*On the Propagation of Waves in Isotropic Incompressible Perfectly Elastic Materials**

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1. Introduction. The problem of the propagation of waves in incompressible, perfectly elastic solids has received little attention, though several authors, including Duhem [1], Hadamard [2], and Tolotti [3], have discussed the corresponding problem for compressible materials. Some important results recently obtained in the theory of large deformations of perfectly elastic materials, namely those of Rivlin [4], Green & Shield [5], apply only to incompressible materials. Furthermore, the theory has been applied most successfully to rubber, which can be considered incompressible. In view of these facts, it seems reasonable to consider the propagation of waves in such materials. When possible, we use Truesdell's notation [6].

At time $t = 0$ let the points of a continuum be assigned coordinates X^α in a stationary Euclidean coordinate system with metric tensor $G_{\alpha\beta}$. At any other time t assign them coordinates x^i in any independently selected coordinate system of the same type with metric tensor g_{ij} . Greek indices always refer to the coordinates X^α , Latin to the coordinates x^i . It is assumed that the mappings

$$(1.1) \quad x^i = x^i(X^\alpha, t)$$

defining the motion of the continuum are single-valued, continuous, and satisfy certain differentiability conditions which will be stated precisely later. It is further assumed that, at each time $t \geq 0$, equations (1.1) possess a unique inverse $X^\alpha = X^\alpha(x^i, t)$. The symbol D/Dt denotes time differentiation with the variables X^α held constant, $\partial/\partial t$ time differentiation with the variables x^i held constant,

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and a superposed dot denotes the material derivative. Then $\dot{x}^i = Dx^i/Dt$ is the velocity vector. The quantities $x^i_{,\alpha} \equiv \partial x^i/\partial X^\alpha$ and $X^\alpha_{,i} \equiv \partial X^\alpha/\partial x^i$ are called displacement gradients.

We consider only homogeneous, isotropic, incompressible, perfectly elastic materials. The equations of motion are then [6]

$$(1.2) \quad t^{ij}_{,j} + \rho f^i = \rho \ddot{x}^i,$$

$$(1.3) \quad t^i_j = -p\delta^i_j + 2 \frac{\partial \Sigma}{\partial I} (c^{-1})^i_j - 2 \frac{\partial \Sigma}{\partial II} c^i_j,$$

$$(1.4) \quad |(c^{-1})^i_j| = 1,$$

where t_{ij} is the stress tensor, f^i the body force per unit mass, ρ the constant density, p is an arbitrary hydrostatic pressure, $|(c^{-1})^i_j|$ is the determinant with elements $(c^{-1})^i_j$,

$$(c^{-1})^{ij} \equiv G^{\alpha\beta} x^i_{,\alpha} x^j_{,\beta},$$

$$I \equiv (c^{-1})^i_i, \quad II \equiv \frac{1}{2}\{I^2 - (c^{-1})^i_j (c^{-1})^j_i\},$$

$$c_{ij} \equiv G_{\alpha\beta} X^\alpha_{,i} X^\beta_{,j} = (c^{-1})_{ik} (c^{-1})^k_j - I(c^{-1})_{ij} + II g_{ij},$$

and $\Sigma = \Sigma(I, II)$ is the strain energy. It is assumed that Σ is single valued and can be differentiated with respect to I and II as many times as may be desired.

According to the classical infinitesimal theory of elasticity, surfaces of weak discontinuity in an incompressible medium are transverse waves and are propagated with the speed $V = \sqrt{\mu/\rho}$, where μ is the Lamé constant. We will show that, according to the theory considered here, such surfaces are again transverse waves and are propagated with a speed V such that

$$(1.5) \quad \rho(V - v)^2 = 2 \frac{\partial \Sigma}{\partial I} \mu^i \nu_i + 2 \frac{\partial \Sigma}{\partial II} (I\nu_i - \mu_i)\mu^i + \frac{1}{2} U^i_i \pm \frac{1}{2} \sqrt{2U^i_j U^j_i - (U^i_i)^2},$$

where ν_i is a unit vector normal to the surface of discontinuity, $\mu^i \equiv (c^{-1})^i_j \nu_j$, $v \equiv \dot{x}^i \nu_i$, and

$$(1.6) \quad \begin{aligned} U^i_k &\equiv 2 \frac{\partial \Sigma}{\partial II} \{\mu^i \mu_k - (c^{-1})^i_k \mu_j \nu^j\} + 4 \frac{\partial^2 \Sigma}{\partial I^2} (\mu^i - \nu^i \mu_j \nu^j) \mu_k \\ &+ 4 \frac{\partial^2 \Sigma}{\partial II^2} \{I(\mu^i - \nu^i \mu_j \nu^j) + \nu^i \mu_j \mu^j - (c^{-1})^i_j \mu^j\} \{I\mu_k - (c^{-1})_{km} \mu^m\} \\ &+ 4 \frac{\partial^2 \Sigma}{\partial I \partial II} \{(\mu^i - \nu^i \mu_j \nu^j)(2I\mu_k - (c^{-1})_{km} \mu^m) \\ &+ \nu^i \mu_k \mu_j \mu^j - (c^{-1})^i_j \mu^j \mu_k\}. \end{aligned}$$

It is shown that, if Σ satisfies certain inequalities derived by Truesdell, the values of $V - v$ given by (1.5) are always real when v_i is an eigenvector of $(c^{-1})^i_j$. These speeds are always real if the strain energy is of the form

$$(1.7) \quad \Sigma = \frac{1}{2}\alpha (\text{I} - 3) + \frac{1}{2}\beta (\text{II} - 3),$$

where α and β are non-negative constants, not both zero. This form has been used by Mooney [7] and Rivlin [4], [8].

2. An alternative form of the equations of motion. Eliminating c^i_j from (1.3) one obtains

$$(2.1) \quad t^i_j = -\bar{p} \delta^i_j + 2 \left(\frac{\partial \Sigma}{\partial \text{I}} + \text{I} \frac{\partial \Sigma}{\partial \text{II}} \right) (c^{-1})^i_j - 2 \frac{\partial \Sigma}{\partial \text{II}} (c^{-1})^i_k (c^{-1})^k_j,$$

where $\bar{p} \equiv p + 2\text{II} \partial \Sigma / \partial \text{II}$ is an arbitrary hydrostatic pressure. It is assumed that $f_i = 0$. Using (2.1) to eliminate t^i_j from (1.2), one obtains

$$(2.2) \quad -\bar{p}^{,i} + T^{ij}_{km} (c^{-1})^{km}_{,j} = \rho \ddot{x}^i,$$

where

$$\begin{aligned} T^{ij}_{km} \equiv & \frac{\partial \Sigma}{\partial \text{I}} (\delta^i_k \delta^j_m + \delta^j_k \delta^i_m) + 2 \frac{\partial^2 \Sigma}{\partial \text{I}^2} g_{km} (c^{-1})^{ij} \\ & + \frac{\partial \Sigma}{\partial \text{II}} \{ 2g_{km} (c^{-1})^{ij} + \text{I} (\delta^i_k \delta^j_m + \delta^j_k \delta^i_m) - \delta^i_k (c^{-1})^j_m \\ (2.3) \quad & - \delta^i_m (c^{-1})^j_k - \delta^j_k (c^{-1})^i_m - \delta^j_m (c^{-1})^i_k \} \\ & + 2 \frac{\partial^2 \Sigma}{\partial \text{I} \partial \text{II}} \{ 2\text{I} g_{km} (c^{-1})^{ij} - (c^{-1})_{km} (c^{-1})^{ij} - g_{km} (c^{-1})^i_n (c^{-1})^{nj} \} \\ & + 2 \frac{\partial^2 \Sigma}{\partial \text{II}^2} \{ \text{I} g_{km} - (c^{-1})_{km} \} \{ \text{I} (c^{-1})^{ij} - (c^{-1})^{in} (c^{-1})^j_n \}. \end{aligned}$$

In the problem considered here, it is convenient to use (1.4), (2.1), (2.2), and (2.3) rather than (1.2), (1.3), and (1.4).

3. Kinematical conditions of compatibility for second order waves. Consider a body which, at time $t = 0$, occupies a region R in space, and let $S(t)$ be a surface separating R into two regions $R_1(t)$ and $R_2(t)$. Let $S(t)$ be given by an equation of the form $f(X^\alpha, t) = 0$, where f is continuously differentiable and $f^{,\alpha} f_{,\alpha} > 0$ in the neighborhood of $S(t)$. Assume that the functions $\bar{p}$, $x^i = x^i(X^\alpha, t)$, $x^i_{,\alpha}$ and Dx^i/Dt are continuous throughout R . The quantities

$$X^\alpha_{,i} = \text{cofactor of } x^i_{,\alpha} \text{ in } |x^j_{,\beta}|/|x^j_{,\beta}|$$

will then be continuous in R . Suppose that the first partial derivatives of these functions with respect to X^α and t exist and are continuous in $R_1(t) \cup S(t)$ and in $R_2(t) \cup S(t)$ but that, at each point of $S(t)$, at least one of these functions is discontinuous. That is, not all the derivatives have the same values on the two sides of $S(t)$. There will then exist functions A and A^i , not all zero, which are defined on $S(t)$ and are such that [2]

$$\begin{aligned} [\bar{p}, \alpha] &= A f_{,\alpha}, \\ \left[\frac{D^2 x^i}{Dt^2} \right] &= A^i \left(\frac{Df}{Dt} \right)^2, \\ \left[\frac{Dx^i_{,\alpha}}{Dt} \right] &= A^i f_{,\alpha} \frac{Df}{Dt}, \\ \left[\frac{\partial^2 x^i}{\partial X^\alpha \partial X^\beta} \right] &= A^i f_{,\alpha} f_{,\beta}, \end{aligned}$$

where the square bracket denotes the jump in the quantity within the bracket. Now f is a scalar so that $Df/Dt = \dot{f}$. Furthermore, it is easily established that $[\dot{x}^i] = [D^2 x^i/Dt^2]$, $[\dot{x}^i_{,\alpha}] = [Dx^i_{,\alpha}/Dt]$ and $[x^i_{,\alpha\beta}] = [\partial^2 x^i/\partial X^\alpha \partial X^\beta]$. For example, if $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols based on $G_{\alpha\beta}$,

$$\begin{aligned} [x^i_{,\alpha\beta}] &= [\partial^2 x^i/\partial X^\alpha \partial X^\beta - x^i_{,\sigma} \Gamma^\sigma_{\alpha\beta}] \\ &= [\partial^2 x^i/\partial X^\alpha \partial X^\beta]. \end{aligned}$$

Thus

$$\begin{aligned} [\bar{p}, \alpha] &= A f_{,\alpha}, \\ [\dot{x}^i] &= A^i \dot{f}^2, \\ [\dot{x}^i_{,\alpha}] &= A^i f_{,\alpha} \dot{f}, \\ [x^i_{,\alpha\beta}] &= A^i f_{,\alpha} f_{,\beta}. \end{aligned}$$

Suppose now that the condition of incompressibility (1.4) is satisfied in R . Differentiating (1.4) with respect to X^β , one obtains

$$x^i_{,\alpha\beta} X^\alpha_{,i} = 0,$$

from which it follows that

$$(3.1) \quad A^i f_{,\alpha} X^\alpha_{,i} = 0$$

on $S(t)$.

Let $R(t)$ denote the region occupied by the body at time t . Since $S(t)$ need not intersect $R(t)$, it seems unlikely that $S(t)$ has any direct physical significance.

On the other hand, if $f(x^i, t) \equiv f(X^\alpha(x^i, t), t)$, the surface $\tilde{S}(t)$ given by the equation $f(x^i, t) = 0$ will separate $R(t)$ into two regions and will be a surface of discontinuity in the first partial derivatives with respect to x^i and t of the functions $\tilde{p}$, $(c^{-1})^{ij}$, and $\dot{x}^i$. Thus $\tilde{S}(t)$ is more likely to be of physical significance. The derivatives $f_{,i}$ and $\partial f/\partial t$ will exist and be continuous on $\tilde{S}(t)$ and

$$f^i f_{,i} = f_{,\alpha} f_{,\beta} X^\alpha_{,i} X^\beta_{,j} g^{ij} > 0$$

since the quantities $g^{ij} X^\alpha_{,i} X^\beta_{,j}$ are coefficients of a positive definite quadratic form. The speed of propagation V of $\tilde{S}(t)$ is given by [2]

$$V = -(\partial f/\partial t)/\sqrt{f^i f_{,i}},$$

while

$$\nu_i = f_{,i}/\sqrt{f^i f_{,i}}$$

is a unit vector normal to $\tilde{S}(t)$. Observe that

$$\dot{f} = \partial f/\partial t + f_{,i} \dot{x}^i = \sqrt{f^i f_{,i}} (v - V), \quad f_{,\alpha} = f_{,i} x^i_{,\alpha}, \quad f_{,i} = f_{,\alpha} X^\alpha_{,i}.$$

The jump conditions given above hold on $\tilde{S}(t)$ when the quantities involved are expressed in terms of x^i and t . Let us now obtain conditions which are more convenient to use. Let $B \equiv A \sqrt{f^i f_{,i}}$, $B^i \equiv A^i f^{ij} f_{,j}$. Then on $\tilde{S}(t)$,

$$\begin{aligned} [\tilde{p}_{,i}] &= [\tilde{p}_{,\alpha}] X^\alpha_{,i} = A f_{,i} = B \nu_i, \\ [\dot{x}^i] &= B^i (v - V)^2, \\ [(c^{-1})^{ij}]_{,k} &= [\partial(c^{-1})^{ij}/\partial x^k] = [(c^{-1})^{ij}]_{,\alpha} X^\alpha_{,k} \\ &= [x^i_{,\beta\alpha}] x^j_{,\gamma} X^\alpha_{,k} G^{\beta\gamma} + [x^j_{,\gamma\alpha}] x^i_{,\beta} G^{\beta\gamma} X^\alpha_{,k} \\ (3.2) \quad &= f_{,\alpha} f_{,\beta} X^\alpha_{,k} G^{\beta\gamma} (A^i x^j_{,\gamma} + A^j x^i_{,\gamma}) \\ &= f_{,k} f_{,m} G^{\beta\gamma} (A^i x^j_{,\gamma} x^m_{,\beta} + A^j x^i_{,\gamma} x^m_{,\beta}) \\ &= \nu_k \nu_m \{B^i (c^{-1})^{jm} + B^j (c^{-1})^{im}\}, \\ &= \nu_k (\mu^i B^j + \mu^j B^i), \end{aligned}$$

while (3.1) becomes

$$(3.3) \quad B^i \nu_i = 0.$$

The *direction of discontinuity* for such a wave is defined to be the direction of the vector B^i . A *transverse (longitudinal)* wave is defined by the condition that the direction of discontinuity be tangent (perpendicular) to the surface of discontinuity. Thus, from (3.3), $\tilde{S}(t)$ is always a transverse wave.

The conditions on $[\dot{x}^i]$, $[(c^{-1})^{ij}]_{,k}$, and B^i are kinematical, holding for any

incompressible material. The quantity $V - v$ can be interpreted as the normal component of velocity of $\tilde{S}(t)$ relative to the material particles of which it is composed.

4. Dynamical conditions of compatibility for second order waves. Here we determine conditions imposed on the quantities B , B^i , ν^i , and $V - v$ by the dynamical equations of motion (2.2) and (2.3). From (2.2) and (3.2)

$$(4.1) \quad -B \nu^i + 2T^{ij}_{km} \nu_j \mu^m B^k = \rho B^i (V - v)^2.$$

Multiplying (4.1) by ν_i , summing on i , and using (3.3), one obtains

$$B = 2T^{ij}_{km} \nu_i \nu_j \mu^m B^k,$$

so that

$$(4.2) \quad 2(T^{ij}_{km} - T^{nj}_{km} \nu_n \nu^i) \nu_j \mu^m B^k = \rho B^i (V - v)^2.$$

Using (2.3) to eliminate T^{ij}_{km} from (4.2) and employing (3.3), one obtains

$$(4.3) \quad U^i_j B^j = U B^i,$$

where U^i_j is given by (1.6) and

$$(4.4) \quad U \equiv \rho(V - v)^2 - 2 \frac{\partial \Sigma}{\partial \mathbf{I}} \mu^i \nu_i - 2 \frac{\partial \Sigma}{\partial \mathbf{II}} (\mathbf{I} \nu^i - \mu_i) \mu^i.$$

As is easily verified, $\nu_i U^i_k = 0$. Thus (3.3) follows from (4.3) except when $U = 0$.

In order that $\tilde{S}(t)$ be a dynamically possible second order wave, there must exist a non-zero vector B^i satisfying (3.3) and (4.3). At time t select a point P on $\tilde{S}(t)$. Choose rectangular Cartesian coordinates so that, at P , $\nu_i = \delta_{i3}$, and consider (3.3) and (4.3) at P . Then $U^3_i = B^3 = 0$ and (4.3) reduces to

$$\begin{aligned} (U^1_1 - U)B^1 + U^1_2 B^2 &= 0, \\ U^2_1 B^1 + (U^2_2 - U)B^2 &= 0, \end{aligned}$$

from which

$$U^2 - U(U^1_1 + U^2_2) + U^1_1 U^2_2 - U^1_2 U^2_1 = 0.$$

Thus

$$U = \frac{1}{2}(U^1_1 + U^2_2) \pm \frac{1}{2} \sqrt{(U^1_1 + U^2_2)^2 - 4(U^1_1 U^2_2 - U^1_2 U^2_1)}.$$

To express this latter equation in invariant form, we note that, since $U^3_i = 0$,

$$\begin{aligned} U^1_1 + U^2_2 &= U^i_i, \\ U^1_1 U^2_2 - U^1_2 U^2_1 &= \frac{1}{2} \{ (U^i_i)^2 - U^i_j U^j_i \}. \end{aligned}$$

Thus, in any Euclidean coordinate system,

$$(4.5) \quad U = \frac{1}{2}U^i{}_i \pm \frac{1}{2}\sqrt{2U^i{}_j U^j{}_i - (U^i{}_i)^2}.$$

It now follows from (4.4) and (4.5) that $V - v$ is given by (1.5), while B^i is a non-trivial solution of the equations

$$U^i{}_j B^j = \frac{1}{2}B^i(U^k{}_k \pm \sqrt{2U^i{}_k U^k{}_i - (U^i{}_i)^2})$$

satisfying the condition $B^i v_i = 0$. For any given set of functions Σ and $(c^{-1})^{ij}$, v_i must be such that the right member of (1.5) is real and non-negative for at least one of the two possible choices of sign. Equation (4.1) imposes no other restrictions on the quantities B , B^i , V , and v_i .

Let $(1 + \delta_1)^2$, $(1 + \delta_2)^2$, and $(1 + \delta_3)^2$ denote the eigenvalues of $(c^{-1})^i{}_j$. Consider the special case where

$$\mu^i = (c^{-1})^{ij} v_j = (1 + \delta_1)^2 v^i$$

so that v^i is an eigenvector of $(c^{-1})^i{}_j$. From (1.6)

$$U^i{}_j = 2 \frac{\partial \Sigma}{\partial \Pi} (1 + \delta_1)^2 \{ (1 + \delta_1)^2 v^i v_j - (c^{-1})^i{}_j \}.$$

Equation (1.5) then yields

$$(4.6) \quad \rho(V - v)^2 = 2(1 + \delta_1)^2 \left\{ \frac{\partial \Sigma}{\partial I} + \frac{\partial \Sigma}{\partial \Pi} (1 + \delta_2)^2 \right\}$$

or

$$(4.7) \quad \rho(V - v)^2 = 2(1 + \delta_1)^2 \left\{ \frac{\partial \Sigma}{\partial I} + \frac{\partial \Sigma}{\partial II} (1 + \delta_3)^2 \right\}$$

depending on which sign is used. Now Truesdell has given a physical argument which shows that the conditions

$$\frac{\partial \Sigma}{\partial I} + \frac{\partial \Sigma}{\partial \Pi} (1 + \delta_i)^2 \geq 0, \quad i = 1, 2, 3,$$

should always be satisfied [6]. If these conditions are satisfied, $V - v$, as determined by (4.6) or (4.7) is real. If $\partial \Sigma / \partial \Pi \neq 0$, the vector B^i corresponding to either of the two possible values of $|V - v|$ is an eigenvector of $(c^{-1})^i{}_j$. If $\partial \Sigma / \partial \Pi = 0$, B^i may be any vector perpendicular to v_i .

5. The Mooney-Rivlin theory of rubber. When Σ is given by (1.5),

$$U_{ik} = U_{ki} = \beta \{ \mu_i \mu_k - (c^{-1})_{ik} \mu_j \mu^j \}.$$

In the special coordinate system introduced in the preceding section,

$$\begin{aligned}
 (U^i_i)^2 - U^i_j U^j_i &= 2(U^1_1 U^2_2 - U^1_2 U^2_1) \\
 &= 2\beta^2(c^{-1})_{33} \{ (c^{-1})_{11}(c^{-1})_{22}(c^{-1})_{33} + 2(c^{-1})_{12}(c^{-1})_{23}(c^{-1})_{31} \\
 &\quad - (c^{-1})_{11}(c^{-1})_{23}^2 - (c^{-1})_{22}(c^{-1})_{23}^2 - (c^{-1})_{33}(c^{-1})_{12}^2 \} \\
 &= 2\beta^2(c^{-1})_{33} | (c^{-1})_{ij} | = 2\beta^2(c^{-1})_{33} \\
 &= 2\beta^2(c^{-1})^{ij} \nu_i \nu_j = 2\beta^2 \mu^i \nu_i,
 \end{aligned}$$

which is a scalar invariant. Thus, from (4.5),

$$U = \frac{1}{2}\beta(\mu_i \mu^i - I\mu_i \nu^i) \pm \frac{1}{2}\beta\sqrt{(\mu_i \mu^i - I\mu_i \nu^i)^2 - 4\mu_i \nu^i}.$$

Note that U , being an eigenvalue of a symmetric matrix, must be real. Thus

$$(5.1) \quad (\mu_i \mu^i - I\mu_i \nu^i)^2 \geq 4\mu_i \nu^i.$$

Using (4.4) and (4.5), one obtains

$$(5.2) \quad \rho(V - v)^2 = \alpha \mu^i \nu_i + \frac{3}{2}\beta(I\nu_i - \mu_i)\mu^i \pm \frac{1}{2}\beta\sqrt{(\mu_i \mu^i - I\mu_i \nu^i)^2 - 4\mu_i \nu^i}.$$

By (5.1), the right member of (5.2) is always real. That it is always non-negative follows from the inequalities

$$\mu^i \nu_i > 0, \quad I\mu^i \nu_i - \mu^i \mu_i > 0.$$

The first of these inequalities follows immediately from the fact that the matrix $\| (c^{-1})^i_j \|$ is positive definite. The latter is easily verified by introducing a rectangular Cartesian coordinate system in which, at the point in question, $(c^{-1})_{ij} = 0$ if $i \neq j$, and using the fact that $\| (c^{-1})^i_j \|$ is positive definite. Thus the values of V determined by (5.2) are always real.

Suppose that $\beta > 0$. Then, corresponding to any motion and any direction ν_i there are two values of $|V - v|$ satisfying the conditions of compatibility. These are distinct unless

$$(\mu^i \mu_i - I\mu^i \nu_i)^2 = 4\mu^i \nu_i.$$

The vectors B^i corresponding to two distinct values of $|V - v|$ are orthogonal. This follows immediately from the fact that they are eigenvectors of U^i_j corresponding to distinct eigenvalues. When $\beta = 0$,

$$\rho(V - v)^2 = \alpha(c^{-1})^{ij} \nu_i \nu_j,$$

where $\alpha > 0$. Let the eigenvalues of $(c^{-1})^i_j$ be ordered so that $(1 + \delta_1)^2 \geq (1 + \delta_2)^2 \geq (1 + \delta_3)^2$. For $(c^{-1})^i_j$ fixed, $|V - v|$ will then be maximum when $(c^{-1})^{ij} \nu_j = (1 + \delta_1)^2 \nu^i$, minimum when $(c^{-1})^{ij} \nu_j = (1 + \delta_3)^2 \nu^i$. These direc-

tions are directions of maximum and minimum strain, as measured by the strain tensor $e_{ij} \equiv \frac{1}{2}(g_{ij} - c_{ij})$.

6. Higher order waves. Suppose that the functions p , $x^i = x^i(X^\alpha, t)$, $x^i_{,\alpha}$, Dx^i/Dt and their first partial derivatives with respect to X^α and t are continuous throughout R , but that, at each point of $S(t)$, there is a finite discontinuity in at least one of the second partial derivatives of these functions. The surface $\tilde{S}(t)$ is then called a *third order wave*. Higher order waves may be defined in a similar way. It is possible to show that any such surface is a transverse wave and is propagated with a speed V satisfying (1.5). We give a brief outline of a proof of these statements for the case of a third order wave. By an analysis similar to that given in section 3, one can show that, on $\tilde{S}(t)$, there exist functions B and B^i , not all zero, such that

$$\begin{aligned} [p_{,ij}] &= B \nu_i \nu_j, \\ [(c^{-1})^{ij}]_{,km} &= \nu_k \nu_m \{B^i \mu^j + B^j \mu^i\}, \\ [\dot{x}^i_{,j}] &= B^i \nu_j (v - V)^2, \\ B^i \nu_i &= 0. \end{aligned} \tag{6.1}$$

The direction of the discontinuity is the direction of the vector B^i . Thus $\tilde{S}(t)$ is a transverse wave. Differentiating (2.2) with respect to x^n and using (6.2), one obtains

$$-B \nu^i + 2T^{ij}_{km} \nu_j \mu^m B^k = \rho B^i (V - v)^2,$$

which is formally identical with (4.1). Thus $V - v$ will be given by (1.5).

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