*A Generalization of the Method of Separation of Variables**

M. H. MARTIN

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland

Introduction. In the application of the method of separation of variables to a partial differential equation for one unknown real function u of two independent real variables x, y one seeks for solutions of the form $u = X(x)Y(y)$ where $X(x), Y(y)$ are real functions¹. For example, if the method is applied to Laplace's equation $u_{xx} + u_{yy} = 0$ we are led to the identity

$$(1) \quad X'' Y + XY'' \equiv 0,$$

which implies $X''/X = -Y''/Y = k^2 = \text{const.}$, to reduce the determination of solutions of the stipulated form to the integration on the ordinary differential equations

$$(2) \quad X'' - k^2 X = 0, \quad Y'' + k^2 Y = 0.$$

There is, however, another line of argument, based on the identity (1), which opens the way to a generalization of the method. Actually (1) is a special case of the identity

$$(3) \quad x_1(x)y_1(y) + x_2(x)y_2(y) \equiv 0,$$

which we lend geometrical significance by introducing the curves

$$M: \quad x_1 = x_1(x), \quad x_2 = x_2(x); \quad N: \quad y_1 = y_1(y), \quad y_2 = y_2(y),$$

and framing the following question.

* This work was supported by the Office of Naval Research, and was presented in part at the Conference on Differential Equations held at the University of Maryland, September 8-9, 1950.

¹ See, however, §3 at the end of this paper.

What can be said about the two curves M, N if the four functions $x_1(x), x_2(x), y_1(y), y_2(y)$ are required to satisfy identity (3)?

If either M or N degenerates into the origin, the other can be taken arbitrarily. Laying these obvious cases aside, we take a point (a_1, a_2) on M and a point (b_1, b_2) on N , both different from the origin. Since (3) is an identity, the following relations must hold

$$(4) \quad b_1 x_1(x) + b_2 x_2(x) \equiv 0, \quad a_1 y_1(y) + a_2 y_2(y) \equiv 0, \quad a_1 b_1 + a_2 b_2 = 0,$$

which assert that M and N must lie on a pair of straight lines perpendicular to each other at the origin. Conversely, if M, N lie on a pair of such straight lines, identity (3) holds. The answer to our question is accordingly

The necessary and sufficient condition for identity (3) to hold is that $M \varepsilon S, N \varepsilon T$ where

$$S: b_1 x_1 + b_2 x_2 = 0, \quad T: a_1 y_1 + a_2 y_2 = 0,$$

are two straight lines perpendicular to each other at the origin, or that one of M, N be the origin itself, in which case the other can be taken arbitrarily.

No continuity assumptions upon the functions entering into (3) are needed for the validity of this result. Actually it remains true if the curves M, N are replaced by arbitrary point sets. It is, however, necessary to insist that the variables x, y in (3) be independent, as may be seen from the example

$$M: x_1 = \cos x, \quad x_2 = \sin x; \quad N: y_1 = \cos y, \quad y_2 = \sin y, \quad y = x + \frac{1}{2}\pi,$$

which shows that when x, y are not independent variables (3) may hold without M, N lying on a pair of straight lines perpendicular to each other at the origin or either of M, N degenerating into the origin.

The differential equations (2) are immediate consequences of this result for we obtain

$$a_2 X'' - a_1 X = 0, \quad a_2 Y'' + a_1 Y = 0,$$

when we set

$$x_1 = X'', \quad x_2 = X, \quad y_1 = Y, \quad y_2 = Y''$$

in the equations of S, T .

As a generalization to (3) we consider the identity

$$(5) \quad \sum_{\alpha=1}^n x_{\alpha}(u_1, \dots, u_r) y_{\alpha}(v_1, \dots, v_s) \equiv 0,$$

with which we associate the manifolds

$$M: x_{\alpha} = x_{\alpha}(u_1, \dots, u_r), \quad N: y_{\alpha} = y_{\alpha}(v_1, \dots, v_s),$$

the u 's and v 's being independent variables. Such an identity is termed *separable*, and with respect to it, we frame the following questions:

I. *What can be said about the manifolds M , N , if the functions $x_\alpha(u_1, \dots, u_r)$, $y_\alpha(v_1, \dots, v_s)$ are required to satisfy a separable identity?*

II. *Can the answer to I be used to obtain particular solutions of partial differential equations², as was the case in the simple example above?*

By way of introducing the third question taken up in this paper let us consider the partial differential equation

$$q(q\psi_q)_q + \psi_{\theta\theta} = 0$$

satisfied by the stream function ψ in plane, irrotational flow of an incompressible fluid when the *speed* q and *direction* θ of the flow are taken for independent variables. For the jet arising when the fluid at rest at infinity is discharged through an aperture from an infinite reservoir bounded by a straight wall the complex potential is³

$$\phi + i\psi = -\frac{1}{2} \log \frac{1}{2}(\bar{h}^{-1} - \bar{h}), \quad h = qe^{i\theta}.$$

On equating imaginary parts, we obtain

$$\psi = -\arctan \left(\tan \theta \cdot \frac{1 + q^2}{1 - q^2} \right),$$

which reduces to the solution

$$(6) \quad \psi = \arctan (\tan \theta \cdot \coth \sigma),$$

of Laplace's equation $\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0$ when we introduce the variable $\sigma = \log q$. This solution has the form $\psi = \psi(\Theta(\theta) \cdot \Sigma(\sigma))$ and suggests the question

III. *Can the usual method of separation of variables, in which solutions of the form $u = X(x)Y(y)$ are sought for, be extended to the search for solutions with the form $u = u(X(x)Y(y))$?*

Investigation of this question will be confined to Laplace's equation $u_{xx} + u_{yy} = 0$.

² For applications of the answer to I see

TIERNEY, J. A., *An Approximation to Transonic Flow of a Polytopic Gas*, Thesis. University of Maryland (1951), *Am. J. Math.*, **75** (1953) pp. 43-56.

MARTIN, M. H. & THICKSTUN, W. R., *An Example of Transonic Flow for the Tricomi gas*, Proceedings IVth Symposium of Applied Mathematics, Am. Math. Soc., to be published by McGraw-Hill.

MARTIN, M. H., *On the Fundamental Solution of $\Delta\psi + e(y)\psi_y = 0$* , *Duke Mathematical Journal*, **18** (1951) pp. 845-858.

—, *The Monge-Ampère Partial Differential Equation $rt - s^2 + \lambda^2 = 0$* , *Pac. J. Math.*, in press.

³ CHAPLYGIN, S. *Gas Jets*, Scientific Memoirs, Moscow University (1902); English translation NACA Technical Memorandum 1063 p. 51.

1. A fundamental lemma. A pair of points x, y will be termed *orthogonal* if their coordinates x_α, y_α are bound together by the bilinear relation.

$$(7) \quad \sum_{\alpha=1}^n x_\alpha y_\alpha = 0.$$

Two point sets M, N are *orthogonal* if each pair of points $x \in M, y \in N$ is orthogonal. For example, any two subsets $M \subset S, N \subset T$ of the two perpendicular straight lines S, T in the previous section are orthogonal.

The answer to question I is contained in the following lemma⁴, inasmuch as the manifolds M, N will be orthogonal point sets, provided the u 's and v 's are independent variables.

Lemma. *The necessary and sufficient conditions for two point sets M, N to be orthogonal is that $M \subset S_p, N \subset T_q$ where S_p, T_q are orthogonal linear subspaces of respective dimensions p, q , with $p + q = n$.*

The linear subspaces S_p, T_q defined by

$$(8) \quad S_p: \sum_{\alpha=1}^n b_{k\alpha} x_\alpha = 0, \quad k = 1, \dots, q; \quad T_q: \sum_{\alpha=1}^n a_{i\alpha} y_\alpha = 0, \quad i = 1, \dots, p$$

$$(p + q = n),$$

where the matrix $\mathbf{A} = \|a_{i\alpha}\|$ has rank p and the matrix $\mathbf{B} = \|b_{k\alpha}\|$ has rank q , will be orthogonal if, and only if,

$$(9) \quad A_{i_1 \dots i_p} / B_{k_1 \dots k_q} = \text{const.},$$

where $A_{i_1 \dots i_p}$ is the minor comprising the columns $i_1, \dots, i_p$ of $\mathbf{A}$ and $B_{k_1 \dots k_q}$ is the minor formed by the columns $k_1, \dots, k_q$ of $\mathbf{B}$, with $i_1 \dots i_p, k_1 \dots k_q$ forming an even permutation of $1, \dots, n$.

Before proving this lemma we examine its significance for $n = 2, 3, 4$, as these cases lend themselves to simple geometric interpretations.

$n = 2$. There are three cases:

(i) $p = 2, q = 0$. S_2 is the entire 2-space and T_0 is the origin.

(ii) $p = 1, q = 1$. Here $S_1: b_{11}x_1 + b_{12}x_2 = 0, T_1: a_{11}y_1 + a_{12}y_2 = 0$ are two straight lines through the origin, perpendicular to each other in view of the relation $a_{11}/b_{12} = -a_{12}/b_{11}$, to which (9) reduces.

(iii) $p = 0, q = 2$. S_0 is the origin and T_2 is the entire 2-space.

For $n = 2$ the lemma only confirms the results in the previous section on the curves M, N .

$n = 3$. There are four cases:

(i) $p = 3, q = 0$. S_3 is the entire 3-space and T_0 is the origin.

⁴ The case $n = 5$ in the lemma has been used by BIRKHOFF, G., "Hydrodynamics" Princeton University Press (1950) pp. 137-139 in dealing with viscous flows. The case $n = 4$ was used by the author in the study of rotational flows. See *Steady, Rotational, Plane Flow of a Gas*, Am. J. Math., **72** (1950) pp. 478-479.

(ii) $p = 2, q = 1$, S_2 is the plane $\sum_{\alpha=1}^3 b_{1\alpha} x_\alpha = 0$ and T_1 is the straight line $\sum_{\alpha=1}^3 a_{i\alpha} y_\alpha = 0, i = 1, 2$ which is the normal to S_2 at the origin, since (9) becomes $A_{23}/b_{11} = A_{31}/b_{12} = A_{12}/b_{13}$.

In the remaining cases only the roles of S, T are interchanged. Thus one of the orthogonal sets M, N must lie on a plane with the other on the normal to this plane at the origin, or else consist of a single point, the origin, in which case the other set can be taken arbitrarily.

If x_1, x_2, x_3 , and $y_1, y_2, -y_3$ be regarded as homogeneous coordinates in a plane of two points with Cartesian coordinates $\xi_1 = x_1/x_3, \xi_2 = x_2/x_3$ and $\eta_1 = -y_1/y_3, \eta_2 = -y_2/y_3$ the bilinear relation (7) is replaced by $\xi_1 \eta_1 + \xi_2 \eta_2 = 1$. Case (i) does not appear and (ii) amounts to requiring that (η_1, η_2) be a fixed point with (ξ_1, ξ_2) any point on the polar line of (η_1, η_2) with respect to the unit circle with center at the origin.

$n = 4$. There are five cases:

(i) $p = 4, q = 0$. S_4 is the entire 4-space and T_0 is the origin.

(ii) $p = 3, q = 1$. S_3 is the hyperplane $\sum_{\alpha=1}^4 b_{1\alpha} x_\alpha = 0$ and T_1 the straight line $\sum_{\alpha=1}^4 a_{i\alpha} y_\alpha = 0, i = 1, 2, 3$, is the normal to S_3 at the origin, in view of the relations $A_{234}/b_{11} = -A_{134}/b_{12} = A_{124}/b_{13} = -A_{123}/b_{14}$.

(iii) $p = 2, q = 2$. Here $S_2: \sum_{\alpha=1}^4 b_{k\alpha} x_\alpha = 0, k = 1, 2; T_2: \sum_{\alpha=1}^4 a_{i\alpha} y_\alpha = 0, i = 1, 2$ are orthogonal 2-spaces with conditions (9) becoming

$$(9') \quad \frac{A_{12}}{B_{34}} = \frac{A_{13}}{B_{42}} = \frac{A_{14}}{B_{23}} = \frac{A_{23}}{B_{14}} = \frac{A_{42}}{B_{13}} = \frac{A_{34}}{B_{12}}.$$

The remaining cases interchange the role of S, T in (i) and (ii).

If we regard x_1, x_2, x_3, x_4 and $y_1, y_2, y_3, -y_4$ as homogeneous coordinates in 3-space of two points with Cartesian coordinates $\xi_\alpha = x_\alpha/x_4$ and $\eta_\alpha = -y_\alpha/y_4, \alpha = 1, 2, 3$ relation (7) becomes $\xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3 = 1$. Case (i) does not appear; case (ii) requires (η_1, η_2, η_3) to be a fixed point with (ξ_1, ξ_2, ξ_3) any point on the polar plane of (η_1, η_2, η_3) with respect to the unit sphere; case (iii) requires $(\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)$ to be points on a pair of straight lines, reciprocal polars to the unit sphere⁵.

We now take up the proof of the lemma. Let p be any one of $0, 1, \dots, n$ and let the indices $i, j; k, l; \alpha$ range as follows:

$$i, j = 1, 2, \dots, p; \quad k, l = 1, 2, \dots, q = n - p, \quad \alpha = 1, \dots, n.$$

Suppose M contains exactly p linearly independent points $a_{i\alpha}$ (distinct from the origin), i.e., the rank of $\mathbf{A} = \|a_{i\alpha}\|$ is p . Clearly $M \subset S_p$, where

$$S_p: x_\alpha = \sum_{i=1}^p \lambda_i a_{i\alpha}, \quad \lambda_i \text{ real},$$

is a linear subspace of dimension p . Furthermore $N \subset T_q$, where

$$T_q: \sum_{\alpha=1}^n a_{i\alpha} y_\alpha = 0,$$

⁵ Cf. MARTIN, M. H., Loc. cit. 4.

is a linear subspace of dimension $q = n - p$. The linear subspaces S_p, T_q are orthogonal for if $x \in S_p, y \in T_q$, we find

$$\sum_{\alpha=1}^n x_{\alpha} y_{\alpha} = \sum_{i=1}^p \sum_{\alpha=1}^n \lambda_i a_{i\alpha} y_{\alpha} = 0.$$

The condition $M \subset S_p, N \subset T_q$ in the first part of the lemma is therefore necessary, and, since it is obviously sufficient, we turn to the proof of the second part.

First we show that the conditions

$$(10) \quad \sum_{\alpha=1}^n a_{i\alpha} b_{k\alpha} = 0,$$

on the matrices **A**, **B** are necessary and sufficient for the linear subspaces S_p, T_q to be orthogonal.

From (8) it follows that $b_{k\alpha}$ are q linearly independent points b_k in T_q and $a_{i\alpha}$ are p linearly independent points a_i in S_p , so that we have the following parametric representations

$$S_p: x_{\alpha} = \sum_{i=1}^p a_{i\alpha} \lambda_i, \quad \lambda_i \text{ real}; \quad T_q: y_{\alpha} = \sum_{k=1}^q b_{k\alpha} \mu_k, \quad \mu_k \text{ real}$$

with

$$\sum_{\alpha=1}^n x_{\alpha} y_{\alpha} = \sum_{i=1}^p \sum_{k=1}^q \sum_{\alpha=1}^n a_{i\alpha} b_{k\alpha} \lambda_i \mu_k.$$

For S_p, T_q to be orthogonal, it is necessary and sufficient that this bilinear form in the independent variables λ_i, μ_k be identically zero, a condition which is equivalent to (10).

Next we prove that numerator and denominator in any ratio (9) must vanish simultaneously, or else both differ from zero. Suppose $A_{1\dots p} \neq 0, B_{p+1\dots n} = 0$. Multipliers μ_k , not all zero, can be found such that

$$(11) \quad \sum_{k=1}^q \mu_k b_{kp+l} = 0,$$

and if we write out conditions (10) as

$$\sum_{j=1}^p a_{ij} b_{kj} + \sum_{l=1}^q a_{ip+l} b_{kp+l} = 0,$$

multiply through by μ_k and sum on k , we obtain

$$\sum_{j=1}^p a_{ij} \left(\sum_{k=1}^q \mu_k b_{kj} \right) = 0.$$

Since $A_{1\dots p} \neq 0$, this implies $\sum_{k=1}^q \mu_k b_{kj} = 0$, from which, and (11) we infer that the rank of $\mathbf{B}$ is less than q , contrary to hypothesis. Thus $A_{1\dots p} \neq 0$, $B_{p+1\dots n} = 0$ is impossible, and a similar argument rules out $A_{1\dots p} = 0$, $B_{p+1\dots n} \neq 0$. Our assertion is demonstrated for the ratio $A_{1\dots p}/B_{p+1\dots n}$ and, since the same argument can be repeated for each ratio in (10), it is completely established.

Since $\mathbf{A}$ has rank p , there is no loss in generality if we assume $A_{1\dots p} \neq 0$, and consequently $B_{p+1\dots n} \neq 0$. By solving the equations of T_q for y_i and the equations of S_p for x_{p+k} we may put (8) in canonical form

$$(12) \quad S_p: \sum_{i=1}^p \tilde{b}_{ki} x_i + x_{p+k} = 0; \quad T_q: y_i + \sum_{k=1}^q \tilde{a}_{ik} y_{p+k} = 0.$$

In matrix notation, if we let $\mathbf{E}_p$ denote the $p \times p$ unit matrix $\mathbf{E}_q$ the $q \times q$ unit matrix and write $\tilde{\mathbf{A}} = \|\tilde{a}_{ik}\|$, $\tilde{\mathbf{B}} = \|\tilde{b}_{ki}\|$, there is a non-singular $p \times p$ matrix $\mathbf{R}$ and a $q \times q$ non-singular matrix $\mathbf{S}$ such that

$$(13) \quad \mathbf{R}\mathbf{A} = \|\mathbf{E}_p \tilde{\mathbf{A}}\| = \mathbf{A}^*, \quad \mathbf{S}\mathbf{B} = \|\tilde{\mathbf{B}}\mathbf{E}_q\| = \mathbf{B}^*.$$

From (12) we find for any $x \in S_p, y \in T_q$

$$\sum_{\alpha=1}^n x_{\alpha} y_{\alpha} = - \sum_{i=1}^p \sum_{k=1}^q (\tilde{a}_{ik} + \tilde{b}_{ki}) x_i y_{p+k}.$$

For S_p, T_q to be orthogonal it is necessary and sufficient that this be identically zero in the independent variables x_i, y_{p+k} . Thus

$$(14) \quad \tilde{a}_{ik} + \tilde{b}_{ki} = 0 \quad \text{or} \quad \tilde{\mathbf{A}} + \tilde{\mathbf{B}}' = \mathbf{O},$$

are necessary and sufficient conditions for S_p, T_q to be orthogonal.

Conditions (14) lead to equivalent conditions

$$A_{i_1 \dots i_p}^* = B_{k_1 \dots k_q}^*,$$

($i_1 \dots i_p, k_1 \dots k_q$ an even permutation of $1, \dots, n$) on the matrices $\mathbf{A}^*, \mathbf{B}^*$.

Conditions (15) imply (14), for

$$A_{1 \dots (i-1)(p+k)(i+1) \dots p}^* = \tilde{a}_{ik}, \quad B_{p+1 \dots (p+k-1)(p+k+1) \dots n}^* = \tilde{b}_{ki},$$

inasmuch as the subscripts of A^* followed by the subscripts of B^* form an odd permutation.

Conversely conditions (14) imply (15). This is certainly true if at most one of $i_1 \dots i_p$ exceeds p . If exactly two of $i_1 \dots i_p$ exceed p , we find

$$A_{1 \dots (i-1)(p+k)(i+1) \dots (j-1)(p+l)(j+1) \dots p}^* = \begin{vmatrix} \tilde{a}_{ik} & \tilde{a}_{il} \\ \tilde{a}_{jk} & \tilde{a}_{jl} \end{vmatrix}, \quad i < j,$$

$$B_{(p+1) \dots (p+k-1)(p+k+1) \dots (p+l-1)(p+l+1) \dots n}^* = \begin{vmatrix} \tilde{b}_{ki} & \tilde{b}_{kj} \\ \tilde{b}_{li} & \tilde{b}_{lj} \end{vmatrix}, \quad k < l,$$

so that (14) implies equality between the first members. Since the subscripts of A^* followed by the subscripts of B^* form an even permutation, (15) is true if at most two of $i_1, \dots, i_p$ exceed p . Continuing in this way step-by-step we establish that (14) implies (15).

From (13) we obtain

$$A_{i_1 \dots i_p}^* = \det \mathbf{R} \cdot A_{i_1 \dots i_p}, \quad B_{k_1 \dots k_q}^* = \det \mathbf{S} \cdot B_{k_1 \dots k_q}$$

and consequently, provided $i_1 \dots i_p k_1 \dots k_q$ is an even permutation,

$$A_{i_1 \dots i_p} / B_{k_1 \dots k_q} = \det \mathbf{S} / \det \mathbf{R}.$$

to complete the demonstration.

2. Laplace's equation $u_{xx} + u_{yy} = 0$. Solutions of the form $u = u(X(x)Y(y))$. As a partial answer to questions II and III in the introduction we shall employ the lemma of the preceding section to obtain solutions $u = u(X(x)Y(y))$ for Laplace's equation.

It is easy to see that any function u of this form is defined implicitly by a relation of the type

$$(16) \quad U(u) = X(x) + Y(y).$$

The problem is to determine the three unknown functions $U(u)$, $X(x)$, $Y(y)$ so that the function $u(x, y)$ defined implicitly by this relation is a solution of Laplace's equation.

A simple calculation shows that for this to be the case the three functions must satisfy the condition.

$$(17) \quad \frac{U''}{U'^2} = \frac{X''}{X'^2} + \frac{Y''}{Y'^2},$$

where differentiation of a function of a single variable is indicated by a prime. The first member of this relation, as a function of u , is, by (16), a function of $X + Y$. Consequently $(X'' + Y'')/(X'^2 + Y'^2)$, $X + Y$ are functionally related and their Jacobian must vanish identically in x, y . This yields the separable identity

$$(18) \quad x_1(x)y_1(y) + x_2(x)y_2(y) + x_3(x)y_3(y) + x_4(x)y_4(y) \equiv 0,$$

where

$$(19) \quad x_1(x) = X'^2, x_2(x) = X'''/X', x_3(x) = X'X''' - 2X''^2, x_4(x) = 1,$$

$$(20) \quad y_1(y) = -Y'''/Y', y_2(y) = Y'^2, y_3(y) = 1, y_4(y) = -(Y'Y''' - 2Y''^2).$$

Conversely, given functions $X(x)$, $Y(y)$ for which (18), (19), (20) hold, the second member of (17) becomes a function $F(X + Y) = F(U)$ and $U(u)$ is obtained by integrating

$$(21) \quad U''/U'^2 = F(U) \quad \text{or} \quad \ddot{u}/\dot{u} = -F(U),$$

where $u = u(U)$ is the inverse function to $U = U(u)$ and the dot denotes differentiation to U . Once $X(x)$, $Y(y)$, $U(u)$ are determined in this manner, (16) defines a solution $u(x, y)$ of Laplace's equation implicitly, as may be verified by calculating the partial derivatives u_{xx} , u_{yy} by the usual rules for differentiating implicit functions.

To determine $X(x)$, $Y(y)$ we use the lemma of the preceding section for $n = 4$, the sets M , N being the curves (19), (20) respectively, with special reference to the discussion for $n = 4$ immediately following the lemma.

Cases (i), (v) require either (19) or (20) to reduce to the origin, and therefore do not arise.

Case (ii). The equations of the normal T_1 yield three simultaneous equations for $Y(y)$

$$\frac{-Y''''Y'}{b_{11}} = \frac{Y'^2}{b_{12}} = \frac{1}{b_{13}} = \frac{-Y'Y''' + 2Y''^2}{b_{14}}.$$

The second implies $Y' = c = \text{const.}$, and (17) yields

$$\frac{U''}{U'^2} = \frac{X''}{X'^2 + c^2} = \text{const.}$$

Thus, either $U'' = 0$, whereupon U is a linear function of u and the only solutions to be obtained from (16) have the form $u = X(x) + Y(y)$, or else $U''/U'^2 = -1/k$, so that $U = U_0 + k \log(u - u_0)$ and we are restricted to solutions of the type $u = X(x)Y(y) + \text{const.}$, obtainable by the ordinary method of separation of variables.

Since case (iv) is not unlike case (ii) we pass on to the remaining case.

Case (iii). If we set

$$(22) \quad \xi = X', \quad \eta = Y',$$

in (19), (20), the equations for S_2 , T_2 become simultaneous differential equations.

$$(23) \quad (b_{12} + b_{13}\xi^2)\xi'' - 2b_{13}\xi\xi'^2 + b_{14}\xi + b_{11}\xi^3 = 0,$$

$$(b_{22} + b_{23}\xi^2)\xi'' - 2b_{23}\xi\xi'^2 + b_{24}\xi + b_{21}\xi^3 = 0,$$

$$(23') \quad (a_{11} + a_{14}\eta^2)\eta'' - 2a_{14}\eta\eta'^2 - a_{13}\eta - a_{12}\eta^3 = 0,$$

$$(a_{21} + a_{24}\eta^2)\eta'' - 2a_{24}\eta\eta'^2 - a_{23}\eta - a_{22}\eta^3 = 0,$$

for ξ , η .

If $B_{23} = 0$, these equations imply that both ξ , η are constant. Indeed when $B_{23} = 0$ multipliers λ_1 , λ_2 not both zero can be found such that

$$\lambda_1 b_{12} + \lambda_2 b_{22} = 0, \quad \lambda_1 b_{13} + \lambda_2 b_{23} = 0,$$

and (23) yields

$$(\lambda_1 b_{14} + \lambda_2 b_{24})\xi + (\lambda_1 b_{11} + \lambda_2 b_{21})\xi^3 = 0.$$

Since $\mathbf{B}$ has rank 2, the coefficients of ξ, ξ^3 cannot both vanish. Consequently $\xi = \text{const.}$ To prove $\eta = \text{const.}$, one recalls that $B_{23} = 0$ implies $A_{14} = 0$ and repeats the argument with (23'). If ξ, η are both constant, X, Y , and therefore, from (17), also U , are linear functions. The solutions defined by (16) are trivial linear functions of x, y .

For $B_{23} \neq 0$, we solve (23), (23') simultaneously for the derivatives to obtain

$$(24) \quad \begin{aligned} 2B_{23} \xi'^2 &= B_{24} + (B_{34} - B_{12})\xi^2 - B_{13} \xi^4, \\ B_{23} \xi'' &= B_{24} \xi - B_{13} \xi^3, \end{aligned}$$

$$(24') \quad \begin{aligned} 2B_{23} \eta'^2 &= B_{24} + (B_{12} - B_{34})\eta^2 - B_{13} \eta^4, \\ B_{23} \eta'' &= B_{12} \eta - B_{13} \eta^3, \end{aligned}$$

provided (9') is used to replace $A_{i_1 i_2}$ by the corresponding $B_{k_1 k_2}$. For a solution of the first equation in either system to be a solution of the second is necessary and sufficient that $B_{12} + B_{34} = 0$. In addition the constants $B_{k_1 k_2}$ must satisfy the Plücker condition $B_{12} B_{34} + B_{13} B_{42} + B_{23} B_{14} = 0$.

These conditions will be met without loss of generality, provided we set

$$\begin{aligned} B_{12} &= -b, & B_{13} &= -c, & B_{14} &= 2(b^2 - ac), \\ & & B_{23} &= \tfrac{1}{2}, & B_{24} &= a, & B_{34} &= b, \end{aligned}$$

where a, b, c denote arbitrary real constants. The systems (24), (24') are replaced by single differential equations

$$(25) \quad \xi'^2 = a + 2b\xi^2 + c\xi^4 = P(\xi^2),$$

$$(25') \quad \eta'^2 = a - 2b\eta^2 + c\eta^4 = Q(\eta^2),$$

so that x, y are given by the elliptic integrals

$$(26) \quad x = \int \frac{d\xi}{\sqrt{P(\xi^2)}}, \quad y = \int \frac{d\eta}{\sqrt{Q(\eta^2)}},$$

and ξ, η are elliptic functions

$$(27) \quad \xi = \xi(x), \quad \eta = \eta(y),$$

of their arguments. The functions $X(x), Y(y)$ in (16) are, from (22), accordingly given by

$$(28) \quad X = \int \xi dx, \quad Y = \int \eta dy,$$

i.e., are integrals of elliptic functions.

To obtain the function $U(u)$, or what is more convenient, the inverse function $u(U)$, we introduce the variables

$$(29) \quad \alpha = \xi^2, \quad \beta = \eta^2,$$

and take X, Y, U as independent variables⁶ in (17) and (25), (25'). One finds (17) replaced by

$$(30) \quad 2 \frac{\ddot{u}}{\dot{u}} + \frac{\dot{\alpha} + \dot{\beta}}{\alpha + \beta} = 0,$$

where the dot denotes differentiation to the new independent variables, and instead of (25), (25'), we obtain

$$\dot{\alpha}^2 = 4P(\alpha), \quad \dot{\beta}^2 = 4Q(\beta).$$

These yield elementary integrals

$$(31) \quad X = \frac{1}{2} \int \frac{d\alpha}{\sqrt{P(\alpha)}}, \quad Y = \frac{1}{2} \int \frac{d\beta}{\sqrt{Q(\beta)}},$$

from which we obtain the six triples of functions in the table below. In this table $\Delta = b^2 - ac$ is the discriminant of the quadratics $P(\alpha)$, $Q(\beta)$ and the integration constants are omitted.

Case	X	Y	$u(U)$
I. $c > 0, \Delta = 0$	$\frac{1}{2\sqrt{c}} \log(\alpha + b/c)$	$\frac{1}{2\sqrt{c}} \log(\beta - b/c)$	$e^{-\sqrt{c}U}$
II. $c = 0, \Delta = 0$	$\alpha/2\sqrt{a}$	$\beta/2\sqrt{a}$	$\log U$
III. $c = 0, \Delta > 0$	$\sqrt{a+2b\alpha}/2b$	$-\sqrt{a-2b\beta}/2b$	$\log U$
IV. $c < 0, \Delta > 0$	$\frac{1}{2\sqrt{-c}} \arcsin \frac{-b - c\alpha}{\sqrt{\Delta}}$	$\frac{1}{2\sqrt{-c}} \arcsin \frac{b - c\beta}{\sqrt{\Delta}}$	$\log \tan \frac{\sqrt{-c}}{2} U$
V. $c > 0, \Delta > 0$	$\frac{1}{2\sqrt{c}} \operatorname{arcosh} \frac{b + c\alpha}{\sqrt{\Delta}}$	$\frac{1}{2\sqrt{c}} \operatorname{arcosh} \frac{-b + c\beta}{\sqrt{\Delta}}$	$\operatorname{arc tan} e^{\sqrt{c}U}$
VI. $c > 0, \Delta < 0$	$\frac{1}{2\sqrt{c}} \operatorname{arsinh} \frac{b + c\alpha}{\sqrt{-\Delta}}$	$\frac{1}{2\sqrt{c}} \operatorname{arsinh} \frac{-b + c\beta}{\sqrt{-\Delta}}$	$\operatorname{arc tanh} e^{\sqrt{c}U}$

The case $c < 0, \Delta < 0$ is excluded for real ξ, η ; the case $c = 0, \Delta < 0$ cannot arise for real b ; the case $c < 0, \Delta = 0$ leads to the trivial result $\xi = \text{const.}$, $\eta = \text{const.}$ To obtain the functions $u(U)$, one computes $\alpha = \alpha(X)$, $\beta = \beta(Y)$ from the columns headed X, Y and substitutes these functions in (30). This

⁶ The introduction of X, Y as independent variables was suggested to me by E. C. WATTERS.

yields the function $F(U)$ in (21) (if we keep (16) in mind), which is readily integrated.

We sum up our results in the following theorem.

Theorem. *The only solutions of Laplace's equation $u_{xx} + u_{yy} = 0$ having the form*

$$u = u(U), \quad U = X(x) + Y(y),$$

in which $u(U)$, $X(x)$, $Y(y)$ are real functions of their arguments, are obtained when $X(x)$, $Y(y)$ are integrals of elliptic functions (or their degenerate cases) and $u(U)$ is one of the elementary functions in the table above.

Since $u = e^{-\sqrt{c}U} = e^{-\sqrt{c}X} \cdot e^{-\sqrt{c}Y}$, the solutions in I fall under those obtained by the usual method of separation of variables.

Among the solutions II we find the fundamental solution $\log \sqrt{x^2 + y^2}$.

By proper choice ($a = 1$, $b = -\frac{1}{2}$) of the constants entering into solutions III, we arrive at the solution $u = \log (\sin x + \cosh y)$, the level lines $u = \text{const.}$ of which yield the stream lines in a flow generated⁷ by a vortex street with vortices equally spaced along the x -axis.

If we take $a = 0$, $b = -2$, $c = 1$ in V, the elliptic functions $\xi(x)$, $\eta(y)$ degenerate into the trigonometric and hyperbolic functions

$$\xi = 2 \csc 2x, \quad \eta = -2 \operatorname{csch} 2y,$$

with the integrals

$$X = \log \tan x, \quad Y = \log \coth y,$$

from which

$$u = \arctan (\tan x \coth y).$$

Thus, apart from notation, we recover the stream function (6) for the jet.

By considering only special cases in which the elliptic functions degenerate we have been led to several of the classic solutions of Laplace's equations and it would be interesting to investigate the remaining cases in the table. It would be particularly worth while to determine the level curves $u = \text{const.}$, as this would give an insight into the nature of the boundary value problems solvable by these solutions of Laplace's equation. It would also be desirable to catalogue the analytic functions $f(z) = u + iv$ for which u falls among these solutions.

3. Further extensions. If the reality conditions be dropped it appears that further solutions may be obtained. Let us take, for example, $a = c = 0$, $b = -\frac{1}{2}$ in (25), (25'). One has

$$\xi = e^{ix}, \quad \eta = e^y; \quad X = -ie^{ix}, \quad Y = e^y$$

so that $u = \log (-ie^{ix} + e^y)$. This is a complex solution of the desired form and, when it is split into its real and imaginary parts, we obtain two real solutions

⁷ LAMB, H., "Hydrodynamics," Cambridge (1924) 5th Ed. p. 207.

$$u = \log (1 - 2e^y \sin x + e^{2y}), \quad u = \arctan \frac{\cos x}{e^y - \sin x},$$

neither of which has the form $u = u(X + Y)$.

Whether or not the dropping of reality conditions would lead to solutions of Laplace's equations of any value in the solution of boundary value problems is an open question.

Another natural extension would be to employ the lemma in Section 2 to the determination of those solutions of Laplace's equation in three dimensions $u_{xx} + u_{yy} + u_{zz} = 0$ defined implicitly by a relation of the form

$$U(u) = X(x) + Y(y) + Z(z),$$

with a view to obtaining solutions in closed form of interest in the solution of boundary value problems.

Finally the lemma should be applicable in the search for solutions $u = X(x)Y(y)$ of any linear partial differential equation of second order in two independent variables x, y , provided the coefficients are sums of products of functions of x by functions of y , inasmuch as this substitution leads to the separable identity dealt with in the lemma.