Uniqueness Theorems in the Unified Theory of Relativity

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I. Introduction.

(1) Denote by X_n an n -dimensional space ($n > 1$). Let $g_{\lambda\mu}$ be a given *real* tensor in X_n having

$$(1.1) \quad h_{\lambda\mu} = g_{(\lambda\mu)}, \quad k_{\lambda\mu} = g_{[\lambda\mu]},$$

as its symmetric and skew-symmetric parts, respectively. Moreover, introduce the notation

$$(1.2a) \quad g \equiv \det ((g_{\lambda\mu})), \quad h \equiv \det ((h_{\lambda\mu})), \quad k \equiv \det ((k_{\lambda\mu})).$$

Throughout this paper we assume

$$(1.3a) \quad h \neq 0,$$

so that the scalars

$$(1.2b) \quad G = g/h, \quad K = k/h,$$

are not meaningless.

We consider $h_{\lambda\mu}$ as a metric tensor in X_n , and denote its Christoffel symbols by $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$, and the covariant derivatives based on $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ by ∇_μ . Lowering and raising of indices are performed by $h_{\lambda\mu}$ and $h^{\rho\sigma}$, respectively.

* Prepared under Army Contract DA33008 ORD-467.

(2) Consider the set of n^3 equations

$$(1.4) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha}.$$

One of the authors has dealt with the solution $\Gamma_{\lambda\mu}^\nu$ of (1.4) in another paper.¹ His results are summarized below.

Introduce the tensors²

$$(1.5) \quad \begin{aligned} S_{\lambda\mu}^\nu &= \Gamma_{[\lambda\mu]}^\nu, \\ K_{\omega\mu\nu} &= \nabla_\omega g_{\nu\mu} + \nabla_\mu g_{\omega\nu} + \nabla_\nu g_{\omega\mu}, \\ X_{\omega\mu\nu}^{\xi\eta\zeta} &= \delta_{[\omega\mu]}^{\xi\eta} \zeta_\nu - 2\delta_\nu^{[\xi} k_{[\omega}^{\eta]} k_{\mu]}^{\zeta]} - 2\delta_{[\mu}^{[\xi} k_{\omega]}^{\eta]} k_\nu^{\zeta]}. \end{aligned}$$

Then

$$(1.6a) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + S_{\lambda\mu}^\nu + 2h^{\nu\alpha} S_{\alpha(\lambda} k_{\mu)\beta},$$

and

$$(1.6b) \quad 2S_{\xi\eta\zeta} X_{\omega\mu\nu}^{\xi\eta\zeta} = K_{\omega\mu\nu}.$$

Supposing that (1.6b) is soluble for $S_{\xi\eta\zeta}$, we substitute the corresponding solution into (1.6a) and thus obtain a solution $\Gamma_{\lambda\mu}^\nu$ of (1.4).

In order to obtain more information about the system (1.6b), we introduce the nomenclature:

$$(1.7) \quad \begin{aligned} 'X_{\omega\mu\nu}^{\xi\eta\zeta} &= 2X_{\omega\mu\nu}^{\xi\eta\zeta}, & \text{for } \omega < \mu, \xi < \eta; \\ 'X_{\omega\mu\nu}^{\xi\eta\zeta} &= 0, & \text{otherwise;} \end{aligned}$$

and denote by X the determinant of $n \left(\frac{n}{2}\right)$ rows, each of whose rows is constituted by $n \left(\frac{n}{2}\right)$ elements $'X_{\omega\mu\nu}^{\xi\eta\zeta}$ with *fixed* ω, μ, ν , ($\omega < \mu$) and *variable* ξ, η, ζ ($\xi < \eta$).

A necessary and sufficient condition for (1.6b) to admit exactly one solution $S_{\xi\eta\zeta}$ is

$$(1.8) \quad X \neq 0.$$

If this condition is satisfied, there is a unique tensor $Y_{\alpha\beta\gamma}^{\omega\mu\nu} = Y_{[\alpha\beta]}^{[\omega\mu]} \gamma^\nu$, which depends solely on $X_{\omega\mu\nu}^{\xi\eta\zeta}$, and in terms of which we may express the solution of (1.6b)

¹ V. HLAVATÝ, "The Elementary Basic Principles of the Unified Theory of Relativity B," *Journal of Rat. Mech. and Analysis* **2**, 1-52 (1953).

² δ_λ^μ is the Kronecker symbol and

$$\delta_{\alpha_1 \dots \alpha_s}^{\beta_1 \dots \beta_s} = \delta_{\alpha_1}^{\beta_1} \dots \delta_{\alpha_s}^{\beta_s}.$$

as follows:

$$(1.6c) \quad 2S_{\alpha\beta\gamma} = K_{\omega\mu\nu} Y^{\omega\mu\nu}_{\alpha\beta\gamma};$$

and, by substituting (1.6c) into (1.6a), we obtain a solution $\Gamma^{\nu}_{\lambda\mu}$ of (1.4) of the form:

$$(1.6d) \quad \Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\alpha\beta\gamma} Y^{\alpha\beta\gamma}_{\xi\eta\zeta} [\delta^{\xi\eta}_{\lambda\mu} h^{\zeta\nu} + 2h^{\xi\nu} \delta^{\eta}_{[\lambda} k^{\zeta}_{\mu]}].$$

The condition (1.8) is obviously a necessary and sufficient one for (1.4) to admit exactly one solution, and this solution is given by (1.6d).³

Our main task is to replace (1.8) by an equivalent condition, expressed in terms of G and K only.

We confine ourselves to the case $n = 4$, when equations (1.4) coincide with the field equations of the Einstein⁴ unified relativity theory.

(3) If $n = 4$, the determinant X contains twenty-four rows, so that its direct computation is out of the question. We have thus derived an alternative procedure which

- (a) requires the computation of a determinant of four rows only and
- (b) at the same time substantially reduces the solution of the twenty-four equations (1.6b) to solution of four equations only.

This method is applied in Parts II and III of the present paper. It is based on the following ideas:

A. Let $h_{\lambda\mu}$ be of signature $---+$, and denote by U^X_{λ} ⁵ four mutually perpendicular covariant unit vector fields:

$$(1.9a) \quad h^{\lambda\mu} U^X_{\lambda} U^Y_{\mu} = \begin{cases} -1, & X = Y = I, II, III, \\ 1, & X = Y = IV, \\ 0, & X \neq Y. \end{cases}$$

Then $h_{\lambda\mu}$ may always be put into the (canonical) form:

$$(1.10a) \quad h_{\lambda\mu} = -U_{\lambda}^I U_{\mu}^I - U_{\lambda}^{II} U_{\mu}^{II} - U_{\lambda}^{III} U_{\mu}^{III} + U_{\lambda}^{IV} U_{\mu}^{IV};$$

while for a suitable choice of the U^Y_{λ} 's $k_{\lambda\mu}$ may be written in one of two mutually exclusive (canonical) forms:

$$(1.11a) \quad k_{\lambda\mu} = 2\alpha U_{[\lambda}^I U_{\mu]}^{II} + 2\beta U_{[\lambda}^{III} U_{\mu]}^{IV},$$

where not both scalars α, β vanish, or

$$(1.11b) \quad k_{\lambda\mu} = 2\gamma U_{[\lambda}^{II} (U_{\mu]}^{III} + \epsilon U_{\mu]}^{IV}), \quad \epsilon \equiv \pm 1.$$

³ The set of all tensors $g_{\lambda\mu}$ for which (1.6a) admits more than one solution is not an empty set. The set of all tensors for which (1.6a) does not admit any solution is not an empty set. Both sets contain tensors $g_{\lambda\mu}$ with $g \neq 0$. For more information see the work cited in footnote 1.

⁴ A. EINSTEIN, *The Meaning of Relativity*, Princeton University Press, 1950, Appendix 2.

⁵ We employ capital Roman letters to designate indices running from I to IV .

Whether (1.11a) or (1.11b) holds depends solely on the scalar density

$$(1.12) \quad D \equiv (\tfrac{1}{2} h k_{\lambda\mu} k^{\lambda\mu})^2 - 4 h k.$$

If $D \neq 0$, we have (1.11a). If $D = 0$, then (1.11b) is valid.⁶

Consider first the case (1.11a). The vector fields U_λ^X satisfying (1.9a), (1.10a), and (1.11a) are defined uniquely, to within a Lorentz transformation, by the given data, namely by $g_{\lambda\mu}$.

As far as the case (1.11b) is concerned, the vector fields U_λ^X satisfying (1.9a), (1.10a), and (1.11b) are not uniquely defined by $g_{\lambda\mu}$,⁷ so that there is a certain arbitrariness in their choice. However, the form (1.11b) is obviously preserved by a special Lorentz transformation of the U_λ^X 's.

B. Since the U_λ^X 's are linearly independent, there is a unique set of contra-variant vector fields U_X^Y such that

$$(1.9b) \quad U_\lambda^X U_X^Y = \delta_\lambda^Y, \quad U_X^\lambda U_\lambda^Y = \delta_X^Y.$$

We may look upon U_X^λ and U_λ^Y as mixed components of the unit tensor, which relate an arbitrary coordinate system of total differentials dx^Y to another system of differentials dx^X (linear Pfaff forms), which is in general non-holonomic:

$$(1.13) \quad dx^Y = U_X^Y dx^X, \quad dx^X = U_\lambda^X dx^\lambda.$$

Any tensor may be represented by means of its components $T_\lambda^{\dots}$ with respect to some holonomic system, in terms of the components

$$(1.14) \quad T_X^{\dots} = T_\lambda^{\dots} U_X^\lambda U_\nu^{\dots},$$

or by the use of intermediate components. In particular, we have:

$$(1.15) \quad \begin{aligned} ((h_{XY})) &= ((h_{\lambda\mu} U_X^\lambda U_Y^\mu)) = \left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \\ & & & +1 \end{pmatrix} \right); \\ ((h^{XY})) &= ((h^{\lambda\mu} U_\lambda^X U_\mu^Y)) = \left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \\ & & & +1 \end{pmatrix} \right); \end{aligned}$$

⁶ If $D \neq 0$, we may still have either $k \neq 0$ or $k = 0$. However, $D = 0$ implies $k = 0$, because the conditions $D = 0$, $k \neq 0$ cannot be satisfied by a *real* tensor $g_{\lambda\mu}$. For more detailed information *cfr.* V. HLAVATÝ, "Elementary Basic Principles of the Unified Theory of Relativity A," *Journal of Rational Mechanics and Analysis* **1**, 539–562 (1952).

⁷ Only the null-vector $U_\lambda^{III} + \epsilon U_\lambda^{IV}$ is uniquely specified by $g_{\lambda\mu}$.

while in cases (1.11a) and (1.11b), respectively, we obtain:

$$(1.16a) \quad ((k_{xy})) = \left(\begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} \right);$$

$$(1.16b) \quad ((k_{xy})) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & \epsilon\gamma \\ 0 & -\gamma & 0 & 0 \\ 0 & -\epsilon\gamma & 0 & 0 \end{pmatrix} \right)$$

C. We now consider the problem of solving the system

$$(1.17) \quad 2S_{uvw} X_{xyz}^{uvw} = K_{xyz} \quad (U, \dots, Z = I, \dots, IV)$$

which is equivalent to (1.6b) because of the results proved in Section B above.

First, let us look at the case (1.11a). The components of $((g_{xy}))$ are specified by

$$(1.18a) \quad ((g_{xy})) = \left(\begin{pmatrix} -1 & \alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & \beta \\ 0 & 0 & -\beta & 1 \end{pmatrix} \right),$$

as follows from (1.15) and (1.16a).

In virtue of (1.18a) the corresponding system (1.17) shows a peculiar symmetry, so that the solution of this system requires essentially a solution of only four of its members, whose determinant enables us to find a solubility condition equivalent to (1.8). This condition is a polynomial in

$$\frac{\det ((g_{xy}))}{\det ((h_{xy}))}, \quad \frac{\det ((k_{xy}))}{\det ((h_{xy}))},$$

which quantities are clearly equal to G and to K , respectively, so that (1.8) may be replaced by a condition involving only these two scalars.

As far as the solution $S_{\omega\mu\lambda}$ of (1.6b) is concerned, we obtain it from that of (1.17) by means of rule (1.14), thus:

$$(1.19) \quad S_{\omega\mu\nu} = S_{xyz} U_{\omega}^x U_{\mu}^y U_{\nu}^z.$$

Since in this case U_{λ}^x is determined uniquely by $g_{\lambda\mu}$ to within a Lorentz transformation, (1.19) expresses $S_{\omega\mu\nu}$ solely in terms of the given data $g_{\lambda\mu}$.

Second, let us look at (1.11b). In this case, we obtain from (1.15) and (1.16b):

$$(1.18b) \quad ((g_{xy})) = \left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & \gamma & \epsilon\gamma \\ 0 & -\gamma & -1 & 0 \\ 0 & -\epsilon\gamma & 0 & 1 \end{pmatrix} \right).$$

Since U_λ^x is not uniquely determined by $g_{\lambda\mu}$, under these circumstances we have to employ a different approach to our problem. *Our new procedure is based on the fact that there exists a subgroup of Lorentz transformations which preserve the form (1.11b) of $k_{\lambda\mu}$, where the new γ has a value smaller than any preassigned positive number ρ .*

The main theorems obtained in this manner may be stated as follows:

Theorem (1.1). *Let $h_{\lambda\mu}$ have signature $---+$, and let*

$$(1.20a) \quad D \neq 0.$$

Then (1.4) admits a unique solution $\Gamma_{\lambda\mu}^$ if and only if*

$$(1.21) \quad G\{[2(1 + 3K) - G]^2 - 4K[3 + K - G]^2\} \neq 0.$$

The condition (1.21) is equivalent either to $GK \neq 0$ or to $G(G - 2) \neq 0$ if $K = 0$. If this condition is satisfied, the actual computation requires the solution of a system of four linear equations only.

Theorem (1.2). *Let $h_{\lambda\mu}$ have signature $---+$, and let*

$$(1.20b) \quad D = 0.$$

Then the system (1.4) admits exactly one solution.⁸

Parts II and III of our paper are devoted to the proof of these theorems, by means of the methods mentioned in Section C of the Introduction.

II. Conditions of Unique Solubility of (1.17) for Cases (1.18a) and (1.18b).

We shall now study the problem of solving the system (1.17) when $((g_{xY}))$ is given by (1.18a) or (1.18b).

We start by introducing the notation⁵

$$(2.1a) \quad \Sigma_{XYZ} \equiv 2S_{UVW} X_{XYZ}^{UVW} - K_{XYZ} \quad (U, \dots, Z = I, \dots, IV):$$

A. Case (1.18a). Under these circumstances, we have

Lemma (2.1). *Define $G \equiv g/h$, $K \equiv k/h$. Then necessary and sufficient conditions for the inequality*

$$(2.2) \quad G\{[2(1 + 3K) - G]^2 - 4K[3 + K - G]^2\} \neq 0$$

to hold are that $GK \neq 0$, or if $K = 0$, then $G \neq 0, 2$.

Proof. The case $K = 0$ is trivial.

⁸ A straightforward elementary computation of the $\Gamma_{\lambda\mu}^*$ from the 64 equations (1.4) would give the same results. On the other hand the direct computation in the *general case* of $g_{\lambda\mu}$ is almost beyond possibility. However, in some *very special* cases of $g_{\lambda\mu}$ this computation has been carried through. Cfr. PAPAPETROU, A., "Static spherically symmetric solutions in the unitary field theory," *Proc. Roy. Irish Acad. (A)* **52**, 69-86 (1948); BANDYOPADHYAY, G., "Particular solutions of Einstein's recent unified theories," *Indian J. Phys.* **25**, 257-261 (1951); and TAKENO, H., IKEDA, M., & ABE, S., "On solutions of new field equations of Einstein and those of Schrödinger," *Progress of Theoretical Physics* **6**, 837-848 (1951).

If $K \neq 0$, our signature assumption regarding $h_{\lambda\mu}$, coupled with (1.15), (1.16a), and the scalar nature of K , implies

$$(2.3) \quad K < 0.$$

From this last condition, we see that the left-hand side of (2.2) vanishes if and only if the simultaneous equations

$$2(1 + 3K) - G = 0, \quad 3 + K - G = 0$$

are satisfied. But these equations imply $K = \frac{1}{3}$, which contradicts (2.3). *q.e.d.*

This lemma plays a central role in the subsequent developments.

We shall now write down the equations of the system (2.1a) for case (1.18a), dividing them into six sets, $S_1, S_2, S_3, S'_1, S'_2, S'_3$. A few words about the nomenclature are required at this point.

1. The initial letters of the Latin alphabet ($a, b, c, d, e, \dots$) range over I, II ; the final ones ($t, u, v, w, x, \dots$) over III, IV . Moreover, we introduce the indices

$$\begin{aligned} \bar{I} &\equiv II, & \bar{II} &\equiv I, & \bar{III} &\equiv IV, & \bar{IV} &\equiv III; \\ I' &\equiv III, & II' &\equiv IV, & III' &\equiv I, & IV' &\equiv II. \end{aligned}$$

2. We define symbols $\epsilon_{XY} = \epsilon_{[XY]}$ as follows:

$$(2.4b) \quad \epsilon_{I \ II} = \epsilon_{III \ IV} = 1, \quad \epsilon_{au} = 0.$$

We thus obtain:

$$S_1. \ \Sigma_{abc} \equiv 2S_{abc} + 2\alpha^2 \epsilon_{ab} \epsilon_{de} S^{[d \cdot e]}_c - K_{abc} = 0.$$

$$S_2. \ \Sigma_{ave} \equiv 2S_{ave} - 2\alpha\beta S^{t \cdot d}_a \epsilon_{dc} \epsilon_{tv} + 2\alpha^2 S^{d \cdot e}_v \epsilon_{ec} \epsilon_{da} + 4\alpha\beta S^{[d \cdot t]}_c \epsilon_{da} \epsilon_{tv} - K_{ave} = 0.$$

$$S_3. \ \Sigma_{uvc} \equiv 2S_{uvc} + 2\beta^2 \epsilon_{uv} \epsilon_{tz} S^{[t \cdot z]}_c - K_{uvc} = 0.$$

$$S'_1. \ \Sigma_{uvw} \equiv 2S_{uvw} + 2\beta^2 \epsilon_{uv} \epsilon_{tz} S^{[t \cdot z]}_w - K_{uvw} = 0.$$

$$S'_2. \ \Sigma_{vaw} \equiv 2S_{vaw} - 2\alpha\beta S^{d \cdot t}_v \epsilon_{tw} \epsilon_{da} + 2\beta^2 S^{t \cdot z}_a \epsilon_{zw} \epsilon_{tv} + 4\alpha\beta S^{[t \cdot d]}_w \epsilon_{tv} \epsilon_{da} - K_{vaw} = 0.$$

$$S'_3. \ \Sigma_{abw} \equiv 2S_{abw} + 2\alpha^2 \epsilon_{ab} \epsilon_{de} S^{[d \cdot e]}_w - K_{abw} = 0.$$

We regard $S_1, \dots, S'_3$ as systems of equations for the unknowns S_{XYZ} , which gives a definite and customary meaning to expressions such as "the determinant $\Delta(S_1, S'_1)$ of the systems S_1 and S'_1 " and so on.

In these equations we raise indices by means of

$$((h^{XY})) = \left(\left(\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \right) \right).$$

From now on, we shall employ h^x as an abbreviation for the diagonal elements of h^{xy} . We emphasize the fact that the h^x 's do not follow the usual summation rule (cfr. for instance (2.9)).

The problem of solving $S_1, \dots, S'_3$ is materially simplified by the following

Lemma 2.2. *Introduce a permutation*

$$(2.5) \quad \Pi \equiv \begin{pmatrix} I & II & III & IV & \alpha & \beta \\ I' & II' & III' & IV' & \beta & \alpha \end{pmatrix}$$

and denote by $\Pi\Sigma_{XYZ}$ the system which one obtains by applying Π to Σ_{XYZ} . Then

$$(2.6a) \quad \Sigma_{X'Y'Z'} = \Pi\Sigma_{XYZ},$$

where $\Sigma_{XYZ} \varepsilon S_i$ and $\Sigma_{X'Y'Z'} \varepsilon S'_i (i = 1, 2, 3)$.

Proof. From (2.4a) we have

$$(2.7) \quad (\bar{X})' = \overline{(X')} \equiv \bar{X}'; \quad (X')' = X, \quad \overline{(\bar{X})} = X; \quad (X = I, \dots, IV).$$

Using (2.7) in conjunction with (2.4a) and (2.4b), we obtain:

$$(2.8) \quad \begin{aligned} \Pi(\alpha^2 S_v^{d \cdot e} \epsilon_{ec} \epsilon_{da}) &= \beta^2 S_{v'}^{t \cdot z} \epsilon_{zc'} \epsilon_{ta'}; \\ \Pi(\alpha\beta S_a^{d \cdot t} \epsilon_{dc} \epsilon_{tv}) &= \alpha\beta S_{a'}^{t \cdot d} \epsilon_{tc'} \epsilon_{dv'}; \\ \Pi(\alpha\beta S_a^{t \cdot d} \epsilon_{dc} \epsilon_{tv}) &= \alpha\beta S_{a'}^{d \cdot t} \epsilon_{tc'} \epsilon_{dv'}; \\ \Pi(\beta^2 \epsilon_{uv} \epsilon_{tz} S_c^{[t \cdot z]}) &= \alpha^2 \epsilon_{u'v'} \epsilon_{da} S_c^{[d \cdot e]}. \end{aligned}$$

Because of (2.8), equation (2.6a) follows by inspection of the sets $S_1, \dots, S'_3$. *q.e.d.*

AA. Determinant of (S_1, S'_1) .

The sets S_1, S'_1 have such simple structures that we shall not have to employ Lemma (2.2).

Let a be any fixed index chosen from I, II . Then (2.4a) and (2.4b) allow us to write S_1, S'_1 thus:

$$(2.9) \quad \begin{aligned} S_1. \quad & \begin{cases} \Sigma_{a\bar{a}a} \equiv 2(1 + \alpha^2 h^a h^{\bar{a}})S_{a\bar{a}a} - K_{a\bar{a}a} = 0, \\ \Sigma_{a\bar{a}\bar{a}} \equiv 2(1 + \alpha^2 h^a h^{\bar{a}})S_{a\bar{a}\bar{a}} - K_{a\bar{a}\bar{a}} = 0. \end{cases} \\ S'_1. \quad & \begin{cases} \Sigma_{a'\bar{a}'a'} \equiv 2(1 + \beta^2 h^{a'} h^{\bar{a}'})S_{a'\bar{a}'a'} - K_{a'\bar{a}'a'} = 0, \\ \Sigma_{a'\bar{a}'\bar{a}'} \equiv 2(1 + \beta^2 h^{a'} h^{\bar{a}'})S_{a'\bar{a}'\bar{a}'} - K_{a'\bar{a}'\bar{a}'} = 0. \end{cases} \end{aligned}$$

In virtue of the identity

$$(2.10a) \quad G = (1 + \alpha^2 h^a h^{\bar{a}})(1 + \beta^2 h^{a'} h^{\bar{a}'}),$$

which is an immediate consequence of (1.18a), and the fact that G is a scalar, we conclude that the determinant $\Delta(S_1, S'_1)$ of $\Sigma_{a\bar{a}a}, \Sigma_{a\bar{a}\bar{a}}, \Sigma_{a'\bar{a}'a'}, \Sigma_{a'\bar{a}'\bar{a}'}$ is

$$(2.11) \quad \Delta(S_1, S'_1) = 2^4 G^2.$$

AB. Determinant of (S_2, S'_3) and (S'_2, S_3) .

We define

$$(2.1b) \quad \begin{aligned} \Sigma_{XYZ}^* &\equiv \Sigma_{XYZ} - \alpha\beta\epsilon_{XU}\epsilon_{YV}\Sigma_Z^{U\cdot V}, \\ K_{XYZ}^* &\equiv K_{XYZ} - \alpha\beta\epsilon_{XU}\epsilon_{YV}K_Z^{U\cdot V}, \end{aligned}$$

where we raise indices by means of h^{XY} in (1.15).

From (2.1b), (2.4b), and Lemma (2.2) we have

$$(2.6b) \quad \Pi\Sigma_{XYZ}^* = \Sigma_{X'Y'Z'}^*.$$

Instead of dealing with (S_2, S'_3) and (S'_2, S_3) , we shall consider (S_2^*, S'_3) and $(S_2'^*, S_3)$, where S_2^* and $S_2'^*$ are defined below:

$$(2.12a) \quad S_2^* \cdot \begin{cases} \Sigma_{avc}^* \equiv 2S_{avc} + 2\alpha^2 S_v^{d\cdot e} \epsilon_{ec} \epsilon_{da} - 2\alpha\beta(S_a^{t\cdot d} \epsilon_{dc} + S_c^{t\cdot d} \epsilon_{da}) \epsilon_{tv} \\ - 2\alpha^3 \beta \epsilon_{da} \epsilon_{tv} \epsilon_{dc} \epsilon_{ef} h^d h^t S_t^{[e\cdot f]} - K_{avc}^* = 0; \end{cases}$$

$$(2.12b) \quad S_2'^* \cdot \begin{cases} \Sigma_{vaw}^* \equiv 2S_{vaw} + 2\beta^2 S_a^{t\cdot z} \epsilon_{zw} \epsilon_{tv} - 2\alpha\beta(S_w^{d\cdot t} \epsilon_{tv} + S_v^{d\cdot t} \epsilon_{tw}) \epsilon_{da} \\ - 2\alpha\beta^3 \epsilon_{da} \epsilon_{tv} \epsilon_{tw} \epsilon_{xy} h^d h^t S_d^{[x\cdot y]} - K_{vaw}^* = 0. \end{cases}$$

Lemma (2.3). *The system $(S_2, S'_3; S'_2, S_3)$ is uniquely soluble if and only if the system $(S_2^*, S_2'^*)$ has a unique solution.*

Proof. From (2.1b) it can be seen that the systems (S_2^*, S'_3) and $(S_2'^*, S_3)$ are equivalent to (S_2, S'_3) and (S'_2, S_3) , respectively. It is also clear, from (2.12a) and (2.12b), that S_2^* and $S_2'^*$ only contain unknowns of the type S_{cud} and S_{vew} , respectively. Since the remaining unknowns, i.e., those of the form S_{cdu} and S_{vwe} , can be uniquely determined from S'_3 and S_3 , respectively, once S_{cud} and S_{vew} are specified, our proof is complete.

Lemma (2.4). *The determinant $\Delta(S_2^*, S_2'^*)$ of the systems S_2^* and $S_2'^*$ is*

$$(2.13) \quad \Delta(S_2^*, S_2'^*) = 2^{16} G^2 \cdot \{[2(1 + 3K) - G]^2 - 4K[3 + K - G]^2\}^2.$$

Proof. Let us first compute Δ_v , the determinant of

$$(2.14) \quad \Sigma_{ava}^*, \Sigma_{av\bar{a}}^*, \Sigma_{a\bar{v}\bar{a}}^*, \Sigma_{\bar{a}\bar{v}a}^*,$$

where a and v are fixed. As we shall see, the value of $\Delta(S_2^*, S_2'^*)$ will follow from this result *without further computation*.

Omitting unessential details, we obtain from (2.12a)

$$\Delta_v = 2^4 \begin{vmatrix} 1 & \alpha^2(h^{\bar{v}})^2 & 2\alpha\beta h^{\bar{a}} h^{\bar{v}} \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} \left[1 + \frac{\alpha^2}{2} h^a h^{\bar{a}}\right] & -\alpha^3 \beta \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} h^a (h^{\bar{a}})^2 h^{\bar{v}} \\ \alpha^2(h^a)^2 & 1 & \alpha^3 \beta \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} (h^a)^2 h^{\bar{a}} h^{\bar{v}} & -2\alpha\beta h^a h^{\bar{v}} \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} \left[1 + \frac{\alpha^2}{2} h^a h^{\bar{a}}\right] \\ \alpha\beta h^a h^v \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} - \alpha\beta h^{\bar{a}} h^v \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} & & 1 & -\alpha^2 h^a h^{\bar{a}} \\ \alpha\beta h^a h^v \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} - \alpha\beta h^{\bar{a}} h^v \epsilon_{a\bar{a}} \epsilon_{v\bar{v}} & & -\alpha^2 h^a h^{\bar{a}} & 1 \end{vmatrix},$$

where we do *not* sum over the repeated indices.

To evaluate Δ_v we perform the following operations:

- (1) We subtract the third row from the fourth one.
- (2) We add the fourth column to the third one.

We thus have

$$(2.15) \quad \Delta_v = 2^4(1 + \alpha^2 h^a h^{\bar{a}})\{(1 - \alpha^2 h^a h^{\bar{a}})^2 - 4K\},$$

so that Δ_v does *not* depend on v . Therefore

$$(2.16) \quad \Delta_v = \Delta_{\bar{v}}, \quad \Delta_v^2 = \Delta_v \Delta_{\bar{v}}.$$

But $\Delta_v \Delta_{\bar{v}}$ is the determinant of

$$\begin{aligned} & \Sigma_{ava}^*, \Sigma_{\bar{a}v\bar{a}}, \Sigma_{a\bar{v}\bar{a}}, \Sigma_{\bar{a}\bar{v}a}^*; \\ & \Sigma_{a\bar{v}a}^*, \Sigma_{\bar{a}\bar{v}\bar{a}}, \Sigma_{av\bar{a}}, \Sigma_{\bar{a}va}^*; \end{aligned}$$

where a and v are *fixed*, i.e., the determinant $\Delta(S_2^*)$ of S_2^* . Hence we have

$$(2.17a) \quad \Delta(S_2^*) = \Delta_v^2 = 2^8(1 + \alpha^2 h^a h^{\bar{a}})^2\{(1 - \alpha^2 h^a h^{\bar{a}})^2 - 4K\}^2.$$

Let us calculate, in the second place, the determinant $\Delta(S_2'^*)$ of $S_2'^*$. From (2.6b) we conclude

$$(2.18) \quad \Delta(S_2'^*) = \Delta(\Pi S_2^*) = \Pi \Delta(S_2^*),$$

which in conjunction with (2.17a) yields

$$(2.17b) \quad \Delta(S_2'^*) = \Delta_{v'}^2 = 2^8(1 + \beta^2 h^{a'} h^{\bar{a}'})^2\{(1 - \beta^2 h^{a'} h^{\bar{a}'})^2 - 4K\}^2.$$

Finally we see that the system S_2^* contains the 8 *unknowns*

$$S_{ava}, S_{\bar{a}v\bar{a}}, S_{a\bar{v}\bar{a}}, S_{\bar{a}\bar{v}a}; \quad S_{a\bar{v}a}, S_{\bar{a}\bar{v}\bar{a}}, S_{av\bar{a}}, S_{\bar{a}va} \quad (a, v: \text{fixed}),$$

while the system $S_2'^*$ contains the 8 *unknowns*

$$S_{a'v'a'}, S_{\bar{a}'v'\bar{a}'}, S_{a'\bar{v}'\bar{a}'}, S_{\bar{a}'\bar{v}'a'}; \quad S_{a'\bar{v}'a'}, S_{\bar{a}'\bar{v}'\bar{a}'}, S_{a'v'\bar{a}'}, S_{\bar{a}'v'a'} \quad (a, v: \text{fixed}),$$

all different from those of S_2^* . Hence

$$(2.19) \quad \Delta(S_2^*, S_2'^*) = \Delta(S_2^*)\Delta(S_2'^*),$$

and (2.13) follows from this equation, in conjunction with (2.17a), (2.17b), and the elementary identity

$$(2.10b) \quad (1 - \alpha^2 h^a h^{\bar{a}})(1 - \beta^2 h^{a'} h^{\bar{a}'}) = 2(1 + K) - G,$$

whose proof runs parallel to that of (2.10a). *q.e.d.*

Coupling (2.11) with Lemmas (2.1), (2.3), and (2.4), we arrive at

Theorem (2.1). *A set of necessary and sufficient conditions for the system (1.17)*

to possess exactly one solution for case (1.18a) is that $GK \neq 0$, or if $K = 0$, then $G \neq 0, 2$,

B. Case (1.18b). In this case we have:

Theorem (2.2). *Let g_{xy} be given by (1.18b). Then the system*

$$(2.20) \quad \Sigma_{xyz} \equiv 2S_{uvw} X_{xyz}^{uvw} - K_{xyz} = 0$$

admits a unique solution, provided

$$(2.21) \quad 0 \leq |\gamma| < \rho > 0,$$

where ρ is a sufficiently small real number.

Proof. Let $\Delta(\gamma)$ be the determinant of (2.20) for a given value of γ in (1.18b). Now, if $\gamma = 0$, we have

$$X_{xyz}^{uvw} = \delta_{[xy]z}^{[uv]w}$$

by (1.5) and (1.18b), so that (2.20) is evidently uniquely soluble for S_{uvw} . Therefore,

$$(2.22) \quad \Delta(0) \neq 0.$$

But $\Delta(\gamma)$ is a polynomial in γ , by (1.5), (1.14), and (1.18b). Consequently (2.22) implies that there exists a positive real ρ , such that $\Delta(\gamma) \neq 0$ if (2.21) holds. *q.e.d.*

III. Completion of the Proof. In this section we shall complete the proof of the main Theorems (1.1) and (1.2) using the results of the previous section. To this effect we shall first consider the case $D \neq 0$ and prove the following

Lemma (3.1). *Suppose $D \neq 0$. Then there is at least one non-holonomic coordinate system for which*

$$(3.1) \quad ((g_{xy})) = \left(\begin{pmatrix} -1 & \alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & \beta \\ 0 & 0 & -\beta & 1 \end{pmatrix} \right),$$

where at least one of the numbers α, β is different from zero. This system is defined by the given data only (i.e. by $g_{\lambda\mu}$) and is known up to an arbitrary transformation of a special Lorentz subgroup, which leaves g_{xy} invariant.

The proof will be accomplished in six steps.⁹

1) Denote by $T_4(P)$ the tangent linear space of X_4 at its generator point P and by $I_3(P)$ the three-dimensional projective ideal space of $T_4(P)$. The tensor $h_{\lambda\mu}$ defines a quadric H in $I_3(P)$. Because $h_{\lambda\mu}$ is of the signature $---+$, H is a real, non-degenerate and non-ruled quadric. The tensor $k_{\lambda\mu}$ defines a linear

⁹ The statements dealt with in this proof are proved in the paper mentioned in footnote 6.

line complex K . This complex is either a general one, if $K \neq 0$, or a special one otherwise.¹⁰

2) Let us consider first the case $K \neq 0$. Then there are exactly two real skew lines, say A and B , in $I_3(P)$ which are harmonic polar lines with respect to H and at the same time conjugate polars with respect to K . They are uniquely defined by $h_{\lambda\mu}$ and $k_{\lambda\mu}$, i.e., by $g_{\lambda\mu}$. Since H is a real nonruled and non-degenerate quadric, only one of the lines A , B , say B , has the property that it is incident with two real tangential planes to H , while the remaining A is incident with two complex conjugate tangential planes to H .

3) Let U_λ^I, U_μ^{II} be two harmonic polar planes with respect to H and incident with A . Let $U_\lambda^{III}, U_\mu^{IV}$ be two harmonic polar planes with respect to H and incident with B . Then the planes $U_\lambda^I, U_\mu^{II}(U_\lambda^{III}, U_\mu^{IV})$ satisfy the last equation (1.9a) for $X, Y = I$ and II ($X, Y = III$ and IV). Moreover, since the lines A, B are harmonic polar with respect to H , the four planes U_λ^X satisfy also the remaining last equations (1.9a). Keeping in mind the signature $--+$ of $h_{\lambda\mu}$ we see that the factors of proportionality of the components U_λ^X may be chosen in such a way that also the first two sets of (1.9a) are satisfied.

4) All couples of harmonic polar planes incident with A constitute an elliptic involution. Hence if U_λ^I, U_μ^{II} is one couple any other couple $'U_\lambda^I, 'U_\mu^{II}$ is given by

$$(3.2a) \quad 'U_\lambda^I + \epsilon i'U_\mu^{II} = e^{i\varphi}(U_\lambda^I + \epsilon iU_\mu^{II}),$$

it being supposed that

$$'U_\lambda^X 'U_\mu^Y h^{\lambda\mu} = h^{XY}, \quad X, Y = I, II.$$

Here φ is an arbitrary real number, $i = \sqrt{-1}$, and $\epsilon = 1$ and $\epsilon = -1$. Hence

$$('U_\lambda^I + \epsilon i'U_\mu^{II})('U_\mu^I - \epsilon i'U_\mu^{II}) = (U_\lambda^I + \epsilon iU_\mu^{II})(U_\mu^I - \epsilon iU_\mu^{II}),$$

or

$$(3.3a) \quad 'U_\lambda^I 'U_\mu^{II} = U_\lambda^I U_\mu^{II}.$$

On the other hand all couples of harmonic polar planes incident with B constitute a hyperbolic involution. Hence if U_λ^{III} and U_μ^{IV} is one couple, any other couple $'U_\lambda^{III}, 'U_\mu^{IV}$ satisfying the equations

$$'U_\lambda^X 'U_\mu^Y h^{\lambda\mu} = h^{XY}, \quad X, Y = III, IV,$$

is given by

$$(3.2b) \quad 'U_\lambda^{III} + \epsilon'U_\mu^{IV} = e^{i\psi}(U_\lambda^{III} + \epsilon U_\mu^{IV}),$$

where ψ is an arbitrary real number and $\epsilon = 1$ and $\epsilon = -1$. These equations

¹⁰ Cf. V. HLAVATÝ, *Differential Line Geometry* (Noordhof, 1953) p. 17, Definition (1.1).

lead again to

$$(3.3b) \quad {}'U_{[\lambda}{}^{III} {}'U_{\mu]}{}^{IV} = U_{[\lambda}{}^{III} U_{\mu]}{}^{IV}.$$

5) The expressions $U_{[\lambda}{}^I U_{\mu]}{}^{II}$ ($U_{[\lambda}{}^{III} U_{\mu]}{}^{IV}$) may be looked upon as the Plücker plane coordinates of the line A (B). The lines A , B being conjugate polar lines with respect to K , the linear line congruence C with the axis A , B is contained in K , and consequently the Plücker coördinates of an arbitrary non-special linear line complex containing C are

$$\rho U_{[\lambda}{}^I U_{\mu]}{}^{II} + \sigma U_{[\lambda}{}^{III} U_{\mu]}{}^{IV},$$

where ρ and σ are arbitrary numbers both different from zero. Hence the equations (1.11a) hold for our complex K , with $\alpha\beta \neq 0$. The transformations (3.2) constitute a group of special Lorentz transformations and the equations (3.3) together with (1.11a) show that α , β are invariant with respect to this group.

The remaining part of Lemma (3.1) is almost obvious.

6) Let us now consider the case $K = 0$. Then the complex K is a special one and the Plücker coordinates of its axis are $k_{\lambda\mu}$. If this axis is incident with two complex conjugate (two real distinct) tangential planes to H we denote it by A (B), and by B (A) its harmonic polar line with respect to H . Repeating step by step the previous considerations, we obtain (1.11a) for $\beta = 0$ (for $\alpha = 0$). The remaining part of the Lemma is almost evident.

Now we are ready to complete the proof of our main Theorem (1.1) for the case $D \neq 0$. If $g_{\lambda\mu}$ is given, then we first determine the U_{λ}^x as indicated, then find the U_x^r by means of (1.9b) and finally define g_{xy} by means of

$$(3.4) \quad g_{xy} = g_{\lambda\mu} U_x^{\lambda} U_y^{\mu}.$$

According to the previous Lemma ((g_{xy})) is given by (1.18a) (for which Theorem (2.1) holds). Applying then (1.19) and substituting for $S_{\omega\mu}^r$ in (1.6a) we obtain $\Gamma_{\lambda\mu}^r$ as a solution of (1.4).

It is obvious from (1.19) that the system (1.6c) is uniquely soluble for $S_{\omega\mu\nu}$ if and only if (1.17) admits exactly one solution S_{xyz} . Necessary and sufficient conditions for the uniqueness of S_{xyz} are stated in Theorem (2.1).

Let us now consider the case $D = 0$.

Lemma (3.2). *Suppose $D = 0$. Then there is at least one non-holonomic coordinate system for which*

$$(3.5a) \quad ((g_{xy})) = \left(\left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & \gamma & \epsilon\gamma \\ 0 & -\gamma & -1 & 0 \\ 0 & -\epsilon\gamma & 0 & 1 \end{pmatrix} \right) \right) (\epsilon = \pm 1)$$

and

$$(3.5b) \quad |\gamma| < \rho,$$

where $\rho > 0$ is any arbitrarily preassigned number.

The proof consists of two parts.

1) If $D = 0$, then we must have also $K = 0$. Hence the complex K is a special one, its axis has the Plücker coordinates $k_{\lambda\mu}$ and is tangent to H at a point Q . Denote by A an *arbitrary* real non-tangent line to H , incident with Q and by B its harmonic polar line with respect to H . A consideration similar to the previous one enables us to find the U_λ^x , as well as to prove that they satisfy (1.9a), (3.2), and (3.3).

2) The unique tangent plane through $k_{\lambda\mu}$ to H is obviously $U_\lambda^{III} + \epsilon U_\lambda^{IV}$, where ϵ is either $+1$ or -1 . Therefore we may assume without loss of generality that the Plücker coordinates $k_{\lambda\mu}$ of the axis are given by (1.11b). Consider now transformation (3.2) with $\varphi = 0$ and $\psi \neq 0$. Then we obtain

$$\begin{aligned}\frac{1}{2}k_{\lambda\mu} &= \gamma U_{[\lambda}^{II} (U_{\mu]}^{III} + \epsilon U_{\mu]}^{IV}), \\ &= {}'\gamma {}'U_{[\lambda}^{II} ({}'U_{\mu]}^{III} + \epsilon {}'U_{\mu]}^{IV}), \\ &= {}'\gamma U_{[\lambda}^{II} (U_{\mu]}^{III} + \epsilon U_{\mu]}^{IV}) e^{\epsilon\psi},\end{aligned}$$

so that

$$(3.6) \quad {}'\gamma = e^{-\epsilon\psi} \gamma.$$

Hence if we choose

$$\epsilon\psi > \ln \frac{|\gamma|}{\rho}$$

we have

$$(3.7) \quad |{}'\gamma| < \rho.$$

Omitting the primes, we obtain very easily the statement of the Lemma (3.2).

We are now ready to complete the proof of our main Theorem (1.2) for the case $D = 0$. If $g_{\lambda\mu}$ is given we find first the U_λ^x and the U_x^r as indicated before and define g_{xyz} by (3.4). Its matrix is by virtue of Lemma (3.2) of the form (1.18b). Choosing conveniently the non-holonomic coordinate system, we obtain (3.7), and, omitting the primes, we have the case mentioned in Theorem (2.2). The remaining part of the proof of Theorem (1.2) follows the same pattern as the corresponding part of the proof of Theorem (1.1).

Remark. As far as the actual calculation of $S_{\omega\mu\lambda}$ is concerned the situation may be described as follows: In the case $D \neq 0$ the U 's and therefore also the components S_{xyz} are expressed in terms of given data $g_{\lambda\mu}$ only. This holds also for the components $S_{\omega\mu\nu}$ given by (1.19). If however $D = 0$, then the construction of the line A (mentioned in the proof of the Lemma (3.2)) brings foreign elements into consideration. Therefore the U_λ^x 's are expressed in terms of $g_{\lambda\mu}$ and these foreign elements. This holds also for the components S_{xyz} , while the components $S_{\omega\mu\nu}$ defined by (1.19) are expressible in terms of given data $g_{\lambda\mu}$ only.