*Integration Formulae and Boundary Conditions for the
Hyperbolic Equation with Three Independent
Variables and Regions Interior to the Cone**

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The use of an integrating factor in solving partial differential equations originated with Green in his study of the Laplace equation. In extending Green's ideas to the hyperbolic equation for $n = 2$, Riemann discovered that characteristics should partially bound the region of integration and that in general the Green potential involves a non-Euclidean distance. Hadamard¹ extended the Green potential to the general linear equation with variable coefficients.

Curiously enough waves in one and three space dimensions were treated by a Green method much earlier than waves in two space dimensions. No doubt one of Volterra's² great contributions was overcoming the difficulty of divergent integrals that are inherent in the latter problem. His method involved integrating the Green potential in a time-like direction, thus distributing the infinity over the axis of the cone. He was then able to enclose the axis of the cone by a cylinder in the Green manner without encountering divergent integrals. The integrating factor used in this paper is obtained by integrating the potential over a surface, thus eliminating the infinity entirely and replacing it by a finite jump discontinuity in one of the partial derivatives.

Integration of potentials naturally suggests the use of integration by parts in the differentiation of improper integrals. The application of differential opera-

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¹ See **II**. (Bold face Roman numerals refer to the bibliography.)

² See **X**.

tors to integrals over arbitrary regions becomes feasible with a set of Leibnitz rules for multiple integrals developed in another paper.¹ Along with this goes the use of basic mixed Pythagorean identities² to prepare the integrals for integration by parts. The need for such a process is emphasized by the fact that Volterra and most other writers on hyperbolic equations leave derivatives of integrals in their integration formulae.

One might expect that in integration by parts in surface integrals the integrations must be done with respect to parameters on the surface. In fact, the above mentioned Pythagorean identities are used to separate the tangential components of the derivative to be integrated. The effect of the process we use, however, is to transfer derivatives in non-tangential directions from one factor to another. The latter is accomplished through the use of other Pythagorean identities, the differential equation, and its adjoint in rewriting the result of the integration by parts. This gives us the freedom, so to speak, to integrate by parts in the same direction on all parts of the bounding surface. This apparently results in more compact formulae, with fewer integrals of lower order.

In this paper we devote considerable space to comparison of integration formulae obtained by using integration by parts in various directions. We find that in case the integration by parts is done in space-like directions the integration formula contains D'Alembert averaging terms. In fact the connection with hyperbolic $n = 2$ goes much deeper than this since we actually quote results from $n = 2$ and also investigate conditions under which our final integration formula reduces to the classical formula for $n = 2$. These connections involve the process of descent.

When we integrate by parts in a time-like direction, we find that the averaging points coalesce in the point where the ray from the vertex of the cone in the time-like direction pierces the portion of the boundary bearing the data of the problem. Although the formulae in this paper are in general more compact they specialize to formulae similar to those previously obtained for the undamped equation in the lecture notes of E. W. Titt and the thesis of W. C. Long.³ Long considered two types of boundaries and demonstrated the passage from a time-like formula to Hadamard's⁴ formula with a line integral using an integration by parts and the mean value theorem.

¹ Besides the paper "Leibnitz Rules for Differentiating Multiple Integrals over Arbitrary Regions—Extended to Cases Where the Classical Rule Yields Divergent Integrals" [see *Bull. Am. Math. Soc.* Abstract 57-4-306] we have occasion to refer to the paper IX published in this volume and a paper "Classification of Partial Differential Equations and Their Associated Non-Euclidean Geometries" in preparation. These papers will be referred to as [L.], [A.P.], and [G.], respectively. The notation of the vector algebras in IX is used throughout this paper. The Office of Naval Research contributed to the support of this work during 1947-48 and 1949-50.

² See IX.

³ See III. One of the time-like formulae in LONG's paper was given independently by M. H. MARTIN; see V.

⁴ See II, p. 208.

Whereas Green's ideas have their origin in the elliptic equation our method definitely has its origin in the hyperbolic equation and thus it is natural that hyperbolic interior $n = 3$ should have played a prominent role in pioneering the work. This paper has appeared in three previous editions¹ over a period of four years. In this particular edition we have included a boundary condition obtained by cross integration by parts in place of the repeated integration by parts used in the preceding edition. The investigation of cross integration was undertaken for comparison purposes with recent results in hyperbolic $n = 4$.

1. Formulation of the Problem and Integrals of Potentials. In this paper we derive integration formulae and boundary conditions for the partial differential equation with constant coefficients as follows:

$$(1.1) \quad \begin{aligned} L(u) &\equiv A^{ij}u_{ij} + B^i u_i + Cu = f(x^i) \quad (i, j, k = 1, 2, 3), \\ L^*(v) &\equiv A^{ij}v_{ij} - B^i v_i + Cv = 0 \quad (A^{ij} = A^{ji}), \end{aligned}$$

where (1.1b) is adjoint to (1.1a). To restrict our discussion to equations of hyperbolic type, we impose the conditions [G.]

$$(1.2) \quad |A^{ij}| \neq 0; \quad A_{ij}x^i x^j = 0 \text{ for some vector } x^i \quad (A^{ij}A_{jk} = \delta_k^i).$$

We use two of the restricted algebras [A.P.], namely, N-P, N-C, and N-P, C. For reference we include the N-P, N-C Pythagorean identity

$$(1.3) \quad A_{ij}x^i y^j = H_{ij}x^i y^j + \frac{(L_i x^i)(L_i y^i)}{\omega}.$$

The results of our derivation are integration formulae giving the value of the solution $u(\xi^i)$, where the ξ^i denote the coordinates of the vertex of the (Greek) characteristic cone $A_{ij}(x^i - \xi^i)(x^j - \xi^j) = 0$. We integrate over a region interior to the cone, bounded partially by one nappe of the cone and partially by an arbitrary surface Q_2 , to obtain formulae for $u(\xi^i)$ in terms of boundary values over Q_2 . The condition that a point x^i lie inside the Greek cone is given by [G.]

$$(1.4) \quad \tilde{A}_{ij}X^i X^j > 0 \text{ for a vector } X^i \text{ inside the cone} \\ (X^i = x^i - \xi^i; \tilde{A}_{ij} = |A^{ij}| A_{ij}).$$

Our integrating factor v has a jump discontinuity in one of its partial derivatives at a fixed, non-characteristic (splitting) plane. This is any plane that cuts the Greek cone in two real lines. We use $L_i(x^i - \xi^i) = 0$ for the equation of this splitting plane, where L_i belongs to an N-P, N-C algebra. The non-Euclidean

¹ See IV.

normal to the splitting plane, namely L^i , must lie outside the cone. This leads to the following restriction on our algebra [see (1.4)].

$$(1.5) \quad \tilde{A}_{ij}L^iL^j = |h^{\alpha\beta}| < 0 \quad (\alpha, \beta = 1, 2).$$

We determine j where $j^2 = 1$ so that

$$(1.6) \quad j|A^{ij}| > 0; \quad \text{hence } |\omega| = -j\omega \quad (\omega = |A_{ij}| |h^{\alpha\beta}|).$$

In view of (1.4) we have

$$(1.7) \quad jA_{ij}X^iX^j > 0, \quad jH_{ij}X^iX^j > 0 \text{ for vectors } X^i \text{ inside the cone.}$$

The latter follows from (1.3) by introducing the factor j and taking $x^i = y^i = X^i$.

To construct our integrating factor we use the Green potential of the second type, given by [A.P.]

$$(1.8) \quad \begin{aligned} V &= \exp[-\alpha_i X^i] \Phi; & \Phi &= \frac{\cosh \delta R}{R}; & R &= \sqrt{jA_{ij}x^ix^j}; \\ S &= \sqrt{jH_{ij}X^iX^j}; & \alpha_i &= -\frac{1}{2}A_{ij}B^j; & \delta &= \sqrt{jJ(-\frac{1}{4}A_{ij}B^iB^i + C)}. \end{aligned}$$

When $J = 1$ in (1.8f) we are concerned with an oscillating function. Using (1.8e) and (1.8f), equations (1.1) become

$$(1.9) \quad \begin{aligned} L(u) &\equiv [A^{ij}(D_{x^i} - \alpha_i)(D_{x^j} - \alpha_j) + jJ\delta^2]u = f, \\ L^*(v) &\equiv [A^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j) + jJ\delta^2]v = 0. \end{aligned}$$

Sign changes are necessary when differentiating with respect to Latin variables in (1.9b). We introduce non-Euclidean rectilinear coordinates and non-Euclidean damping factors as follows:

$$(1.10) \quad \begin{aligned} p^\alpha &= l_i^\alpha X^i; & P &= \frac{\sqrt{-j\omega}}{\omega} L_i X^i; & R^2 &= S^2 - P^2 \\ & & & & & \text{(Pythagorean theorem);} \\ a_\beta &= l_\beta^i \alpha_i; & a_n &= \frac{L^i \alpha_i}{\sqrt{-j\omega}}. \end{aligned}$$

In our method we first construct an integrating factor, φ , for equation (1.1a) without exponential damping, that is, $B^i = 0$. The method of construction involves integrating the potential Φ over a region Q lying in the splitting plane. This region Q is bounded by the Greek cone $A_{ij}(\bar{x}^i - \xi^i)(\bar{x}^j - \xi^j) = 0$ and the Latin cone $A_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j) = 0$, where the $\bar{x}^i$ denote current coordinates and the x^i (Latin vertex) denote the coordinates of any point interior to the Greek cone. The splitting plane intersects the Greek cone in two real lines, and the Latin cone in a hyperbola with center at c^i . The point denoted by c^i is the foot

of the non-Euclidean perpendicular from x^i to the splitting plane, with coordinates [G.]

$$(1.11) \quad c^i = x^i - \frac{L^i P}{\sqrt{-j\omega}} = \xi^i + l_\alpha^i p^\alpha.$$

The region Q is a portion of the non-Euclidean square with opposite vertices at c^i and ξ^i , and having sides of zero length.

The function ϕ is a weighted non-Euclidean area [G.] of the above-described region Q , the weight function being Φ . In the integrand we replace the variables X^i by $x^i - \bar{x}^i$ in Φ of (1.8b). In symbols

$$(1.12) \quad \phi = \frac{1}{2} \iint_Q \Phi(x^i - \bar{x}^i) dA.$$

In treating the integral (1.12) we introduce, in the splitting plane, a coordinate system referred to the two generators of the Greek cone. For characterizing the directions of these generators here and elsewhere, we use an N-P, C algebra having basic vectors N^α , n_1^α , with metric $h^{\alpha\beta}$. Then $N^\alpha l_\alpha^i$ and $n_1^\alpha l_\alpha^i$ lie in the cone since $A_{ij}(N^\alpha l_\alpha^i)(N^\beta l_\beta^j) = h_{\alpha\beta} N^\alpha N^\beta = 0$, the latter being a property of an N-P, C algebra; a similar statement applies to n_1^α . In order to direct the vectors $N^\alpha l_\alpha^i$, $n_1^\alpha l_\alpha^i$ into the same nappe of the Greek cone we must determine the sign in our rules of indices as follows [A.P. (4.12)]:

$$(1.13) \quad h^{\alpha\beta} \frac{N_\beta}{\bar{\omega}} = j \sqrt{-|h^{\alpha\beta}|} n_1^\alpha; \quad h^{\alpha\beta} n_{\beta 1} = j \sqrt{-|h^{\alpha\beta}|} N^\alpha.$$

To verify (1.13) we note that $jA_{ij}(N^\alpha l_\alpha^i)(n_1^\beta l_\beta^j) = jh_{\alpha\beta} N^\alpha n_1^\beta > 0$ is the condition for the vectors $N^\alpha l_\alpha^i$, $n_1^\alpha l_\alpha^i$ to lie in the same nappe [see (1.7)]. This condition can be compared with (1.13b) by introducing the factor $h_{\alpha\gamma} n_1^\gamma$ in the latter. In labeling our drawings we use the fact that the vectors L^i , $N^\alpha l_\alpha^i$, $n_1^\alpha l_\alpha^i$ of the compound basic algebra are positively oriented.

For parameter λ^2 in the splitting plane we take the non-Euclidean length of the vector $x^i - \bar{x}^i$, and for parameter λ^1 we use a scalar product.

$$(1.14) \quad \begin{aligned} \lambda^1 &= \lambda = \frac{N_\alpha}{\bar{\omega}} l_\alpha^i (x^i - \bar{x}^i), \\ \lambda^2 &= \mu = \sqrt{jA_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j)}, \quad L_i(\bar{x}^i - \xi^i) = 0. \end{aligned}$$

Using the N-P, C Pythagorean identity, namely

$$(1.15) \quad jh_{\alpha\beta} l_\alpha^i (x^i - \bar{x}^i) l_\beta^j (x^j - \bar{x}^j) = 2 \sqrt{-|h_{\alpha\beta}|} \frac{N_\alpha}{\bar{\omega}} l_\alpha^i (x^i - \bar{x}^i) n_{\beta 1}^j (x^j - \bar{x}^j),$$

the Pythagorean identity (1.3) can be expressed in terms of λ , μ as follows:

$$(1.16) \quad \mu^2 = 2 \sqrt{-|h_{\alpha\beta}|} n_1^\alpha l_\alpha^i (x^i - \bar{x}^i) \lambda - P^2.$$

FIGURE 1.1

The relation (1.16) is used in mapping the boundaries of Q .

$$\begin{aligned}
 A_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j) &= 0 \rightarrow \mu = 0, \\
 jA_{ij}(\bar{x}^i - \xi^i)(\bar{x}^j - \xi^j) \\
 (1.17) \quad &= 2\sqrt{-|h_{\alpha\beta}|} \frac{N_\alpha}{\bar{\omega}} l_i^\alpha (\bar{x}^i - x^i + X^i) n_\beta^1 l_j^\beta (\bar{x}^j - x^j + X^j) \\
 &\rightarrow (\lambda - p) \left(\frac{\mu^2 + P^2}{2\lambda\sqrt{-|h_{\alpha\beta}|}} - q \right) = 0.
 \end{aligned}$$

The notation

$$(1.18) \quad p = \frac{N_\alpha}{\bar{\omega}} l_i^\alpha X^i, \quad q = n_\alpha^1 l_i^\alpha X^i,$$

introduced here is used only in the present discussion. Thus, the boundaries of the region Q in Figure 1.1b are $\mu = 0$, $\lambda = p$, and $\lambda = (\mu^2 + P^2)/2q \sqrt{-|h_{\alpha\beta}|}$. For convenience in sketching we assume the vectors $N^\alpha l_\alpha^i$, $n_1^\alpha l_\alpha^i$ are directed into the nappe containing x^i . Here this means $p > 0$, $q > 0$; for example,

$$(1.19) \quad n_1^\alpha l_\alpha^i X^i = j \sqrt{-|h^{\alpha\beta}|} h_{\alpha\beta} N^\alpha l_\alpha^i X^i = j \sqrt{-|h^{\alpha\beta}|} A_{ij} N^\alpha l_\alpha^i X^j > 0,$$

where the last inequality is the condition for acuteness.

The element of area for a plane is given by

$$(1.20) \quad dA = \pm \frac{|^{\alpha}h_{\beta}|}{\sqrt{-|h^{\alpha\beta}|}} d\lambda d\mu,$$

where $^{\alpha}h_{\beta} = l_i^\alpha \lambda_\beta^i$, $\lambda_\beta^i = \bar{x}^i_{\lambda\beta}$. To compute $|^{\alpha}h_{\beta}|$ we use (1.14) and (1.16).

$$(1.21) \quad |^{\alpha}h_{\beta}| = \left| \frac{N_\alpha}{\bar{\omega}} \right| \left| l_i^\alpha \lambda_1^i \right| = \left| \frac{N_\alpha}{\bar{\omega}} l_i^\alpha \lambda_\beta^i \right| = \left| \begin{array}{cc} -1 & 0 \\ \cdots & -\mu \\ \lambda \sqrt{-|h_{\alpha\beta}|} & \end{array} \right| = \frac{\mu}{\lambda \sqrt{-|h_{\alpha\beta}|}}.$$

In terms of λ , μ parameters, $\Phi(x^i - \bar{x}^i) = \frac{\cosh}{\cosh} \delta\mu/\mu$, and (1.12) becomes

$$(1.22) \quad \begin{aligned} \phi &= \int_0^R \int_{\frac{\mu^2 + P^2}{2q\sqrt{-|h_{\alpha\beta}|}}}^p \frac{\frac{\cosh}{\cosh} \delta\mu}{2\lambda} d\lambda d\mu = \int_0^R \ln \frac{S}{\sqrt{\mu^2 + P^2}} \frac{\cosh}{\cosh} \delta\mu d\mu \\ &= \int_{|P|}^S \nu \ln \frac{S}{\nu} \frac{\frac{\cosh}{\cosh} \delta \sqrt{\nu^2 - P^2}}{\sqrt{\nu^2 - P^2}} d\nu, \end{aligned}$$

since $S^2 = 2pq \sqrt{-|h_{\alpha\beta}|}$ by (1.15).

In our method we require an integrating factor v with the following principal properties:

Property I: v is a function of X^i which satisfies the adjoint (1.1b) with respect to the Latin variables.

Property II: v and its first partial derivatives are zero when the point x^i lies on the Greek cone.

Property III: The directional (non-Euclidean normal) derivative of v , namely $(L^i/\sqrt{-j\omega})(D_{x^i} + \alpha_i)v$, has a jump discontinuity as the point x^i crosses the splitting plane. The amount of the discontinuity involves a distortionless Green potential for hyperbolic $n = 2$.

Property IV: The operator $\Delta = jH^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j)$ turns v into V .

For our integrating factor v we take

$$(1.23) \quad v = e^{-\alpha_i X^i} \int_0^R \ln \frac{S}{\sqrt{\mu^2 + P^2}} \frac{\cosh}{\cosh} \delta\mu d\mu.$$

In discussing Property I we observe that R , S , and P are functions of X^i and hence v given by (1.23) is also a function of differences X^i . By postponing the proof that v satisfies the adjoint equation the proof can be inferred from the same property of one of the upper index symbols treated below.

Since $R = 0$ when x^i lies on the Greek cone, the proof of Property II is immediate from (1.23).

To demonstrate Property III we compute the derivative of v in the direction of the non-Euclidean normal to the splitting plane and integrate by parts.

$$\begin{aligned}
 (1.24) \quad & \frac{L^i}{\sqrt{-j\omega}} (D_{x^i} + \alpha_i) v = e^{-\alpha_i X^i} \int_0^R \frac{-P}{\mu^2 + P^2} \frac{\cosh}{\cosh} \delta\mu \, d\mu \\
 & = e^{-\alpha_i X^i} \left\{ \int_0^R \arctan \frac{P}{\mu} J \delta \frac{\sinh}{\sinh} \delta\mu \, d\mu + \arctan \frac{P}{R} \frac{\cosh}{\cosh} \delta R - \frac{\pi}{2} \frac{|P|}{P} \right\}.
 \end{aligned}$$

Letting $P \rightarrow 0$ in (1.24) we obtain $\pi e^{-\alpha_i X^i}$ for the amount of the discontinuity.

$$(1.25) \quad \lim_{P \rightarrow 0} \frac{L^i}{\sqrt{-j\omega}} (D_{x^i} + \alpha_i) v = \lim_{P \rightarrow 0} -\frac{\pi}{2} \frac{|P|}{P} e^{-\alpha_i X^i}.$$

In order to treat Property IV we need to know the behavior of the Δ operator when applied to a damped function of S and P . [See **A.P.** (9.8).]

$$\begin{aligned}
 (1.26) \quad & \Delta e^{-\alpha_i X^i} \phi(S, P) = j H^{ij} (D_{\xi^i} - \alpha_i) (D_{\xi^j} - \alpha_j) e^{-\alpha_i X^i} \phi \\
 & = e^{-\alpha_i X^i} j H^{ij} D_{\xi^i} D_{\xi^j} \phi = e^{-\alpha_i X^i} \left(D_{ss} + \frac{1}{S} D_s \right) \phi.
 \end{aligned}$$

When applied to (1.23) this yields

$$\begin{aligned}
 (1.27) \quad & \Delta v = e^{-\alpha_i X^i} \left(D_{ss} + \frac{1}{S} D_s \right) \int_0^R \ln \frac{S}{\sqrt{\mu^2 + P^2}} \frac{\cosh}{\cosh} \delta\mu \, d\mu \\
 & = e^{-\alpha_i X^i} \left\{ \int_0^R \left(D_{ss} + \frac{1}{S} D_s \right) \ln \frac{S}{\sqrt{\mu^2 + P^2}} \frac{\cosh}{\cosh} \delta\mu \, d\mu + \frac{1}{S} \frac{S}{R} \frac{\cosh}{\cosh} \delta R \right\} \\
 & = e^{-\alpha_i X^i} \frac{1}{R} \frac{\cosh}{\cosh} \delta R = V.
 \end{aligned}$$

We now consider further integrals of potentials which we indicate by upper indices P , p , and s . At the same time we introduce L , M , and m operators, and

the upper index mentioned above is used to indicate which of the operators is inverse to the integrations performed on the potential. In order to define these operators, together with the variables P, p, s , we introduce a second compound algebra with contravariant basis $L^i, M^\alpha l_\alpha^i, m_1^\alpha l_\alpha^i$. Whereas the compound algebra with basis $L^i, N^\alpha l_\alpha^i, n_1^\alpha l_\alpha^i$ used above has two of its basic vectors in the cone, the algebra introduced here has the vector $m_1^\alpha l_\alpha^i$ inside the cone, and the other two outside. The M algebra is taken to be N-P, N-C with metric $h_{\alpha\beta}$, while the N algebra is N-P, C with the same metric. The condition that the vector $m_1^\alpha l_\alpha^i$ lie inside the cone is given by $jA_{ij}(m_1^\alpha l_\alpha^i)(m_1^\beta l_\beta^j) = jh_{\alpha\beta}m_1^\alpha m_1^\beta = jg_{11} > 0$. The metric $h_{\alpha\beta}$ being of the type hyperbolic $n = 2$, the vector M is subject to $jh_{\alpha\beta}M^\alpha M^\beta = j\bar{\omega} < 0$, and the vector $M^\alpha l_\alpha^i$ lies outside the cone. For convenience in computation and in sketching we direct the vector $m_1^\alpha l_\alpha^i$ into the nappe of the cone containing the region of integration, which entails $m_1^\alpha l_\alpha^i X^i > 0$.

Throughout the rest of the paper we use variables defined as follows:

$$\begin{aligned}
 P &= \frac{\sqrt{-j\bar{\omega}}}{\omega} L_i X^i; & p &= \frac{\sqrt{-j\bar{\omega}}}{\bar{\omega}} M_\alpha l_\alpha^i X^i; & s &= \sqrt{jg_{11}} m_1^\alpha l_\alpha^i X^i; \\
 (1.28) \quad R &= \sqrt{jA_{ij}X^iX^j} = \sqrt{S^2 - P^2} = \sqrt{\bar{S}^2 - p^2} = \sqrt{s^2 - \bar{S}^2} \\
 &= \sqrt{s^2 - p^2 - P^2} \\
 S &= \sqrt{s^2 - p^2}; & \bar{S} &= \sqrt{s^2 - P^2}; & \bar{S} &= \sqrt{p^2 + P^2}.
 \end{aligned}$$

Introducing L, M , and m operators, we indicate their behavior when applied to damped functions of P, p, s .

$$\begin{aligned}
 (1.29) \quad \frac{L^i}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_i) e^{-\alpha_i X^i} F(s, p, P) &= e^{-\alpha_i X^i} [-D_P F(s, p, P)]; \\
 \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_i) e^{-\alpha_i X^i} F(s, p, P) &= e^{-\alpha_i X^i} [-D_p F(s, p, P)]; \\
 \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (D_{\xi^i} - \alpha_i) e^{-\alpha_i X^i} F(s, p, P) &= e^{-\alpha_i X^i} [-D_s F(s, p, P)].
 \end{aligned}$$

All the upper index symbols used in this paper satisfy the adjoint equation, and the limits of integration are so chosen that these symbols vanish on $R = 0$. In addition v, V^{ps} , and V^{pP} have first derivatives that vanish on $R = 0$. Each time we introduce an upper index P or p we introduce a jump discontinuity at the planes $P = 0$ or $p = 0$. Such is not the case for upper indices s , since the plane $s = 0$ lies outside the region.

We define the upper index symbols and tabulate certain of their properties.

$$\begin{aligned}
 V &= e^{-\alpha_i X^i} \frac{\cosh \delta R}{R}; & v &= e^{-\alpha_i X^i} \int_0^R \ln \frac{S}{\sqrt{\mu^2 + P^2}} \frac{\cosh \delta \mu}{\cosh \delta \mu} d\mu; \\
 V^P &= \frac{|P|}{P} e^{-\alpha_i X^i} \int_{|P|}^S \frac{\cosh \delta \sqrt{S^2 - \nu^2}}{\sqrt{S^2 - \nu^2}} d\nu = \frac{|P|}{P} e^{-\alpha_i X^i} \int_0^R \frac{\cosh \delta \mu}{\sqrt{S^2 - \mu^2}} d\mu; \\
 V^p &= \frac{|p|}{p} e^{-\alpha_i X^i} \int_{|p|}^S \frac{\cosh \delta \sqrt{S^2 - \nu^2}}{\sqrt{S^2 - \nu^2}} d\nu = \frac{|p|}{p} e^{-\alpha_i X^i} \int_0^R \frac{\cosh \delta \mu}{\sqrt{S^2 - \mu^2}} d\mu; \\
 V^s &= e^{-\alpha_i X^i} \int_s^{\bar{s}} \frac{\cosh \delta \sqrt{\nu^2 - \bar{S}^2}}{\sqrt{\nu^2 - \bar{S}^2}} d\nu = e^{-\alpha_i X^i} \int_0^R -\frac{\cosh \delta \mu}{\sqrt{\mu^2 + \bar{S}^2}} d\mu; \\
 V^{ps} &= \frac{|p|}{p} e^{-\alpha_i X^i} \int_0^R \int_{|p|}^{\sqrt{s^2 - \mu^2 - P^2}} -\frac{\cosh \delta \mu}{\sqrt{\nu^2 + \mu^2 + P^2}} d\nu d\mu \\
 (1.30) \quad &= \frac{|p|}{p} e^{-\alpha_i X^i} \int_0^R -\frac{\cosh \delta \mu}{\cosh \delta \mu} \\
 &\quad \cdot \left[\operatorname{arccosh} \frac{s}{\sqrt{\mu^2 + P^2}} - \operatorname{arcsinh} \frac{|p|}{\sqrt{\mu^2 + P^2}} \right] d\mu; \\
 V^{Pp} &= \frac{|Pp|}{Pp} e^{-\alpha_i X^i} \int_0^R \int_{|p|}^{\sqrt{s^2 - \mu^2 - P^2}} \frac{\cosh \delta \mu}{\sqrt{s^2 - \nu^2 - \mu^2}} d\nu d\mu \\
 &= \frac{|Pp|}{Pp} e^{-\alpha_i X^i} \int_0^R \frac{\cosh \delta \mu}{\cosh \delta \mu} \\
 &\quad \cdot \left[\arctan \frac{\sqrt{s^2 - \mu^2 - P^2}}{|P|} - \arctan \frac{|p|}{\sqrt{s^2 - p^2 - \mu^2}} \right] d\mu; \\
 v^p &= \frac{|p|}{p} e^{-\alpha_i X^i} \int_0^R \int_{|p|}^{\sqrt{s^2 - \mu^2 - P^2}} \frac{1}{2} \ln \frac{s^2 - \lambda^2}{\mu^2 + P^2} \frac{\cosh \delta \mu}{\cosh \delta \mu} d\lambda d\mu; \\
 \Delta v &= V; & \Delta v^p &= V^p.
 \end{aligned}$$

We now proceed to the proof that V^{ps} is a solution of the adjoint equation. Referring to table (1.30) we introduce the following symbolism in the single integral form of V^{ps} .

$$\begin{aligned} F &= \frac{\cos}{\cosh} \delta\mu; & F_{\mu\mu} + J\delta^2 F &= 0; \\ (1.31) \quad G &= -\operatorname{arccosh} \frac{s}{\sqrt{\mu^2 + P^2}}; & G_{ss} - G_{PP} - G_{\mu\mu} &= 0; & sG_s + PG_P + \mu G_\mu &= 0; \\ H &= \operatorname{arcsinh} \frac{|p|}{\sqrt{\mu^2 + P^2}}; & H_{\mu\mu} + H_{pp} + H_{PP} &= 0; & pH_p + PH_P + \mu H_\mu &= 0 \end{aligned}$$

In computing the adjoint expression we use all five of the above differential equations.

$$\begin{aligned} \frac{|p|}{p} e^{\alpha_s x^i} j L^*(V^{ps}) &= (D_{ss} - D_{pp} - D_{PP} + J\delta^2) \int_0^R F(G + H) d\mu \\ &= \int_0^R [F(G_{ss} - G_{PP} - H_{pp} - H_{PP}) + J\delta^2 F(G + H)] d\mu \\ (1.32) \quad &+ \left[F \left(\frac{s}{R} G_s + \frac{P}{R} G_P + \frac{p}{R} H_p + \frac{P}{R} H_P \right) \right]_{\mu=R} \\ &= \int_0^R [F(G_{\mu\mu} + H_{\mu\mu}) - F_{\mu\mu}(G + H)] d\mu + [F(-G_\mu - H_\mu)]_{\mu=R} \\ &= [F(G_\mu + H_\mu) - F_\mu(G + H)]_{\mu=0}^{\mu=R} - [F(G_\mu + H_\mu)]_{\mu=R} = 0. \end{aligned}$$

The discussion of V^{Pp} is almost identical to the above.

In showing that $L^*(v^p) = 0$ we obtain $\Delta v^p = V^p$ as a by-product. We introduce the symbolism

$$\begin{aligned} F &= \frac{\cos}{\cosh} \delta\mu; & F_{\mu\mu} + J\delta^2 F &= 0; \\ (1.33) \quad G &= \ln \sqrt{s^2 - \lambda^2}; & G_{ss} - G_{\lambda\lambda} &= 0; & sG_s + \lambda G_\lambda &= 1; \\ H &= -\ln \sqrt{\mu^2 + P^2}; & H_{\mu\mu} + H_{PP} &= 0; & \mu H_\mu + PH_P &= -1. \end{aligned}$$

The computation of Δv^p utilizes F and G differential equations.

$$\begin{aligned}
 \frac{|p|}{p} e^{\alpha_i x^i} \Delta v^p &= (D_{ss} - D_{pp}) \int_0^R \int_{|p|}^{\sqrt{S^2 - \mu^2}} F(G + H) d\lambda d\mu \\
 &= D_s \int_0^R \int_{|p|}^{\sqrt{S^2 - \mu^2}} F G_s d\lambda d\mu + \frac{|p|}{p} D_p \int_0^R [F(G + H)]_{\lambda=|p|} d\mu \\
 (1.34) \quad &= \int_0^R \left\{ [F G_\lambda]_{\lambda=|p|}^{\sqrt{S^2 - \mu^2}} + \left[\frac{F s G_s}{\sqrt{S^2 - \mu^2}} \right]_{\lambda=\sqrt{S^2 - \mu^2}} + [F G_\lambda]_{\lambda=|p|} \right\} d\mu \\
 &= \int_0^R \frac{\cos \delta \mu}{\sqrt{S^2 - \mu^2}} d\mu = \frac{|p|}{p} e^{\alpha_i x^i} V^p.
 \end{aligned}$$

In computing the remainder of the adjoint expression we use F and H differential equations.

$$\begin{aligned}
 (-D_{pp} + J\delta^2) \int_{|p|}^S \int_0^{\sqrt{S^2 - \lambda^2}} F(G + H) d\mu d\lambda \\
 = \int_{|p|}^S \int_0^{\sqrt{S^2 - \mu^2}} [F H_{\mu\mu} - F_{\mu\mu}(G + H)] d\mu d\lambda \\
 (1.35) \quad + \int_{|p|}^S \left[\frac{F P H_p}{\sqrt{S^2 - \lambda^2}} \right]_{\mu=\sqrt{S^2 - \lambda^2}} d\lambda \\
 = \int_{|p|}^S \left\{ [F H_\mu - F_\mu(G + H)]_{\mu=0}^{\sqrt{S^2 - \lambda^2}} + \left[\frac{F P H_p}{\mu} \right]_{\mu=\sqrt{S^2 - \lambda^2}} \right\} d\lambda \\
 = \int_{|p|}^S - \frac{\cos \delta \sqrt{S^2 - \lambda^2}}{\sqrt{S^2 - \lambda^2}} d\lambda = - \frac{|p|}{p} e^{\alpha_i x^i} V^p.
 \end{aligned}$$

Since (1.34) and (1.35) have opposite signs we have $L^*(v^p) = 0$. By interchange of order of operations all the other v symbols in table (1.30) satisfy the adjoint.

In some of our work we use the upper index symbol $\tilde{V}^p$ which is continuous at $p = 0$ and which does not vanish on $R = 0$. This symbol is defined by

$$(1.36) \quad \tilde{V}^p = V^p - V_0^p, \quad \text{where} \quad V_0^p = \frac{|p|}{p} e^{-\alpha_i x^i} \int_0^S \frac{\cos \delta \mu}{\sqrt{S^2 - \mu^2}} d\mu.$$

Note that we obtain V_0^p from (1.30) by putting $p = 0$ in the integral but not the exponential factor. Thus, the M operator annihilates the upper p index in the barred symbol just as in the unbarred symbol. That $\tilde{V}^p$ satisfies the adjoint equation follows from

$$(1.37) \quad L^*(V_0^p) = 0 \quad \text{or} \quad \left(D_{ss} + \frac{1}{S} D_s + J\delta^2 \right) \frac{|p|}{p} e^{\alpha_i x^i} V_0^p = 0.$$

From (1.37) it follows that V_0^p is a damped Bessel function of order zero, and we use table (1.42) below to obtain the exact relationship.

$$(1.38) \quad \frac{|p|}{p} e^{\alpha_i x^i} V_0^p = \int_0^s \frac{\cosh \delta \mu}{\sqrt{\bar{S}^2 - \mu^2}} d\mu = \frac{\pi}{2} \left\{ \frac{J_0(\delta \bar{S})}{I_0(\delta \bar{S})} \right\}.$$

That V_0^p is obtained by putting $p = 0$ is a manifestation of descent, and we see that descent furnishes an interesting approach to integral representations of the Bessel functions in (1.38).

Since the Riemann function for hyperbolic $n = 2$ is a damped Bessel function we can express (1.38) as a descent relationship with the symbolism $(|p|/p)V_0^p \searrow (\pi/2)V$. In Section 2 we have occasion to use the similar relation $(|p|/p)v_0^p \searrow (\pi/2)v$ which connects the integrating factors of our theory for hyperbolic interior $n = 3$ and $n = 2$. To prove the latter relation we start with v for $n = 2$ and reletter according to the correspondences of ascent: $R \nearrow \bar{S}$, $P \nearrow P$, $S \nearrow s$. We wish to show that

$$(1.39) \quad \frac{|p|}{p} e^{\alpha_i x^i} v_0^p = \frac{\pi}{2} \int_{|P|}^s (s - \nu) \left\{ \frac{J_0}{I_0} \right\} (\delta \sqrt{\nu^2 - P^2}) d\nu,$$

the right member¹ being the relettered v of $n = 2$. From (1.38) we have

$$(1.40) \quad \frac{\pi}{2} \left\{ \frac{J_0}{I_0} \right\} (\delta \sqrt{\nu^2 - P^2}) = \int_0^{\sqrt{\nu^2 - P^2}} \frac{\cosh \delta \mu}{\sqrt{\nu^2 - \mu^2 - P^2}} d\mu.$$

We substitute from (1.40) into (1.39) and interchange order of integration:

$$(1.41) \quad \begin{aligned} \frac{|p|}{p} e^{\alpha_i x^i} v_0^p &= \int_0^s \int_s^{\sqrt{\mu^2 + P^2}} \frac{\nu - s}{\sqrt{\nu^2 - \mu^2 - P^2}} \cosh \delta \mu d\nu d\mu \\ &= \int_0^s \left[-\sqrt{s^2 - \mu^2 - P^2} + s \operatorname{arccosh} \frac{s}{\sqrt{\mu^2 + P^2}} \right] \cosh \delta \mu d\mu. \end{aligned}$$

The relation (1.41) can be verified by performing a λ integration in v_0^p obtained from table (1.30).

Next we include a table of series expansions for the v symbols and certain of their derivatives. These expansions are useful in testing integrals for convergence in the derivation of integration formulae and in setting up such a formula for

¹ See VIII.

a particular example. In this table we use abbreviations for the operators in (1.29).

$$\begin{aligned}
 V^p &= \frac{|p|}{p} e^{-\alpha_i x^i} \left[\arctan \frac{R}{|p|} + \frac{J\delta^2}{4} \left(|p| R - \bar{S}^2 \arctan \frac{R}{|p|} + \dots \right) \right]; \\
 \left\{ \begin{matrix} L \\ m \end{matrix} \right\} V^p &= \frac{|p|}{p} e^{-\alpha_i x^i} \left\{ \begin{matrix} -P \\ s \end{matrix} \right\} \left[\frac{-|p|}{\bar{S}^2 R} + \frac{J\delta^2}{2} \arctan \frac{R}{|p|} + \dots \right]; \\
 V^s &= e^{-\alpha_i x^i} \left[-\operatorname{arcsinh} \frac{R}{\bar{S}} + \frac{J\delta^2}{4} \left(sR - \bar{S}^2 \operatorname{arcsinh} \frac{R}{\bar{S}} \right) + \dots \right]; \\
 \left\{ \begin{matrix} L \\ M \end{matrix} \right\} V^s &= e^{-\alpha_i x^i} \left\{ \begin{matrix} P \\ p \end{matrix} \right\} \left[-\frac{s}{\bar{S}^2 R} + \frac{J\delta^2}{2} \operatorname{arcsinh} \frac{R}{\bar{S}} + \dots \right]; \\
 V^{ps} &= \frac{|p|}{p} e^{-\alpha_i x^i} \left[-s \arctan \frac{R}{|p|} + |p| \operatorname{arcsinh} \frac{R}{\bar{S}} \right. \\
 &\quad \left. + |P| \arctan \frac{sR}{|Pp|} - |P| \arctan \frac{|p| R}{s|P|} + \delta^2(\dots) \right]; \\
 (1.42) \quad \{L\} V^{ps} &= \frac{|Pp|}{Pp} e^{-\alpha_i x^i} \left[\arctan \frac{|p| R}{s|P|} - \arctan \frac{sR}{|Pp|} + \delta^2(\dots) \right]; \\
 V^{pp} &= \frac{|Pp|}{Pp} e^{-\alpha_i x^i} \left[-|P| \arctan \frac{R}{|p|} - |p| \arctan \frac{R}{|P|} \right. \\
 &\quad \left. + s \arctan \frac{|P| R}{s|p|} + s \arctan \frac{|p| R}{s|P|} + \delta^2(\dots) \right]; \\
 \{m\} V^{pp} &= \frac{|Pp|}{Pp} e^{-\alpha_i x^i} \left[-\arctan \frac{|P| R}{s|p|} - \arctan \frac{|p| R}{s|P|} + \delta^2(\dots) \right]; \\
 v &= e^{-\alpha_i x^i} \left[R - |P| \arctan \frac{R}{|P|} + \delta^2(\dots) \right]; \\
 v^p &= \frac{|p|}{p} e^{-\alpha_i x^i} \left[-\frac{|p| R}{2} + \frac{s^2 + P^2}{2} \arctan \frac{R}{|p|} \right. \\
 &\quad \left. + |Pp| \arctan \frac{R}{|P|} - s|P| \arctan \frac{sR}{|Pp|} + \delta^2(\dots) \right].
 \end{aligned}$$

2. Integration Formulae with Integration by Parts in Space-like Direction. In deriving our integration formulae we integrate over a region Q_3 within one nappe of the cone with vertex at ξ^i . The region of integration is partially bounded by this nappe and partially by a surface Q_2 , independent of ξ^i , which may or

may not contain characteristic patches. The boundary of the region thus contains ξ^i , but the surface Q_2 does not pass through ξ^i . The boundary of the region Q_3 is such as to permit multiple integration.

FIGURE 2.1

Combining equations (1.1) and integrating over Q_3 we have

$$(2.1) \quad \iint_{\text{boundary of } Q_3} \left[\frac{A^{ij} \Lambda_i}{\Omega} (v u_j - v_j u) + \frac{B^i \Lambda_i}{\Omega} v u \right] = \iiint_{Q_3} v f.$$

In terms of the damping constants this can be written

$$(2.2) \quad \iint_{\text{boundary of } Q_3} \frac{A^{ij} \Lambda_i}{\Omega} [v(u_j - \alpha_j u) - u(v_j + \alpha_j v)] = \iiint_{Q_3} v f.$$

The vector Λ_i/Ω denotes the Euclidean normal to the boundary surface and belongs to an algebra determined by tangent vectors to the surface, namely $\lambda_\alpha^i = x_{\lambda^\alpha}^i$, where λ^α denote parameters on the surface. This algebra is N-P, N-C or N-P, C according to the portion of the boundary in question. To accommodate both algebras we do not raise indices on Λ_i .

We first consider the left member of (2.2) over the surface of the cone. Here we are concerned with an N-P, C algebra where $A^{ij}\Lambda_j = r^\alpha\lambda_\alpha^i$, and the integrand of (2.2) involves $v(\lambda^\alpha)$ and its tangential derivatives, namely v_{λ^α} . By Property II we have $v(\lambda^\alpha) = 0$, and the surface integral over the cone makes no contribution.

Since the non-Euclidean normal derivative of v has a jump discontinuity at the splitting plane, we must take account of such surface integrals. For this purpose we use the representation

$$(2.3) \quad x^i = e^i + l_\alpha^i \lambda^\alpha, \quad x^i - \xi^i = e^i - \xi^i + l_\alpha^i \lambda^\alpha.$$

Here we assume that the e^i are functions of $L_i \xi^i$ only, so that although the point e^i varies with the splitting plane, its position is fixed relative to variations of ξ^i in the direction of the splitting plane. For x^i in the splitting plane we have

$$(2.4) \quad P = 0; \quad p^\alpha = \lambda^\alpha - \lambda_0^\alpha, \quad \text{where } \lambda_0^\alpha = l_\alpha^i (\xi^i - e^i).$$

Over the splitting plane the Λ algebra is taken to be the L algebra (N-P, N-C) of Section 1. In combining the splitting plane integrals, we take the sense where $P > 0$, as indicated in Figure 2.1.

$$(2.5) \quad \iint_{P=0} \left[-\lim_{P \rightarrow 0^+} \frac{L^i}{\omega} (v_i + \alpha_i v) u + \lim_{P \rightarrow 0^-} \frac{L^i}{\omega} (v_i + \alpha_i v) u \right] \\ = \iint_{P=0} \frac{-j\pi}{\sqrt{-j\omega}} e^{-\alpha_\alpha p^\alpha} u (e^i + l_\alpha^i \lambda^\alpha).$$

Here we have used the limit (1.25).

Substituting (2.5) into the left member of (2.2) and transposing the surface integral over Q_2 , we have

$$(2.6) \quad \iint_{P=0} \frac{-j\pi}{\sqrt{-j\omega}} e^{-\alpha_\alpha p^\alpha} u (e^i + l_\alpha^i \lambda^\alpha) \\ = \iint_{Q_2} \frac{A^{ij}\Lambda_j}{\Omega} [-v(u_i - \alpha_i u) + u(v_i + \alpha_i v)] + \iiint_{Q_3} v f.$$

The fact that the surface integrals in (2.6) do not account for discontinuities in u at surfaces interior to the region implies that we are seeking a solution u which is continuous throughout the region Q_3 . Since the surface integral over a characteristic surface involves only tangential derivatives of u , the formula (2.6) does allow solutions having discontinuities in non-tangential derivatives at a characteristic surface. Thus our final formulae are applicable to boundary

values u and u_i which are defined and continuous over a finite number of closed subregions of Q_2 and which are sufficiently differentiable at such points to allow the integration by parts used in subsequent steps. Of these boundary values u is continuous at all points of Q_2 .

FIGURE 2.2. *Left*: Map of $P = 0^+$ section of Q_3 . *Right*: Relettering according to descent.

To obtain a formula for $u(\xi^i)$ we apply the inverse operator Δ to both members of equation (2.6). For a function of λ_0^α this operator is equivalent to $j h^{\alpha\beta} (D_{\lambda_0^\alpha} - a_\alpha) (D_{\lambda_0^\beta} - a_\beta)$. We now show that this operator applied to the left member of (2.6) yields

$$(2.7) \quad \Delta \iint_{P=0} \frac{-j\pi}{\sqrt{-j\omega}} e^{-a_\alpha p^\alpha} u(e^i + l_\alpha^i \lambda^\alpha) = 2j\pi \sqrt{j} |A^{ij}| u(\xi^i).$$

For variations of ξ^i in the directions of l_α^i the splitting plane remains fixed. Hence Q_1 , the intersection of Q_2 and the splitting plane, also remains fixed.

The problem in (2.7) amounts to showing that the non-homogeneous term of the integration formula for distortionless hyperbolic $n = 2$ is a solution of that differential equation. To verify (2.7) we apply a method of descent in which we rephrase the problem in the latter form. This entails a mapping of the splitting plane region using the representation (2.3) and relettering according to the correspondence $\lambda^\alpha \searrow x^i$. [See Figure 2.2.] Since the distortionless potential is a solution of the differential equation $h^{\alpha\beta} (D_{\lambda_0^\alpha} - a_\alpha) (D_{\lambda_0^\beta} - a_\beta) = 0$ we are led to adopt the metric $h^{\alpha\beta}$ for $n = 2$ and reletter according to $h^{\alpha\beta} \searrow A^{ij}$. The complete descent table is as follows:

$$(2.8) \quad \begin{array}{llll} h^{\alpha\beta} \searrow A^{ij}; & \lambda^\alpha \searrow x^i; & \lambda_0^\alpha \searrow \xi^i; & a_\alpha p^\alpha \searrow \alpha_i X^i; \\ u \searrow f; & N^\alpha \searrow N^i; & n_1^\alpha \searrow n_1^i. \end{array}$$

Setting aside the factor $\pi / \sqrt{-j\omega}$, the problem (2.7) becomes

$$(2.9) \quad A^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j) \iint_{Q_2} -e^{-\alpha_i x^i} f(x^i) = 2j \sqrt{-|A^{ij}|} f(\xi^i) \\ (i, j = 1, 2).$$

In comparing with the integration formula for $n = 2$ we see that the minus sign in (2.9) is compensated for by the negative sense in Figure 2.2.

The N^i algebra being N-P, C we can factor the A operator in (2.9) using the Pythagorean identity

$$(2.10) \quad A^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j) = 2j \sqrt{-|A^{ij}|} N^i(D_{\xi^i} - \alpha_i) n_1^j (D_{\xi^j} - \alpha_j).$$

Applying the ordinary Leibnitz rule in (2.9) the double integral is annihilated and we obtain no line integral along the fixed boundary Q_1 . For the generators of Figure 2.2 we use the representations $x^i = \xi^i + N^i s$ and $x^j = \xi^j + n_1^j s$. The left member of (2.9) becomes

$$(2.11) \quad 2j \sqrt{-|A^{ij}|} N^i(D_{\xi^i} - \alpha_i) \left\{ \int_{s_1}^0 -e^{-\alpha_i x^i} f(\xi^i + N^i s) \right. \\ \left. + \int_0^{s_2} -e^{-\alpha_i x^i} f(\xi^i + n_1^j s) \right\} \left| \frac{n_1^j}{x_s^j} \right|.$$

The second term of (2.11) contains a determinant with two rows alike and we need $N^i D_{\xi^i}(s_1) = -1$ where $s_1 = N_i X^i / \bar{\omega}$ in order to complete the computation in the first term. The expression (2.11) becomes

$$(2.12) \quad 2j \sqrt{-|A^{ij}|} \left\{ [e^{-\alpha_i x^i} f]_{s=s_1} + \int_{s_1}^0 D_s(e^{-\alpha_i x^i} f) ds \right\} \\ = 2j \sqrt{-|A^{ij}|} f(\xi^i) \quad (i, j = 1, 2).$$

This completes the demonstration of (2.9). When we reverse the correspondence (2.8a) and combine the coefficient of the result (2.9) with the factor set aside we have $(\pi / \sqrt{-j\omega}) 2j \sqrt{-|h^{\alpha\beta}|} = 2j\pi \sqrt{j|A^{ij}|}$ for the coefficient in (2.7).

In operating on the right member of (2.6) we integrate by parts and it would seem that integrations must be done with respect to surface parameters in the case of surface integrals. It is our practice to use the basic mixed Pythagorean identity,

$$(2.13) \quad |^{\alpha} h_{\beta} | x_i y^i = {}^{\alpha} \tilde{h}^{\beta} (l_i^{\alpha} y^i) (\lambda_{\beta}^i x_i) + (L^i x_i) \frac{\Lambda_i y^i}{\Omega},$$

to separate the tangential components of the derivative to be integrated. In effect, however, derivatives in other directions are transferred from one factor to another by the use of other Pythagorean identities and the u and v differential equations during the process.

Throughout subsequent work, the L algebra of (2.13) is that of Section 1. For the purpose of integration by parts in directions other than L we use two other basic algebras designated by $\bar{L}$ and $\bar{\bar{L}}$, which are obtained by re-arrangement of the vectors of the compound $M l$ algebra. These algebras are as follows:

$$(2.14) \quad \begin{vmatrix} \bar{L}^i \\ l_1^i \\ l_2^i \end{vmatrix} = \begin{vmatrix} M^\alpha l_\alpha^i \\ m_1^\alpha l_\alpha^i \\ L^i \end{vmatrix}; \quad \begin{vmatrix} \bar{\bar{L}}^i \\ \bar{l}_1^i \\ \bar{l}_2^i \end{vmatrix} = \begin{vmatrix} m_1^\alpha l_\alpha^i \\ L^i \\ M^\alpha l_\alpha^i \end{vmatrix}.$$

The Pythagorean identities used with (2.14) are obtained from (2.13) by putting bars over the h 's and the l_i^α , but the L vector is written in explicit compound form.

In applying the Δ operator to the right member of (2.6) our plan is to introduce an upper p index in each term and use the second type Leibnitz rule for the first part of the Δ operation. Although in our case this results in more integrations by parts than are necessary, this plan has the advantage of enabling us to apply results from hyperbolic interior $n = 2$ in differentiating the line integral. The integration formula with an upper p index in each term is interesting in itself because of its connection with $n = 2$ and it is a simple matter to reverse certain of the integrations and to obtain from this formula another formula having fewer derivatives on u .

In preparing the right member of (2.6) for integration by parts, we replace v by one M operator on v^p . With (2.13) we then replace each term of the surface integral by two terms. In one of the terms resulting from the second Pythagorean identity, we use the fact that v^p satisfies the adjoint equation. The right member of (2.6) becomes

$$(2.15) \quad \begin{aligned} & \Delta \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) \frac{M^\alpha l_\alpha^k}{\sqrt{-j\bar{\omega}}} D_{\xi^k} - \alpha_k \right\} v^p \\ & - u \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi^j} - \alpha_j) \frac{M^\alpha l_\alpha^k}{\sqrt{-j\bar{\omega}}} (D_{\xi^k} - \alpha_k) v^p \Big\} + \Delta \iiint_{Q_3} f \frac{M^\alpha l_\alpha^k}{\sqrt{-j\bar{\omega}}} (D_{\xi^k} - \alpha_k) v^p \\ & = \Delta \iint_{Q_2} \left\{ {}^\alpha \bar{h}^\beta \frac{A^{ij} j_l^\alpha}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) (-D_\beta - \alpha_k \lambda_\beta^k) v^p - \frac{|{}^\alpha \bar{h}_\beta|}{\sqrt{-j\bar{\omega}}} \right. \\ & \quad A^{ij} (u_i - \alpha_i u) (D_{\xi^j} - \alpha_j) v^p + u {}^\alpha \bar{h}^\beta (-D_\beta - \alpha_k \lambda_\beta^k) \frac{A^{ij} j_l^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^j} - \alpha_j) v^p \\ & \quad \left. + \frac{|{}^\alpha \bar{h}_\beta|}{\sqrt{-j\bar{\omega}}} u j J \delta^2 v^p \right\} + \Delta \iiint_{Q_3} f \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (-D_{\xi^i} - \alpha_i) v^p \end{aligned}$$

In performing the first part of the Δ operation we use the second type Leibnitz

rule on the integral over Q_3 and on the terms involving ${}^{\alpha}\tilde{h}^{\beta}$. On the remaining terms we use the ordinary Leibnitz rule.

$$\begin{aligned}
 & jH^{ik}(D_{\xi^i} - \alpha_i) \left[\iint_{Q_2} \left\{ {}^{\alpha}\tilde{h}^{\beta}(D_{\beta} - \alpha_k \lambda_{\beta}^k) \frac{A^{ij}l_i^{\alpha}}{\sqrt{-j\tilde{\omega}}} (u_j - \alpha_j u) D_{\xi^k} - \alpha_k v^p \right. \right. \\
 & - \frac{|{}^{\alpha}\tilde{h}_{\beta}|}{\sqrt{-j\tilde{\omega}}} A^{ij}(u_i - \alpha_i u)(D_{\xi^i} - \alpha_j)(D_{\xi^k} - \alpha_k)v^p + {}^{\alpha}\tilde{h}^{\beta}(D_{\beta} - \alpha_k \lambda_{\beta}^k)u \\
 & \cdot \frac{A^{ij}l_i^{\alpha}}{\sqrt{-j\tilde{\omega}}} (D_{\xi^i} - \alpha_j)(D_{\xi^k} - \alpha_k)v^p + \frac{|{}^{\alpha}\tilde{h}_{\beta}|}{\sqrt{-j\tilde{\omega}}} u j J \delta^2(D_{\xi^k} - \alpha_k)v^p \Big\} \\
 & + \iiint_{Q_3} \frac{M^{\alpha}l_{\alpha}^i}{\sqrt{-j\tilde{\omega}}} (f_i - \alpha_i f)(D_{\xi^k} - \alpha_k)v^p + \int_{\substack{(p=0) \\ (P=0)}} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) \right. \\
 (2.16) \quad & \cdot \frac{M^{\alpha}l_{\alpha}^i}{\sqrt{-j\tilde{\omega}}} (D_{\xi^i} - \alpha_i)v^p - u \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) \frac{M^{\alpha}l_{\alpha}^i}{\sqrt{-j\tilde{\omega}}} (D_{\xi^i} - \alpha_i)v^p \Big\} \left| \frac{\lambda_{\xi^k}^{\beta}}{\lambda_t^{\beta}} \right| \\
 & + \int_{\substack{(p=0) \\ (P=0)}} \left\{ -\frac{A^{ij}l_i^{\alpha}}{\sqrt{-j\tilde{\omega}}} (u_j - \alpha_j u)(D_{\xi^k} - \alpha_k)v^p \right. \\
 & - u \frac{A^{ij}l_i^{\alpha}}{\sqrt{-j\tilde{\omega}}} (D_{\xi^i} - \alpha_j)(D_{\xi^k} - \alpha_k)v^p \Big\} \left| \frac{{}^{\alpha}\tilde{h}^{\beta}}{\lambda_t^{\beta}} \right| + \iint_{\substack{(Q_2) \\ (p=0) \\ (P=0)}} f v x_{\xi^k}^i \frac{\Lambda_i}{\Omega} \\
 & + \iint_{\substack{(Q_2) \\ (p=0) \\ (P=0)}} -f(D_{\xi^k} - \alpha_k)v^p \frac{M^{\alpha}l_{\alpha}^i}{\sqrt{-j\tilde{\omega}}} \frac{\Lambda_i}{\Omega} \Big\} \Big].
 \end{aligned}$$

In writing the lower order integrals resulting from the Leibnitz rules, we indicate those manifolds which might conceivably contribute taking into account the boundary of the region and the nature of the v symbols. Although the integrals make no contribution over the manifolds enclosed in parentheses, these possibilities are included to help the reader in following the discussion. Since most of the v symbols vanish on the cone, the possibility $R = 0$ is in general not included. In the case of (2.16) the lowest order terms in the second type Leibnitz rules, which include sums over vertices and line integrals along edges, are not indicated since v^p and $A^{ij}l_i^{\alpha}(D_{\xi^i} - \alpha_j)v^p$ are continuous by quadrants.

All other possible terms resulting from the Leibnitz rules are indicated in (2.16). The line integral with $\lambda_{\xi^k}^{\beta}$ determinant makes no contribution over $p = 0$ since v_0^p does not contain p . The same term drops out on $P = 0$ if we restrict our discussion to representations of $P = 0$, which depend on the combination $L_i \xi^i$ only. Then this term vanishes in view of the N-P, N-C rule $H^{ik}L_k = 0$. Of the f double integral terms, the first goes out by continuity of the inte-

grand and the fixed nature of the bounding surface Q_2 . The other f term makes no contribution over $P = 0$ where $M^\alpha l_\alpha^i \Lambda_i = 0$.

To complete the discussion of (2.16) we have yet to consider the line integral on $P = 0$ containing the $\tilde{h}$ determinant. The first term of this integral drops out at the total derivative stage, due to the continuity of v^p at $P = 0$. The second term does not vanish until we perform the second half of the Δ operation. At the total derivative stage the discontinuity in $A^{ij} l_\alpha^i (D_{\xi i} - \alpha_j) v^p$ is in the L component and the amount of the discontinuity is $\pi(s - |p|)$. This function being linear the integrand is annihilated on completion of the Δ operation. To see that there is no contribution from the limits of integration on the second part of the Δ operation we let either limit be designated by t_1 where $x^i[\lambda^\alpha(t_1, L_{\xi}^i)]$ is subject to $s - |p| = 0$. Thus

$$(2.17) \quad l_\alpha^i D_{\xi i} (s - |p|) = \left(s_{x^i} - \frac{|p|}{p} p_{x^i} \right) x_i^j l_\alpha^i D_{\xi i}(t_1) - l_\alpha^i \left(s_{x^i} - \frac{|p|}{p} p_{x^i} \right) = 0.$$

Then the contribution through the limit of integration contains the factor

$$(2.18) \quad H^{ik} \left(s_{x^i} - \frac{|p|}{p} p_{x^i} \right) \left(s_{x^k} - \frac{|p|}{p} p_{x^k} \right) = H^{ik} \left[j g_{11} m_\alpha^1 l_\alpha^i m_\beta^1 l_\beta^k - \frac{j}{\tilde{\omega}} M_\alpha l_\alpha^i M_\beta l_\beta^k \right] = 0.$$

It should be remarked that (2.18) shows that $s - |p| = 0$ is a characteristic plane and that (2.17) contains a zero times infinity indeterminate form when x_i^i is tangent to the cone at t_1 .

The operation on the $p = 0$ integrals of (2.16) is taken up later with the aid of descent. In collecting the Q_2 and Q_3 integrals of (2.16) we use two Pythagorean identities in the double integral. Referring to the collected form (2.19) we see that the first term of (2.16) gives rise to two terms and that the second and third terms of (2.16) collect. We have

$$(2.19) \quad \begin{aligned} & j H^{ik} (D_{\xi i} - \alpha_i) \left[\iint_{Q_2} \left\{ \frac{-A^{ij} \Lambda_i}{\Omega} (D_{x i} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\tilde{\omega}}} (u_i - \alpha_i u) (D_{\xi k} - \alpha_k) v^p \right. \right. \\ & - M^\alpha l_\alpha^i (u_i - \alpha_i u) \frac{A^{ij} \Lambda_i}{\Omega} (D_{\xi i} - \alpha_j) (D_{\xi k} - \alpha_k) v^p \\ & + \frac{|{}^\alpha \tilde{h}_\beta|}{\sqrt{-j\tilde{\omega}}} [A^{ij} (D_{x i} - \alpha_i) (u_j - \alpha_j u) + j J \delta^2 u - f] (D_{\xi k} - \alpha_k) v^p \Big\} \\ & \left. + \iiint_{Q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\tilde{\omega}}} (f_i - \alpha_i f) (D_{\xi k} - \alpha_k) v^p \right]. \end{aligned}$$

The terms with $|{}^\alpha \tilde{h}_\beta|$ drop out in view of the u differential equation.

When we complete the Δ operation on (2.19) and make use of $\Delta v^p = V^p$ the resulting Q_2 and Q_3 integrals agree with the corresponding terms of the integration formula (2.26) below. To see that no lower integrals result from this operation we write out such terms.

$$\begin{aligned}
 (2.20) \quad & \int_{\substack{(p=0) \\ (P=0)}} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) (D_{\xi^k} - \alpha_k) v^p \right. \\
 & \left. - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) (D_{\xi^k} - \alpha_k) v^p \right\} \left| \frac{jH^{ik}\lambda_{\xi^i}^\beta}{\lambda_t^\beta} \right| \\
 & + \iint_{\substack{(Q_2) \\ (p=0) \\ (P=0)}} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) (D_{\xi^k} - \alpha_k) v^p jH^{ik} x_t^i \frac{\Lambda_j}{\Omega}.
 \end{aligned}$$

Of the above integrals, those over Q_2 make no contribution because of the fixed nature of the boundary and those over $P = 0$ go out for the restricted representation. With similar restrictions on the representation of $p = 0$ the operators on v^p contain at least one M component and the terms go out by continuity of the v symbol.

The remaining terms of the right member of the integration formula (2.26) come from completing the operations on the $p = 0$ integrals in (2.16). After rephrasing these terms by descent the result of this operation can be quoted from the operation in hyperbolic $n = 2$ which corresponds to our own operation on the right member of (2.6) now in progress. Since v_0^p is independent of p we can drop the M components of the H operator and change the H operator to a G operator in the case of the $p = 0$ integrals of (2.16).

In the descent process we adopt the representation $\lambda^\alpha = l_\alpha^i x^i$ in the double integral over $p = 0$ in (2.16). In the line integral over $p = 0$ we use the second one of the following determinant relations to express the integral in a form which is independent of the Q_2 representation.

$$\begin{aligned}
 (2.21) \quad & \left| \frac{{}_\alpha \bar{h}^\beta}{\lambda_t^\beta} \right| = \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_t^i} \right| = \left| \frac{L^i}{l_\alpha^i} \right|; \quad \left| \frac{{}_\alpha \bar{h}^\beta}{\lambda_t^\beta} \right| = \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_t^i} \right| = \left| \frac{M^\beta l_\beta^i}{l_\alpha^i} \right|; \\
 & \left| \frac{{}_\alpha \bar{h}^\beta}{\lambda_t^\beta} \right| = \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_t^i} \right| = \left| \frac{m_1^\beta l_\beta^i}{l_\alpha^i} \right|.
 \end{aligned}$$

Thus in the rephrasing of all integrals we use the correspondence $l_\alpha^i x^i \searrow x^i$. Since v_0^p is independent of p we insert l basic Kronecker deltas in the adjoint for v_0^p

obtaining $\bar{h}^{\alpha\beta} = A^{ij}l_i^\alpha l_j^\beta$ for the appropriate metric in $n = 2$. The descent table is

$$(2.22) \quad \begin{aligned} l_i^\alpha x^i &\searrow x^i; & \frac{M^\alpha l_\alpha^i \Lambda_i}{\Omega} &= 1; & \left| \frac{\delta_\alpha^\beta}{l_i^\beta x^i} \right| &\searrow \frac{\Lambda_i}{\Omega}; & \bar{h}^{\alpha\beta} &\searrow A^{ij}; \\ l_i^\alpha L^i &\searrow -L^i; & \omega &\searrow \omega; & P &\searrow -P; & s &\searrow S; & \frac{|p|}{p} v_0^p &\searrow \frac{\pi}{2} v; \\ f &\searrow f; & jG^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j) &\searrow jH^{ij}(D_{\xi^i} - \alpha_i)(D_{\xi^j} - \alpha_j) &= \Delta. \end{aligned}$$

Computation shows that $|\bar{h}^{\alpha\beta}| = g^{11}/\omega < 0$ and hence our $n = 2$ metric is hyperbolic. If we use the correspondence $l_i^\alpha L^i \searrow -L^i$ to determine an N-P, N-C algebra in $n = 2$ it turns out that $l_i^\alpha m_1^\beta l_\beta^i \searrow l_1^i$ as indicated in the relettering of the map in Figure 2.3.

FIGURE 2.3. *Left*: $p = 0^+$ section of Q_3 . *Right*: Relettered mapping according to descent.

When we set aside the factor $\pi/2\sqrt{-j\bar{\omega}}$ the G operation on the $p = 0$ integrals of (2.16) can be rephrased as follows:

$$(2.23) \quad \begin{aligned} &jH^{ik}(D_{\xi^i} - \alpha_i) \left[\int_{Q_1} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u)(D_{\xi^k} - \alpha_k)v \right. \right. \\ &\quad \left. \left. - u \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi^j} - \alpha_j)(D_{\xi^k} - \alpha_k)v \right\} + \iint_{Q_2} f(D_{\xi^k} - \alpha_k)v \right] \quad (i, j = 1, 2). \end{aligned}$$

Since the orientation in Figure 2.3a is taken from the surface Q_2 in Figure 2.1, this sense agrees with the $p = 0^+$ line integral of (2.16). Since the $p = 0^+$ double integral of (2.16) has the opposite orientation, we change the sign of the double integral in (2.23) when associating all integrals with Figure 2.3b.

The operation (2.23) is the Δ operation of hyperbolic $n = 2$ with the first part of the operation already performed. Referring to the right member of the integration formula for hyperbolic $n = 2$, we quote the result¹ of the completed operation (2.23).

$$(2.24) \quad j\sqrt{-|A^{ij}|} \{[e^{-\alpha_i x^i} u](\sigma^i) + [e^{-\alpha_i x^i} u](\bar{\sigma}^i)\} \\ + \int_{q_1} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u)V - u \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi i} - \alpha_j)V \right\} + \iint_{q_2} fV \\ (i, j = 1, 2).$$

In ascending from (2.24) to the corresponding terms of the integration formula (2.26) we use the descent table (2.22) with the additional correspondence,

$$(2.25) \quad \frac{\pi}{2} V \nearrow \frac{|p|}{p} V_0^p.$$

In the correspondence for Λ_i/Ω in (2.22) we use the third order determinant form in (2.21). The double integral requires a sign change corresponding to the sign change in descent. In descent we drop the factor $\lim_{p \rightarrow 0} |p|/p$ in the v symbol and compensating for this with a change in sense, the result (2.24) is applicable to integration over either side of the plane $p = 0$. Thus, in ascent we double the averaging terms and taking into account the factor which was set aside, we have the result $(\pi/2 \sqrt{-j\bar{\omega}}) 2j \sqrt{-|h^{\alpha\beta}|} = \pi j \sqrt{j|A^{ij}|}$ for the coefficients of the averaging terms in (2.26).

Combining the results (2.7), the completed form of (2.19), and the ascended form of (2.24), the result of the Δ operation on (2.26) yields the *integration formula*

$$(2.26) \quad 2\pi j \sqrt{j|A^{ij}|} u(\xi^i) = \pi j \sqrt{j|A^{ij}|} \{[e^{-\alpha_i x^i} u](\sigma^i) + [e^{-\alpha_i x^i} u](\bar{\sigma}^i)\} \\ + \int_{p=0} \left\{ -(u_j - \alpha_j u)V_0^p - u(D_{\xi i} - \alpha_j)V_0^p \right\} \begin{vmatrix} M^\beta l_\beta^i / \sqrt{-j\bar{\omega}} \\ A^{ij} \\ x_i^i \end{vmatrix} \\ + \iint_{p=0} -fV_0^p \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} \frac{\Lambda_i}{\Omega} + \iint_{q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u)V^p \right. \\ \left. - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi i} - \alpha_j)V^p \right\} + \iiint_{q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f)V^p.$$

Since the derivation of formula (2.26) involved quoting results from hyper-

¹ See VIII.

bolic $n = 2$, it is natural that this formula should throw light on the question of when the integration formula for $n = 3$ reduces to one for $n = 2$. If f does not vary in the space like direction $M^{\alpha l_i^i}$ and if the damping constant $M^{\alpha l_i^i} \alpha_i = 0$ then the Q_3 integral in (2.26) drops out. If, furthermore, the first and second derivatives of u having at least one derivative in the direction of $M^{\alpha l_i^i}$ vanish on Q_2 then the integration formula (2.26) reduces to the integration formula for $n = 2$ where integrations are taken over the $p = 0$ section of Q_3 .

Using the relation $V^p = \tilde{V}^p + V_0^p$ of Section 1 it is possible to modify formula (2.26) so that line integrals are taken over the intersection of Q_2 and $R = 0$ instead of Q_2 and $p = 0$. To obtain this modification we make the substitution $V^p = \tilde{V}^p + V_0^p$ in the Q_2 and Q_3 integrals of (2.26). A reverse integration by parts is possible in the terms involving V_0^p since this symbol involves a first type Bessel function. In rewriting these terms we use two Pythagorean identities as follows:

$$\begin{aligned}
 (2.27) \quad & \iint_{Q_2} \left\{ {}^{\alpha} \tilde{h}^{\beta} (D_{\beta} - \alpha_i \lambda_{\beta}^i) \frac{A^{ij} l_i^{\alpha}}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) V_0^p \right. \\
 & - \frac{|{}^{\alpha} \tilde{h}_{\beta}|}{\sqrt{-j\bar{\omega}}} A^{ij} (D_{xi} - \alpha_i) (u_j - \alpha_j u) V_0^p + {}^{\alpha} \tilde{h}^{\beta} (D_{\beta} - \alpha_i \lambda_{\beta}^i) u \frac{A^{ij} l_i^{\alpha}}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_j) V_0^p \\
 & \left. - \frac{|{}^{\alpha} \tilde{h}_{\beta}|}{\sqrt{-j\bar{\omega}}} A^{ij} (u_i - \alpha_i u) (D_{xi} - \alpha_j) V_0^p \right\} + \iiint_{Q_3} \frac{M^{\alpha l_i^i}}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) V_0^p.
 \end{aligned}$$

When we integrate by parts in the $\tilde{h}$ terms and the Q_3 integral of (2.27) the term resulting from the first collects with the fourth and the term resulting from the third is replaced by two. At the same time we use u and v differential equations.

$$\begin{aligned}
 (2.28) \quad & \iint_{Q_2} \left\{ -\frac{A^{ij} \Lambda_i}{\Omega} (u_j - \alpha_j u) \frac{M^{\alpha l_i^i}}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) V_0^p - u \frac{A^{ij} \Lambda_i}{\Omega} (D_{xi} - \alpha_j) \right. \\
 & \cdot \frac{M^{\alpha l_i^i}}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) V_0^p + \frac{|{}^{\alpha} \tilde{h}_{\beta}|}{\sqrt{-j\bar{\omega}}} [(jJ\delta^2 u - f) V_0^p - u(jJ^2\delta^2 V_0^p)] \Big\} \\
 & + \iiint_{Q_3} f \frac{M^{\alpha l_i^i}}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) V_0^p \\
 & + \int_{\substack{p=0 \\ R=0}} \left\{ \frac{A^{ij} l_i^{\alpha}}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) V_0^p + u \frac{A^{ij} l_i^{\alpha}}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_j) V_0^p \right\} \left| \frac{{}^{\alpha} \tilde{h}^{\beta}}{\lambda_i^{\beta}} \right| \\
 & + \iint_{\substack{Q_2 \\ p=0 \\ R=0}} f V_0^p \frac{M^{\alpha l_i^i}}{\sqrt{-j\bar{\omega}}} \frac{\Lambda_i}{\Omega}.
 \end{aligned}$$

In (2.28) all Q_2 and Q_3 integrals drop out since the δ^2 and f terms cancel and in the other terms V_0^p does not involve p . Replacing the V_0^p terms in the Q_2 and Q_3 integrals of (2.26) with (2.28) the line integrals over $p = 0$ cancel. *In effect we obtain a second integration formula from (2.26) by placing bars over V^p in the Q_2 and Q_3 integrals and by changing the lower order integrals from $p = 0$ to $R = 0$ with opposite signs in the integrands.*

As mentioned above the derivation of (2.26) involves more integration by parts than is necessary and a formula with fewer derivatives on u can be obtained by using reverse integrations in the first Q_2 term and the Q_3 term of (2.26). The results of this operation can be obtained by selecting the appropriate terms from (2.27) and (2.28) with V_0^p changed to V^p . These terms are

$$\begin{aligned}
 (2.29) \quad & \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_i) V^p \right. \\
 & + \frac{|\alpha \bar{h}_\beta|}{\sqrt{-j\bar{\omega}}} [A^{ij}(u_i - \alpha_i u)(D_{\xi^j} - \alpha_j) V^p + (jJ\delta^2 u - f) V^p] \Big\} \\
 & + \iiint_{Q_3} f \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_i) V^p + \int_{p=0} \frac{A^{ij} l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) V^p \left| \frac{\bar{h}^\beta}{\lambda_t^\beta} \right| \\
 & + \iint_{Q_2, p=0} f V^p \frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega}.
 \end{aligned}$$

Replacing the first Q_2 term and the Q_3 term of (2.26) with (2.29) we have another form of the *integration formula*

$$\begin{aligned}
 (2.30) \quad & 2\pi j \sqrt{j|A^{ij}|} u(\xi^i) = \pi j \sqrt{j|A^{ij}|} \{[e^{-\alpha_i X^i} u](\sigma^i) + [e^{-\alpha_i X^i} u](\bar{\sigma}^i)\} \\
 & + \int_{p=0} -u(D_{\xi^i} - \alpha_j) V_0^p \left| \begin{array}{c} M^\alpha l_\alpha^i / \sqrt{-j\bar{\omega}} \\ A^{ij} \\ x_t^i \end{array} \right| \\
 & + \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) V - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi^j} - \alpha_j) V^p \right. \\
 & + \frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega} [A^{ij}(u_i - \alpha_i u)(D_{\xi^j} - \alpha_j) V^p + jJ\delta^2 u V^p] \Big\} + \iiint_{Q_3} f V.
 \end{aligned}$$

In applying the various integration formulae in this paper one must first determine whether the differential equation is of the hyperbolic type. For this we use the test (1.2) which involves showing that $|A^{ij}| \neq 0$ and that $A_{ij}X^iX^j = 0$ represents a real cone. This might be accomplished by a sketch which would

also take care of the test (1.4) for the interior. The test for j is $j | A^{ij} | > 0$, and J is determined by the sign of the discriminant in (1.8). The vectors L^i , $M^\alpha l_\alpha^i$, and $m_1^\alpha l_\alpha^i$ constitute a non-Euclidean orthogonal system with $m_1^\alpha l_\alpha^i$ directed into the nappe of the cone containing the region of integration. The choice of these vectors is a matter of determining a convenient coordinate system. The vector Λ_i/Ω is the cross-product vector for the tangent vectors to Q_2 , namely $\lambda_\alpha^i = x_{\lambda\alpha}^i$. In computing the integrands it is convenient to insert basic Kronecker deltas in the directional derivatives and refer to table (1.42) for the derivatives of the V symbols. In the distortionless case ($\delta = 0$) the line integral in (2.30) is missing.

3. Integration Formulae with Integration by Parts in Time-like Direction. We next derive a form of the integration formula, containing no averaging terms, but rather a $2\pi u$ term evaluated at the point of intersection of the surface Q_2 and the planes $p = 0$ and $P = 0$. Our procedure is to integrate by parts in the integration formula (2.26), introducing upper s indices, and then to remove the upper p indices with reverse integrations by parts. In this connection we observe that an upper index s indicates an integration in a time-like direction, while an upper index p or P indicates an integration in a space-like direction.

Our first step is to replace V^p with one m operator on V^{ps} in the Q_2 and Q_3 integrals of (2.26). In preparing for integration by parts the pattern of Pythagorean identities is like that of (2.15).

$$\begin{aligned}
 & \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (D_{xi} - \alpha_i) V^{ps} \right. \\
 & \quad \left. - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (D_{xi} - \alpha_i) V^{ps} \right\} \\
 & \quad + \iiint_{Q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (D_{xi} - \alpha_i) V^{ps} \\
 (3.1) \quad & = \iint_{Q_2} \left\{ \frac{\bar{h}^\beta}{\alpha} \frac{A^{ij}\bar{l}_i^\alpha}{\sqrt{jg_{11}}} (D_{xi} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{j\bar{\omega}}} (u_i - \alpha_i u) (-D_\beta - \alpha_i \lambda_\beta^i) V^{ps} \right. \\
 & \quad \left. - \frac{|\bar{h}_\beta|}{\sqrt{jg_{11}}} A^{ij} (D_{xi} - \alpha_i) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) (D_{xi} - \alpha_j) V^{ps} \right. \\
 & \quad \left. + \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \left[\frac{\bar{h}^\beta}{\alpha} \frac{A^{ij}\bar{l}_i^\alpha}{\sqrt{-jg_{11}}} (D_{xi} - \alpha_j) (-D_\beta - \alpha_i \lambda_\beta^i) V^{ps} \right. \right. \\
 & \quad \left. \left. + \frac{|\bar{h}_\beta|}{\sqrt{-jg_{11}}} jJ\delta^2 V^{ps} \right] \right\} + \iiint_{Q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (-D_{xi} - \alpha_i) V^{ps}.
 \end{aligned}$$

In the next step we integrate by parts in the Q_3 integral and in the $\bar{\hbar}$ terms of the Q_2 integral. After integration by parts we collect the Q_2 integral using two Pythagorean identities. The resulting second and third terms combine. The resulting first term yields two terms, one of which combines with the fourth in a u differential equation.

$$\begin{aligned}
 & \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) V^{ps} \right. \\
 & \quad - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) V^{ps} \\
 & \quad \left. + \frac{|\alpha \bar{h}_\beta|}{\sqrt{jg_{11}}} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) V^{ps} \right\} + \iiint_{Q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{xi} - \alpha_i) \\
 (3.2) \quad & \cdot \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^{ps} + \int_{\substack{p=0 \\ (P=0)}} \left\{ -\frac{A^{ij}\bar{l}_i^\alpha}{\sqrt{jg_{11}}} (D_{xi} - \alpha_j) \right. \\
 & \cdot \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) V^{ps} - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij}\bar{l}_i^\alpha}{\sqrt{jg_{11}}} (D_{xi} - \alpha_j) V^{ps} \Big\} \\
 & \cdot \left| \frac{\bar{\hbar}^\beta}{\lambda_s^\beta} \right| + \iint_{\substack{Q_2 \\ (p=0) \\ (P=0)}} -\frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) V^{ps} \frac{m_1^\alpha l_\alpha^i \Lambda_i}{\sqrt{jg_{11}} \Omega}.
 \end{aligned}$$

Of the lower order integrals indicated above, those over $P = 0$ drop out by continuity of the V symbol. In the case of $A^{ij}\bar{l}_i^\alpha (D_{xi} - \alpha_j) V^{ps}$, the L component vanishes on $P = 0$ and the M component is continuous. The f terms over Q_2 cancel and the f term over $p = 0$ vanishes due to $m_1^\alpha l_\alpha^i \Lambda_i = 0$. In the case of the line integral over $p = 0$ the determinant vanishes for $\alpha = 1$, as can be seen by comparing (2.21) and (2.14). When $\alpha = 2$ the $\bar{l}$ operator is an M operator and the second term goes out by continuity. Thus, we are left with the first term over $p = 0$ in (3.2).

Our next procedure is to perform reverse integration by parts in the Q_2 and Q_3 integrals of (3.2). Lower order integrals resulting from this integration by parts also require reverse integration which is accomplished by descent. In the process we obtain integrals which cancel with the integrals over $p = 0$ of the integration formula (2.26).

In preparing the Q_2 and Q_3 integrals of (3.2) for reverse integration by parts, each term is replaced by two terms using Pythagorean identities involving the l algebra.

$$\begin{aligned}
(3.3) \quad & \iint_{Q_2} \left\{ \tilde{\alpha} \tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{x^i} - \alpha_j) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) V^{ps} \right. \\
& + \frac{|\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} \left[j J \delta^2 \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) - \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) \right] V^{ps} \\
& + \tilde{\alpha} \tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V^{ps} \\
& - \frac{|\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} A^{ij} (D_{x^i} - \alpha_i) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) (D_{\xi^i} - \alpha_j) V^{ps} \Big\} \\
& + \iiint_{Q_3} \frac{M^{\alpha j} l_\alpha^i}{\sqrt{-j\bar{\omega}}} (D_{x^i} - \alpha_i) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^{ps}.
\end{aligned}$$

The integration by parts in (3.3) is similar to that of (3.1) except that we integrate the factor containing u or f . After integration by parts we collect the Q_2 integral using a pattern of Pythagorean identities similar to that used in going from (3.1) to (3.2). As before we obtain a cancellation of f terms over Q_2 as shown below.

$$\begin{aligned}
(3.4) \quad & \iint_{Q_2} \left\{ \frac{-A^{ij} \Lambda_i}{\Omega} (D_{x^i} - \alpha_j) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) V^s - \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \right. \\
& \cdot \frac{A^{ij} \Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) V^s - \frac{|\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^{ps} \Big\} \\
& + \iiint_{Q_3} \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^s + \int_{p=0} \left\{ \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{x^i} - \alpha_i) \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} \right. \\
& \cdot (u_i - \alpha_i u) V^{ps} + \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V^{ps} \Big\} \left| \frac{\tilde{\alpha} \tilde{h}^\beta}{\lambda_s^\beta} \right| \\
& + \iint_{p=0} \frac{m_1^{\alpha j} l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^{ps} \frac{M^{\alpha j} l_\alpha^i \Lambda_i}{\sqrt{-j\bar{\omega} \Omega}}.
\end{aligned}$$

Because of continuity, integrals over $P = 0$ are not indicated. The Q_2 and Q_3 integrals of (3.4) appear in the final integration formula (3.11).

In order to reverse the integration by parts in the integrals over $p = 0$ in (3.4), we use the descent table (2.22). The only new element is the correspondence

$$(3.5) \quad \frac{|p|}{p} V_0^{ps} \searrow \frac{\pi}{2} V^s; \quad V^s = e^{-\alpha_i x^i} \int_s^{|P|} J_0(\delta \sqrt{\nu^2 - P^2}) d\nu.$$

Setting aside the factor $\pi/2\sqrt{-j\bar{\omega}}$ the $p = 0$ integrals of (3.4) can be rephrased

as shown below in (3.7). Then we replace each term of the line integral by two terms using Pythagorean identities involving an l algebra in $n = 2$ as follows:

$$(3.6) \quad \left| \begin{array}{c} \bar{L}^i \\ l_1^i \end{array} \right| = \left| \begin{array}{c} l_1^i \\ -L^i \end{array} \right| ; \quad \left| \begin{array}{c} \bar{L}_i/\bar{\omega} \\ l_1^1 \end{array} \right| = \left| \begin{array}{c} l_1^1 \\ -L_i/\omega \end{array} \right|.$$

The $p = 0$ integrals of (3.4) can be written

$$(3.7) \quad \begin{aligned} & \int_{Q_1} \left\{ \frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) \frac{l_1^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) V^s \right. \\ & + \left. \frac{l_1^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) V^s \right\} + \iint_{Q_2} - \frac{l_1^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^s \\ & = \int_{Q_1} \left\{ - {}_a\tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) \frac{A^{ij}l_i^\alpha}{\sqrt{jg_{11}}} (u_j - \alpha_j u) V^s \right. \\ & + \left. \frac{|{}^a\tilde{h}_\beta|}{\sqrt{jg_{11}}} A^{ij} (D_{xi} - \alpha_i) (u_j - \alpha_j u) V^s - {}_a\tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) u \frac{A^{ij}l_i^\alpha}{\sqrt{jg_{11}}} (D_{xi} - \alpha_j) V^s \right. \\ & + \left. \frac{|{}^a\tilde{h}_\beta|}{\sqrt{jg_{11}}} A^{ij} (u_i - \alpha_i u) (D_{xi} - \alpha_j) V^s \right\} + \iint_{Q_2} - \frac{l_1^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^s. \end{aligned}$$

($i, j = 1, 2$)

In passing from (3.7) to (3.8) we integrate by parts, integrating the u or f factor. In the subsequent collection of the line integral the term resulting from the first collects with the fourth. The resulting third term is replaced by two, one of which is rewritten with the aid of a v differential equation. The expression (3.7) becomes

$$(3.8) \quad \begin{aligned} & \int_{Q_1} \left\{ \frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) V + u \frac{A^{ij}\Lambda_i}{\Omega} (D_{xi} - \alpha_j) V \right. \\ & + \left. \frac{l_1^i\Lambda_i}{\sqrt{jg_{11}}\Omega} [A^{ij}(D_{xi} - \alpha_i)(u_j - \alpha_j u) + jJ\delta^2 u] V^s \right\} \\ & + \iint_{Q_2} - fV + \sum_\rho \left[- \frac{A^{ij}l_i^1}{\sqrt{jg_{11}}} (u_j - \alpha_j u) V^s - u \frac{A^{ij}l_i^1}{\sqrt{jg_{11}}} (D_{xi} - \alpha_j) V^s \right]_{t_{2\rho-1}}^{t_{2\rho}} \\ & + \int_{Q_1 \atop (P=0)} - \frac{l_1^i\Lambda_i}{\sqrt{jg_{11}}\Omega} fV^s \quad ({}_1\tilde{h}^1 = 1; \quad i, j = 1, 2). \end{aligned}$$

The lower order terms of (3.8) must be examined for discontinuities at $P = 0$ and for contributions at the end points of Q_1 [see Figure 2.3b]. From (3.5) we see that V^s is continuous at $P = 0$ and vanishes at the end points while

$$A^{ij}l_i^1(D_{xi} - \alpha_j)V^s$$

yields contributions at both places; the latter involves a L operator by (3.6). These contributions are computed after ascent. In ascending from (3.8) to (3.9) we use the correspondences (2.22), (2.25), and (3.5). For Λ_i/Ω we use the third order determinant form in (2.21). Taking into account the factor previously set aside we have

$$\begin{aligned}
 & \int_{p=0} \left\{ \left[(u_j - \alpha_j u) V_0^p + u(D_{\xi^i} - \alpha_j) V_0^p \right] \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j\bar{\omega}} \\ A^{ij} \\ x_i^i \end{vmatrix} \right. \\
 (3.9) \quad & \left. + \bar{H}^{ij}(D_{x^i} - \alpha_i)(u_j - \alpha_j u) + jJ\delta^2 u - f \right] V_0^p \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j\bar{\omega}} \\ m_1^\alpha l_\alpha^i / \sqrt{jg_{11}} \\ x_i^i \end{vmatrix} \Bigg\} \\
 & + \iint_{p=0} f V_0^p \frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega} + \frac{2}{\sqrt{-j\bar{\omega}}} \sum_p \left[-u \frac{L^i}{\omega \sqrt{jg_{11}}} (D_{\xi^i} - \alpha_i) \frac{|p|}{p} V_0^p \right]_{t_{2p-1}}^{t_{2p}}.
 \end{aligned}$$

The lower order terms resulting from the integration by parts begun in (3.1) consist of the first $p = 0$ term of (3.2) and all terms in (3.9). Our problem is to show that these terms combine with the lower order terms of the integration formula (2.26) to give the $2\pi u$ term in the integration formula (3.11) below. Referring to (2.21) for the third order form of the determinant in the odd term of (3.2), we see that this term involves two M operators on u and supplies the missing component of the u differential equation in (3.9). The remaining $p = 0$ integrals of (3.9) cancel with corresponding terms of (2.26).

To evaluate the finite sum in (3.9) we refer to table (1.42) for one L operator on V_0^p . With the orientation in Figure 2.3a we have

$$\begin{aligned}
 & \sum \left[\frac{2ju}{\sqrt{-j\bar{\omega}} \sqrt{-j\bar{\omega}} \sqrt{jg_{11}}} \frac{L^i}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_i) \frac{|p|}{p} V_0^p \right] \\
 (3.10) \quad & = \sum 2j \sqrt{j|A^{ij}|} u \left(-\frac{\pi}{2} \frac{|P|}{P} e^{-\alpha_i x^i} \right) \\
 & = \pi j \sqrt{j|A^{ij}|} \{ [2e^{-\alpha_i x^i} u](o^i) - [e^{-\alpha_i x^i} u](\sigma^i) - [e^{-\alpha_i x^i} u](\bar{\sigma}^i) \}.
 \end{aligned}$$

Combining (3.10) with the averaging terms of (2.26) and using (3.4) for the form of the Q_2 and Q_3 integrals we have the *integration formula*

$$\begin{aligned}
 & 2\pi j \sqrt{j|A^{ij}|} u(\xi^i) = 2\pi j \sqrt{j|A^{ij}|} [e^{-\alpha_i x^i} u](o^i) \\
 (3.11) \quad & + \iint_{Q_2} \left\{ -\frac{A^{ij} \Lambda_i}{\Omega} (D_{x^i} - \alpha_j) \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) V^s \right. \\
 & \left. - \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij} \Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) V^s \right\} + \iint_{Q_3} \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (f_i - \alpha_i f) V^s.
 \end{aligned}$$

An idea of the nature of this formula can be gained by considering the wave equation in $n = 3$. If the time-like damping constant is zero and f does not vary

in the time direction the triple integral of (3.11) drops out. Furthermore, if first and second derivatives of u involving at least one time derivative vanish on Q_2 then u is constant in the time direction.

It is possible to perform reverse integrations by parts in (3.11) similar to that used in going from (2.26) to (2.30). In the present case no lower order terms result from the integration by parts since $V^s = 0$ on $R = 0$ and since the infinity in V^s at $P = p = 0$ is logarithmic. Thus we obtain *another form* of (3.11) as follows:

$$\begin{aligned}
 (3.12) \quad & 2\pi j \sqrt{j|A^{ij}|} u(\xi^i) = 2\pi j \sqrt{j|A^{ij}|} [e^{-\alpha_i x^i} u](o^i) \\
 & + \iint_{Q_2} \left\{ -\frac{A^{ij}\Lambda_i}{\Omega} (u_j - \alpha_j u) V - \frac{m_1^\alpha l_i^\alpha}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \frac{A^{ij}\Lambda_i}{\Omega} (D_{\xi i} - \alpha_j) V^s \right. \\
 & \left. + \frac{m_1^\alpha l_i^\alpha \Lambda_i}{\sqrt{jg_{11}\Omega}} [A^{ij}(u_i - \alpha_i u)(D_{\xi i} - \alpha_j) V^s + jJ\delta^2 u V^s] \right\} + \iiint_{Q_3} f V.
 \end{aligned}$$

4. Boundary Conditions. Boundary conditions are derived from the integration formula and its derivatives by allowing the point ξ^i to approach the fixed boundary of Q_2 . Unless the region of integration shrinks to zero, this process leads to conditions on the boundary values. These conditions are used to obtain boundary values not given in the data of the problem, but which are needed in its solution.

We use the terminology *first type boundary condition* to denote the result of applying the above mentioned process to the integration formula itself. Since the integration formulae in this paper contain integrals which permit a formal limit process, no first type boundary conditions are explicitly shown. The second type boundary condition in this paper is derived by applying a differential operator to the integration formula (2.30). The fact that (2.30) as it stands permits a formal limit process can be seen by integrating by parts in the form of the integrals in (4.1) below, passing to the limit, and reversing the integration by parts. In case ξ^i coalesces with one of the averaging points in deriving the first type boundary condition, the averaging term merely transposes and collects with the left member.

To write the Q_2 and Q_3 integrals of (2.30) in the form given below we introduce an upper p index in the first term of the double integral and in the Q_3 integral. In the second term of the Q_2 integral we introduce an upper P index. The latter term is replaced by two terms and the first and third terms of (2.30) collect. The Q_2 and Q_3 integrals of (2.30) become

$$\begin{aligned}
 (4.1) \quad & N^k (D_{\xi k} - \alpha_k) \left[\iint_{Q_2} \left\{ \frac{\tilde{\alpha} \tilde{h}^\beta}{\sqrt{-j\tilde{\omega}}} \frac{A^{ij} l_i^\alpha}{\sqrt{-j\tilde{\omega}}} (u_j - \alpha_j u) (-D_\beta - \alpha_k \lambda_\beta^k) V^p \right. \right. \\
 & + \frac{|\alpha \tilde{h}_\beta|}{\sqrt{-j\tilde{\omega}}} jJ\delta^2 u V^p + \frac{M^\alpha l_i^\alpha}{\sqrt{-j\tilde{\omega}}} (u_i - \alpha_i u) \left[\frac{\tilde{\alpha} \tilde{h}^\beta}{\sqrt{-j\tilde{\omega}}} (-D_\beta - \alpha_k \lambda_\beta^k) \right. \\
 & \left. \left. \frac{A^{ij} l_i^\alpha}{\sqrt{-j\tilde{\omega}}} (D_{\xi i} - \alpha_j) V^{pP} + \frac{|\alpha \tilde{h}_\beta|}{\sqrt{-j\tilde{\omega}}} jJ\delta^2 V^{pP} \right] \right\} \\
 & \left. + \iiint_{Q_3} f \frac{M^\alpha l_i^\alpha}{\sqrt{-j\tilde{\omega}}} (-D_{x^i} - \alpha_i) V^p \right].
 \end{aligned}$$

The N operator displayed in (4.1) contains a fixed but arbitrary contravariant vector N^k .

In performing the operation indicated in (4.1) we use second type Leibnitz rules in the Q_3 integral and in the terms of the Q_2 integral containing $\tilde{h}$. On the remaining terms we use the ordinary Leibnitz rule.

$$\begin{aligned}
 & \iint_{Q_2} \left\{ {}^\alpha \tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) N^k (D_{\xi^k} - \alpha_k) V^p \right. \\
 & + \frac{|{}^\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} [-A^{ij} (D_{xi} - \alpha_i) (u_j - \alpha_j u) + f] N^k (D_{\xi^k} - \alpha_k) V^p \\
 & + {}^\alpha \tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) N^k (D_{\xi^k} - \alpha_k) V^{pP} \\
 & + \frac{|{}^\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) j J \delta^2 N^k (D_{\xi^k} - \alpha_k) V^{pP} \Big\} \\
 & + \iiint_{Q_3} \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) N^k (D_{\xi^k} - \alpha_k) V^p \\
 & + \int_{\substack{p=0 \\ (P=0)}} \left\{ \frac{-M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij} \Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) V_0^p \right. \\
 (4.2) \quad & + \frac{|{}^\alpha \tilde{h}_\beta|}{\sqrt{-j\bar{\omega}}} [A^{ij} (u_i - \alpha_i u) (D_{\xi^i} - \alpha_j) V_0^p + j J \delta^2 u V_0^p] \Big\} \left| \begin{matrix} N^k \lambda_{\xi^k}^\beta \\ \lambda_t^\beta \end{matrix} \right| \\
 & + \int_{\substack{p=0 \\ (P=0)}} - \frac{A^{ij} l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) N^k (D_{\xi^k} - \alpha_k) V^p \left| \begin{matrix} {}^\alpha \tilde{h}^\beta \\ \lambda_t^\beta \end{matrix} \right| \\
 & + \int_{\substack{(p=0) \\ (P=0)}} \frac{-M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) N^k (D_{\xi^k} - \alpha_k) V^{pP} \left| \begin{matrix} {}^\alpha \tilde{h}^\beta \\ \lambda_t^\beta \end{matrix} \right| \\
 & + \iint_{\substack{(Q_2) \\ (p=0) \\ (P=0)}} f V N^k x_{\xi^k}^i \frac{\Lambda_i}{\Omega} + \iint_{\substack{Q_2 \\ (p=0) \\ (P=0)}} - f N^k (D_{\xi^k} - \alpha_k) V^p \frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega} \\
 & + \sum_p \left\{ - \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (u_j - \alpha_j u) V^p \left| \begin{matrix} {}^\alpha \tilde{h}^\beta \\ N^k D_{\xi^k}(\lambda^\beta) \end{matrix} \right| - \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \right. \\
 & \cdot \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V^{pP} \left| \begin{matrix} {}^\alpha \tilde{h}^\beta \\ N^k D_{\xi^k}(\lambda^\beta) \end{matrix} \right| \Big\}_{t_{2p-1}}^{t_{2p}} + \int_{\text{edges}} - f V^p \left| \begin{matrix} M^\alpha l_\alpha^i / \sqrt{-j\bar{\omega}} \\ N^k D_{\xi^k}(x^i) \\ D_i(x^i) \end{matrix} \right|.
 \end{aligned}$$

In the above work we have written the lower order integrals with N determinant in the original form (2.30) and because of continuity this integral does not con-

tribute for $P = 0$. In fact all eliminations of integrals on $P = 0$ can be accounted for by continuity. In the term with $\tilde{h}$ determinant taken over $p = 0$ we must have $\alpha = 1$ and the term drops out by continuity. The first f integral drops out by continuity on $p = 0$ and by the fixed nature of Q_2 . The computation of the finite sum in (4.2) is postponed until after the treatment of line integrals. The Q_2 and Q_3 integrals of (4.2) appear in the final boundary condition (4.16).

Next we operate on the line integral of (2.30) and because of the dependence of the representation on $M_\alpha l_i^\alpha \xi^i$ we must differentiate through both Greek and Latin arguments. We have

$$\begin{aligned}
 & N^k(D_{\xi^k} - \alpha_k) \int_{p=0} -u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| \\
 &= \int_{p=0} \left\{ -u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) N^k(D_{\xi^k} - \alpha_k) V_0^p \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| \right. \\
 (4.3) \quad &+ (N^k \lambda_{\xi^k}^\beta) D_\beta \left[-u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \right] \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| \\
 &+ \left[-u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \right] \left| \frac{\delta_\alpha^\beta}{l_i^\beta N^k x_{i\xi^k}^i} \right| \Bigg\} \\
 &+ \left\{ -u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| N^i t_{\xi^i} \right\}_{t=\sigma}^{t=\bar{\sigma}}
 \end{aligned}$$

Our next step involves integration by parts in the term of (4.3) with second derivatives in the determinant. To collect the result we need the determinant relation

$$(4.4) \quad N^i \lambda_{\xi^i}^\gamma \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| - \lambda_i^\gamma \left| \frac{\delta_\alpha^\beta}{l_i^\beta N^j x_{i\xi^j}^i} \right| = {}_\alpha \tilde{h}^\gamma \left| \frac{N^i \lambda_{\xi^i}^\beta}{\lambda_i^\beta} \right|.$$

We integrate by parts in (4.3) and use (4.4) to collect two terms in the line integral. In performing the λ^β differentiation in the latter result we add and subtract a term involving the factor $\alpha_i \lambda_\beta^i$. Then (4.3) becomes

$$\begin{aligned}
 & \int_{p=0} \left\{ -u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) N^k(D_{\xi^k} - \alpha_k) V_0^p \left| \frac{\delta_\alpha^\beta}{l_i^\beta x_i^i} \right| \right. \\
 &+ \left[-{}_a \tilde{h}^\beta (D_\beta - \alpha_k \lambda_\beta^k) u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \right. \\
 (4.5) \quad &+ \left. \left. u {}_a \tilde{h}^\beta (-D_\beta - \alpha_i \lambda_\beta^i) \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \right] \left| \frac{N^k \lambda_{\xi^k}^\beta}{\lambda_i^\beta} \right| \right\} \\
 &+ \sum \left\{ -u \frac{A^{ij} l_i^\alpha}{\sqrt{-j\bar{\omega}}} (D_{\xi^i} - \alpha_j) V_0^p \left| \frac{\delta_\alpha^\beta}{l_i^\beta N^k D_{\xi^k}(x^i)} \right| \right\}.
 \end{aligned}$$

When we rewrite with Pythagorean identities the $\tilde{h}$ terms of (4.5) cancel the line integral with N determinant in (4.2). The remaining $P = 0$ and $p = 0$ integrals of (4.2) and (4.5) appear in the final boundary condition (4.16).

The averaging terms to be collected are to be found in (4.2), (4.5), and the terms of (2.30) yet to be differentiated. In order to clarify the following discussion it is expedient to make a definite choice for the vectors of zero length in Section 1, namely $N^\alpha l_\alpha^i$ and $n_1^\alpha l_\alpha^i$, and similarly for generators in $p = 0$ which we let have directions $\bar{N}^\alpha l_\alpha^i$ and $\bar{n}_1^\alpha l_\alpha^i$. We take

$$(4.6) \quad \begin{aligned} \left\{ \begin{matrix} N^\alpha l_\alpha^i \\ n_1^\alpha l_\alpha^i \end{matrix} \right\} &= \frac{1}{\rho} \left[\pm \frac{M^\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} + \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} \right], & \text{where } \rho &= \sqrt[4]{-4|A^{ij}|\omega}; \\ \left\{ \begin{matrix} \bar{N}^\alpha l_\alpha^i \\ \bar{n}_1^\alpha l_\alpha^i \end{matrix} \right\} &= \frac{1}{\bar{\rho}} \left[\mp \frac{L^i}{\sqrt{-j\bar{\omega}}} + \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} \right], & \text{where } \bar{\rho} &= \sqrt[4]{-4|A^{ij}|\bar{\omega}}. \end{aligned}$$

The factors ρ can be checked by the basic postulates. Note that the $N^\alpha l_\alpha^i$ and $\bar{N}^\alpha l_\alpha^i$ issue to θ^i and σ^i respectively in Figure 2.1.

Next we take up the computation of the total derivative $N^k D_{\xi^k}(x^i)$ appearing in (4.2) and (4.5). At the point θ^i we have $P = 0$ and $n_\alpha^\alpha l_\alpha^i X^i = 0$ and implicit differentiation gives

$$(4.7) \quad \begin{aligned} L_i N^k D_{\xi^k}(x^i) &= L_i N^i; \\ n_\alpha^\alpha l_\alpha^i N^k D_{\xi^k}(x^i) &= n_\alpha^\alpha l_\alpha^i N^i; \\ \Lambda_i N^k D_{\xi^k}(x^i) &= 0. \end{aligned}$$

The solution of these equations is given by

$$(4.8) \quad N^k D_{\xi^k}(x^i) = N^i - \frac{N^\alpha l_\alpha^i N^k \Lambda_k}{N^\alpha l_\alpha^i \Lambda_i} \quad (\text{for } \bar{\theta}^i \text{ replace } N^\alpha \text{ by } n_1^\alpha).$$

Equations (4.8) hold for σ^i and $\bar{\sigma}^i$ if we use the bar vectors in (4.6b).

After these preliminaries we begin the computation of the coefficient of $u_j - \alpha_j u$ at σ^i in (4.16). A part of this term comes from differentiating the averaging terms in (2.30) as follows:

$$(4.9) \quad \pi j \sqrt{j|A^{ij}|} \{ [e^{-\alpha_i x^i} (u_i - \alpha_i u) N^k D_{\xi^k}(x^i)](\sigma^i) + [e^{-\alpha_i x^i} (u_i - \alpha_i u) \bar{N}^k D_{\xi^k}(x^i)](\bar{\sigma}^i) \}.$$

Combining (4.9) with the finite sum in (4.2) we have all the terms at σ^i involving $(u_j - \alpha_j u)$. The coefficient is

$$\begin{aligned}
(4.10) \quad & \pi e^{-\alpha_i x^i} j \sqrt{j |A^{ij}|} \left[N^j - \frac{\bar{N}^\alpha l_\alpha^j N^k \Lambda_k}{\bar{N}^\alpha l_\alpha^i \Lambda_i} \right] \\
& + \left\{ - \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j \bar{\omega}} \\ A^{ij} \\ N^i \end{vmatrix} + \frac{N^k \Lambda_k}{\bar{N}^\alpha l_\alpha^i \Lambda_i} \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j \bar{\omega}} \\ A^{ij} \\ \bar{N}^\alpha l_\alpha^i \end{vmatrix} \right\} V^p \\
& + \frac{M^\beta l_\beta^j}{\sqrt{-j \bar{\omega}}} \left\{ - \begin{vmatrix} L^i / \sqrt{-j \bar{\omega}} \\ A^{ik} \\ N^i \end{vmatrix} + \frac{N^k \Lambda_k}{\bar{N}^\alpha l_\alpha^i \Lambda_i} \begin{vmatrix} L^i / \sqrt{-j \bar{\omega}} \\ A^{ik} \\ \bar{N}^\alpha l_\alpha^i \end{vmatrix} \right\} (D_{\xi^k} - \alpha_k) V^{pP} \quad (p = 0).
\end{aligned}$$

The last determinant in (4.10) has the direction $M^\beta l_\beta^j$ and the term is continuous at $p = 0$. The second and fourth terms of (4.10) cancel since the determinant in the fourth term has the value $-j \sqrt{j |A^{ij}|} \bar{N}^\alpha l_\alpha^i$ and V^p contributes $-\pi e^{-\alpha_i x^i}$ at σ^i . In the third term of (4.10) we use the relation

$$(4.11) \quad A^{ij} = \frac{M^\alpha l_\alpha^i M^\beta l_\beta^j}{\bar{\omega}} + j \sqrt{-|\bar{h}^{\alpha\beta}|} [\bar{n}_1^\alpha l_\alpha^i \bar{N}^\beta l_\beta^j + \bar{N}^\alpha l_\alpha^i \bar{n}_1^\beta l_\beta^j].$$

The fifth term contributes for the m component of A^{ik} . Thus (4.10) becomes

$$\begin{aligned}
(4.12) \quad & \pi e^{-\alpha_i x^i} j \sqrt{j |A^{ij}|} \left\{ N^j - \frac{N^i \bar{N}^\alpha l_\alpha^i \bar{N}^\beta l_\beta^j}{\bar{\omega}} + N^i \bar{n}_1^\alpha l_\alpha^i \bar{n}_1^\beta l_\beta^j - \frac{N^i M^\alpha l_\alpha^i}{\bar{\omega}} M^\beta l_\beta^j \right\} \\
& = \pi e^{-\alpha_i x^i} j \sqrt{j |A^{ij}|} 2(N^i \bar{n}_1^\alpha l_\alpha^i) \bar{n}_1^\beta l_\beta^j
\end{aligned}$$

which agrees with (4.16) for the coefficient of $u_j - \alpha_j u$ at σ^i .

The contribution at θ^i on $P = 0$ involves only the V^{pP} term in (4.2). Comparing with (4.10) and (4.12) we have the coefficient

$$\begin{aligned}
(4.13) \quad & \frac{M^\beta l_\beta^j}{\sqrt{-j \bar{\omega}}} \left\{ - \begin{vmatrix} L^i / \sqrt{-j \bar{\omega}} \\ A^{ik} \\ N^i \end{vmatrix} + \frac{N^k \Lambda_k}{\bar{N}^\alpha l_\alpha^i \Lambda_i} \begin{vmatrix} L^i / \sqrt{-j \bar{\omega}} \\ A^{ik} \\ \bar{N}^\alpha l_\alpha^i \end{vmatrix} \right\} (D_{\xi^k} - \alpha_k) V^{pP} \\
& = \frac{M^\beta l_\beta^j}{\sqrt{-j \bar{\omega}}} j \sqrt{j |A^{ij}|} \left\{ \frac{N^i N_\alpha l_\alpha^i N^\beta l_\beta^k}{\bar{\omega}} \right. \\
& \quad \left. - N^i \bar{n}_1^\alpha l_\alpha^i \bar{n}_1^\beta l_\beta^k - \frac{N^k \Lambda_k}{\bar{N}^\alpha l_\alpha^i \Lambda_i} N^\alpha l_\alpha^k \right\} (D_{\xi^k} - \alpha_k) V^{pP}.
\end{aligned}$$

The following derivative formula is obtained from (1.42).

$$\begin{aligned}
(4.14) \quad & \left\{ \frac{N^\alpha l_\alpha^i}{\bar{n}_1^\alpha l_\alpha^i} \right\} (D_{\xi^i} - \alpha_i) V^{pP} = \frac{e^{-\alpha_i x^i}}{\rho} \left\{ \pm \frac{|P|}{P} \arctan \frac{R}{|P|} \right. \\
& \quad \left. - \frac{|Pp|}{Pp} \left[\arctan \frac{|P| R}{s |p|} + \arctan \frac{|p| R}{s |P|} \right] + \dots \right\}.
\end{aligned}$$

Thus the discontinuity in (4.14) at θ^i or $P = 0$ is in the $n^\alpha l_\alpha^i$ component, and the middle term of (4.13) is the coefficient of $(u_j - \alpha_j u)$ at θ^i in the final formula (4.16).

The sum in (4.5) contributes the δ^2 term of (4.16). Here we are concerned with the second direction in (4.10) together with the following derivative from (1.42).

$$(4.15) \quad l_\alpha^i (D_{\xi^i} - \alpha_i) V^p = \frac{|p|}{p} e^{-\alpha_i x^i} (j A_{ij} l_\alpha^i X^j) \left[-\frac{|p|}{S^2 R} + \frac{J \delta^2}{2} \arctan \frac{R}{|p|} + \dots \right].$$

The fourth term of (4.10) makes no contribution since $\bar{n}_\alpha^1 l_\alpha^i X^i = 0$ at σ^i . The third term accounts for the δ^2 term in (4.16).

Our *second type boundary condition* is

$$\begin{aligned} & 2\pi j \sqrt{j |A^{ij}|} [N^i (u_i - \alpha_i u)] (\xi^i) \\ &= \pi e^{-\alpha_i x^i} \left\{ 2j \sqrt{j |A^{ij}|} (\bar{n}_\alpha^1 l_\alpha^i N^i) \bar{n}_\beta^j l_\beta^j (u_j - \alpha_j u) \right. \\ & \quad \left. + \frac{j J \delta^2}{2} u \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j \bar{\omega}} \\ X^i \\ N^i \end{vmatrix} \right\} (\sigma^i) \\ &+ \pi e^{-\alpha_i x^i} \left\{ 2j \sqrt{j |A^{ij}|} \left(\frac{\bar{N}_\alpha l_\alpha^i N^i}{\bar{\omega}} \right) \bar{N}^\beta l_\beta^j (u_j - \alpha_j u) \right. \\ & \quad \left. - \frac{j J \delta^2}{2} u \begin{vmatrix} M^\alpha l_\alpha^i / \sqrt{-j \bar{\omega}} \\ X^i \\ N^i \end{vmatrix} \right\} (\bar{\sigma}^i) \\ &- \left[\pi e^{-\alpha_i x^i} \frac{2j \sqrt{j |A^{ij}|}}{\rho} (n_\alpha^1 l_\alpha^i N^i) \frac{M^\beta l_\beta^j}{\sqrt{-j \bar{\omega}}} (u_j - \alpha_j u) \right] (\theta^i) \\ (4.16) \quad &+ \left[\pi e^{-\alpha_i x^i} \frac{2j \sqrt{j |A^{ij}|}}{\rho} \left(\frac{N_\alpha l_\alpha^i N^i}{\bar{\omega}} \right) \frac{M^\beta l_\beta^j}{\sqrt{-j \bar{\omega}}} (u_j - \alpha_j u) \right] (\bar{\theta}^i) \\ &+ \int_{P=0} \left\{ -\frac{M^\alpha l_\alpha^i}{\sqrt{-j \bar{\omega}}} (u_i - \alpha_i u) (D_{\xi^i} - \alpha_i) N^k (D_{\xi^k} - \alpha_k) V^{pP} \right\} \begin{vmatrix} L^i / \sqrt{-j \bar{\omega}} \\ A^{ij} \\ x_i^i \end{vmatrix} \\ &+ \int_{P=0} \{ -(u_j - \alpha_j u) N^k (D_{\xi^k} - \alpha_k) V_0^p - u (D_{\xi^i} - \alpha_i) N^k (D_{\xi^k} - \alpha_k) V_0^p \} \end{aligned}$$

$$\begin{aligned}
& \left| \begin{array}{c} M^{\alpha} l_{\alpha}^i / \sqrt{-j\bar{\omega}} \\ A^{ij} \\ x_i^i \end{array} \right| + \iint_{p=0} -f N^k (D_{\xi^k} - \alpha_k) V_0^p \frac{M^{\alpha} l_{\alpha}^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega} \\
& + \iint_{Q_2} \left\{ -\frac{A^{ij} \Lambda_i}{\Omega} (D_{x^i} - \alpha_j) \frac{M^{\alpha} l_{\alpha}^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) N^k (D_{\xi^k} - \alpha_k) V^p \right. \\
& - \frac{L^i}{\sqrt{-j\bar{\omega}}} (D_{x^i} - \alpha_i) \frac{M^{\alpha} l_{\alpha}^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \frac{A^{ij} \Lambda_i}{\Omega} (D_{\xi^i} - \alpha_j) N^k \\
& \cdot (D_{\xi^k} - \alpha_k) V^{pp} + \frac{L^i \Lambda_i}{\sqrt{-j\bar{\omega}} \Omega} \left[A^{ij} (D_{x^i} - \alpha_i) \frac{M^{\alpha} l_{\alpha}^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \right. \\
& \cdot (D_{\xi^i} - \alpha_j) N^k (D_{\xi^k} - \alpha_k) V^{pp} + \frac{M^{\alpha} l_{\alpha}^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) j J \delta^2 N^k \\
& \cdot (D_{\xi^k} - \alpha_k) V^{pp} \left. \right] \left. \right\} + \iiint_{Q_3} \frac{M^{\alpha} l_{\alpha}^i}{\sqrt{-j\bar{\omega}}} (f_i - \alpha_i f) N^k (D_{\xi^k} - \alpha_k) V^p.
\end{aligned}$$

This formula allows a formal limit process as ξ^i approaches Q_2 .

5. Illustrative Example. We illustrate the application of the above theory by setting up the integration formulae (2.30) and (3.12) for the undamped vibrating drumhead equation

$$(5.1) \quad -c^2 u_{xx} - c^2 u_{yy} + u_{tt} = f.$$

As shown in Figure 5.1 we take the case where data is assigned over $t = 0$ and $y = 0$.

Referring to the discussion at the end of Section 2, we compute the A_{ij} and determine j .

$$(5.2) \quad |A^{ij}| = \begin{vmatrix} -c^2 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}; \quad |A_{ij}| = \begin{vmatrix} -1/c^2 & 0 & 0 \\ 0 & -1/c^2 & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

$$j = 1; \quad j\sqrt{j|A^{ij}|} = c^2.$$

Next we compute quantities connected with the compound algebra. The choice of the vectors L^i and $M^{\alpha} l_{\alpha}^i$ must be consistent with Figure 2.1, the positive orientation in the present case being determined by the ordering of the axes x, y, t .

FIGURE 5.1

$$\begin{aligned}
 L^i &= [0, 1, 0]; & L_i &= \left[0, \frac{-1}{c^2}, 0\right]; & \omega &= -\frac{1}{c^2}; & \sqrt{-j\omega} &= \frac{1}{c}; \\
 M^\alpha l_\alpha^i &= [1, 0, 0]; & M_\alpha l_i^\alpha &= \left[\frac{-1}{c^2}, 0, 0\right]; & \bar{\omega} &= -\frac{1}{c^2}; \\
 \sqrt{-j\bar{\omega}} &= \frac{1}{c}; & m_1^\alpha l_\alpha^i &= [0, 0, -1]; & m_\alpha^1 l_i^\alpha &= [0, 0, -1]; \\
 g_{11} &= A_{ij}(m_1^\alpha l_\alpha^i)(m_1^\beta l_\beta^j) = 1; & \sqrt{jg_{11}} &= 1; & R &= \frac{\sqrt{c^2 T^2 - X^2 - Y^2}}{c}; \\
 P &= \frac{Y}{c}; & p &= \frac{X}{c}; & s &= -T; & S &= \frac{\sqrt{c^2 T^2 - X^2}}{c}; \\
 \bar{S} &= \frac{\sqrt{c^2 T^2 - Y^2}}{c}; & \tilde{S} &= \frac{\sqrt{X^2 + Y^2}}{c}; & V &= \frac{1}{R}.
 \end{aligned}
 \tag{5.3}$$

Using representations $(\lambda^1, \lambda^2, 0)$ and $(\lambda^1, 0, \lambda^2)$ for the boundaries, we compute quantities connected with the λ algebra.

Base plane:

$$\lambda_\alpha^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \frac{\Lambda_i}{\Omega} = [0, 0, 1]; \quad \frac{L^i \Lambda_i}{\sqrt{-j\omega\Omega}} = 0;$$

$$(5.4) \quad \frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\omega\Omega}} = 0; \quad \frac{m_1^\alpha l_\alpha^i \Lambda_i}{\sqrt{jg_{11}\Omega}} = -1.$$

Side plane:

$$\lambda_\alpha^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \frac{\Lambda_i}{\Omega} = [0, -1, 0]; \quad \frac{L^i \Lambda_i}{\sqrt{-j\omega\Omega}} = -c;$$

$$\frac{M^\alpha l_\alpha^i \Lambda_i}{\sqrt{-j\omega\Omega}} = 0; \quad \frac{m_1^\alpha l_\alpha^i \Lambda_i}{\sqrt{jg_{11}\Omega}} = 0.$$

In setting up (2.30) we note that the line integral and two terms of the double integral do not contribute since $\delta = 0$ and $M^\alpha l_\alpha^i \Lambda_i = 0$. In computing the second term of the double integral we insert a basic Kronecker delta and use table (1.42).

$$(5.5) \quad \frac{A^{ij} \Lambda_i}{\Omega} \left[\frac{L_j L^k}{\omega} + m_1^\alpha l_j^\alpha m_{1\beta}^{\beta k} \right] (D_{\xi^k} - \alpha_k) V^p$$

$$= \frac{-L^i \Lambda_i}{\sqrt{-j\omega\Omega}} \left(\frac{pP}{\bar{S}^2 R} \right) + \frac{m_1^\alpha l_\alpha^i \Lambda_i}{\sqrt{jg_{11}\Omega}} \left(\frac{-sp}{\bar{S}^2 R} \right).$$

This result is used in the u_x terms of the following formula. The sense of integration indicated by the sign in front of the double integral is determined by mapping in the λ^α plane. The integration formula is

$$(5.6) \quad 2\pi c^2 u(\xi, \eta, \tau) = \pi c^2 \left[u \left(\xi, 0, \tau - \frac{\eta}{c} \right) + u(\xi, \eta + c\tau, 0) \right]$$

$$- \int_0^{\eta+c\tau} \int_{\xi-\sqrt{c^2\tau^2-Y^2}}^{\xi+\sqrt{c^2\tau^2-Y^2}} \left\{ -u_t \frac{c}{\sqrt{c^2\tau^2-X^2-Y^2}} \right.$$

$$\left. - cu_x \frac{c^2\tau X}{(c^2\tau^2-Y^2) \sqrt{c^2\tau^2-X^2-Y^2}} \right\} dx dy$$

$$+ \int_0^{\tau-\eta/c} \int_{\xi-\sqrt{c^2T^2-\eta^2}}^{\xi+\sqrt{c^2T^2-\eta^2}} \left\{ -c^2 u_y \frac{c}{\sqrt{c^2T^2-X^2-\eta^2}} \right.$$

$$\left. - cu_x \frac{-c^2\eta X}{(c^2T^2-Y^2) \sqrt{c^2T^2-X^2-\eta^2}} \right\} dx dt + \iiint \frac{f}{R}.$$

In setting up formula (3.12) we must compute the first three terms of the double integral. In place of (5.5) we have

$$\begin{aligned}
 & \frac{A^{ij}\Lambda_i}{\Omega} \left[\frac{L_j L^k}{\omega} + m_\alpha^1 l_j^\alpha m_1^\beta l_\beta^k \right] (D_{\xi^k} - \alpha_k) V^s \\
 & = -j \frac{L^i \Lambda_i}{\sqrt{-j\omega\Omega}} \left(-\frac{sP}{\bar{S}^2 R} \right) + j \frac{m_1^\alpha l_\alpha^i \Lambda_i}{\sqrt{jg_{11}\Omega}} \left(\frac{1}{R} \right); \\
 (5.7) \quad & A^{ij}(u_i - \alpha_i u) \left[\frac{L_j L^k}{\omega} + \frac{M_\alpha l_j^\alpha M^\beta l_\beta^k}{\bar{\omega}} + m_\alpha^1 l_j^\alpha m_1^\beta l_\beta^k \right] (D_{\xi^k} - \alpha_k) V^s \\
 & = -j \frac{L^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \left(-\frac{sP}{\bar{S}^2 R} \right) - j \frac{M_\alpha l_\alpha^i}{\sqrt{-j\bar{\omega}}} (u_i - \alpha_i u) \left(-\frac{sp}{\bar{S}^2 R} \right) \\
 & \quad + j \frac{m_1^\alpha l_\alpha^i}{\sqrt{jg_{11}}} (u_i - \alpha_i u) \left(\frac{1}{R} \right).
 \end{aligned}$$

On the side plane only the first quantity in (5.7) contributes; on the base plane the two terms involving $1/R$ cancel, and the other two terms in (5.7b) contribute. Formula (3.12) becomes

$$\begin{aligned}
 (5.8) \quad & 2\pi c^2 u(\xi, \eta, \tau) = 2\pi c^2 u(\xi, \eta, 0) - \int_0^{\eta+c\tau} \int_{\xi-\sqrt{c^2\tau^2-Y^2}}^{\xi+\sqrt{c^2\tau^2-Y^2}} \\
 & \cdot \left\{ -u_t \frac{c}{\sqrt{c^2\tau^2-X^2-Y^2}} + \frac{cu_y(-c^2\tau Y) + cu_x(-c^2\tau X)}{(X^2+Y^2)\sqrt{c^2\tau^2-X^2-Y^2}} \right\} dx dy \\
 & + \int_0^{\tau-\eta/c} \int_{\xi-\sqrt{c^2T^2-\eta^2}}^{\xi+\sqrt{c^2T^2-\eta^2}} \left\{ -c^2 u_y \frac{c}{\sqrt{c^2T^2-X^2-\eta^2}} \right. \\
 & \quad \left. + u_t \frac{-c^3 T \eta}{(X^2+\eta^2)\sqrt{c^2T^2-X^2-\eta^2}} \right\} dx dt + \iiint \frac{f}{R}.
 \end{aligned}$$

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