Errors of Numerical Approximation for Analytic Functions

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1. Introduction. It has become traditional in works on numerical methods to express errors committed in the use of approximate formulas of integration, differentiation, interpolation, etc., in terms of the higher derivatives of the function operated upon. The error consists of two parts: a multiplicative constant which depends solely upon the numerical rule employed, and a higher derivative $f^{(n)}(\xi)$ evaluated at some intermediate point of the interval I . In most cases, the exact value of ξ will be unknown, and the estimate $\max_{\xi \in I} |f^{(n)}(\xi)|$ is employed instead. Such expressions of error are valid for the class of real functions which are sufficiently differentiable and therefore possess wide applicability. On the other hand, they have several drawbacks. In the first place, since the error term of different rules may contain different orders of derivatives, the respective accuracy of the rules may not be compared on a common basis. Secondly, data on the higher derivatives may be unavailable or may be difficult to obtain. Such may be the case when operating with functions given in closed form but which are highly composite.

In the present note, we deal with functions which are analytic on an interval of the real axis, and which, therefore, may be considered as analytic functions of a complex variable in an appropriate region of the $z = x + iy$ plane. We shall derive our estimates of error by introducing a Hilbert space of analytic functions and using essentially the Riesz representation of bounded linear functionals. The error $E(f)$ committed in the use of formulas of numerical approximation applied to f may be estimated in the form $|E(f)| \leq \sigma_E \|f\|$. This estimate is a Schwarz inequality. The quantity σ_E is the norm of the error functional; it depends solely upon the approximation rule employed, but is independent of the particular function operated upon. The quantity $\|f\|$ is the

norm of f in the Hilbert space. The norm of f may be estimated from a knowledge of the values of the function over the interval in question or, indeed, from a knowledge only of the values which the function assumes on a set of points which has an accumulation point interior to the interval. That such estimates may in principal be obtained follows from classical function theory which tells us that this data serves to fix the function completely and hence its norm. In the cases considered below, the norm may also be estimated from the maximum modulus of the function along an appropriate contour or higher dimensional manifold in the case of functions of several variables. This method is particularly convenient when dealing with functions given in closed form.

In §2 and §3, the essentials of the method are explained and some numerical values of σ_E are derived for cases of interest. In §4, methods for estimating $\|f\|$ are discussed in some detail, while §5 considers the case of several variables.

Finally, it should be pointed out that the present work is intimately related to the theory of interpolation series for functions of one and several variables in arbitrary regions.

2. Essentials of the Method. In order that the details of the method not be encumbered by too great generality, we shall deal initially with analytic functions which are regular and single-valued in $|z| < 1$ and which belong to the class H^2 there.¹ That is to say, if $f(z)$ possesses the Taylor expansion

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1,$$

then it will belong to H^2 if and only if

$$(2) \quad \|f\|^2 = 2\pi \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

If $f(z)$ is analytic in $|z| < 1$ and continuous on $|z| = 1$, it will surely belong to this class. Each function of H^2 possesses a representation of the form

$$(3) \quad f(z) = \frac{1}{2\pi} \int_{|w|=1} \frac{f(w)}{1 - z\bar{w}} ds_w.$$

For general $f \in H^2$, this representation requires special interpretation as follows: as $r \rightarrow 1^-$, $f(re^{i\theta})$ possesses radial limits for almost all values of θ . Representation (3) holds with these limiting values taken for $f(e^{i\theta})$. The kernel $(1 - z\bar{w})^{-1}$ will be recognized as the Szegő kernel function for the unit circle. If we operate on $f(z)$ within a closed region interior to $|z| < 1$ by the usual linear numerical processes of approximation, these processes will be bounded linear functionals over H^2 . We shall designate such linear functionals generically by $E(f)$, and think of E as yielding an error in a fixed numerical process. Thus, for example,

¹ See, e. g., WALSH [12], p. 141.

if it is desired to discuss integration over $0 \leq x \leq \frac{1}{2}$ and its approximation by the trapezoidal rule, it would be appropriate to introduce the linear functional

$$(4) \quad E_T(f) = \int_0^{\frac{1}{2}} f(x) dx - 4^{-1}(f(0) + f(\tfrac{1}{2})).$$

For an arbitrary bounded linear functional E , we have from (3),

$$(5) \quad E(f) = \frac{1}{2\pi} \int_{|w|=1} E\left(\frac{1}{1-z\bar{w}}\right) f(w) ds_w$$

where the E behind the integral sign is to be interpreted as acting on the variable z . Applying the Schwarz inequality to (5), we obtain

$$(6) \quad |E(f)|^2 \leq (4\pi^2)^{-1} \int_{|w|=1} \left| E\left(\frac{1}{1-z\bar{w}}\right) \right|^2 ds_w \int_{|w|=1} |f(w)|^2 ds_w$$

so that

$$(7) \quad |E(f)| \leq \sigma_E \|f\|,$$

where we have set

$$(8) \quad \sigma_E^2 = (4\pi^2)^{-1} \int_{|w|=1} \left| E\left(\frac{1}{1-z\bar{w}}\right) \right|^2 ds_w.$$

The quantity σ_E is generally termed the norm of the functional E , and is independent of the particular function operated upon.² It may be written in an alternate form which is more convenient for computation in specific instances. We have

$$(9) \quad \sigma_E^2 = (4\pi^2)^{-1} \int_{|w|=1} \left| E \sum_{n=0}^{\infty} z^n \bar{w}^n \right|^2 ds_w = (2\pi)^{-1} \sum_{n=0}^{\infty} |E(z^n)|^2.$$

It should be stressed that the constant σ_E which appears in the estimate (7) is a best possible one. That is, there will surely be an $f^* \in H^2$ for which the inequality becomes an equality. Interpreted in the language of errors, this means that there will be a function f^* of class H^2 for which the error $E(f^*)$ is given exactly by $E(f^*) = \sigma_E \|f^*\|$. To show this, we need merely observe that the function

$$(10) \quad f^*(w) = \overline{E((1-z\bar{w})^{-1})} = \sum_{n=0}^{\infty} \overline{E(z^n)} w^n$$

is of class H^2 and that by (5),

² Cf. A. SARD [9] and MEYERS & SARD [6], where such norms have been studied for spaces of n times differentiable functions.

$$(11) \quad E(f^*) = \sum_{n=0}^{\infty} E(w^n) \overline{E(w^n)} = \sigma_E \|f^*\|.$$

The maximizing function f^* will be realized within the subclass of functions which are real on the real axis providing the quantities $E(z^n)$ ($n = 0, 1, \dots$) are real. This will, of course, be the case in all examples of interest. One implication of the fact that σ_E is best possible is that the usual method of estimating errors by obtaining $\max_{\xi \in I} |f^{(j)}(\xi)|$ and the method which is used in (7), are, in general, independent. That is, for most rules of interest, there will be a class of functions for which the former method gives a more optimistic estimate of error, and there will be a class of functions for which the latter method gives the more optimistic estimate.

It should be stressed that for a fixed selection of the abscissas λ_n ($n = 0, 1, \dots, N$) in a rule, the usual $N + 1$ point rules need not yield a minimum value for the corresponding σ . The minimizing weights corresponding to a fixed (complex) set of abscissas may be found by the process of orthogonalizing the functionals $\{L_n\}$, $L_n(f) = f(\lambda_n)$. There is, moreover, the problem of varying both the (complex) abscissas and the weights so as to yield a still lower value for the corresponding σ . This problem may be posed for an arbitrary bounded linear functional L and an arbitrary domain B . (Cf. §5). We shall not be able to discuss these interesting extremal problems at the present time.

3. Some Examples of Functional Norms. The several specific examples which will now be given are intended merely to be suggestive, and the list may be augmented at will. Let us investigate the error committed in approximating the integral $\int_0^1 f(x) dx$ by a weighted combination of $N + 1$ functional values at the abscissas $\lambda_0, \dots, \lambda_N$. We write

$$(12) \quad E(f) = \int_0^1 f(x) dx - \sum_{n=0}^N a_n f(\lambda_n).$$

Equation (9) then yields

$$(13) \quad \sigma_E^2 = (2\pi)^{-1} \sum_{n=0}^{\infty} \left| \frac{1}{(n+1)2^{n+1}} - \sum_{j=0}^N a_j \lambda_j^n \right|^2$$

In particular, the trapezoidal rule (4), Simpson's rule,

$$(14) \quad E_s(f) = \int_0^1 f(x) dx - (12)^{-1} [f(0) + 4f(1/4) + f(1/2)],$$

and Weddle's rule,

$$(15) \quad E_w(f) = \int_0^1 f(x) dx - (40)^{-1} \cdot [f(0) + 5f(\frac{1}{12}) + f(\frac{1}{6}) + 6f(\frac{1}{4}) + f(\frac{1}{3}) + 5f(\frac{5}{12}) + f(\frac{1}{2})]$$

lead to the following values of σ :

$$(16a) \quad (2\pi)^{\frac{1}{2}}\sigma_T = 2.08 \times 10^{-1}$$

$$(16b) \quad (2\pi)^{\frac{1}{2}}\sigma_s = 5.1 \times 10^{-4}$$

$$(16c) \quad (2\pi)^{\frac{1}{2}}\sigma_w = 4.28 \times 10^{-7}$$

If it is desired to interpolate to f at a point z_0 , $|z_0| < 1$, by a fixed combination of functional values, we may take

$$(17) \quad E(f) = f(z_0) - \sum_{n=0}^N a_n f(\lambda_n).$$

As specific examples, we shall cite linear interpolation

$$(20) \quad E_L(f) = f\left(\frac{1}{4}\right) - \frac{1}{2}[f(0) + f\left(\frac{1}{2}\right)]$$

and a Lagrange interpolation formula

$$(21) \quad E_{LA}(f) = f\left(\frac{1}{4}\right) - \left[-\frac{1}{16}f(0) + \frac{9}{16}f\left(\frac{1}{8}\right) + \frac{9}{16}f\left(\frac{3}{8}\right) - \frac{1}{16}f\left(\frac{1}{2}\right)\right].$$

These rules lead to the following values of σ :

$$(22a) \quad (2\pi)^{\frac{1}{2}}\sigma_L = 8.45 \times 10^{-2}$$

$$(22b) \quad (2\pi)^{\frac{1}{2}}\sigma_{LA} = 8.20 \times 10^{-4}.$$

To approximate to the values of the derivative of f at a point z_0 , $|z_0| < 1$, we may take

$$(23) \quad E(f) = f'(z_0) - \sum_{n=0}^N a_n f(\lambda_n).$$

As specific examples, we shall cite a threepoint differentiation formula

$$(24) \quad E_{DT}(f) = f'\left(\frac{1}{4}\right) - 2(f\left(\frac{1}{2}\right) - f(0)),$$

and a five point formula

$$(25) \quad E_{DF}(f) = f'\left(\frac{1}{4}\right) - \frac{2}{3}[f(0) - 8f\left(\frac{1}{8}\right) + 8f\left(\frac{3}{8}\right) - f\left(\frac{1}{2}\right)]$$

leading to the following values of σ :

$$(26a) \quad (2\pi)^{\frac{1}{2}}\sigma_{DT} = 1.02 \times 10^{-1},$$

$$(26b) \quad (2\pi)^{\frac{1}{2}}\sigma_{DF} = 2.20 \times 10^{-3}.$$

4. Methods for Estimating $\|f\|$. Having discussed the manner in which the norm σ_E of a rule E may be computed, we turn next to the question of how the norm of the function operated upon may be computed. Once these two quan-

tities have been obtained, the error itself can be estimated through (7). The norm of f has been defined by (2); hence, if a Taylor expansion of f is known, its norm is easily computed. If f is continuous on $|z| = 1$, we have, as an alternate expression,

$$(27) \quad \|f\|^2 = \int_{|z|=1} |f|^2 ds.$$

The norm may therefore be computed from the modulus of f on the unit circle. If M designates the maximum modulus of f on $|z| = 1$, we have

$$(28) \quad \|f\| \leq (2\pi)^{\frac{1}{2}} M.$$

Formula (28) is particularly convenient to apply when f possesses a closed form expression, for then M may be estimated directly by inspection. By way of illustration, let $f(x) = [1 - (x+2)^{\frac{1}{2}} - (x+3)^{\frac{1}{2}}]^{\frac{1}{2}}$. Then $f(z)$ is regular in $|z| < 2$ and hence is of class H^2 . We have

$$M = \max_{|z|=1} |f(z)| \leq [1 + (2+1)^{\frac{1}{2}} + (3+1)^{\frac{1}{2}}] = 1.68.$$

Therefore $\|f\| \leq (2\pi)^{\frac{1}{2}}(1.68)$. If now f is integrated over $0 \leq x \leq \frac{1}{2}$ by means of Weddle's rule (15), the error committed will be less than $(1.68)(4.28 \times 10^{-7}) = 7.19 \times 10^{-7}$.

Let us assume, however, that we possess only a knowledge of the values $f(\beta_n)$ ($n = 1, 2, \dots$) for a sequence of points $\{\beta_n\}$, $|\beta_n| < 1$ which has an accumulation point interior to $|z| = 1$. The norm of f may be expressed in terms of $f(\beta_n)$ in the following way. We introduce the following sequence of functions

$$(29) \quad \phi_n^*(z) = \frac{(1 - |\beta_{n+1}|^2)^{\frac{1}{2}}(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)}{(2\pi)^{\frac{1}{2}}(1 - z\bar{\beta}_1)(1 - z\bar{\beta}_2) \cdots (1 - z\bar{\beta}_{n+1})} \quad (n = 0, 1, \dots),$$

which are regular in $|z| \leq 1$ and orthonormal in the sense of the inner product $(f, g) = \int_{|z|<1} f\bar{g} ds$. We then set up the interpolation series³ for f

$$(30) \quad f(z) \sim \frac{a_0}{1 - z\bar{\beta}_1} + a_1 \frac{z - \beta_1}{(1 - z\bar{\beta}_1)(1 - z\bar{\beta}_2)} + a_2 \frac{(z - \beta_1)(z - \beta_2)}{(1 - z\bar{\beta}_1)(1 - z\bar{\beta}_2)(1 - z\bar{\beta}_3)} + \cdots,$$

where the quantities a_n have been obtained from $f(z)$ by successive interpolation at the points $z = \beta_i$. The following theorem now holds⁴: a function $f(z)$ is of class H^2 if and only if we have

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{1 - |\beta_{n+1}|^2} < \infty.$$

³ This series is intimately related to the BLASCHKE product and has been studied extensively by TAKENAKA [11], MALMQUIST [5], WALSH [12]. For generalizations to arbitrary regions and arbitrary sets of interpolating conditions, see WALSH & DAVIS [13].

⁴ See e.g., WALSH [12], p. 304.

If this inequality holds, then we have

$$(31) \quad \|f\|^2 = (2\pi)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{|a_n|^2}{1 - |\beta_{n+1}|^2}.$$

By substituting $z = \beta_i$ ($i = 1, 2, \dots$) successively in (27) we obtain

$$(32) \quad \begin{aligned} f(\beta_1) &= \frac{a_0}{1 - |\beta_1|^2} \cdot \\ f(\beta_2) &= \frac{a_0}{1 - \beta_2 \bar{\beta}_1} + a_1 \frac{\beta_2 - \beta_1}{(1 - \beta_2 \bar{\beta}_1)(1 - \beta_2 \bar{\beta}_2)} \cdot \\ &\vdots \end{aligned}$$

The infinite system (32) may be solved in a step-by-step fashion for the quantities a_n . Each a_n is a fixed finite linear combination of $f(\beta_1), f(\beta_2), \dots, f(\beta_{n+1})$. The coefficients in each linear combination depend solely upon the points β_n and are independent of f . For a fixed disposition of points, these coefficients can be computed once for all.

5. Several Independent Variables. In the present section we shall consider a number of generalizations of the method of §2. In the first place, it is not necessary to limit ourselves to functions which are analytic in a circle. We may, for instance, operate within the class of functions analytic in any simply connected domain with a smooth boundary by introducing the set of polynomials which are orthonormal in the sense of some given inner product. A frequently used inner product is

$$(33) \quad (f, g)_B = \iint_B f \bar{g} \, dx \, dy; \quad z = x + iy,$$

which leads to the Bergman theory⁵ and to the error-functional norm

$$(34) \quad \sigma_E^2 = \sum_{n=0}^{\infty} |E(p_n(z))|^2 = E_z E_{\bar{z}} K_B(z, \bar{z}).$$

Here $\{p_n(z)\}$ is a set of polynomials which are orthonormal over B in the sense of (33) and $K_B(z, \bar{z})$ is the Bergman kernel function of B . In dealing with functions which are analytic over $[-1, 1]$, it may be particularly appropriate to introduce the inner product

$$(35) \quad (f, g)_E = \int_E f(z) \overline{g(z)} |1 - z^2|^{-\frac{1}{2}} ds.$$

Here E designates a small ellipse possessing foci at $z = \pm 1$. The resulting theory would then be related to the theory of Tchebycheff polynomials.⁶ In Walsh-

⁵ S. BERGMAN [2].

⁶ Cf., e.g., WALSH [12], p. 134, PRIVALOFF [8].

Davis [13] will be found a general theory of expansions of analytic functions in orthogonal series pertinent to a preassigned set of linear functionals. Additional generalizations may be based upon this work.

A generalization which is perhaps more significant from our present point of view is the consideration of approximate methods in several independent variables. These include such rules as numerical cubature, higher dimensional interpolation, partial differentiation, *etc.* Of particular interest here is the approximation of the Laplacian $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ by linear combinations

$$(36) \quad \Delta f = \sum_{n=0}^N a_n f(x_n, y_n) + R(f).$$

The remainder term R is usually expressed in terms of the higher derivatives of f and is valid for all functions which are sufficiently many times differentiable. By operating within the class of analytic functions of several complex variables, and introducing an appropriate Hilbert space, it will be possible again to obtain expressions for error directly in terms of functional values.

In what follows, we shall restrict the discussion to two independent variables and introduce the space⁷ $L^2(B^4)$ of functions $f(z_1, z_2)$ which are single-valued and analytic in the bicylinder B^4 : $|z_1| < 1, |z_2| < 1; z_1 = x + ix^*, z_2 = y + iy^*$ and are such that

$$(37) \quad \|f\|_{B^4} = \int_{B^4} |f|^2 dV < \infty; \quad dV = dx dx^* dy dy^*.$$

If f possesses the Taylor expansion $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$, (37) is equivalent to

$$(38) \quad \|f\|_{B^4}^2 = \pi^2 \sum_{m,n=0}^{\infty} (m+1)^{-1} (n+1)^{-1} |a_{mn}|^2 < \infty.$$

Each function $f \in L^2(B^4)$ possesses a representation of the form

$$(39) \quad f(z_1, z_2) = \int_{B^4} K_{B^4}(z_1, z_2; \bar{t}_1, \bar{t}_2) f(t_1, t_2) dV_t,$$

where

$$(40) \quad \begin{aligned} K_{B^4}(z_1, z_2; \bar{t}_1, \bar{t}_2) &= \pi^{-2} (1 - z_1 \bar{t}_1)^{-2} (1 - z_2 \bar{t}_2)^{-2} \\ &= \pi^{-2} \sum_{m,n=0}^{\infty} (m+1)(n+1) z_1^m z_2^n \bar{t}_1^m \bar{t}_2^n. \end{aligned}$$

If now E designates a linear functional which is bounded over $L^2(B^4)$, there is obtained

$$(41) \quad E(f) = \int_{B^4} E(K_{B^4}(z_1, z_2; \bar{t}_1, \bar{t}_2)) f(t_1, t_2) dV_t$$

⁷ Cf. S. BERGMAN [2].

where the E behind the integral sign is to be applied to the variables z_1, z_2 . This leads immediately to the estimate

$$(42) \quad |E(f)|^2 \leq \sigma_E^2 \|f\|_{B^4},$$

where

$$(43) \quad \sigma_E^2 = \int_{B^4} |EK_B(z_1, z_2; \bar{l}_1, \bar{l}_2)|^2 dV,$$

or, in view of (40) and the orthonormality of the functions $(m+1)^{\frac{1}{2}}(n+1)^{\frac{1}{2}} \cdot \pi^{-1} z_1^m z_2^n$,

$$(44) \quad \sigma_E^2 = \pi^{-2} \sum_{m, n=0}^{\infty} (m+1)(n+1) |E(z_1^m z_2^n)|^2.$$

Again, the estimate (42) is best possible, there being functions of class $L^2(B^4)$ for which the inequality becomes an equality. For real rules E , such a function exists which is real in the real plane $x^* = 0, y^* = 0$. The norm $\|f\|_{B^4}$ of the function operated upon, may be obtained directly from (38) if its Taylor expansion is available. In the absence of the Taylor expansion, the norm may be obtained from a sequence of functional values in the following way.

Let us consider the linear functional

$$(45) \quad L(f) = f(\alpha, \beta).$$

If the point $P \equiv (\alpha, \beta)$, $\alpha = x + ix^*, \beta = y + iy^*$ lies interior to B^4 , then L is bounded over $L^2(B^4)$. If we are given a sequence of points $\{P_n\}$ which have an accumulation point interior to B^4 , but do not lie on an analytic hypersurface, the set of functionals $\{L_n\}$, $L_n(f) = f(P_n)$, will be complete for $L^2(B^4)$. In particular, if $\lim_{j \rightarrow \infty} \alpha_j = 0, \lim_{k \rightarrow \infty} \beta_k = 0, (\alpha_j, \beta_k) \in B^4$, and if P_n designates the points (α_j, β_k) as j, k run through the positive integers independently, then the set of functionals $\{L_n\}$ will be complete.⁸ Under these circumstances, it is known⁹ that the set of functions

$$(46) \quad \begin{aligned} \phi_n(z_1, z_2) &= L_{n,i} K_B(z_1, z_2; \bar{l}_1, \bar{l}_2) \\ &= \pi^{-2} (1 - z_1 \bar{\alpha}_n)^{-2} (1 - z_2 \bar{\beta}_n)^{-2} \quad (n = 0, 1, \dots) \end{aligned}$$

is complete for $L^2(B^4)$. Moreover, if the set (46) is orthogonalized in the sense of the inner product

$$(47) \quad (f, g)_{B^4} = \int_{B^4} f \bar{g} dV,$$

⁸ OSGOOD [7], p. 39. We call $\{L_n\}$ complete for $L^2(B^4)$ if $L_n(f) = 0$ ($n = 0, 1, \dots$), $f \in L^2(B^4)$ implies $f \equiv 0$.

⁹ WALSH & DAVIS [13]. A generic argument for the case of analytic functions of one complex variable will be found. The extension to the present case is immediate.

and if the resulting orthonormal functions are designated by $\phi_n^*(z_1, z_2)$, each $f \in L^2(B^4)$ will possess an expansion of the form

$$(48) \quad f(z_1, z_2) = \sum_{n=0}^{\infty} L_n^*(f) \phi_n^*(z_1, z_2)$$

with

$$(49) \quad \|f\|_{B^4}^2 = \sum_{n=0}^{\infty} |L_n^*(f)|^2.$$

The functionals L_n^* are certain fixed linear combinations of L_n , and hence $L_n^*(f)$ is expressible directly in terms of $f(\alpha_0, \beta_0), \dots, f(\alpha_n, \beta_n)$. The series (48) is an extension to two complex variables of the Takenaka interpolation series (29). Note, however, that the related inner product (47) is a volume and not a surface integral. The orthonormal set $\{\phi_n^*\}$ is a set of rational functions in z_1 and z_2 . The Gram-Schmidt process yields

$$(50) \quad \phi_n^*(z_1, z_2) = \begin{vmatrix} (\phi_0, \phi_0) & (\phi_0, \phi_1) & \cdots & (\phi_0, \phi_n) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ (\phi_{n-1}, \phi_0) & (\phi_{n-1}, \phi_1) & \cdots & (\phi_{n-1}, \phi_n) \\ \phi_0 & \phi_1 & & \phi_n \end{vmatrix} \bigg/ D_{n-1}^{\frac{1}{2}} D_n^{\frac{1}{2}}$$

with

$$(51) \quad D_n = |(\phi_i, \phi_j)| \quad (i, j = 1, 2, \dots, n)$$

and

$$(52) \quad (\phi_i, \phi_j) = L_{i, i} L_{j, i} K_{B^4}(z_1, z_2; \bar{t}_1, \bar{t}_2) = \pi^{-2} (1 - \alpha_i \bar{\alpha}_j)^{-2} (1 - \beta_i \bar{\beta}_j)^{-2}.$$

The values $L_n^*(f)$ may be obtained from (46) by successive interpolation in the points $(z_1, z_2) = (\alpha_n, \beta_n)$ ($n = 0, 1, \dots$). We obtain in analogy to (32),

$$(53) \quad \begin{aligned} f(\alpha_0, \beta_0) &= L_0^*(f) \phi_0^*(\alpha_0, \beta_0), \\ f(\alpha_1, \beta_1) &= L_0^*(f) \phi_0^*(\alpha_1, \beta_1) + L_1^*(f) \phi_1^*(\alpha_1, \beta_1), \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

As specific examples of functional norms σ_E in several variables we shall cite two approximations to the Laplacian, the five point rule,

$$(54) \quad E_{\Delta_1}(f) = \Delta f|_{0,0} - 4(f(\tfrac{1}{2}, 0) + f(0, \tfrac{1}{2}) + f(-\tfrac{1}{2}, 0) + f(0, -\tfrac{1}{2}) - 4f(0, 0))$$

and the nine point rule,

$$\begin{aligned}
 E_{\Delta_2}(f) = & \Delta f|_{0,0} - \frac{4}{3}[-60f(0,0) \\
 (55) \quad & + 16\{f(\tfrac{1}{4},0) + f(0,\tfrac{1}{4}) + f(-\tfrac{1}{4},0) + f(0,-\tfrac{1}{4})\} \\
 & - \{f(\tfrac{1}{2},0) + f(0,\tfrac{1}{2}) + f(-\tfrac{1}{2},0) + f(0,-\tfrac{1}{2})\}],
 \end{aligned}$$

leading respectively to

$$(56a) \quad \sigma_{E_{\Delta_1}} = 5.27 \times 10^{-1},$$

$$(56b) \quad \sigma_{E_{\Delta_1}} = 3.99 \times 10^{-2}.$$

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