On an Inequality for Minimal Surfaces $z = z(x, y)$

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Consider a minimal surface S of the form $z = z(x, y)$ defined for a circle $(x - x_0)^2 + (y - y_0)^2 < R^2$, i.e. a solution of the equation

$$(1 + p^2)r - 2pqrs + (1 + q^2)t = 0$$

of class C'' in that circle. We prove that there is a constant κ , independent of R and of the special minimal surface, such that in the center of circle

$$(1) \quad |K| \leq \kappa \frac{\zeta^2(1 + \zeta^2)}{R^2}, \quad \zeta = \frac{1}{\sqrt{1 + p^2 + q^2}},$$

holds for the Gaussian curvature K of every such surface S .

This inequality contains the remarkable inequality which was recently proved by E. Heinz¹ and in which the slope factor is absent. (1) shows that K_0 is small not only if R is large but also if the slope is large. A conjecture about the best value of κ in (1) is stated at the end of this note.

We may, of course, assume that $x_0 = y_0 = 0$. The proof of (1) makes use of the following facts.

a) If S is a minimal surface $z(x, y)$ of class C'' above a closed circle $x^2 + y^2 \leq R_1^2$ then there exists a one-to-one and continuous mapping of S upon the plane unit circle $\alpha^2 + \beta^2 \leq 1$,

$$(2) \quad x = y = 0 \quad \text{if} \quad \alpha = \beta = 0,$$

such that $x(\alpha, \beta)$, $y(\alpha, \beta)$, $z(\alpha, \beta)$ are harmonic in $\alpha^2 + \beta^2 < 1$ and that (conformality)

$$(3) \quad ds^2 = dx^2 + dy^2 + dz^2 = E(\alpha, \beta)(d\alpha^2 + d\beta^2), \quad (E > 0).$$

¹ E. HEINZ: Über die Lösungen der Minimalflächengleichung. Nachrichten der Akad. d. Wiss. in Göttingen, Math.-Phys. Klasse 1952 Nr. 8, p. 51-56.

This means that

$$(4) \quad \begin{aligned} x_\alpha^2 + y_\alpha^2 + z_\alpha^2 &= x_\beta^2 + y_\beta^2 + z_\beta^2 = E \\ x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta &= 0. \end{aligned}$$

b) *Basic lemma of Heinz.* If $x(\alpha, \beta), y(\alpha, \beta)$ are harmonic in $\alpha^2 + \beta^2 < 1$, continuous in $\alpha^2 + \beta^2 \leq 1$,

$$x(0, 0) = y(0, 0) = 0,$$

and if these functions map the circle $\alpha^2 + \beta^2 \leq 1$ in a one-to-one way upon a circle $x^2 + y^2 \leq R_1^2$, then

$$x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2 \geq \lambda R_1^2$$

where $\lambda > 0$ is independent of R_1 and of the special mapping.

c) The unit normals of a minimal surface $S, n = (\xi, \eta, \zeta)$

$$(5) \quad \xi = -\frac{p}{\sqrt{1+p^2+q^2}}, \quad \eta = -\frac{q}{\sqrt{1+p^2+q^2}}, \quad \zeta = \frac{1}{\sqrt{1+p^2+q^2}}$$

drawn from a fixed point O map S conformally upon part of the unit sphere,

$$(6) \quad dn^2 = d\xi^2 + d\eta^2 + d\zeta^2 = |K| ds^2.$$

That the factor must equal the absolute value of the Gauss curvature follows from the well known fact that $K = d\Sigma/ds$ = quotient of corresponding elements of area on $\xi^2 + \eta^2 + \zeta^2 = 1$ and on S . The simple reason for the well known fact c) itself is this. An infinitesimal circle on S is mapped on an infinitesimal ellipse on the unit sphere with its semi-axes proportional to the principal curvatures. But these are equal in absolute value since S has mean curvature zero.

We also use two more facts the first of which is elementary whereas the second one is the well-known geometric form (due to Pick) of Schwarz's lemma on analytic functions.

d) The stereographic projection of the unit sphere from $\xi = \eta = 0, \zeta = -1$, upon a u - v -plane

$$u = \frac{\xi}{1+\zeta}, \quad v = \frac{\eta}{1+\zeta}, \quad u^2 + v^2 = \frac{1-\zeta}{1+\zeta}$$

is one-to-one and conformal. It maps the upper hemisphere $\zeta > 0$ upon the circle $u^2 + v^2 < 1$. We have $dn^2 = (1+\zeta)^2 (du^2 + dv^2)$.

e) If $w = w(\gamma)$ is regular analytic in $|\gamma| < 1$ and if $|w| < 1$, then

$$\frac{|dw|}{1-|w^2|} \leq \frac{|d\gamma|}{1-|\gamma|^2}.$$

Proof of (1). We consider the portion of the minimal surface above $x^2 + y^2 \leq R_1^2 < R^2$. a), c) and d) induce a conformal mapping of the circle $\alpha^2 + \beta^2 < 1$ upon the circle $u^2 + v^2 < 1$ which may be described by a single-valued complex function

$$w = w(\gamma), \quad \gamma = \alpha + i\beta, \quad w = u + iv.$$

w is regular and analytic in γ or in $\bar{\gamma}$. In either case e) and d) imply that

$$\frac{|dn|}{|d\gamma|} \leq \frac{2\xi}{1 - |\gamma|^2}.$$

From (3) and (6),

$$(7) \quad |K| = \frac{dn^2}{ds^2} = \frac{dn^2}{|d\gamma|^2} \cdot \frac{|d\gamma|^2}{ds^2} \leq \frac{4\xi^2}{E(1 - \gamma\bar{\gamma})^2}.$$

A lower bound for E is found in the following way. (3) induces the corresponding area quotient

$$\frac{dS}{d\alpha d\beta} = E.$$

But

$$\frac{dS}{dx dy} = 1 + p^2 + q^2 = \frac{1}{\xi}.$$

Hence

$$(8) \quad E = \frac{1}{\xi} \frac{dx dy}{d\alpha d\beta} = \frac{1}{\xi} |x_\alpha y_\beta - x_\beta y_\alpha|.$$

On the other hand, by means of

$$(x_\alpha y_\beta - x_\beta y_\alpha)^2 + (x_\alpha x_\beta + y_\alpha y_\beta)^2 = (x_\alpha^2 + y_\alpha^2)(x_\beta^2 + y_\beta^2)$$

z_α, z_β can be eliminated from (4). The result is that

$$(9) \quad E = \frac{x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2}{1 + \xi^2}.$$

On applying b) we now obtain from (7) and (9) at $\alpha = \beta = \gamma = 0$, i.e., at $x = y = 0$

$$|K| \leq \frac{4\xi^2(1 + \xi^2)}{\lambda R_1^2}.$$

On letting $R_1 \rightarrow R$ we infer that (1) holds with the constant

$$\kappa = \frac{4}{\lambda}$$

λ being the constant in Heinz's lemma b).

Remarks. The greatest λ for which b) is generally valid is presumably obtained for a degenerate harmonic mapping which maps $\alpha^2 + \beta^2 < 1$ upon a diameter of $x^2 + y^2 < R_1^2$, for instance, if

$$x = \frac{2R_1}{\pi} I\left(\ln \frac{\gamma - 1}{\gamma + 1}\right), \quad y = 0.$$

In this case

$$x_\alpha^2 + x_\beta^2 + y_\alpha^2 + y_\beta^2 = \frac{16}{\pi^2} R_1^2, \quad \alpha = \beta = 0.$$

So the conjecture is that $\lambda = 16/\pi^2$ is the best value in b). This would give $\kappa = \pi^2/2$ in (1) which is presumably the best value of κ . The question of the best κ in (1) does not seem to be of great interest. Minimal surfaces which approach this hypothetical value of κ presumably approach a vertical plane. Of more interest is the question of the best constant μ in Heinz's inequality $|K| \leq \mu/R^2$ and of the corresponding extremal surface S . Does such an extremizing S exist in $x^2 + y^2 < R^2$? An obvious exact inference from (1) is at least that a sequence of admissible surfaces S which approach the best μ must have bounded slopes at $x = y = 0$.