Bounds for Solutions of Laplace's Equation Satisfying Mixed Conditions

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In this paper upper and lower bounds are established for the integrals of the Laplace equation which satisfy mixed conditions on the frontier of the open set considered. These bounds are obtained by introducing two auxiliary functions whose existence is assumed. Two suitable auxiliary functions are then exhibited for a particular boundary value problem.

a. Let us indicate by x a general point of a Euclidean space S_r of r dimensions, with coordinates $(x_1, x_2, \dots, x_r)$. Let A be an open set of S_r and $\mathfrak{F}A$ its boundary, and further let $f(x)$ be a real function in A and $\alpha(x)$, $\beta(x)$, $\gamma(x)$ be three real functions defined in $\mathfrak{F}A$. Let us further suppose $|\alpha(x)| + |\beta(x)| > 0$ for $x \in \mathfrak{F}A$, and that the set H of the points of $\mathfrak{F}A$ for which $\beta(x) \neq 0$ is not empty, and also that through every point x of $\mathfrak{F}A$ for which $\alpha(x) \neq 0$ there is defined an axis l which penetrates into A .¹

Let us write Γ for the class of functions continuous in $A + \mathfrak{F}A$, having first and second partial derivatives continuous in A , and having for every point x of $\mathfrak{F}A$ for which $\alpha(x) \neq 0$ a directional derivative, with respect to the axis l which is defined at such a point. With this as a premiss, we have the following theorem:

Theorem: *Let the function $u(x)$, belonging to the class Γ , be an integral in A of the Laplace equation*

¹ An axis l which originates at a point x on the boundary $\mathfrak{F}A$ is said to penetrate into A if there exists on it a point y such that the interior points of the segment xy belong to A and the sense of the vector xy is the same as that of the axis l .

$$(1) \quad \Delta_2 u(x) \equiv \sum_{h=1}^r \frac{\partial^2 u}{\partial x_h^2} = f(x),$$

and let it satisfy the boundary conditions

$$(2) \quad Lu(x) \equiv \alpha(x) \frac{du}{dl} + \beta(x)u = \gamma(x), \quad \text{for } x \in \mathfrak{F}A,$$

where it is assumed that

$$(3) \quad \alpha(x)\beta(x) \leq 0 \quad \text{for } x \in \mathfrak{F}A.$$

If there exist two functions $w_1(x)$, $w_2(x)$ of the class Γ such that

$$(4) \quad \Delta_2 w_2(x) < f(x) < \Delta_2 w_1(x) \quad \text{for } x \in A,$$

$$(5) \quad \frac{dw_2}{dl} < \frac{\gamma(x)}{\alpha(x)} < \frac{dw_1}{dl} \quad \text{for } x \in \mathfrak{F}A - H,$$

then the following bounds for $u(x)$ hold in $A + \mathfrak{F}A$:

$$(6) \quad w_1(x) + \min_H \left[\frac{\gamma - Lw_1}{\beta} \right] \leq u(x) \leq w_2(x) + \max_H \left[\frac{\gamma - Lw_2}{\beta} \right].^3$$

Proof. The function

$$(7) \quad v_1(x) = u(x) - w_1(x)$$

belongs to the class Γ .

$$\Delta_2 v_1(x) = f(x) - \Delta_2 w_1(x) < 0 \quad \text{for } x \in A$$

is true for equation (4), and hence the function $v_1(x)$ cannot attain its minimum value at a point in A , because at such a point we would have $\Delta_2 v_1 > 0$.

We have from equation (2)

$$(8) \quad \alpha(x) \frac{dv_1}{dl} + \beta(x)v_1 = \gamma(x) - Lw_1(x) \quad \text{for } x \in \mathfrak{F}A.$$

Recalling that $\beta(x) = 0$ for $x \in \mathfrak{F}A - H$, we have from (5)

$$\frac{dv_1}{dl} = \frac{\gamma(x)}{\alpha(x)} - \frac{dw_1}{dl} < 0 \quad \text{for } x \in \mathfrak{F}A - H$$

It follows hence that $v_1(x)$ cannot assume its minimum value on $\mathfrak{F}A - H$; if we indicate by y the point of $A + \mathfrak{F}A$ in which $v_1(x)$ assumes its minimum value, y must then belong to H .

² Since H is the set of points X of $\mathfrak{F}A$ for which $\beta(x) \neq 0$, $\beta(x) = 0$ in $\mathfrak{F}A - H$, and hence $\alpha(x) \neq 0$ in $\mathfrak{F}A - H$.

³ We have written $\min_H [\gamma - Lw_1/\beta]$ for the lower bound of the values assumed by $(\gamma(x) - Lw_1(x)/\beta(x))$ in the set H . The number $\max_H [\gamma - Lw_1/\beta]$ is defined in a like manner.

From equation (8) it follows that

$$v_1(y) = \frac{\gamma(y) - [Lw_1]_{x=y}}{\beta(y)} - \frac{\alpha(y)}{\beta(y)} \left[\frac{dv_1}{dl} \right]_{x=y}.$$

Since y is a minimum point for $v_1(x)$, $\left[\frac{dv_1}{dl} \right]_{x=y}$ is less than or equal to zero, and so for equation (3) we obtain

$$v_1(y) \geq \frac{\gamma(y) - [Lw_1]_{x=y}}{\beta(y)} \geq \min_H \left[\frac{\gamma - Lw_1}{\beta} \right].$$

From equation (7) it follows that, for $x \in A + \mathfrak{F}A$,

$$u(x) \geq w_1(x) + \min_{A+\mathfrak{F}A} [v] \geq w_1(x) + \min_H \left[\frac{\gamma - Lw_1}{\beta} \right].$$

In a similar manner the validity of the upper bound in (6) can be demonstrated, by considering the function $v_2(x) = u(x) - w_2(x)$.

b. The limiting condition obtained above has been arrived at on the assumption of the existence of two convenient auxiliary functions. Let us establish these functions for a simple case.

Put $r = 2$ and $x_1 = x = \rho \cos \theta$, $x_2 = y = \rho \sin \theta$. Let us indicate by A the set of points of the plane for which $\rho < 1$. The function $u(x, y)$ belonging to the class Γ is an integral in A of the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

satisfying the boundary conditions

$$u(\cos \theta, \sin \theta) = 1 \quad \text{for } \frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4},$$

$$\left[\frac{\partial u(\rho \cos \theta, \rho \sin \theta)}{\partial \rho} \right]_{\rho=1} = 0 \quad \text{for } -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

Let us define two real numbers k_1 and k_2 such that

$$k_1 > \max \left\{ \frac{1}{4} \max_A [f], 0 \right\},$$

$$k_2 < \min \left\{ \frac{1}{4} \min_A [f], 0 \right\},$$

and let us write

$$w_1(x, y) = k_1[(x-2)^2 + y^2], \quad w_2(x, y) = k_2[(x-2)^2 + y^2].$$

⁴ In this case, therefore, the set H is the circumference arc $\rho = 1$, $\pi/4 \leq \theta \leq 7\pi/4$, and the axis l is the normal to the boundary.

We obtain

$$\Delta_2 w_2 = 4k_2 < f(x, y) < \Delta_2 w_1 = 4k_1 \quad \text{for } (x, y) \in A;$$

further, since $w_1(\rho \cos \theta, \rho \sin \theta) = k_1(\rho^2 + 4 - 4\rho \cos \theta)$, we obtain

$$\left[\frac{\partial w_1}{\partial(-\rho)} \right]_{\rho=1} = k_1(4 \cos \theta - 2) > 0 \quad \text{for } -\frac{\pi}{4} < \theta < \frac{\pi}{4},$$

and in the same way,

$$\left[\frac{\partial w_2}{\partial(-\rho)} \right]_{\rho=1} = k_2(4 \cos \theta - 2) < 0 \quad \text{for } -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

The functions $w_1(x, y)$ and $w_2(x, y)$ hence satisfy the two inequalities (4) and (5).

We have further

$$1 - k_1(5 - 4 \cos \theta) \geq 1 - 9k_1 \quad \frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4},$$

$$1 - k_2(5 - 4 \cos \theta) \geq 1 - 9k_2 \quad \frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4},$$

$$k_1 > 0, \quad k_2 < 0.$$

It follows from the theorem proved above that the following limiting condition in $A + \mathfrak{F}A$ on the function $u(x, y)$ holds:

$$1 - k_1[9 - (x - 2)^2 - y^2] \leq u(x, y) \leq 1 - k_2[9 - (x - 2)^2 - y^2],$$

or, more simply,

$$1 - 8k_1 \leq u(x, y) \leq 1 - 8k_2.$$