Vector Algebras and Potentials

E. W. TITT, W. S. McCULLEY, FLETCHER W. DONALDSON,
ROGER OSBORN, L. G. WORTHINGTON, & W. C. LONG¹

The Mathematical Research Laboratory at The University of Texas, Austin, Texas

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Green, in treating Laplace's equation, was the first to base integrating factors on solutions which are functions of the distance between two points. Other writers have extended the idea to equations of the hyperbolic and parabolic types and to damped equations. Riemann was the first to discover that in the case of a damped equation the Green potential should involve a Bessel function of the above mentioned distance. Hadamard² has extended the Green potential

¹ Assisted over shorter periods by: O. B. FAIRCLOTH, MRS. MARJORIE EASON, HAL D. SMITH, JOHN S. MILLER, PHILIP J. MILLER, C. B. WILLIAMS, L. N. ROWELL, R. L. J. PENN, & F. R. YETT.

This paper constitutes a part of a theory of linear partial differential equations developed under the direction of E. W. TITT, and is one of a series of joint papers which present the parts of the theory which are common to all partial differential equations. The students listed as contributing authors, in the order mentioned, entered the work before the general theory began to materialize and took over the tremendous load of computation required when the knowledge of one equation contributed relatively little to the knowledge of another. In the order mentioned the assisting personnel, each working for approximately one year, have, over a period of years, kept the work up to date in the many types of equations to be covered which in turn has contributed to the development of the theory as a whole. The Office of Naval Research under contract with The University of Texas contributed to the support of the work during 1947-48 and 1949-50. Part of the work on this paper was done at Indiana University in 1952-53 under Army Ordnance contract DA-33-008 ORD-454.

² HADAMARD, J., *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, New Haven: Yale University Press, 1923, Bk. II, Ch. III. Compare with T. Y. THOMAS & E. W. TITT, "On the Elementary Solution of the General Linear Differential Equation of the Second Order with Analytic Coefficients," *Journal de Mathématiques Pures et Appliquées*, Tome XVIII, 1939, pp. 217-248. The reader is referred to the former for a more complete bibliography of the subject.

(elementary solution) to the most general linear equation with analytic coefficients.

In this paper we present a second type of solution which we call the retarded potential. Our treatment covers the general second order equation with constant coefficients and at the same time we compare the Green and the retarded potentials for such an equation. Non-parabolic retarded potentials depend not only on the non-Euclidean length R of the Greek-to-Latin vector but also on the length S of the non-Euclidean projection of this vector in a non-characteristic plane. Although this non-Euclidean geometry is taken up in a separate paper it is necessary to develop various vector algebras in the present treatment in order to present the retarded potential in its full generality.

In developing these algebras we use the tensor notation, and their power results largely from Lagrange-type identities¹ and some extensions which we call Pythagorean identities. The basic algebra developed in Section 1 deals with two sets of n vectors each in n dimensions and its properties are concerned with scalar and cross product relationships together with identities of the above mentioned types. In a restricted algebra we associate a symmetric matrix A^{ij} with our set of $2n$ vectors and the matrix A^{ij} is used to place further restrictions on the vectors of the basic algebra. In classifying the restricted algebras we used the terms parabolic and non-parabolic, characteristic and non-characteristic² in accordance with the terminology of partial differential equations.

Besides being an appropriate tool for studying the above mentioned non-Euclidean geometry, these algebras are useful in classifying boundary value problems, they serve as an alternative to the classical theory of characteristics, and they are also useful in the various parts of the theory involving integration processes. In another paper we present a Leibnitz rule for the differentiation of a multiple integral and at the same time an extension to improper integrals where the classical rule yields divergent results. This extension involves integration by parts and the algebra of this paper enters again in preparing the integrand for application of the latter rule. These two operations combined form a workable alternative to Hadamard's³ finite part of an improper integral.

In treating parabolic potentials we use Hadamard's idea of obtaining the parabolic as the limit of the non-parabolic. The parabolic potentials are functions of S and P which are analogous to non-parabolic distances S and P . The quantity P is the non-Euclidean length of the third side of the R, S non-Euclidean right triangle mentioned above, P being the distance from point to plane. One of the above mentioned Pythagorean identities leads to the appropriate Py-

¹ One of the authors has used a non-Euclidean Lagrange type identity in the study of a boundary value problem for non-linear equations. See E. W. TITT, "An Initial Value Problem for All Hyperbolic Partial Differential Equations of Second Order with Three Independent Variables," *Annals of Mathematics*, **40**, 1939, pp. 862-891.

² We use the abbreviations P and $N-P$ for parabolic and non-parabolic; also, C and $N-C$ for characteristic and non-characteristic.

³ HADAMARD, J., *op. cit.* Bk. III, Ch. I.

thagorean theorems for the R, S, P right triangle in our various non-Euclidean geometries (this accounts for the terminology Pythagorean identity).

In deriving an exterior formula for the vibrating drumhead equation, Volterra¹ employed the logarithmic potential $(1/R) \ln (R^2/S)$. When $n = 3$ the retarded and Green potentials coincide and Volterra's logarithmic quantity can be regarded as the second solution. In this paper we extend Volterra's logarithmic potential to the damped equation. In our method the integrating factor is obtained from a weighted non-Euclidean area of a portion of a non-characteristic plane, the weight function being the potential. In case the potential is retarded the plane of integration is the plane which enters in the construction of the retarded potential. For $n \neq 3$ the retarded potential has so far played the role of the initial potential from which integrating factors are derived and not the final potential on which integration formulae are based.

1. Basic Vector Algebra. A basic vector algebra deals with two sets of n vectors each in n dimensions. In conformity with the tensor notation one of the sets, consisting of *covariant* vectors, will be designated by L_i/ω and l_i^α where $i = 1, \dots, n$ and $\alpha = 1, \dots, n - 1$. The other set, consisting of *contravariant* vectors, will be designated by L^i, l_i^α . Throughout the work indices i, j, k have a range $1, \dots, n$ while indices α, β, γ have a range $1, \dots, n - 1$.

Such a set of $2n$ vectors shall constitute a basic algebra when they satisfy the postulates

$$(1.1) \quad \text{Basic Postulates:} \quad \left\{ \begin{array}{l} \frac{L_i}{\omega} L^i = 1; \quad \frac{L_i}{\omega} l_i^\alpha = 0; \quad L^i l_i^\alpha = 0; \quad l_i^\alpha l_i^\beta = \delta_\alpha^\beta; \\ \text{either} \quad \left| \begin{array}{c} L_i/\omega \\ l_i^1 \\ \vdots \\ l_i^{n-1} \end{array} \right| \quad \text{or} \quad \left| \begin{array}{c} L^i \\ l_1^i \\ \vdots \\ l_{n-1}^i \end{array} \right| \quad \text{has the value unity.} \end{array} \right.$$

In the above column notation for determinants the index i indicates the column. In the scalar product relations or *row rules of indices* the *repeated* or *summed* index i occurring once as a subscript and once as a superscript indicates a sum of n terms. As is customary the Kronecker delta is defined as follows:

$$(1.2) \quad \delta_\alpha^\beta = \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

That a set of $2n$ vectors satisfying the postulates (1.1) exists is evident from the theory of determinants. For the covariant vectors we could take the rows of any determinant with value unity. For the contravariant vectors we could use

¹ HADAMARD, J., *op. cit.*, pp. 201-203.

the rows of the cofactor matrix. The row rules of indices merely state the scalar product relations between the various rows of a determinant and the various rows of the cofactor matrix. Subsequent arguments show that any basic algebra consists of the row vectors of a unit determinant and the row vectors of the cofactor matrix. However, the postulatory approach (1.1) gives us a more convenient test for a basic algebra and developing the properties of the algebra from the postulates serves to familiarize the reader with our notation.

To illustrate we consider an example with $n = 3$. We let

$$(1.3) \quad \begin{vmatrix} L_i/\omega \\ l_i^1 \\ l_i^2 \end{vmatrix} = \begin{vmatrix} L_1/\omega & L_2/\omega & L_3/\omega \\ l_1^1 & l_1^2 & l_1^3 \\ l_2^1 & l_2^2 & l_2^3 \end{vmatrix} = \begin{vmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -1 & 2 & 3 \\ 2 & -2 & 1 \end{vmatrix} = 1.$$

In our notation the computation of the cofactors of the first row can be arranged in the following way.

$$(1.4) \quad L^i = \begin{vmatrix} \delta_j^i \\ l_j^1 \\ l_j^2 \end{vmatrix} = \begin{vmatrix} \delta_1^i & \delta_2^i & \delta_3^i \\ -1 & 2 & 3 \\ 2 & -2 & 1 \end{vmatrix} = (8, 7, -2).$$

Here i is a *free index* indicating the component of L^i . The *column index* j appears as a subscript in each row. Letting i have the values 1, 2, 3 we obtain three determinants whose values are the respective cofactors of the elements of the first row of (1.3). Similar computations for the vectors l_1^i and l_2^i lead to the result

$$(1.5) \quad \begin{vmatrix} L^i \\ l_1^i \\ l_2^i \end{vmatrix} = \begin{vmatrix} L^1 & L^2 & L^3 \\ l_1^1 & l_1^2 & l_1^3 \\ l_2^1 & l_2^2 & l_2^3 \end{vmatrix} = \begin{vmatrix} 8 & 7 & -2 \\ -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{vmatrix}.$$

The postulates (1.1) are easily verified in the example (1.3) and (1.5).

As a first consequence of the postulates (1.1) we show that *both determinants, contravariant and covariant, have the value unity*. For this we need the rule for multiplying determinants translated into column notation. Since the column index is a *dummy index* (that is the letter selected from the appropriate range is immaterial) we chose the same column index in the two determinants to be multiplied together. We illustrate with $n = 3$.

$$(1.6) \quad \begin{vmatrix} \frac{L_i}{\omega} \\ l_i^1 \\ l_i^2 \end{vmatrix} \begin{vmatrix} L^i \\ l_1^i \\ l_2^i \end{vmatrix} = \begin{vmatrix} \frac{L_i}{\omega} L^i & \frac{L_i}{\omega} l_1^i & \frac{L_i}{\omega} l_2^i \\ l_i^1 L^i & l_i^1 l_1^i & l_i^1 l_2^i \\ l_i^2 L^i & l_i^2 l_1^i & l_i^2 l_2^i \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Since one of the determinants, contravariant or covariant, has the value unity and their product is unity, then both determinants have the value unity.

Next we show each of the covariant vectors is the *cross product vector* for $n - 1$ contravariant vectors and similarly for the other n vectors. We wish to show that

$$(1.7) \quad L^i = \begin{vmatrix} \delta_j^i \\ l_j^1 \\ \vdots \\ l_j^{n-1} \end{vmatrix}; \quad l_\alpha^i = \begin{vmatrix} \frac{L_j}{\omega} \\ l_j^1 \\ \vdots \\ \delta_j^i \\ l_j^{\alpha+1} \\ \vdots \\ l_j^{n-1} \end{vmatrix}; \quad \frac{L_i}{\omega} = \begin{vmatrix} \delta_i^j \\ l_1^j \\ \vdots \\ l_{n-1}^j \end{vmatrix}; \quad l_i^\alpha = \begin{vmatrix} L^j \\ l_1^j \\ \vdots \\ l_{\alpha-1}^j \\ \delta_i^j \\ l_{\alpha+1}^j \\ \vdots \\ l_{n-1}^j \end{vmatrix}.$$

To demonstrate (1.7a) we select a system of n equations from (1.1) in which we regard L^i as unknown. Let us consider the system

$$(1.8) \quad \frac{L_i}{\omega} L^i = 1; \quad l_i^\alpha L^i = 0.$$

Since the determinant of the coefficients is unity, the system (1.8) has a unique solution L^i . Since $n - 1$ of the equations are homogeneous we are led to try for a solution of the form

$$(1.9) \quad L^i = k \begin{vmatrix} \delta_j^i \\ l_j^1 \\ \vdots \\ l_j^{n-1} \end{vmatrix}.$$

To show that $k = 1$ we compute the left member of (1.8a) using the expression (1.9) for L^i .

$$(1.10) \quad \frac{L_i}{\omega} L^i = \frac{L_i}{\omega} k \begin{vmatrix} \delta_j^i \\ l_j^1 \\ \vdots \\ l_j^{n-1} \end{vmatrix} = k \begin{vmatrix} \frac{L_i}{\omega} \delta_j^i \\ l_j^1 \\ \vdots \\ l_j^{n-1} \end{vmatrix} = k \begin{vmatrix} \frac{L_j}{\omega} \\ l_j^1 \\ \vdots \\ l_j^{n-1} \end{vmatrix} = k.$$

The computation (1.10) is an illustration of an important sequence of operations on determinants. In each of n determinants the corresponding factor L_i/ω is introduced into the first row; the determinants are then added according to the summation convention; and, finally, we use the *contraction principle*, $\delta_i^i L_i/\omega = L_j/\omega$.

From (1.10) we see that we must have $k = 1$ if (1.9) is to satisfy (1.8a). Computations of the type (1.10) show that (1.9) also satisfies (1.8b) and we have completed the demonstration of (1.7a). The other cross product relations (1.7) are demonstrated in similar fashion.

To complete the properties of the basic algebra we wish to develop the *column rule of indices*, namely

$$(1.11) \quad \frac{L_i}{\omega} L^j + l_i^\alpha l_\alpha^j = \delta_i^j \quad \text{or} \quad l_i^\alpha l_\alpha^j = \delta_i^j - \frac{L_i}{\omega} L^j.$$

The form (1.11b) can be looked upon as the rule for summing l vectors on a Greek index while the form (1.11a) is useful in separating a vector into components. In demonstrating (1.11) we illustrate with $n = 4$.

$$(1.12) \quad \begin{aligned} \frac{L_i}{\omega} L^j &= \begin{vmatrix} \delta_i^k & \delta_k^j \\ l_1^k & l_k^j \\ l_2^k & l_k^2 \\ l_3^k & l_k^3 \end{vmatrix} = \begin{vmatrix} \delta_i^k \delta_k^j & \delta_i^k l_k^\alpha \\ l_1^k \delta_k^j & l_1^k l_k^\alpha \\ l_2^k \delta_k^j & l_2^k l_k^\alpha \\ l_3^k \delta_k^j & l_3^k l_k^\alpha \end{vmatrix} = \begin{vmatrix} \delta_i^j & l_i^\alpha \\ l_1^j & \delta_1^\alpha \\ l_2^j & \delta_2^\alpha \\ l_3^j & \delta_3^\alpha \end{vmatrix} \\ &= \delta_i^j - l_1^j \begin{vmatrix} l_i^\alpha \\ \delta_2^\alpha \\ \delta_3^\alpha \end{vmatrix} + l_2^j \begin{vmatrix} l_i^\alpha \\ \delta_1^\alpha \\ \delta_3^\alpha \end{vmatrix} - l_3^j \begin{vmatrix} l_i^\alpha \\ \delta_1^\alpha \\ \delta_2^\alpha \end{vmatrix} \\ &= \delta_i^j - l_1^j l_i^1 - l_2^j l_i^2 - l_3^j l_i^3. \end{aligned}$$

This completes the demonstration of (1.11).

It is customary to base the distinction between contravariant and covariant vectors on their laws of transformation. If we assume that the l_i^α transform as covariant vectors then it follows that their cross product vector L^i transforms as a relative contravariant vector of weight one. However, our chief concern is not the transformation of vectors which in our case would mean a change from one vector basis to another. At the same time one of the applications of our algebras is the selection of a suitable coordinate system for a given computation and the rules of the algebra expedite the computation itself. In turn the application of the algebras to the treatment of non-Euclidean geometry brings out interesting distinctions between contravariant and covariant vectors. If we take L_i of an N - P , N - C algebra to be the Euclidean normal to a plane it turns out that L^i is the non-Euclidean normal to the same plane. This also gives an indication of the reason for singling out preferred L vectors.

In a great deal of our work we need relations involving vectors from two different basic algebras. To distinguish the two algebras we use base letters l , L and λ , Λ . In applications the l algebra is ordinarily fixed and the λ algebra is associated with the variable tangent plane to a surface. Most of the useful relations involving two algebras stem from one *basic mixed Pythagorean identity* which we write

$$(1.13) \quad |{}^{\alpha}h_{\beta}| x_i y^i = {}_{\alpha}\tilde{h}^{\beta} (l_i^{\alpha} y^i) (\lambda_{\beta}^i x_i) + (L^i x_i) \left(\frac{\Lambda_i y^i}{\Omega} \right) \quad [{}^{\alpha}h_{\beta} = l_i^{\alpha} \lambda_{\beta}^i].$$

In developing this identity we see that the ${}_{\alpha}\tilde{h}^{\beta}$ turn out to be cofactors of the determinant $|{}^{\alpha}h_{\beta}|$. This notation for a determinant can be distinguished from absolute value since the latter is used only in connection with scalar quantities where any indices would be repeated or summed.

The proof of the identity (1.13) is much like (1.12). We express the last term of (1.13) in determinant form and multiply determinants

$$(1.14) \quad \begin{aligned} (L^i x_i) \left(\frac{\Lambda_i y^i}{\Omega} \right) &= x_i y^j \begin{vmatrix} \delta_k^i & \delta_j^k \\ l_k^1 & \lambda_1^k \\ l_k^2 & \lambda_2^k \\ l_k^3 & \lambda_3^k \end{vmatrix} = \begin{vmatrix} x_k & y^k \\ l_k^1 & \lambda_1^k \\ l_k^2 & \lambda_2^k \\ l_k^3 & \lambda_3^k \end{vmatrix} = \begin{vmatrix} x_k y^k & x_k \lambda_{\alpha}^k \\ l_k^1 y^k & l_k^1 \lambda_{\alpha}^k \\ l_k^2 y^k & l_k^2 \lambda_{\alpha}^k \\ l_k^3 y^k & l_k^3 \lambda_{\alpha}^k \end{vmatrix} \\ &= |{}^{\alpha}h_{\beta}| x_k y^k - l_k^1 y^k \begin{vmatrix} x_k \lambda_{\alpha}^k \\ {}^2h_{\alpha} \\ {}^3h_{\alpha} \end{vmatrix} + l_k^2 y^k \begin{vmatrix} x_k \lambda_{\alpha}^k \\ {}^1h_{\alpha} \\ {}^3h_{\alpha} \end{vmatrix} - l_k^3 y^k \begin{vmatrix} x_k \lambda_{\alpha}^k \\ {}^1h_{\alpha} \\ {}^2h_{\alpha} \end{vmatrix} \\ &= |{}^{\alpha}h_{\beta}| x_k y^k - (l_k^1 y^k) (\lambda_{\gamma}^k x_k) {}_1\tilde{h}^{\gamma} - (l_k^2 y^k) (\lambda_{\gamma}^k x_k) {}_2\tilde{h}^{\gamma} \\ &\quad - (l_k^3 y^k) (\lambda_{\gamma}^k x_k) {}_3\tilde{h}^{\gamma}. \end{aligned}$$

This completes the demonstration of (1.13). The expansion of bordered determinants similar to that in (1.14) occurs over and over in our work and the details will be omitted in the future.

When the l and λ algebras are identical we have ${}^{\alpha}h_{\beta} = \delta_{\beta}^{\alpha}$ and the identity (1.13) is equivalent to the column rule of indices (1.11). The reason for the term Pythagorean identity is that a similar identity in the N - P , N - C algebra leads to the appropriate Pythagorean theorems in our various non-Euclidean geometries.

Other useful identities can be obtained by particular choices of the arguments x_i and y^j in (1.13). No doubt the most familiar of these is the *Lagrange identity* obtained by putting $x_i = L_i/\omega$ and $y^i = L^i$. We have

$$(1.15) \quad |{}^{\alpha}h_{\beta}| \left(\frac{L_i}{\omega} L^i \right) = {}_{\alpha}\tilde{h}^{\beta} (l_i^{\alpha} L^i) \left(\lambda_{\beta}^i \frac{L_i}{\omega} \right) + \left(L^i \frac{L_i}{\omega} \right) \left(\frac{\Lambda_i L^i}{\Omega} \right).$$

Using the rules of indices (1.1) this becomes

$$(1.16) \quad |^{\alpha}h_{\beta}| = \frac{L^i \Lambda_i}{\Omega} \quad \text{or} \quad |l_i^{\alpha} \lambda_{\beta}^i| = \frac{L^i \Lambda_i}{\Omega} \quad (\text{Lagrange identity}).$$

In the case of $n = 3$ this identity would appear as follows in the Gibbs notation.

$$(1.17) \quad \begin{vmatrix} l \cdot \lambda & l \cdot \mu \\ m \cdot \lambda & m \cdot \mu \end{vmatrix} = (l \times m) \cdot (\lambda \times \mu).$$

Note that the notation (1.16) requires a distinction between contravariant and covariant vectors. However, the Lagrange identity does not involve the full $2n$ vectors of each basic algebra but only the vectors l_i^{α} and λ_{β}^i together with their cross-product vectors.

In some applications of the restricted algebras taken up below, it is desirable to use an algebra in which the basic vectors l_{α}^i are chosen in a particular fashion relative to the non-Euclidean geometry of the partial differential equation. This can usually be accomplished by first choosing any L algebra of the proper type and then effecting a change of basic vectors l_{α}^i by *compounding* the original restricted algebra with another restricted M algebra of one lower dimension (perhaps differing in type from the former). The process of compounding an algebra having basic vectors L^i and l_{α}^i with another algebra having basic vectors M^{α} and m_{α}^a (where $a, b, c = 1, \dots, n-2$) consists of forming a third basic algebra with basic vectors

$$(1.18) \quad \begin{vmatrix} \frac{L_i}{\omega} \\ \frac{M_{\alpha}}{\bar{\omega}} \\ m_{\alpha}^1 \\ m_{\alpha}^2 \end{vmatrix} \begin{vmatrix} l_{\alpha}^i \\ l_{\alpha}^{\alpha} \\ l_{\alpha}^{\alpha} \end{vmatrix}; \quad \begin{vmatrix} L^i \\ M^{\alpha} \\ m_1^{\alpha} \\ m_2^{\alpha} \end{vmatrix} \begin{vmatrix} l_{\alpha}^i \\ l_{\alpha}^{\alpha} \\ l_{\alpha}^i \end{vmatrix}.$$

Given that the L algebra and the M algebra are both basic our problem is to show that the algebra (1.18) is also basic.

For clarity we write out the postulates for each basic algebra.

$$(1.19) \quad \begin{aligned} \frac{L_i}{\omega} L^i &= 1; & \frac{L_i}{\omega} l_{\alpha}^i &= 0; & l_i^{\alpha} L^i &= 0; & l_i^{\alpha} l_{\beta}^i &= \delta_{\beta}^{\alpha}; \\ \frac{M_{\alpha}}{\bar{\omega}} M^{\alpha} &= 1; & \frac{M_{\alpha}}{\bar{\omega}} m_{\alpha}^a &= 0; & m_{\alpha}^a M^{\alpha} &= 0; & m_{\alpha}^a m_b^{\alpha} &= \delta_b^a. \end{aligned}$$

The postulates for a basic algebra consist of the row scalar product rules in (1.19) together with the assumption that one of the determinants, contravariant or covariant, is unity. In testing the row scalar product rules on the compound algebra (1.18) it is important to change every repeated index to a distinct letter before removing parentheses to rearrange factors.

$$\begin{aligned}
 \frac{L_i}{\omega} L^i &= 1; \\
 \left(\frac{M_\alpha}{\bar{\omega}} l_\alpha^i \right) (M^\alpha l_\alpha^i) &= \frac{M_\alpha}{\bar{\omega}} l_\alpha^i M^\beta l_\beta^i = \frac{M_\alpha}{\bar{\omega}} M^\beta l_\alpha^i l_\beta^i = \frac{M_\alpha}{\bar{\omega}} M^\beta \delta_\beta^\alpha = \frac{M_\alpha}{\bar{\omega}} M^\alpha = 1; \\
 (1.20) \quad (m_\alpha^a l_\alpha^i) (m_b^\beta l_b^i) &= m_\alpha^a m_b^\beta l_\alpha^i l_b^i = m_\alpha^a m_b^\beta \delta_\beta^\alpha = m_\alpha^a m_b^\alpha = \delta_b^a; \\
 \frac{L_i}{\omega} (M^\alpha l_\alpha^i) &= \left(\frac{L_i}{\omega} l_\alpha^i \right) M^\alpha = 0; \quad \frac{L_i}{\omega} (m_\alpha^a l_\alpha^i) = \left(\frac{L_i}{\omega} l_\alpha^i \right) m_\alpha^a = 0; \\
 \left(\frac{M_\alpha}{\bar{\omega}} l_\alpha^i \right) (m_\alpha^\beta l_\beta^i) &= \frac{M_\alpha}{\bar{\omega}} m_\alpha^\beta l_\alpha^i l_\beta^i = \frac{M_\alpha}{\bar{\omega}} m_\alpha^\beta \delta_\beta^\alpha = \frac{M_\alpha}{\bar{\omega}} m_\alpha^\alpha = 0.
 \end{aligned}$$

The last three rules should be checked with indices inverted.

To show that one of the compound determinants is unity we multiply determinants

$$(1.21) \quad \begin{vmatrix} M^\alpha \\ m_1^\alpha \\ m_2^\alpha \end{vmatrix} \begin{vmatrix} L^i \\ l_1^i \\ l_2^i \\ l_3^i \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & M^\alpha \\ 0 & m_1^\alpha \\ 0 & m_2^\alpha \end{vmatrix} \begin{vmatrix} L^1 & l_\alpha^1 \\ L^2 & l_\alpha^2 \\ L^3 & l_\alpha^3 \\ L^4 & l_\alpha^4 \end{vmatrix} = \begin{vmatrix} L^i \\ M^\alpha & l_\alpha^i \\ m_1^\alpha & l_\alpha^i \\ m_2^\alpha & l_\alpha^i \end{vmatrix} = 1.$$

This completes the checking of our postulates on the compound algebra.

2. Classification of Restricted Algebras. In a restricted algebra we deal not only with a set of $2n$ vectors, but also with a symmetric matrix A^{ij} , that is, a matrix such that $A^{ij} = A^{ji}$. We use the notation $\tilde{A}_{ij}$ for the cofactor of the element A^{ij} in the determinant $|A^{ij}|$. In our column notation we express this relationship as follows:

$$(2.1) \quad \tilde{A}_{ij} = \begin{vmatrix} \delta_j^i \\ A^{2i} \\ A^{3i} \end{vmatrix}; \quad \tilde{A}_{2j} = \begin{vmatrix} A^{1i} \\ \delta_j^i \\ A^{3i} \end{vmatrix}; \quad \tilde{A}_{3j} = \begin{vmatrix} A^{1i} \\ A^{2i} \\ \delta_j^i \end{vmatrix}.$$

As in the development of the basic algebra we compute the scalar products of the rows of the original matrix with rows of the cofactor matrix.

$$(2.2) \quad A^{kj} \tilde{A}_{1j} = A^{kj} \begin{vmatrix} \delta_j^i \\ A^{2i} \\ A^{3i} \end{vmatrix} = \begin{vmatrix} A^{kj} \delta_j^i \\ A^{2i} \\ A^{3i} \end{vmatrix} = \begin{vmatrix} A^{ki} \\ A^{2i} \\ A^{3i} \end{vmatrix} = \begin{cases} |A^{ij}| & \text{if } k = 1, \\ 0 & \text{if } k \neq 1. \end{cases}$$

These results together with those involving cofactors of other rows can be summarized as follows:

$$(2.3) \quad A^{kj} \tilde{A}_{ij} = \delta_i^k |A^{ij}|.$$

When the determinant $|A^{ij}| \neq 0$ we can divide (2.3) by the value of the determinant and using the notation A_{ij} for the normalized cofactor the result becomes

$$(2.4) \quad A^{kj} A_{ij} = \delta_i^k, \quad \text{where } A_{ij} = \frac{\tilde{A}_{ij}}{|A^{ij}|}.$$

When i, j, k have a range of one we take (2.3) and (2.4) as defining relationships leading to $\tilde{A}_{11} = 1$ and $A_{11} = 1/A^{11}$. To show that the determinants $|A^{ij}|$ and $|A_{ij}|$ are always reciprocals we multiply determinants and use the relations (2.4)

$$(2.5) \quad |A^{ij}| |A_{ij}| = \begin{vmatrix} A^{1j} \\ A^{2j} \\ A^{3j} \end{vmatrix} \begin{vmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{vmatrix} = \begin{vmatrix} A^{1j} A_{ij} \\ A^{2j} A_{ij} \\ A^{3j} A_{ij} \end{vmatrix} = |\delta_i^k| = 1.$$

In conformity with the usage in partial differential equations we classify matrices A^{ij} into two types.

$$(2.6) \quad \begin{array}{ll} \text{Non-parabolic:} & |A^{ij}| \neq 0; \\ \text{Parabolic:} & |A^{ij}| = 0, \end{array} \text{rank } (n-1).$$

Our treatment excludes matrices A^{ij} of lower rank.

For both parabolic and non-parabolic A^{ij} we separate all vectors L_i/ω into mutually exclusive classes as follows:

$$(2.7) \quad \begin{array}{ll} \text{Non-characteristic:} & A^{ij} \frac{L_i}{\omega} \frac{L_j}{\omega} \neq 0; \\ \text{Characteristic:} & A^{ij} \frac{L_i}{\omega} \frac{L_j}{\omega} = 0, \end{array} \left\{ \begin{array}{l} \text{Type I:} \\ \text{(Parabolic only) all } A^{ij} \frac{L_j}{\omega} = 0, \\ \text{Type II:} \quad \text{not all } A^{ij} \frac{L_j}{\omega} = 0. \end{array} \right.$$

To show that non-parabolic A^{ij} do not admit characteristic type I vectors L_i/ω we multiply the relation $A^{ij}(L_j/\omega) = 0$ by A_{ik} and sum on i which leads to

$$(2.8) \quad A_{ik} A^{ij} \frac{L_j}{\omega} = \delta_k^j \frac{L_j}{\omega} = \frac{L_k}{\omega} = 0.$$

This contradicts the assumption that L_i/ω is a vector.

In addition to the impossibility of characteristic type I vectors for non-parabolic A^{ij} we now plan to show that some, but not all, non-parabolic A^{ij} admit characteristic type II vectors; similarly some, but not all, parabolic A^{ij} admit characteristic type II vectors. Furthermore, all types of A^{ij} allow non-characteristic vectors; all parabolic A^{ij} allow characteristic type I vectors.

To see that any matrix A^{ij} of types (2.6) admits a non-characteristic L_i/ω we first remark that some element $A^{ij} \neq 0$. If some diagonal element is not zero, say A^{11} , then $L_i/\omega = (1, 0, \dots, 0)$ leads to $A^{ij}(L_i/\omega)(L_j/\omega) = A^{11} \neq 0$. If all diagonal elements vanish and for example $A^{12} \neq 0$, then $L_i/\omega = (1, 1, 0, \dots, 0)$ leads to $A^{ij}(L_i/\omega)(L_j/\omega) = 2A^{12} \neq 0$.

To show that all parabolic A^{ij} admit characteristic type I vectors we note that if $|A^{ij}| = 0$ then the system $A^{ij}(L_j/\omega) = 0$ always has a non-trivial solution L_i/ω . Then $A^{ij}(L_j/\omega) = 0$ implies $A^{ij}(L_i/\omega)(L_j/\omega) = 0$ and characteristic type I vectors are possible.

To exhibit a non-parabolic matrix A^{ij} which does not admit characteristic type II vectors we take $A^{ij} = \delta^{ij}$ and $A^{ij}(L_i/\omega)(L_j/\omega)$ is a sum of squares and cannot vanish for a vector L_i/ω . On the other hand if we alter the example $A^{ij} = \delta^{ij}$ by making a sign change in one diagonal element a characteristic L_i/ω is clearly possible.

For an example of a parabolic matrix A^{ij} which does not admit a characteristic type II vector consider $A^{\alpha\beta} = \delta^{\alpha\beta}$, $A^{in} = 0$. Then

$$A^{ij}(L_i/\omega)(L_j/\omega) = A^{\alpha\beta}(L_\alpha/\omega)(L_\beta/\omega)$$

is a sum of squares and a characteristic vector must be of the form $L_i/\omega = (0, \dots, 0, L_n/\omega)$. However, $A^{ij}(L_j/\omega) = A^{in}(L_n/\omega) = 0$ and our vector is type I and not type II. On the other hand for $n \geq 3$ the choice,

$$A^{11} = \dots = A^{(n-2)(n-2)} = 1, A^{(n-1)(n-1)} = -1, A^{nn} = 0$$

and off the diagonal $A^{ij} = 0$, admits the characteristic vector $L_i/\omega = (0, \dots, 1, 1, 0)$. Since $A^{ij}(L_j/\omega) = 1$ for $i = n-2$ the vector is type II.

At this point it is expedient to include two Lagrange type identities as follows:

$$(2.9) \quad |A^{ij} l_i^\alpha l_j^\beta| = \tilde{A}_{ij} L^i L^j; \quad |A_{ij} l_\alpha^i l_\beta^j| = \tilde{A}^{ij} \frac{L_i}{\omega} \frac{L_j}{\omega}.$$

Although these identities involve the basic algebra together with sets of A^{ij} or A_{ij} , the proof we give does not depend on the type of A 's or the type of l 's. In particular the A_{ij} need not be normalized cofactors of A^{ij} .

In demonstrating (2.9a) our plan is to express $\tilde{A}_{ij} L^i L^j$ as a bordered determinant and to multiply by determinants with value unity.

$$\begin{aligned}
 \tilde{A}_{ij} L^i L^j = & - \begin{vmatrix} 1 & 0 \\ 0 & \frac{L_i}{\omega} \\ 0 & l_i^1 \\ 0 & l_i^2 \\ 0 & l_i^3 \end{vmatrix} \begin{vmatrix} 0 & L^i \\ L^1 & A^{1i} \\ L^2 & A^{2i} \\ L^3 & A^{3i} \\ L^4 & A^{4i} \end{vmatrix} = - \begin{vmatrix} 0 & L^j \\ \frac{L_i}{\omega} L^i & \frac{L_i}{\omega} A^{ji} \\ l_i^1 L^i & l_i^1 A^{ji} \\ l_i^2 L^i & l_i^2 A^{ji} \\ l_i^3 L^i & l_i^3 A^{ji} \end{vmatrix} \\
 = & \begin{vmatrix} \frac{L_j}{\omega} \\ l_j^1 \\ l_j^2 \\ l_j^3 \end{vmatrix} \begin{vmatrix} L^j \\ A^{ji} l_i^1 \\ A^{ji} l_i^2 \\ A^{ji} l_i^3 \end{vmatrix} = \begin{vmatrix} \frac{L_j}{\omega} L^j & \frac{L_j}{\omega} A^{ji} l_i^\alpha \\ l_j^1 L^j & l_j^1 A^{ji} l_i^\alpha \\ l_j^2 L^j & l_j^2 A^{ji} l_i^\alpha \\ l_j^3 L^j & l_j^3 A^{ji} l_i^\alpha \end{vmatrix} = |A^{ij} l_i^\alpha l_j^\beta|.
 \end{aligned}
 \tag{2.10}$$

3. Non-Parabolic Non-Characteristic Algebra. $|A^{ij}| \neq 0$, $A^{ij}(L_i/\omega)(L_j/\omega) \neq 0$. We are given a non-parabolic matrix A^{ij} and a non-characteristic vector L_i/ω . We wish to show that we can always set up a basic algebra containing the given vector L_i/ω and having the additional property

$$(3.1) \quad N-P, N-C \text{ postulate:} \quad L^i = A^{ij} L_j.$$

Such an algebra will be called an $N-P, N-C$ restricted algebra.

Since $\omega = L^i L_i = A^{ij} L_i L_j$ we can determine ω by the relation

$$1/\omega = A^{ij}(L_i/\omega)(L_j/\omega).$$

For the l_i^α we can take any set of vectors having L_i/ω for cross-product vector according to (1.7). In view of the $N-P, N-C$ postulate (3.1) the determinant of the contravariant vectors has the value $L^i L_i/\omega = A^{ij} L_i L_j/\omega = 1$. Determining the other vectors l_i^α by the cross-product rules the remaining basic postulates are easily demonstrated.

In order to complete our rules of indices we define quantities $h^{\alpha\beta}$ and $h_{\alpha\beta}$ as follows:

$$(3.2) \quad \text{Definitions:} \quad h^{\alpha\beta} = A^{ij} l_i^\alpha l_j^\beta; \quad h_{\alpha\beta} = A_{ij} l_\alpha^i l_\beta^j.$$

We multiply (3.2a) by (3.2b) and sum on a Greek index and then make use of (1.11), (2.4), (3.1), and (1.1) to obtain

$$(3.3) \quad h^{\alpha\beta} h_{\beta\gamma} = \delta_\gamma^\alpha; \quad |h^{\alpha\beta}| \neq 0.$$

In a similar fashion we introduce factors into the $N-P, N-C$ postulate (3.1) and the definitions (3.2) to demonstrate the $N-P, N-C$ rules of indices

$$(3.4) \quad A^{ij}L_j = L^i; \quad A_{ij}L^j = L_i; \quad A^{ij}l_j^\alpha = h^{\alpha\beta}l_\beta^i; \quad A_{ij}l_\alpha^j = h_{\alpha\beta}l_i^\beta$$

(A rules of indices for N - P , N - C).

With these rules of indices and the Lagrange identity (2.9a) we have the following forms of ω

$$(3.5) \quad \omega = L^jL_j = A^{ij}L_iL_j = A_{ij}L^iL^j = |A_{ij}| |h^{\alpha\beta}|.$$

The N - P , N - C Pythagorean identities involve the quantities

$$(3.6) \quad \text{Definitions:} \quad H^{ij} = h^{\alpha\beta}l_\alpha^i l_\beta^j; \quad H_{ij} = h_{\alpha\beta}l_i^\alpha l_j^\beta.$$

Using the various rules of indices we have

$$(3.7) \quad \begin{aligned} H^{ij} &= A^{ik}l_k^\beta l_\beta^j = A^{ik} \left(\delta_k^j - \frac{L^j L_k}{\omega} \right) = A^{ij} - \frac{L^i L^j}{\omega}; \\ H_{ij} &= A_{ik}l_\beta^k l_j^\beta = A_{ik} \left(\delta_j^k - \frac{L^k L_j}{\omega} \right) = A_{ij} - \frac{L_i L_j}{\omega}. \end{aligned}$$

An N - P , N - C Pythagorean identity is obtained by introducing the factor $x_i y_j$ into (3.7a).

$$(3.8) \quad \begin{aligned} A^{ij}x_i y_j &= H^{ij}x_i y_j + \frac{1}{\omega} (L^i x_i)(L^j y_j); \\ A_{ij}x^i y^j &= H_{ij}x^i y^j + \frac{1}{\omega} (L_i x^i)(L_j y^j). \end{aligned} \quad (\text{N-P, N-C Pythagorean identities})$$

The H rules of indices follow immediately from the definitions (3.6) using only the basic rules (1.1).

$$(3.9) \quad H^{ij}L_j = 0; \quad H_{ij}L^j = 0; \quad H^{ij}l_j^\alpha = h^{\alpha\beta}l_\beta^i; \quad H_{ij}l_\alpha^j = h_{\alpha\beta}l_i^\beta.$$

(H rules of indices for all restricted algebras)

These rules are valid in any restricted algebra since the definitions (3.6) are universal.

The following relations between A and H are found useful:

$$(3.10) \quad \begin{aligned} A^{ij}H_{jk} &= A_{jk}H^{ij} = H^{ij}H_{jk} = \delta_k^i - \frac{L^i L_k}{\omega}; \\ A^{ij}H_{ij} &= H^{ij}H_{ij} = n - 1; \quad A^{ij}H_{ik}H_{jm} = H_{km}. \end{aligned}$$

To see that the matrix H_{ij} has rank $n - 1$ we make use of $|h_{\alpha\beta}| \neq 0$.

$$(3.11) \quad \begin{vmatrix} 0 & 0 \\ 0 & h_{1\beta} \\ 0 & h_{2\beta} \\ 0 & h_{3\beta} \end{vmatrix} \begin{vmatrix} L_1 & l_1^\beta \\ L_2 & l_2^\beta \\ L_3 & l_3^\beta \\ L_4 & l_4^\beta \end{vmatrix}^2 = \begin{vmatrix} 0 & h_{\alpha\beta} l_1^\beta \\ 0 & h_{\alpha\beta} l_2^\beta \\ 0 & h_{\alpha\beta} l_3^\beta \\ 0 & h_{\alpha\beta} l_4^\beta \end{vmatrix} \begin{vmatrix} L_1 & l_1^\alpha \\ L_2 & l_2^\alpha \\ L_3 & l_3^\alpha \\ L_4 & l_4^\alpha \end{vmatrix} = \begin{vmatrix} h_{\alpha\beta} l_i^\alpha l_1^\beta \\ h_{\alpha\beta} l_i^\alpha l_2^\beta \\ h_{\alpha\beta} l_i^\alpha l_3^\beta \\ h_{\alpha\beta} l_i^\alpha l_4^\beta \end{vmatrix} = |H_{ij}|.$$

Both (3.10) and (3.11) allow the inversion of indices.

4. Non-Parabolic Characteristic Algebra. $|A^{ij}| \neq 0$, $A^{ij}(L_i/\omega)(L_j/\omega) = 0$. We are given a non-parabolic matrix A^{ij} and a characteristic vector L_i/ω . We wish to show that we can set up a basic algebra containing the vector L_i/ω and having the additional property

$$(4.1) \quad N-P, C \text{ postulate:} \quad A_{ij}L^iL^j = 0.$$

To show that such a vector L^i exists we first select an $N-P$, $N-C$ algebra with vectors $\bar{L}$ and l such that $\bar{L}^iL_i \neq 0$ or in other words such that the vector $\bar{L}^i$ does not lie in the plane with normal L_i . We express the vector $A^{ij}L_j/\omega$ in terms of the $N-P$, $N-C$ basis and make the following choice of L^i .

$$(4.2) \quad A^{ij}\frac{L_j}{\omega} = a\bar{L}^i + b^\alpha l_\alpha^i; \quad L^i = a\bar{L}^i - b^\alpha l_\alpha^i.$$

Both vectors (4.2) satisfy the postulate (4.1) since

$$(4.3) \quad 0 = A_{ij}\left(A^{ik}\frac{L_k}{\omega}\right)\left(A^{jm}\frac{L_m}{\omega}\right) = a^2\bar{\omega} + \bar{h}_{\alpha\beta}b^\alpha b^\beta = A_{ij}L^iL^j.$$

We have yet to test the basic postulate $L_iL^i \neq 0$. However,

$$(4.4) \quad \frac{L_iL^i}{\omega} = A_{ij}L^j\frac{A^{ik}L_k}{\omega} = A_{ij}(a\bar{L}^j - b^\alpha l_\alpha^j)(a\bar{L}^i + b^\beta l_\beta^i) = a^2\bar{\omega} - \bar{h}_{\alpha\beta}b^\alpha b^\beta.$$

Since $a\bar{\omega} = \bar{L}^iL_i/\omega$ by (4.2a) it follows from the choice of $\bar{L}^i$ that $a \neq 0$ and (4.4) together with (4.3) implies that $L^iL_i \neq 0$. The vectors l_α^i and l_i^α are then chosen with proper attention to the cross product rules (1.7).

In the $N-P$, C algebra the A rules of indices involve the quantities

$$(4.5) \quad \begin{aligned} h^{\alpha\beta} &= A^{ij}l_i^\alpha l_j^\beta; & h_{\alpha\beta} &= A_{ij}l_\alpha^i l_\beta^j; \\ r^\alpha &= A^{ij}L_i l_j^\alpha; & r_\alpha &= A_{ij}L^i l_\alpha^j. \end{aligned}$$

Using the Lagrange identities (2.9) and the rules (1.11) and (1.1) we have

$$(4.6) \quad |h^{\alpha\beta}| = 0, \quad |h_{\alpha\beta}| = 0, \quad r^\alpha r_\alpha = \omega.$$

The A rules of indices follow from the definitions (4.5).

$$\begin{aligned}
 A^{ij}L_j &= r^\alpha l_\alpha^i; & A_{ij}L^j &= r_\alpha l_i^\alpha; \\
 (4.7) \qquad & & & \text{(A rules of indices for N-P, C)} \\
 A^{ij}l_j^\alpha &= h^{\alpha\beta}l_\beta^i + \frac{r^\alpha L^i}{\omega}; & A_{ij}l_\alpha^j &= h_{\alpha\beta}l_i^\beta + \frac{r_\alpha L_i}{\omega}.
 \end{aligned}$$

With the H matrices defined as in N - P , N - C the Pythagorean identities are as follows:

$$\begin{aligned}
 A^{ij}x_{ij} &= H^{ij}x_{ij} + \frac{A^{ij}L_i x_j}{\omega} (L^i y_i) + \frac{A^{ij}L_i y_j}{\omega} (L^i x_i), \\
 (4.8) \qquad A_{ij}x^i y^j &= H_{ij}x^i y^j + A_{ij}L^i x^j \left(\frac{L_i y^i}{\omega} \right) + A_{ij}L^i y^j \left(\frac{L_i x^i}{\omega} \right). \\
 & \qquad \qquad \qquad (N\text{-}P, C \text{ Pythagorean identities})
 \end{aligned}$$

In order to investigate the ranks of the h and H matrices we use the following basic Pythagorean identities valid for any A matrices and associated h matrices defined as in (4.5).

$$\begin{aligned}
 |h^{\alpha\beta}| A^{ij}x_{ij} &= \tilde{h}_{\alpha\beta} (A^{ij}l_i^\alpha x_j) (A^{ij}l_j^\beta y_i) + (L^i x_i) (L^i y_i) |A^{ij}|, \\
 (4.9) \qquad & \qquad \qquad \text{(Basic)}
 \end{aligned}$$

$$|h_{\alpha\beta}| A_{ij}x^i y^j = \tilde{h}^{\alpha\beta} (A_{ij}l_\alpha^i x^j) (A_{ij}l_\beta^j y^i) + \frac{(L_i x^i) (L_i y^i)}{\omega^2} |A_{ij}|.$$

These extensions of (3.8) for N - P , N - C can be proved by multiplying three determinants together; for example, $|A^{ij}|(L^i x_i)(L^i y_i)$ with the latter factors expressed in determinant form. Next we use arguments $x_i = A_{ik}l_\gamma^k$ and $y_j = A_{jm}l_\delta^m$ in (4.9a) and similar arguments in (4.9b) to obtain

$$(4.10) \qquad \tilde{h}_{\alpha\beta} = -|A^{ij}| r_\alpha r_\beta, \qquad \tilde{h}^{\alpha\beta} = -\frac{|A_{ij}|}{\omega^2} r^\alpha r^\beta.$$

These relations together with the definitions (4.5) lead to the following h , r rules of indices.

$$(4.11) \quad h^{\alpha\beta}r_\beta = 0, \quad h_{\alpha\beta}r^\beta = 0, \quad \tilde{h}^{\alpha\beta}r_\beta = -\frac{|A_{ij}|}{\omega} r^\alpha, \quad \tilde{h}_{\alpha\beta}r^\beta = -\omega |A^{ij}| r_\alpha.$$

Since r^α and r_α are vectors with non-zero components, it follows from (4.10) that the matrices $h^{\alpha\beta}$ and $h_{\alpha\beta}$, and hence also H^{ij} and H_{ij} , all have rank $n - 2$.

When $n = 2$ we have $h^{11} = h_{11} = H^{ij} = H_{ij} = 0$ and $\tilde{h}^{11} = \tilde{h}_{11} = 1$ in the

above formulae. Thus for $n = 2$ we can use (4.10) to put the *rules of indices* (4.7) in the more convenient form

$$(4.12) \quad A^{ij} \frac{L_j}{\omega} = \pm \sqrt{-|A^{ij}|} l_i^j; \quad A^{ij} l_j^1 = \pm \sqrt{-|A^{ij}|} L^i; \quad (n = 2)$$

$$A_{ij} L^j = \pm \sqrt{-|A_{ij}|} l_i^1; \quad A_{ij} l_1^j = \pm \sqrt{-|A_{ij}|} \frac{L_i}{\omega}.$$

5. Parabolic Non-Characteristic Algebra. $|A^{ij}| = 0$ (rank $n - 1$), $A^{ij}(L_i/\omega)(L_j/\omega) \neq 0$. Here the postulate is

$$(5.1) \quad P, N-C \text{ postulate:} \quad L^i = A^{ij} L_j,$$

and the existence of such an algebra is exactly the same as in $N-P, N-C$.

However, the definition of the $h_{\alpha\beta}$ must be altered. We take

$$(5.2) \quad \text{Definitions:} \quad h^{\alpha\beta} = A^{ij} l_i^\alpha l_j^\beta, \quad h_{\alpha\beta} = \tilde{A}_{ij} l_\alpha^i l_\beta^j.$$

Using the Lagrange identities (2.9) we have

$$(5.3) \quad |h^{\alpha\beta}| = 0 \quad (n \geq 2), \quad |h_{\alpha\beta}| = 0 \quad (n > 2).$$

The rules of indices are

$$(5.4) \quad A^{ij} L_j = L^i, \quad \tilde{A}_{ij} L^j = 0, \quad A^{ij} l_j^\alpha = h^{\alpha\beta} l_\beta^i, \quad \tilde{A}_{ij} l_\alpha^j = h_{\alpha\beta} l_\alpha^i.$$

(A rules of indices for $P, N-C$)

Since $H_{ij} = \tilde{A}_{ij}$ by (5.4d) we have the Pythagorean identities

$$(5.5) \quad A^{ij} x_i y_j = H^{ij} x_i y_j + \frac{(L^i x_i)(L^j y_j)}{\omega}; \quad (P, N-C \text{ Pythagorean identities})$$

$$\tilde{A}_{ij} x^i y^j = H_{ij} x^i y^j = \frac{1}{\omega} \tilde{h}_{\alpha\beta} (l_\alpha^i x^i) (l_\beta^j y^j).$$

The latter form of (5.5) is obtained by multiplying bordered determinants as follows:

$$(5.6) \quad \begin{vmatrix} A^{1j} & y^1 \\ A^{2j} & y^2 \\ A^{3j} & y^3 \\ x^j & 0 \end{vmatrix} \begin{vmatrix} \frac{L_i}{\omega} & 0 \\ l_i^1 & 0 \\ l_i^2 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} \frac{L_j}{\omega} & 0 \\ l_j^1 & 0 \\ l_j^2 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\omega} \begin{vmatrix} h^{\alpha\beta} & l_\alpha^i x^i \\ l_\beta^j y^j & 0 \end{vmatrix}.$$

From (5.5b) we have $h_{\alpha\beta} = \tilde{h}_{\alpha\beta}/\omega$. Also we see that $h^{\alpha\beta}$ and H^{ij} have rank $n - 2$, while $h_{\alpha\beta}$ and H_{ij} have rank one. When $n = 2$ we have $h^{11} = H^{11} = 0$, $\tilde{h}_{11} = 1$, and $h_{11} = 1/\omega$ in the above formulae.

6. Parabolic Characteristic Type I Algebra. $|A^{ij}| = 0$ (rank $n - 1$), $A^{ij}(L_i/\omega)(L_j/\omega) = 0$, $A^{ij}(L_j/\omega) = 0$. The P, C type I postulate is

$$(6.1) \quad P, C \text{ type I postulate: } \tilde{A}_{ij}L^iL^j \neq 0.$$

Having chosen a vector L^i which satisfies the condition $\tilde{A}_{ij}L^iL^j \neq 0$ we first show that $L_iL^i \neq 0$. Since $A^{ij}L_j/\omega = 0$ we know that the L_j/ω are proportional to the cofactors of some row of A^{ij} . Thus for each value of i we have

$$(6.2) \quad \tilde{A}_{ij} = k \frac{L_j}{\omega} \quad \text{or} \quad \tilde{A}_{ij} = 0 \quad (\text{for each } i).$$

But $L_iL^i = 0$ would entail $\tilde{A}_{ij}L^j = 0$ which is contrary to the choice of L^i . To simplify the rules of the algebra we adjust the length of L_i so that $L_iL^i = \omega = \tilde{A}_{ij}L^iL^j$.

Since $\tilde{A}_{ij}L^j = 0$ in this algebra we define h 's as follows:

$$(6.3) \quad \text{Definitions: } h^{\alpha\beta} = A^{ij}l_i^\alpha l_j^\beta, \quad h^{\alpha\beta}h_{\beta\gamma} = \delta_\gamma^\alpha.$$

The rules of indices are

$$(6.4) \quad A^{ij}L_j = 0, \quad \tilde{A}_{ij}L^j = L_i, \quad A^{ij}l_j^\alpha = h^{\alpha\beta}l_{\beta}^i, \quad \tilde{A}_{ij}l_\alpha^j = 0.$$

(A rules of indices for P, C type I)

In proving (6.4b) we use (1.11). Using (6.4) and (1.11) we have the Pythagorean identities

$$(6.5) \quad \begin{aligned} A^{ij}x_iy_j &= H^{ij}x_iy_j, \\ \tilde{A}_{ij}x^iy^j &= \frac{(L_ix^i)(L_jy^j)}{\omega}. \end{aligned} \quad (P, C \text{ type I Pythagorean identities})$$

For handy reference we include the forms of

$$(6.6) \quad \omega = L_iL^i = \tilde{A}_{ij}L^iL^j = |h^{\alpha\beta}|.$$

The following sums are the same as in $N-P, N-C$ but the inversion of indices is not permissible in P, C type I.

$$(6.7) \quad A^{ij}H_{jk} = \delta_k^i - \frac{L^iL_k}{\omega}; \quad A^{ij}H_{ij} = n - 1; \quad A^{ij}H_{ik}H_{jm} = H_{km}.$$

The $N-P, N-C$ rules (3.7) are not valid here but it is nevertheless important to

obtain an expression for the H_{ij} in terms of L^i . To do this we consider the system

$$(6.8) \quad \begin{aligned} A^{mj}H_{ij} &= \delta_i^m - \frac{L_i L^m}{\omega}, \\ L^j H_{ij} &= 0. \end{aligned}$$

Since $\tilde{A}_{ij}L^j \neq 0$ for some value of i the system (6.8) contains n independent equations for the H_{ij} (i fixed). The solution is given by

$$(6.9) \quad H_{ij} = -\frac{1}{\omega} \begin{vmatrix} A^{1k} & L^1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A^{(i-1)k} & L^{i-1} \\ \delta_j^k & 0 \\ A^{(i+1)k} & L^{i+1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A^{nk} & L^n \\ L^k & 0 \end{vmatrix}.$$

Verification of (6.8b) is immediate. In verifying (6.8a) we multiply (6.9) by A^{mj} and sum on j . When $i \neq m$ we subtract the i^{th} from the m^{th} row, interchange the i^{th} and m^{th} rows, then interchange the i^{th} and the last row and expand by the last row. When $i = m$ we separate into a sum of determinants introducing sums in the i^{th} row.

Introducing the factor $x^i y^j$ in (6.9) we have $H_{ij}x^i y^j$ in the form of a twice bordered determinant.

$$(6.10) \quad H_{ij}x^i y^j = \frac{1}{\omega} \begin{vmatrix} 0 & L^j & 0 \\ L^i & A^{ij} & x^i \\ 0 & y^j & 0 \end{vmatrix} = -\frac{1}{\omega} \begin{vmatrix} h^{\alpha\beta} & l_i^\alpha x^i \\ l_i^\beta y^i & 0 \end{vmatrix}$$

$$(P, C \text{ type I}; \quad P, N-C; \quad N-P, N-C).$$

The equivalence of the latter forms of (6.10) can be shown directly using multiplication by bordered L, l determinants. This method of approach shows that (6.10) applies in $P, N-C$ and also in $N-P, N-C$ if we introduce the factor $|A_{ij}|$ in the second and third members.

7. Parabolic Characteristic Type II Algebra. $|A^{ij}| = 0$ (rank $n - 1$), $A^{ij}(L_i/\omega)(L_j/\omega) = 0$, $A^{ij}L_j/\omega$ not all zero. In setting up this algebra we use a single bar to designate a P, C type I algebra and a double bar for a $P, N-C$ algebra.

$$(7.1) \quad P, C \text{ type II postulate:} \quad \tilde{A}_{ij}L^i = 0;$$

$$\bar{H}_{ij}L^iL^j = 0 \quad \text{for all } P, C \text{ type I matrices } \bar{H}_{ij}.$$

The vector $A^{ij}L_j/\omega$ satisfies the conditions (7.1) but this vector cannot be used for L^i since it violates the basic condition $L^iL_i \neq 0$. As in the discussion of $N-P, C$ we represent in terms of $P, N-C$ basis.

$$(7.2) \quad \frac{L_i}{\omega} = a\bar{L}_i + b_\alpha \bar{l}_i^\alpha; \quad A^{ij}\frac{L_j}{\omega} = a\bar{L}^i + b_\alpha A^{ij}\bar{l}_j^\alpha; \quad L^i = a\bar{L}^i - b_\alpha A^{ij}\bar{l}_j^\alpha;$$

where $\bar{L}^iL_i \neq 0$. The property $\tilde{A}_{ij}L^i = 0$ is immediate and for the second part of (7.1) we have

$$(7.3) \quad 0 = \bar{H}_{ij}\left(\frac{A^{ik}L_k}{\omega}\right)\left(\frac{A^{jm}L_m}{\omega}\right) = a^2\bar{\omega} + \bar{h}^{\alpha\beta}b_\alpha b_\beta = \bar{H}_{ij}L^iL^j.$$

From (7.2) we have

$$(7.4) \quad \frac{L_iL^i}{\omega} = a^2\bar{\omega} - \bar{h}^{\alpha\beta}b_\alpha b_\beta.$$

Since $a \neq 0$ by (7.2a) and the choice of the $\bar{L}$ algebra, we have $L_iL^i \neq 0$ and the construction of the P, C type II algebra can be completed in the usual way.

In P, C type II the A rules of indices involve

$$(7.5) \quad \text{Definitions: } h^{\alpha\beta} = A^{ij}l_i^\alpha l_j^\beta; \quad h_{\alpha\beta} = \tilde{A}_{ij}l_i^\alpha l_j^\beta; \quad r^\alpha = A^{ij}L_i l_j^\alpha.$$

The rules of indices are

$$(7.6) \quad A^{ij}L_j = r^\alpha l_\alpha^i; \quad \tilde{A}_{ij}L^j = 0; \quad (A \text{ rules of indices for } P, C \text{ type II})$$

$$A^{ij}l_j^\alpha = h^{\alpha\beta}l_\beta^i + \frac{r^\alpha L^i}{\omega}; \quad \tilde{A}_{ij}l_\alpha^j = h_{\alpha\beta}l_i^\beta.$$

The Pythagorean identities are

$$(7.7) \quad A^{ij}x_i y_j = H^{ij}x_i y_j + \frac{A^{ij}L_i x_j}{\omega} (L^i y_i) + \frac{A^{ij}L_i y_j}{\omega} (L^i x_i),$$

$$\tilde{A}_{ij}x^i y^j = H_{ij}x^i y^j. \quad (P, C \text{ type II Pythagorean identities})$$

The matrices $h_{\alpha\beta}$ and H_{ij} have rank one. To investigate the rank of $h^{\alpha\beta}$ we use the relation

$$(7.8) \quad \bar{H}_{ij} L^i L^j = \frac{1}{\omega} \bar{h}_{\alpha\beta} (l_i^\alpha \bar{L}^i) (l_j^\beta \bar{L}^j).$$

This relation is obtained from $\bar{H}_{ij} L^i L^j$ in the determinant form (6.10) using multiplication by bordered L, l determinants. Since (7.8) vanishes for arbitrary P, C type I vectors $\bar{L}^i$, we have $\bar{h}_{\alpha\beta} = 0$ and the matrix $h^{\alpha\beta}$ has at most rank $n - 3$. That the rank is in fact $n - 3$ can be seen from the first of the relations

$$(7.9) \quad H_{ij} x^i y^j = - \begin{vmatrix} 0 & \frac{r^\beta}{\omega} & 0 \\ \frac{r^\alpha}{\omega} & h^{\alpha\beta} & l_i^\alpha x^i \\ 0 & l_i^\beta y^i & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & L^j & 0 \\ 0 & 0 & A^{jk} L_k / \omega & 0 \\ L^i & A^{ik} L_k / \omega & A^{ij} & x^i \\ 0 & 0 & y^j & 0 \end{vmatrix}$$

(P, C type II; $N-P, C$)

which is obtainable from the multiplication scheme (5.6). It is interesting to observe that (7.9) is also applicable in $N-P, C$ if we introduce the factor $|A_{ij}|$ in the second and third members.

8. Classification of the Partial Differential Equations. The general linear second order equation with constant coefficients and its homogeneous adjoint can be written

$$(8.1) \quad \begin{aligned} L(u) &\equiv A^{ij} u_{ij} + B^i u_i + C u = f(x^i) & (A^{ij} = A^{ji}); \\ L^*(v) &\equiv A^{ij} v_{ij} - B^i v_i + C v = 0. \end{aligned}$$

Our treatment covers all non-parabolic equations and those parabolic equations which come under the following classification.

$$(8.2) \quad \begin{aligned} \text{Non-parabolic:} & \quad |A^{ij}| \neq 0; \\ \text{Parabolic:} & \quad |A^{ij}| = 0, \quad \tilde{A}_{ij} B^i B^j \neq 0. \end{aligned}$$

The potentials studied in this paper are functions of $X^i = x^i - \xi^i$ which satisfy equation (8.1b). A subscript i indicates differentiation with respect to Latin arguments while Greek differentiations are always explicitly indicated. A function of X^i which satisfies (8.1b) with respect to Latin arguments also satisfies the homogeneous equation (8.1a) with respect to Greek arguments.

Since all of our potentials are exponentially damped we try for a solution of the form

$$(8.3) \quad v = \exp(-\alpha_i X^i) \Phi(X^i).$$

For a function (8.3) equation (8.1b) becomes

$$(8.4) \quad \exp(-\alpha_i X^i) [A^{ij}(\alpha_i \alpha_j \Phi - \alpha_i \Phi_j - \alpha_j \Phi_i + \Phi_{ij}) - B^i(-\alpha_i \Phi + \Phi_i) + C\Phi] = 0.$$

Dropping the exponential factor (8.4) can be written

$$(8.5) \quad A^{ij} \Phi_{ij} + \Phi [-2A^{ij} \alpha_j - B^i] + \Phi [A^{ij} \alpha_i \alpha_j + B^i \alpha_i + C] = 0.$$

For parabolic and non-parabolic equations the damping constants α_i are determined differently.

9. Non-Parabolic Potentials. In the non-parabolic case of (8.5), analogy with ordinary equations leads us to determine damping constants α_i and δ as follows:

$$(9.1) \quad \begin{aligned} \alpha_i &= -\frac{1}{2} A_{ij} B^j, \\ jJ\delta^2 &= A^{ij} \alpha_i \alpha_j + B^i \alpha_i + C = -\frac{1}{4} A_{ij} B^i B^j + C. \end{aligned}$$

The symbols j and J , more fully explained later on, are subject to $j^2 = J^2 = 1$. With the damping constants (9.1) equation (8.5) becomes

$$(9.2) \quad A^{ij} \Phi_{ij} + jJ\delta^2 \Phi = 0.$$

The original equation (8.1b) can be written

$$(9.3) \quad [A^{ij}(D_{x^i} + \alpha_i)(D_{x^j} + \alpha_j) + jJ\delta^2]v = 0.$$

For the non-parabolic potentials in this paper, the function Φ depends on R, S, P which are related to the terms of the Pythagorean identity (3.8b) as follows.

$$(9.4) \quad R = \sqrt{jA_{ij}X^iX^j}, \quad S = \sqrt{jH_{ij}X^iX^j}, \quad P = \frac{\sqrt{|\omega|}}{\omega} L_i X^i.$$

Here the algebra is N - P , N - C and $j^2 = \tilde{j}^2 = 1$. The sign of j is to be chosen so that R is real. Thus, j depends on the position of the vector X^i relative to the cone $A_{ij}X^iX^j = 0$. Similarly, the sign of $\tilde{j}$ depends on the position of X^i relative to the cylinder $H_{ij}X^iX^j = 0$.

For a function $\Phi(R, S, P)$ equation (9.2) becomes

$$(9.5) \quad \begin{aligned} \Phi_{RR} A^{ij} R_i R_j + \Phi_{SS} A^{ij} S_i S_j + \Phi_{PP} A^{ij} P_i P_j + 2\Phi_{RS} A^{ij} R_i S_j + 2\Phi_{RP} A^{ij} R_i P_j \\ + 2\Phi_{SP} A^{ij} S_i P_j + \Phi_R A^{ij} R_{ij} + \Phi_S A^{ij} S_{ij} + jJ\delta^2 \Phi = 0. \end{aligned}$$

Using (9.4) we compute the various sums appearing in (9.5).

$$\begin{aligned}
RR_i &= jA_{ij}X^j, & SS_i &= jH_{ij}X^j, \\
A^{ij}R_iR_j &= R^{-2}A^{ij}A_{ik}X^kA_{jm}X^m = R^{-2}A_{km}X^kX^m = j, \\
A^{ij}S_iS_j &= S^{-2}A^{ij}H_{ik}X^kH_{jm}X^m = S^{-2}H_{km}X^kX^m = \tilde{j}, \\
A^{ij}R_iS_j &= j\tilde{j}(RS)^{-1}A^{ij}A_{ik}X^kH_{jm}X^m = j\tilde{j}(RS)^{-1}H_{km}X^kX^m = \frac{jS}{R}, \\
(9.6) \quad A^{ij}P_iP_j &= \frac{|\omega|}{\omega^2}A^{ij}L_iL_j = \frac{|\omega|}{\omega}, \\
A^{ij}R_iP_j &= A^{ij}j\frac{A_{ik}X^k}{R}\frac{\sqrt{|\omega|}}{\omega}L_j = j\frac{\sqrt{|\omega|}}{\omega}\frac{L_iX^i}{R} = \frac{jP}{R}, \\
A^{ij}S_iP_j &= A^{ij}\tilde{j}\frac{H_{ik}X^k}{S}\frac{\sqrt{|\omega|}}{\omega}L_j = 0.
\end{aligned}$$

We compute the sums involving second derivatives using the results on first derivatives.

$$\begin{aligned}
RR_{ij} + R_iR_j &= jA_{ij}, & SS_{ij} + S_iS_j &= jH_{ij}, \\
(9.7) \quad A^{ij}R_{ij} &= R^{-1}[jA^{ij}A_{ij} - A^{ij}R_iR_j] = \frac{j(n-1)}{R}, \\
A^{ij}S_{ij} &= S^{-1}[\tilde{j}A^{ij}H_{ij} - A^{ij}S_iS_j] = \frac{\tilde{j}(n-2)}{S}.
\end{aligned}$$

Substituting from (9.6) and (9.7), equation (9.5) becomes

$$\begin{aligned}
(9.8) \quad j\Phi_{RR} + 2j\frac{S}{R}\Phi_{RS} + 2j\frac{P}{R}\Phi_{RP} + \tilde{j}\Phi_{SS} + \frac{|\omega|}{\omega}\Phi_{PP} + \frac{j(n-1)}{R}\Phi_R \\
+ \frac{\tilde{j}(n-2)}{S}\Phi_S + jJ\delta^2\Phi = 0.
\end{aligned}$$

In the solutions which we designate as the Green potentials and the retarded potentials, the function Φ has the form

$$(9.9) \quad \Phi = S^\mu F(R).$$

Since Φ does not involve P explicitly, Φ_{RP} and Φ_{PP} are missing in (9.8). If we use the Φ_S and Φ_{SS} terms to determine μ , we are left with an ordinary differential equation for F as follows:

$$\begin{aligned}
(9.10) \quad \mu(n-2) + \mu(\mu-1) &= 0; & \mu &= 0, 3-n; \\
F'' + \frac{(n-1) + 2\mu}{R}F' + J\delta^2F &= 0.
\end{aligned}$$

In treating the differential equation

$$(9.11) \quad F'' + \frac{k}{R} F' + J\delta^2 F = 0,$$

we make the substitution $F(R) = \phi(\delta R) = \phi(\rho)$, and equation (9.11) becomes

$$(9.12) \quad \phi'' + \frac{k}{\rho} \phi' + J\phi = 0.$$

The substitution $\phi = \rho^{(1-k)/2} Z(\rho)$ shows that Z is a Bessel function of order $(1 - k)/2$.

$$(9.13) \quad \rho^{(-1-k)/2} \left\{ \rho Z'' + Z' + Z \left[J\rho - \left(\frac{1-k}{2} \right)^2 \frac{1}{\rho} \right] \right\} = 0.$$

The Bessel function is oscillating or non-oscillating according as $J = 1$ or $J = -1$.

The Green and retarded potentials are obtained by inserting (9.9) into (8.3) with two sets of values of μ and k , namely

$$(9.14) \quad \begin{aligned} \mu = 0, \quad k = n - 1, \quad \frac{1-k}{2} &= \frac{2-n}{2}, \\ \mu = 3 - n, \quad k = 5 - n, \quad \frac{1-k}{2} &= \frac{n-4}{2}. \end{aligned}$$

The potentials are

$$(9.15) \quad \begin{aligned} v &= \exp(-\alpha_i X^i) \frac{Z_{(n-2)/2}(\delta R)}{(\delta R)^{(n-2)/2}} && \text{(Green),} \\ v &= \exp(-\alpha_i X^i) \frac{(\delta R)^{(n-4)/2} Z_{(n-4)/2}(\delta R)}{S^{n-3}} && \text{(retarded).} \end{aligned}$$

When $n = 3$ the potentials (9.15) become the same, and for a retarded potential we look for a solution of the form

$$(9.16) \quad v = \exp(-\alpha_i X^i) \Phi(R, S), \quad \Phi = F(R) \ln S + G(R) \quad (\text{retarded } n = 3).$$

Again Φ does not involve P explicitly and when we substitute in (9.8) the Φ_s and Φ_{ss} terms cancel. When we equate terms with $\ln S$ and terms without $\ln S$ to zero we are left with two ordinary differential equations

$$(9.17) \quad \begin{aligned} F'' + \frac{2}{R} F' + J\delta^2 F &= 0, \\ G'' + \frac{2}{R} G' + J\delta^2 G + \frac{2}{R} F' &= 0. \end{aligned}$$

Equation (9.17a) is of the type (9.11). Corresponding to the two solutions F , we can have particular solutions G of (9.17b) as follows:

$$(9.18) \quad \begin{aligned} F_1 &= k \frac{\cosh \rho}{\rho}, & G_1 &= k \left[\frac{-2 \ln \rho}{\rho} \cosh \rho - J \int_0^\rho \frac{2}{\rho} \ln r \sinh(2r - \rho) dr \right], \\ F_2 &= k \frac{\sinh \rho}{\rho}, & G_2 &= k \left[\frac{-2 \ln \rho}{\rho} \sinh \rho + \int_0^\rho \frac{2}{\rho} \ln r \cosh(2r - \rho) dr \right]. \end{aligned}$$

There is no gain in including the complementary function in (9.18) since this would merely add terms involving the Green potential to the retarded potential in (9.16).

In the case $n = 1$ our differential equation (8.1) is ordinary, and we have only the Green potential (9.15a). Here

$$(9.19) \quad \begin{aligned} A_{11} &= \frac{1}{A^{11}}, & R &= \sqrt{jA_{11}(x^1 - \xi^1)^2}, \\ \alpha_1 &= \frac{-B^1}{2A^{11}}, & \delta &= \sqrt{\frac{-jJ[(B^1)^2 - 4A^{11}C]}{4A^{11}}}. \end{aligned}$$

For $J = 1$ we have the familiar result that $(B^1)^2 - 4A^{11}C < 0$ corresponds to the oscillating case.

In case $\delta = 0$ it is customary to call (8.1) a distortionless equation. When not only $\delta = 0$ but also $\alpha_i = 0$, then the equation (8.1) is said to be undamped. The distortionless potentials and the undamped potentials are identical except for the exponential damping factor. For $\delta = 0$ and $\mu = 0$ equation (9.11) becomes

$$(9.20) \quad \left. \begin{aligned} R^{n-1}F'' + (n-1)R^{n-2}F' &= 0, & R^{n-1}F' &= \text{constant}, \\ v = F = \frac{a}{R^{n-2}} + b & \quad (n \neq 2) \\ v = a \ln R + b & \quad (n = 2) \end{aligned} \right\} \text{(undamped Green potentials).}$$

Φ FOR GREEN POTENTIALS

	FIRST TYPE	$\delta = 0$	SECOND TYPE	$\delta = 0$
$n = 2$	$J_0(\delta R)$	k	$Y_0(\delta R) - \ln \delta J_0(\delta R)$	$k \ln R$
$n = 3$	$\left(\frac{1}{\delta R}\right)^{\frac{1}{2}} J_{\frac{1}{2}}(\delta R)$	k	$\left(\frac{\delta}{R}\right)^{\frac{1}{2}} Y_{\frac{1}{2}}(\delta R)$	$\frac{k}{R}$
$n = 4$	$\left(\frac{1}{\delta R}\right) J_1(\delta R)$	k	$\left(\frac{\delta}{R}\right) Y_1(\delta R)$	$\frac{k}{R^2}$
$n = 5$	$\left(\frac{1}{\delta R}\right)^{\frac{3}{2}} J_{\frac{3}{2}}(\delta R)$	k	$\left(\frac{\delta}{R}\right)^{\frac{3}{2}} Y_{\frac{3}{2}}(\delta R)$	$\frac{k}{R^3}$
$n = 6$	$\left(\frac{1}{\delta R}\right)^2 J_2(\delta R)$	k	$\left(\frac{\delta}{R}\right)^2 Y_2(\delta R)$	$\frac{k}{R^4}$

Φ FOR RETARDED POTENTIALS

	FIRST TYPE	$\delta = 0$	SECOND TYPE	$\delta = 0$
$n = 2$	$S \left(\frac{\delta}{R} \right) Y_1(\delta R)$	$\frac{kS}{R^2}$	$S \left(\frac{1}{\delta R} \right) J_1(\delta R)$	kS
$n = 3$	$\frac{1}{R} H(R, S)$	$\frac{k}{R} \ln \frac{R^2}{S}$	$\frac{1}{\delta R} \bar{H}(R, S)$	$k \ln S$
$n = 4$	$\frac{1}{S} [Y_0(\delta R) - \ln \delta J_0(\delta R)]$	$\frac{k}{S} \ln R$	$\frac{1}{S} J_0(\delta R)$	$\frac{k}{S}$
$n = 5$	$\frac{1}{S^2} \left(\frac{R}{\delta} \right)^{\frac{1}{2}} J_{\frac{1}{2}}(\delta R)$	$\frac{kR}{S^2}$	$\frac{1}{S^2} (\delta R)^{\frac{1}{2}} Y_{\frac{1}{2}}(\delta R)$	$\frac{k}{S^2}$
$n = 6$	$\frac{1}{S^3} \left(\frac{R}{\delta} \right) J_1(\delta R)$	$\frac{kR^2}{S^3}$	$\frac{1}{S^3} (\delta R) Y_1(\delta R)$	$\frac{k}{S^3}$

$$H(R, S) = \frac{\cosh \delta R}{\cosh \delta R} \ln \frac{R^2}{S} + J \int_0^{\delta R} \ln r^2 \frac{\sin}{\sinh} (2r - \delta R) dr;$$

$$\bar{H}(R, S) = \frac{\cosh \delta R}{\cosh \delta R} \left[\ln \frac{(\delta R)^2}{S} - 2 \right] - \int_0^{\delta R} \ln r^2 \frac{\sin}{\sinh} (2r - \delta R) dr.$$

Table 2.1

For $\delta = 0$ and $\mu = 3 - n$ the equation (9.11) becomes

$$(9.21) \quad \left. \begin{aligned} R^{5-n} F'' + (5-n) R^{4-n} F' &= 0, & R^{5-n} F' &= \text{constant}, \\ v = S^{3-n} F &= \frac{1}{S^{n-3}} [a R^{n-4} + b] \quad (n \neq 3, 4) \\ v &= \frac{1}{S} [a \ln R + b] \quad (n = 4) \end{aligned} \right\} \quad \begin{array}{l} \text{(undamped retarded} \\ \text{potentials).} \end{array}$$

For the undamped retarded potential $n = 3$, the equations (9.17) become

$$(9.22) \quad \begin{aligned} (RF)'' &= 0, & (RG)'' &= -2F' = \frac{2a}{R^2}, \\ F &= \frac{a}{R} + b, & G &= \frac{-2a \ln R}{R} + \frac{c}{R} + d, \\ v &= \ln S \left[\frac{a}{R} + b \right] + \left[\frac{-2a \ln R}{R} + \frac{c}{R} + d \right] \end{aligned}$$

(undamped retarded potential, $n = 3$).

If we choose the arbitrary constants $b = c = d = 0$, the function (9.22) becomes the quantity upon which Volterra¹ based his integrating factor for integrating the drumhead equation over regions exterior to the cone.

¹ HADAMARD, J., *op. cit.*, p. 201.

The tables of Green potentials and retarded potentials are useful for comparison purposes in the case of small values of n . Throughout the tables, the exponential damping factor is not shown. These tables apply to the oscillating case, $J = 1$, and for $J = -1$ we change all oscillating functions to non-oscillating functions.

Our use of retarded potential is not to be confused with Kirchhoff's¹ use of the same terminology. Kirchhoff's quantities are of a more special nature in that they exist for only a few types of differential equations. For particular choices of the arbitrary functions appearing in Kirchhoff's quantities the latter reduce to functions in the table of retarded potentials. Kirchhoff used the terminology in a physical sense. Our reason for the terminology is algebraic, referring to the fact that in the last columns the exponents of S in the retarded table lag the exponents of R in the Green table.

Numerous other potentials involving the damping constants (9.1) can be obtained by solving the differential equation (9.8). As mentioned in the preface the potentials (9.15) are already known to be useful in treating partial differential equations and we include a second type because of their connection with potentials encountered in the next section. We let $\Phi = S^\mu F(P)$ in (9.8). The R derivatives are missing, the S derivatives determine μ , and F is determined by the differential equation $F_{PP} + (|\omega|/\omega)jJ\delta^2 F = 0$. Here the oscillatory character is determined by the sign of $jJ\omega$. There are four such potentials

$$(9.23) \quad \begin{aligned} v &= \exp(-\alpha_i X^i) S^\mu \frac{\cos}{\cosh} \delta P, \\ v &= \exp(-\alpha_i X^i) S^\mu \frac{\sin}{\sinh} \delta P \end{aligned} \quad (\mu = 0, 3 - n).$$

10. Parabolic Potentials. Although these functions occurred much earlier in the literature, a parabolic potential was first used as an integrating factor by Volterra² in his study of the heat equation. Following Hadamard we shall obtain the parabolic potentials as the limit of non-parabolic potentials. This is of importance since the functions are of such a peculiar nature that we cannot be sure that our terminology "Green and retarded potentials" is appropriate when applied to the parabolic case. In passing to the limit our N - P , N - C algebra is replaced by a P , C type I algebra.

In considering the parabolic equation as the limit of a non-parabolic equation, we will suppose that $|A^{ij}| \rightarrow 0$ and that $\tilde{A}_{ij}B^iB^j$ has a limit which is finite and different from zero. The latter condition insures that the parabolic equation contains partial derivatives with respect to all the variables.

Under the above conditions we will be concerned with the limit of a non-parabolic potential containing a non-oscillating Bessel function. To see this we

¹ HADAMARD, J., *op. cit.*, p. 67.

² VOLTERRA, V. *Leçons sur l'intégration des équations différentielles aux dérivées partielles* (Stockholm lectures). Paris: Hermann, 1912.

note that our non-parabolic potential is defined in a region where $jA_{ij}X^iX^j > 0$. Since in the limit $\tilde{A}_{ij}x^ix^j$ has a definite sign for all x^i (see the Pythagorean identity P , C type I), then we must also have $jA_{ij}B^iB^j > 0$ throughout the limiting process. Thus $jA_{ij}B^iB^j \rightarrow +\infty$, and we see from the form of δ in (9.1) that $J = -1$ throughout the limiting process and we are concerned with non-oscillating Bessel functions.

We will consider the limit as $|A^{ij}| \rightarrow 0$ of the following non-parabolic potential:

$$(10.1) \quad u = \frac{\exp[-\alpha_i X^i] Z_{(n-2)/2}(\delta R)}{[|A^{ij}|^2]^{(n-1)/4} (\delta R)^{(n-2)/2}}.$$

The key to this limiting process lies in applying the Pythagorean identity to δ , R and $-\alpha_i X^i$. We use an N - P , N - C Pythagorean identity with an algebra determined by the vector L^i where $\tilde{A}_{ij}L^iL^j$ has a limit finite and different from zero as we approach the parabolic case. The vector L^i may or may not coincide with the vector B^i .

Before passing to the limit we will express our potential in terms of the vector L^i alone and after passing to the limit the vector L^i is used to determine a P , C type I algebra in terms of which we express the final result. From (3.7b), we might expect the quantities H_{ij} to become infinite in the limit. That this is not the case follows from the fact that if we replace ω by $\tilde{A}_{ij}L^iL^j$ in (6.9) this form for H_{ij} applies in P , C type I and N - P , N - C . With this information about the H_{ij} we can use the Pythagorean identity

$$(10.2) \quad \tilde{A}_{ij}X^iX^j = |A^{ij}| H_{ij}X^iX^j + \frac{(\tilde{A}_{ij}L^iX^j)^2}{\tilde{A}_{ij}L^iL^j} \quad (N-P, N-C),$$

to show that the limit of $\tilde{A}_{ij}L^iL^j$ is not zero if the same is true of $\tilde{A}_{ij}X^iX^j$.

In rewriting (10.1) we use the following N - P , N - C identities:

$$(10.3) \quad \begin{aligned} R &= \sqrt{jA_{ij}X^iX^j} = \left[jH_{ij}X^iX^j + \frac{j(\tilde{A}_{ij}L^iX^j)^2}{|A^{ij}| \tilde{A}_{ij}L^iL^j} \right]^{\frac{1}{2}}, \\ \delta &= \sqrt{j(\frac{1}{4}A_{ij}B^iB^j - C)} = \left[\frac{j}{4} H_{ij}B^iB^j + \frac{j}{4} \frac{(\tilde{A}_{ij}L^iB^j)^2}{|A^{ij}| \tilde{A}_{ij}L^iL^j} - jC \right]^{\frac{1}{2}}, \\ -\alpha_i X^i &= \frac{A_{ij}B^iX^j}{2} = \frac{1}{2} H_{ij}B^iX^j + \frac{(\tilde{A}_{ij}L^iB^j)(\tilde{A}_{ij}L^iX^j)}{2|A^{ij}| \tilde{A}_{ij}L^iL^j}. \end{aligned}$$

Using the binomial theorem, we have

$$(10.4) \quad \delta R = \left[\frac{(\tilde{A}_{ij}L^iB^j)^2(\tilde{A}_{ij}L^iX^j)^2}{4|A^{ij}|^2(\tilde{A}_{ij}L^iL^j)^2} \right]^{\frac{1}{2}} \left[1 + \frac{2|A^{ij}| \tilde{A}_{ij}L^iL^j}{(\tilde{A}_{ij}L^iB^j)^2} (\frac{1}{4}H_{ij}B^iB^j - C) + \frac{|A^{ij}| \tilde{A}_{ij}L^iL^j}{2(\tilde{A}_{ij}L^iX^j)^2} H_{ij}X^iX^j + \dots \right].$$

When we let the determinant $|A^{ij}| \rightarrow 0$ in (10.1) the argument δR becomes positively infinite as seen from (10.4). Thus, in obtaining the limit of (10.1) we shall use the following asymptotic relationship valid for non-oscillating Bessel functions of any order.

$$(10.5) \quad Z(u) = \frac{Ae^u + Be^{-u}}{\sqrt{u}}.$$

In taking the limit in (10.1) we will be concerned with combining the limiting exponents (10.3c) and (10.4).

The last term in the right member of (10.3c) and the first term in (10.4) become infinite in the limit. In order to show that these terms make no contribution, we let $\bar{j}$ indicate the sign in the last term of (10.3c).

$$(10.6) \quad \frac{\bar{j}(\tilde{A}_{ij}L^iB^j)(\tilde{A}_{ij}L^iX^j)}{2|A^{ij}|\tilde{A}_{ij}L^iL^j} > 0 \quad (\bar{j}^2 = 1).$$

With the proper combination of the Bessel functions in (10.5) it is possible to obtain an asymptote containing either e^u or e^{-u} alone. In our case we need an asymptote involving $\exp[-j\delta R]$ in order to make the above mentioned terms in (10.3c) and (10.4) have opposite signs. On passing to the limit, we shall use a P , C type I algebra, and the remaining terms in our exponential can be written

$$(10.7) \quad \exp \left[\frac{1}{2}H_{ij}B^iX^j - \bar{j}^2 \frac{L_iX^i}{L_iB^i} \left(\frac{1}{4}H_{ij}B^iB^j - C \right) - \bar{j}^2 \frac{L_iB^i}{4L_iX^i} H_{ij}X^iX^j \right].$$

In (10.7) the factor $-\bar{j}$ was introduced in the asymptote and the other factor $\bar{j}$ is determined by (10.6).

In passing to the limit in (10.1) the denominators in (10.1) and (10.5) can be written

$$(10.8) \quad \begin{aligned} [|A^{ij}|^2]^{(n-1)/4}(\delta R)^{(n-1)/2} &= [|A^{ij}|^2(\frac{1}{4}A_{ij}B^iB^j - C)A_{ij}X^iX^j]^{(n-1)/4} \\ &= [(\frac{1}{4}\tilde{A}_{ij}B^iB^j - |A^{ij}|C)\tilde{A}_{ij}X^iX^j]^{(n-1)/4} \rightarrow \left[\frac{(L_iB^i)^2(L_iX^i)^2}{4\omega^2} \right]^{(n-1)/4} \end{aligned}$$

Here we have used the Pythagorean identity (6.5b).

Combining (10.7) and (10.8) the limit of (10.1) can be written

$$(10.9) \quad \left[\frac{4\omega^2}{(L_iB^i)^2(L_iX^i)^2} \right]^{(n-1)/4} \exp \left[\frac{1}{2}H_{ij}B^iX^j - \frac{L_iX^i}{L_iB^i} \left(\frac{1}{4}H_{ij}B^iB^j - C \right) - \frac{L_iB^i}{4L_iX^i} H_{ij}X^iX^j \right].$$

Thus we have obtained the parabolic potential (10.9) as the limit of the Green potential (10.1). Guided by (10.9) we shall now try for solutions of the parabolic equation (8.1) having a certain specified form. In so doing we shall arrive at

other parabolic potentials all of which can be obtained as the limit of non-parabolic potentials.

In trying for a solution of the form $v = \exp(-\alpha_i X^i) \Phi(X^i)$ for the parabolic equation (8.1b) we use a different determination of the damping constants in (8.5). As above we use a P, C type I algebra and guided by (10.9) we define

$$(10.10) \quad \beta = \frac{j}{4} \frac{\sqrt{|\omega|}}{\omega} L_i B^i; \quad \alpha_i = -\frac{1}{2} H_{ij} B^j - \frac{j}{4\beta} \frac{\sqrt{|\omega|}}{\omega} L_i [-\frac{1}{4} H_{ij} B^i B^j + C].$$

With these damping constants we compute the coefficients of (8.5).

$$(10.11) \quad \begin{aligned} -2A^{ik} \alpha_k - B^i &= A^{ik} H_{kj} B^j - B^i = -L_j B^j \frac{L^i}{\omega} = -4j\beta \frac{L^i}{\sqrt{|\omega|}}, \\ A^{ij} \alpha_j + B^i \alpha_i + C &= \frac{1}{4} H_{ij} B^i B^j - \frac{1}{2} H_{ij} B^i B^j \\ &\quad + \frac{1}{4} H_{ij} B^i B^j - C + C = 0. \end{aligned}$$

Then (8.5) becomes

$$(10.12) \quad A^{ij} \Phi_{ij} - 4j\beta \frac{L^i}{\sqrt{|\omega|}} \Phi_i = 0.$$

The original equation (8.1b) when written in terms of the damping constants (10.10) becomes

$$(10.13) \quad \left[A^{ij} (D_{xi} + \alpha_i) (D_{xj} + \alpha_j) - 4j\beta \frac{L^i}{\sqrt{|\omega|}} (D_{xi} + \alpha_i) \right] v = 0.$$

In view of condition (8.2b) and the P, C type I Pythagorean identity we see that β is never zero. The vanishing of the discriminant $-\frac{1}{4} H_{ij} B^i B^j + C$ implies no contribution from the $\beta \alpha_i$ term in (10.13).

For parabolic potentials we look for functions $\Phi(S, P)$, where S and P are given by (9.4). Here we replace $\tilde{j}$ by j and the algebra is P, C type I as used above. In modifying (9.5) to fit the parabolic case, we drop those terms with R derivatives and those terms containing $A^{ij} P_{,i}$. For a function of S and P , equation (10.12) becomes

$$(10.14) \quad 0 = \Phi_{ss} A^{ij} S_i S_j + \Phi_s A^{ij} S_{ij} - \frac{4j\beta}{\sqrt{|\omega|}} (\Phi_s L^i S_i + \Phi_P L^i P_{,i}).$$

Using (9.6) and (9.7) this becomes

$$(10.15) \quad \Phi_{ss} + \frac{(n-2)}{S} \Phi_s - 4\beta \Phi_P = 0.$$

Guided by (10.9) we try for a solution of the form

$$(10.16) \quad \Phi = |P|^\lambda S^\mu \exp \left[\frac{-\beta S^2}{P} \right].$$

Substituting in (10.15) and equating terms with like powers to zero, we arrive at four sets of values for λ , μ , and $\bar{\beta}$.

$$\begin{aligned} 4\bar{\beta}^2 - 4\beta\bar{\beta} &= 0, & \bar{\beta} &= \beta, \quad \lambda = \frac{1-n}{2} - \mu, \\ (10.17) \quad -4\beta\lambda - 2\bar{\beta}(n-2) - 2\bar{\beta}(2\mu+1) &= 0, & \bar{\beta} &= 0, \quad \lambda = 0, \\ \mu(n-2) + \mu(\mu-1) &= 0, & \mu &= 0, \quad 3-n. \end{aligned}$$

Introducing Φ of (10.16) into the v of (8.3), these four potentials can be written

$$\begin{aligned} v &= |P|^{(1-n)/2} \exp \left[-\alpha_i x^i - \beta \frac{S^2}{P} \right] & (\text{Green}), \\ (10.18) \quad v &= |P|^{(n-5)/2} S^{3-n} \exp \left[-\alpha_i X^i - \beta \frac{S^2}{P} \right] & (\text{retarded}), \\ v &= \exp [-\alpha_i X^i]; \quad v = S^{3-n} \exp [-\alpha_i X^i]. \end{aligned}$$

In (10.9) we have seen that the first of the potentials (10.18) is the limit of the Green potential (9.15a). In a similar fashion, the retarded potential (10.18) is the limit of the retarded potential (9.15b). The remaining two potentials of (10.18) can be obtained as the limit of potentials (9.23). In the latter work one merely has to replace R by $|P|$ in (10.3) and (10.4).

When $n = 3$, the Green and retarded potentials of (10.18) become the same and we look for a second solution of (10.15) of the form

$$(10.19) \quad \Phi = [|P|^\lambda \ln S + F(P)] \exp \left[-\bar{\beta} \frac{S^2}{P} \right].$$

Substituting into (10.15) and equating terms with like powers to zero, we have

$$\begin{aligned} (10.20) \quad \bar{\beta} &= \beta, \quad \lambda = -1; \quad F' + \frac{F}{P} + \frac{|P|^\lambda}{P} = 0, \quad F = \frac{-1}{|P|} \ln |P| + \frac{K}{|P|}; \\ \bar{\beta} &= 0, \quad \lambda = 0; \quad F' = 0, \quad F = K. \end{aligned}$$

Inserting Φ of (10.19) into v of (8.3) we have the following potentials for $n = 3$:

$$\begin{aligned} (10.21) \quad v &= \frac{1}{|P|} \ln \frac{S}{|P|} \exp \left[-\alpha_i X^i - \beta \frac{S^2}{P} \right] & (\text{retarded } n = 3), \\ v &= \ln S \exp [-\alpha_i X^i]. \end{aligned}$$

Taking $K = 0$ in (10.20) drops terms already included in (10.18).