \rho + \dot{\rho}v^2 \\ -dv & du & 0 & 0 \\ 0 & 0 & -dv & du \end{vmatrix} = (\rho + \dot{\rho}u^2) du^2 + 2\dot{\rho}uv du dv + (\rho + \dot{\rho}v^2) dv^2.$$

We have further to compute

$$\begin{vmatrix} d\rho u & d\rho v \\ d\xi & d\eta \end{vmatrix} = \begin{vmatrix} (\rho + \dot{\rho}u^2) du + \dot{\rho}uv dv, & \dot{\rho}uv du + (\rho + \dot{\rho}v^2) dv \\ \xi_u du + \xi_v dv, & \eta_u du + \eta_v dv \end{vmatrix}.$$

By using equations (10) one easily brings it into the form

$$(55) \quad \begin{vmatrix} d\rho u & d\rho v \\ d\xi & d\eta \end{vmatrix} = \frac{\eta_u - \xi_v}{2} \cdot (-F).$$

We have, therefore,

Lemma 4: *The assumption of Theorem 8 is satisfied if*

$$(56) \quad \frac{\eta_u - \xi_v}{2} \neq 0$$

in the whole domain Δ .

The verification of inequality (56) is facilitated if system (10) governing the conservation laws is brought into a more suitable form. The quantities u and v being components of a vector, the equations (1) are, separately, invariant under motions in the x - y -plane. If a conservation law $\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} = 0$ is subjected to such a transformation, ξ and η are transformed also like vector-components. Hence,

if ξ and η are written in the form

$$(57') \quad \xi = \lambda(-v) + \mu\rho u, \quad \eta = \lambda u + \mu\rho v$$

expressing the vector (ξ, η) as linear combination of the vector $(-v, u)$ and the vector $(\rho u, \rho v)$, the coefficients λ and μ will transform as *scalars*, and we have to expect them to satisfy simpler equations than do ξ and η . We introduce further polar coordinates q and θ in the u - v -plane so that $v = q \sin \theta$, $u = q \cos \theta$. A simple computation which may be left to the reader leads then to the equations

$$(57) \quad \mu_q = \frac{\rho + \dot{\rho}q^2}{\rho^2 q} \lambda_\theta, \quad \mu_\theta = -\frac{q}{\rho} \lambda_q.$$

One computes further from (57') by use of the last equations

$$\frac{\eta_u - \xi_v}{2} = (\lambda q)_q.$$

We have, therefore, in polar coordinates,

$$(55') \quad \left| \begin{array}{cc} d\rho u & d\rho v \\ d\xi & d\eta \end{array} \right| = (\lambda q)_q \{(\rho + \dot{\rho}q^2) dq^2 + \rho q^2 d\theta^2\}.$$

the bracket expression representing $-F$ in polar coordinates. We formulate explicitly

Lemma 4: *The assumption of Theorem 8 is satisfied if*

$$(56') \quad (\lambda q)_q \neq 0$$

in the whole domain Δ .

The dependence of ρ on q in all fluid dynamical applications is such that ρ decreases monotonically in its domain of definition $0 \leq q \leq q_*$ from a positive value ρ_0 to zero whereas ρq first increases until it reaches its maximum at a value $q = q_*$ and then decreases monotonically. For values of q less than q_* , the sound velocity, the form $-F$ is positive definite, in the ring domain $q_* < q < q_*$ it is indefinite and, finally, on the sonic circle $q = q_*$ semidefinite. This corresponds to the well known ellipticity of system (1) in the subsonic region, hyperbolicity in the supersonic region and parabolicity on the sonic circle.

We now compute particular solutions of system (57) and determine regions where the condition of Lemma 4' is satisfied. These solutions will then be utilized in the last section.

We consider solutions for which λ and μ can be written as products of functions of q alone and θ alone:

$$(58) \quad \lambda = A_1(q)B_1(\theta), \quad \mu = A_2(q)B_2(\theta).$$

Substitution into (57) gives

$$(59) \quad \begin{aligned} A_2' B_2 &= \frac{(\rho q)'}{\rho^2 q} A_1 B_1' \\ A_2 B_2' &= -\frac{q}{\rho} A_1' B_1. \end{aligned}$$

From (59) we conclude by separation of variables that B_1 and B_2 are either trigonometric or hyperbolic or linear functions. Restricting ourselves to the first case we may set, after suitable rotation of the coordinate system,

$$(60) \quad B_1 = \cos \beta \theta, \quad B_2 = \sin \beta \theta \quad (\beta > 0),$$

and one obtains for A_1 and A_2 the system of ordinary differential equations

$$(61) \quad A_1' = -\beta \frac{\rho}{q} A_2, \quad A_2' = -\beta \frac{(\rho q)'}{\rho^2 q} A_1.$$

For the sake of simplicity we restrict ourselves in the following discussions of equations (61) to functions ρ analytic in $\frac{1}{2}q^2$ and we may assume that

$$(61') \quad \rho(0) = 1.$$

The equations (61) are then at $q = 0$ of the Fuchsian type, and the characteristic equation is $\lambda^2 = \beta^2$. There exists, therefore, a solution defined in the whole region $q < q_e$ which allows an expansion

$$(62') \quad A_1 = a_1 q^\beta P_1(q), \quad A_2 = a_2 q^\beta P_2(q)$$

with power series $P_1(q)$ and $P_2(q)$ such that

$$(62'') \quad P_1(0) = P_2(0) = 1$$

and constants a_1 and a_2 different from zero. The solution corresponding to $\lambda = -\beta$ we shall not consider here.

By comparing corresponding coefficients after substituting (62') into (61) one finds $a_1 = -a_2$, and one may set $a_1 = 1$ so that

$$(62) \quad A_1 = q^\beta P_1(q), \quad A_2 = -q^\beta P_2(q).$$

We are interested in regions where

$$(65) \quad (\lambda q)_q = (A_1 q)' \cos \beta \theta \neq 0.$$

By restricting ourselves to the sector

$$(66) \quad -\frac{\pi}{2\beta} < \theta < \frac{\pi}{2\beta}$$

we insure that $\cos \beta\theta > 0$ and we have to study only the behavior of $(A_1 q)'$. But by equations (61) we compute

$$(A_1 q)' = A_1' q + A_1 = A_1 - \beta \rho A_2.$$

Setting

$$(67) \quad A_1 - \beta \rho A_2 = U$$

we compute further

$$\begin{aligned} U' &= A_1' - \beta \rho A_2' - \beta \dot{\rho} q A_2 \\ &= -\beta \frac{\rho}{q} A_2 + \beta^2 \frac{(\rho q)'}{\rho q} A_1 - \beta \dot{\rho} q A_2 \\ &= \beta^2 \frac{(\rho q)'}{\rho q} A_1 - \beta \frac{(\rho q)'}{q} A_2 \\ &= \beta \frac{(\rho q)'}{\rho q} (\beta A_1 - \rho A_2). \end{aligned}$$

Setting

$$(67') \quad \beta A_1 - \rho A_2 = V,$$

we compute finally

$$\begin{aligned} V' &= \beta A_1' - \rho A_2' - \dot{\rho} q A_2 \\ &= -\beta^2 \frac{\rho}{q} A_2 + \beta \frac{(\rho q)'}{\rho q} A_1 - \dot{\rho} q A_2 \\ &= \beta \frac{(\rho q)'}{\rho q} A_1 - \frac{\beta^2 \rho + \dot{\rho} q^2}{q} A_2. \end{aligned}$$

We consider first the general case $\beta \neq 1$. The expression for V' can then be put into the form $\beta U/q + \dot{\rho} q V/\rho$, and we have for U and V the system of differential equations

$$(68) \quad U' = \beta \frac{(\rho q)'}{\rho q} V, \quad V' = \frac{\beta}{q} U + \frac{\dot{\rho} q}{\rho} V$$

from which all further conclusions will be drawn. We *restrict* ourselves from now on to the subsonic region $0 \leq q \leq q_s$, and we shall show that U increases monotonically in this interval from $U(0) = 0$ on and is therefore *positive* there. From (62) and (67') we conclude first that $U' > 0$ for sufficiently small positive q . If our assertion were false there would exist a smallest value $q = q' < q_s$ where $U' = 0$. On account of the first equation (68) also $V = 0$ at $q = q'$. The second

equation (68) would give then $V' = \beta U/q > 0$ for $q = q'$. But since $(\rho q)' > 0$ in the subsonic region we see from the first equation (68) that for $q < q'$ not only U' but also $V > 0$. This would give $V'(q') \leq 0$, which contradicts the previously derived inequality $V' > 0$ at $q = q'$.

In case $\beta = 1$ we have $U = V$, and the first equation (68) gives

$$U = \rho q$$

increasing in the subsonic region. We finally arrive at

Theorem 9: *For the solution*

$$\lambda = A_1 \cos \beta \theta, \quad \mu = A_2 \sin \beta \theta$$

with A_1 and A_2 given by (62) the condition of Lemma 4' is satisfied in the sector

$$-\frac{\pi}{2\beta} < \theta < \frac{\pi}{2\beta}, \quad 0 < q < q_*$$

The differential form (55') is positive definite there.

Remark: The case $\beta = 1$ corresponds, as one easily sees, to the second of the conservation laws of momentum (5).

3. The Concept of Strong Ellipticity. If the mapping (47) is *elliptic* in all points of a domain D of the x - y -plane the quadratic form (51) is *definite* everywhere in D . Since its coefficients are continuous it cannot change sign, and we may assume that it is positive definite everywhere. From the positiveness of (51) we conclude that for a pair of different points which lie in a sufficiently small neighborhood of a point of D the determinant

$$(69) \quad \begin{vmatrix} \Delta f_x & \Delta f_y \\ \Delta g_x & \Delta g_y \end{vmatrix}$$

of the increments of the derivatives is also positive. This will, in general, not be the case for an arbitrary pair of points in D . We, therefore, introduce a concept of *strong ellipticity* by the following definitions:

Definition 4: A mapping (47) will be called *strongly elliptic* in D if the determinant (69) considered for all pairs of points in D does not oscillate in sign.

Definition 5: A system of partial differential equations (50) will be called *strongly elliptic* if determinant (69) considered for all pairs of arguments satisfying (50) does not oscillate in sign.

If a mapping is associated with a pair of conservation laws (ξ_i, η_i) ($i = 1, 2$) defined in a domain Δ of the u - v -plane, it will be strongly elliptic if the determinant

$$(70) \quad \begin{vmatrix} \Delta \xi_1 & \Delta \eta_1 \\ \Delta \xi_2 & \Delta \eta_2 \end{vmatrix}$$

considered for all points of Δ does not oscillate in sign. We add, therefore,

Definition 6: A pair of conservation laws (ξ_i, η_i) ($i = 1, 2$) will be called *strongly elliptic* in a domain Δ if determinant (70) considered for all pairs of points of Δ does not oscillate in sign.

If in one of the preceding three definitions the determinant vanishes only for coinciding points we will speak of *strong ellipticity in the strict sense*.

The concept of strong ellipticity of systems (50) introduced by Lavrentiev is more restrictive than ours insofar as it contains also conditions on the first derivatives rather than, as in our case, only on their increments. His definition also contains specific constants restricting the values of (69).

We shall now prove several properties of strongly elliptic mappings or pairs of conservation laws which will be used later. First we mention

Lemma 3: A mapping (47) which is strongly elliptic remains so if the x - y -plane or the f - g -plane is subjected to an affine mapping.

Lemma 4: If the strongly elliptic mapping (41) has an inverse and its Jacobian never vanishes, then the inverse mapping is also strongly elliptic.

The simple proofs of these two lemmas may be left to the reader.

One might ask for simple criteria which allow us to assert that a mapping which was already proved to be elliptic everywhere in D is strongly elliptic there. One such criterion will now be derived. We may assume that form (51) is positive everywhere in D . Consider the mapping by

$$(71') \quad u_1 = f_x, \quad u_2 = f_y$$

and assume that it transforms D into a domain Γ of the u_1 - u_2 -plane in a *one-to-one* manner. Each function defined in D may then be rewritten as a function of u_1 and u_2 in Γ . We make further the assumption that Γ is *convex*. The inequality

$$(72) \quad \begin{vmatrix} df_x & df_y \\ dg_x & dg_y \end{vmatrix} = \begin{vmatrix} du_1 & du_2 \\ dg_x & dg_y \end{vmatrix} > 0$$

expressing positive ellipticity of mapping (47) may then be integrated between any two points of Γ along the connecting straight line segment and leads evidently to

$$(72') \quad \begin{vmatrix} \Delta f_x & \Delta f_y \\ \Delta g_x & \Delta g_y \end{vmatrix} > 0$$

for any two points of D , i.e., the positive *strong* ellipticity holds in D . Since strong ellipticity is not destroyed if the f - g -plane is subjected to an affine transformation we arrive at

Theorem 10: Suppose that mapping (47) is *positive (negative) elliptic* in D and that there exists a linear combination $h = af + bg$ with constant coefficients a and b not both zero such that

$$(71) \quad u_1 = h_x, \quad u_2 = h_y$$

maps D onto a convex domain Γ of the u_1 - u_2 -plane in a one-to-one manner. Then the mapping (47) is strongly positive (negative) elliptic in the strict sense.

We add

Theorem 10' : Suppose that D is bounded by a Jordan curve J and that the first derivatives of f and g have boundary values everywhere on J . If then (47) is positive (negative) elliptic in D and J is mapped by a suitable (71) onto a convex curve in a one-to-one manner then (47) is strongly positive (negative) elliptic in D in the strict sense.

Proof. From the assumption of ellipticity it follows that the differentials du_1 and du_2 are linearly independent. The mapping (71) is, therefore, one-to-one in the small. This fact together with the assumption about the boundary mapping is, as is well known, sufficient to insure the mapping of the closure of D to be one-to-one in the large.

In exactly the same way as we proved Theorems 10 and 10' one can prove the following two theorems on pairs of conservation laws:

Theorem 11: Let the pair of conservation laws (ξ_i, η_i) ($i = 1, 2$) be positive (negative) elliptic in a domain Δ of the u - v -plane. Assume further that there exists a linear combination

$$(72) \quad \xi = a\xi_1 + b\xi_2, \quad \eta = a\eta_1 + b\eta_2 \quad (a, b \text{ constant and not both zero})$$

which maps Δ onto a convex domain Γ of the ξ - η -plane in a one-to-one manner. Then the pair of conservation laws is strongly positive (negative) elliptic in Δ in the strict sense.

Theorem 11' : Suppose that Δ is bounded by a Jordan curve J and that the ξ_i and η_i have boundary values everywhere on J . If the pair of conservation laws (ξ_i, η_i) is positive (negative) elliptic in Δ and if there exists a linear combination (72) which maps J onto a convex curve Γ in a one-to-one manner then the pair of conservation laws is strongly positive (negative) elliptic in Δ in the strict sense.

We shall apply Theorem 11 to a pair of conservation laws of (1), the first of which is the conservation law of mass $(\rho u, \rho v)$. As domain Δ we choose an arbitrary convex sector of the sonic circle. This domain is mapped by the pair $\xi = \rho u$, $\eta = \rho v$ itself onto a convex domain in a one-to-one manner. Indeed, the point (q, θ) is transformed into the point $(\rho q, \theta)$, and ρq is a monotonically increasing function of q . We have, therefore,

Theorem 12: If a pair of conservation laws of (1), the first of which is the conservation law of mass, is positive (negative) elliptic in a convex sector of the sonic circle then it is strongly positive (negative) elliptic in the sector in the strict sense.

4. Some Inequalities in Compressible Fluid Flow. We shall apply the results of the previous sections to obtain some inequalities concerning the behavior of a flow around an obstacle on its boundary. They all have in common the characteristic feature that they give a bound for a mean value of a quantity expressible in terms of the speed distribution on the boundary of the obstacle.

We make use of two conservation laws of the system (1) describing a steady

irrotational compressible flow in the plane, the first being the conservation law of mass ($\rho u, \rho v$), the second (ξ, η) being left to a certain extent arbitrary. We think them defined in a domain Δ of the u - v -plane and assume that ξ and η have boundary values along the whole boundary of Δ . We further make the fundamental assumption *that the pair of conservation laws is strongly positive elliptic in Δ .*

We consider a flow around an obstacle whose exterior we call D . We make the following assumption about the flow:

- (a) It is uniform at infinity in the direction of the positive x -axis.
- (b) It is free of circulation.
- (c) The velocity components give points (u, v) falling into Δ .
- (d) The velocity components have continuous boundary values along the obstacle.

We now introduce the mapping associated with the pair of conservation laws

$$(73) \quad \psi = \int (-\rho v \, dx + \rho u \, dy), \quad \chi = \int (-\eta \, dx + \xi \, dy),$$

the first function ψ representing the stream function of the flow. From assumption (c) we conclude that the mapping is strongly positive elliptic in D . This leads in particular to the inequality

$$(74) \quad \begin{vmatrix} \psi_x - \psi_x^\infty & \psi_y - \psi_y^\infty \\ \chi_x - \chi_x^\infty & \chi_y - \chi_y^\infty \end{vmatrix} \geq 0$$

for each point (x, y) in D or on the perimeter of the obstacle. The superscript ∞ indicates the value of the respective function at infinity. Let us now integrate this inequality over the part of the stream-field contained in a big circle $x^2 + y^2 \leq R^2$. In order to write the result in a simple way, we introduce the functions

$$(75) \quad \begin{aligned} \psi^* &= \psi - \psi_x^\infty x - \psi_y^\infty y = \psi - \rho^\infty u^\infty y \\ \chi^* &= \chi - \chi_x^\infty x - \chi_y^\infty y = \chi + \eta(u^\infty, 0)x - \xi(u^\infty, 0)y. \end{aligned}$$

By transforming the double integral into a line integral over the whole boundary we obtain

$$(76) \quad \int_{C_R} \psi^* \, d\chi^* - \int_B \psi^* \, d\chi^* \geq 0,$$

C_R being the perimeter of the introduced circle and B the boundary curve of the obstacle both traversed in the positive sense. The following considerations rest on

Lemma 5:

$$(77) \quad \lim_{R \rightarrow \infty} \int_{C_R} \psi^* \, d\chi^* = 0.$$

Proof. We make essential use of a result of Shiffman [III] which asserts that the Dirichlet integral of ψ^* extended over D is *finite*. We conclude from it that also the Dirichlet integral of χ^* over D is finite. Indeed, both statements are evidently equivalent to the assertion of finiteness of the integral

$$\iint_D \{(u - u^\infty)^2 + v^2\} dx dy.$$

We now estimate the integrand of (77). We introduce polar coordinates (R, Φ) in the x - y -plane and operate in the exterior of a circle $R \geq R_0$. By using the Schwarz inequality we obtain first

$$(78) \quad \left| \int_{c_R} \psi^* d\chi^* \right|^2 = \left| \int_0^{2\pi} \psi^* \frac{\partial \chi^*}{\partial \Phi} d\Phi \right|^2 \leq \int_0^{2\pi} \psi^{*2} d\Phi \cdot \int_0^{2\pi} \left(\frac{\partial \chi^*}{\partial \Phi} \right)^2 d\Phi.$$

By integration along a radial segment we obtain

$$\begin{aligned} \psi^*(R, \Phi) - \psi^*(R_0, \Phi) &= \int_{R_0}^R \frac{\partial \psi^*}{\partial r} dr, \\ \psi^*(R, \Phi) &= \psi^*(R_0, \Phi) + \int_{R_0}^R \frac{\partial \psi^*}{\partial r} dr, \\ (\psi^*(R, \Phi))^2 &\leq 2 \left[(\psi^*(R_0, \Phi))^2 + \left(\int_{R_0}^R \frac{\partial \psi^*}{\partial r} dr \right)^2 \right]. \end{aligned}$$

Again using the Schwarz inequality we obtain

$$(\psi^*(R, \Phi))^2 \leq 2(\psi^*(R_0, \Phi))^2 + 2 \log \frac{R}{R_0} \int_{R_0}^R r \left(\frac{\partial \psi^*}{\partial r} \right)^2 dr,$$

and by integration with respect to Φ

$$\int_0^{2\pi} \psi^{*2}(R, \Phi) d\Phi \leq 2 \int_0^{2\pi} (\psi^*(R_0, \Phi))^2 d\Phi + 2 \log \frac{R}{R_0} \int_0^{2\pi} \int_{R_0}^R r \left(\frac{\partial \psi^*}{\partial r} \right)^2 dr d\Phi.$$

The double integral in the last formula is not greater than the Dirichlet integral of ψ^* over the whole D . We see therefore that there exist positive constants a and b such that

$$(79) \quad \int_0^{2\pi} (\psi^*)^2 d\Phi \leq a \log \frac{R}{b} \quad (R \geq R_0).$$

Calling the left hand side of (78) for the moment $F(R)$ we can write by utilizing (79)

$$(80) \quad \frac{F(R)}{\log \frac{R}{b}} \leq a \int_0^{2\pi} \left(\frac{\partial \chi^*}{\partial \Phi} \right)^2 d\Phi.$$

Multiplying finally (80) by $1/R$ and integrating with respect to R we get

$$(81) \quad \int_{R_0}^{\infty} \frac{F(R)}{R \log \frac{R}{b}} dR \leq a \int_0^{2\pi} \int_{R_0}^{\infty} \frac{1}{R} \left(\frac{\partial \chi^*}{\partial \Phi} \right)^2 dR d\Phi.$$

Again the double integral of the last inequality is not greater than the Dirichlet integral of χ^* over D and therefore finite. We see, therefore, that the integral on the left hand side of (81) is finite, from which fact we draw further conclusions. The integral

$$\int_{R_0}^{\infty} \frac{dR}{R \log \frac{R}{b}} = \infty.$$

There exists, therefore, a sequence of radii R_n approaching infinity such that

$$(82') \quad \lim_{n \rightarrow \infty} F(R_n) = 0$$

from which it follows also that

$$(82'') \quad \lim_{n \rightarrow \infty} \int_{C_{R_n}} \psi^* d\chi^* = 0.$$

But the integral $\int_{C_R} \psi^* d\chi^*$ is a monotonically increasing function of R as one immediately sees by integrating the inequality (74) over a ring domain $R_1 \leq R \leq R_2$. Equation (82') therefore has (77) as consequence.

Using the proved Lemma 5 we let in inequality (76) R approach infinity and obtain the *fundamental inequality*

$$(83') \quad \int_B \psi^* d\chi^* \leq 0.$$

Using the expressions (75) and the facts that $\psi = 0$ on B and that $u^\infty > 0$ we can easily bring it into the form

$$(83'') \quad \int_B y d\chi \geq \chi_x^\infty \int y dx.$$

Integral $\int_B x dy$ represents the area A of the obstacle. Thus we have arrived finally at

Theorem 13: *Under the conditions (a) to (d) concerning the character of the flow the inequality*

$$(84) \quad \int_B y d\chi + \chi_x^\infty A \geq 0$$

holds.

Remark: In order to rewrite inequality (84) in a sometimes more suitable form, we now show that χ is a single-valued function. By subjecting the differential $d\chi$ say for $R \geq R_0$ to a rotation by an arbitrary angle α a differential $d\chi_\alpha$ is obtained. We form then the average

$$(85) \quad d\bar{\chi} = \frac{1}{2\pi} \int d\chi_\alpha d\alpha.$$

Suppose now that χ has a period $\omega \neq 0$. All functions χ_α have then the same period and so has $\bar{\chi}$. On the other hand all χ_α have the same Dirichlet integral over $R \geq R_0$ and since it is finite $\bar{\chi}$ has also a finite Dirichlet integral in the same domain. But from the averaging process (84) it follows that $\bar{\chi}$ must have the form

$$(86) \quad \bar{\chi} = \chi_1(r) + \frac{\omega}{2\pi} \Phi,$$

$\chi_1(r)$ being a function of r alone, and the Dirichlet integral of (85) is infinite if $\omega \neq 0$. We arrived thus at a contradiction to the finiteness of the Dirichlet integral of χ .

Using the single valuedness of χ we may write (84) *also in the form*

$$(84') \quad \int_B \chi dy \leq \chi_x^\infty A.$$

We shall now consider special cases of Theorem 13. Take first $(\xi, \eta) = (-v, u)$ leading to $\chi = -\varphi$, φ being the potential function of the flow. Here we have strong positive ellipticity in the whole subsonic circle. This leads to

Theorem 14: *For any purely subsonic circulation-free flow around an obstacle in the direction of the positive x-axis the inequality*

$$(87) \quad \int_B \varphi dy \geq u^\infty A$$

holds.

In all further cases we restrict ourselves to obstacles which are *symmetric with respect to the x-axis and convex in the y-direction*, i.e., *each line parallel to the y-axis intersects the obstacle in one straight line segment*. We shall first show that the velocity component u in the x -direction is positive in the whole stream field. Indeed consider the stream function ψ in the part D' of the stream field which lies in the upper half plane. On its boundary $\psi = 0$. On the other hand $\partial\psi/\partial y$ approach the *positive* value $\rho^\infty u^\infty$ if one goes to infinity in this domain. These two facts together show that ψ is positive if one is sufficiently far away from the origin. Applying the minimum principle to ψ one sees that ψ positive in the upper half plane and therefore $\partial\psi/\partial y \geq 0$, i.e., $u \geq 0$ on the whole boundary of D' . Applying again the minimum principle now to u which is, as is well known, possible, we obtain the desired result.

We now combine the conservation law of mass with that occurring in Theorem 9. We know that this pair is positively strong elliptic in the greatest convex sector Δ of the sonic circle contained in the sector described in the theorem. For $\beta \leq 1$ this sector is the half circle lying to the right of the v -axis, for $\beta > 1$ it is identical with the sector of the theorem. We compute easily that on B

$$d\chi = -\lambda d\varphi = -A_1(q) \cos \beta\theta d\varphi.$$

We consider first the case $\beta \leq 1$. Since the flow around the obstacle satisfies the conditions under which Theorem 13 is valid we obtain

Theorem 15: *For $\beta \leq 1$ a purely subsonic circulation-free flow around an obstacle satisfies the inequality*

$$(88) \quad \int_B y A_1(q) \cos \beta\theta d\varphi - A_1(u^\infty) u^\infty A \leq 0.$$

For $\beta > 1$ we have the more restricted theorem

Theorem 16: *For $\beta \geq 1$ and a purely subsonic circulation-free flow around an obstacle with velocity directions restricted to the interval $-\pi/(2\beta) \leq \theta \leq \pi/(2\beta)$ the inequality (88) holds.*

Of particular interest is the case $\beta = 1$ because it leads to an inequality regarding the pressure distribution around the obstacle. Since we assume a circulation-free flow the pressure values in the intersection points of a vertical line with the boundary are the same. We may express them as functions of the area α of the obstacle to the left of the vertical line. One sees easily that (88) yields here

Theorem 17: *For a purely subsonic circulation-free flow around an obstacle*

$$\frac{1}{A} \int_0^A p d\alpha \leq p^\infty.$$

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