*Comparison Methods in the Theory of Subsonic Flows**

DAVID GILBARG

Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana

Introduction. We shall be concerned with several qualitative results in the theory of subsonic flows. Our methods will be based on comparison of stream functions for flows having a common subregion of definition and certain general properties in the large. These comparison methods have been singularly successful when applied to the free boundary problems of hydrodynamics (as in [1], [2], [3]) and recommend themselves to the problems of compressible fluids because the difficulties raised by nonlinearity can be circumvented by these means without recourse to the hodograph plane, and also because the simplicity and generality of the arguments are well suited to a theory so lacking in rigorously established general theorems.

In this paper we first prove several theorems on the monotonic dependence of the boundary flow speed on the free stream velocity in flow past a wall and past a symmetric body (Theorems 1, 2, 3). By means of a new theorem of Phragmén-Lindelöf type on the behavior at infinity of bounded solutions of elliptic partial differential equations (Theorem 4), we then prove the uniqueness with respect to free stream Mach number and maximum local Mach number of subsonic flows past a certain class of walls, including as special case the symmetric flow past a finite body (Theorems 6, 7). Theorem 8 extends to subsonic flows a comparison theorem first stated by Lavrentieff [1] for incompressible fluids. We see in Theorems 9 and 10 that all our results for flows past a wall carry over, more simply and more generally, to flows in a channel. Finally we consider flows with axial symmetry—that we can do so at all is one of the distinctive features of the comparison method—and show that the theorems on monotonic dependence and on flows in a channel remain true without essential modification.

As application of these theorems we obtain immediately the extension to subsonic flows of a number of the previous results on free boundaries in incompressible fluids. In a later work, in collaboration with M. Shiffman, we shall apply

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these methods to the determination of bodies achieving extreme values of the critical Mach number.

With regard to the details, we call attention to the elementary character of the proofs, which consist for the most part of simple applications of the maximum principle.

Several of the results contained here have been found independently by L. Bers using different methods; his work is as yet unpublished.

Preliminaries. We consider the steady irrotational plane flow of an ideal compressible fluid. Let ρ and p be the non-dimensional density and pressure referred to their stagnation values so that $\rho = 1$, $p = 1$ for the fluid at rest, and let q be the non-dimensional flow speed referred to the limit speed of the flow. In non-dimensional form Bernoulli's equation becomes

$$(1) \quad \int_{\rho_0}^{\rho} c^2 \frac{d\rho}{\rho} = \frac{1}{2}(1 - q^2), \quad c^2 = \frac{dp}{d\rho} > 0,$$

where the sound speed c is also normalized with respect to the limit speed of the flow. This equation defines ρ as a single valued function of q . In particular, for a polytropic flow, in which $p = \rho^\gamma$, $\gamma > 1$, we have

$$\rho = (1 - q^2)^{1/(\gamma-1)}, \quad c^2 = \frac{\gamma p}{\rho} = \frac{\gamma - 1}{2} (1 - q^2).$$

The velocity potential ϕ and stream function ψ are connected by the relations,

$$\rho \phi_x = \psi_y, \quad \rho \phi_y = -\psi_x,$$

from which we obtain for the non-dimensional mass flow Q , normalized with respect to the limit speed,

$$(2) \quad \begin{aligned} Q &\equiv \sqrt{\psi_x^2 + \psi_y^2} = q\rho(q) \\ &= q(1 - q^2)^{1/(\gamma-1)} \quad \text{for a polytropic flow.} \end{aligned}$$

Since from (1),

$$(3) \quad \frac{dQ}{dq} = \rho \left(1 - \frac{q^2}{c^2} \right) \geq 0 \quad \text{for} \quad q/c \leq 1,$$

it follows that q is a single valued function of the mass flow Q in the subsonic and also in the supersonic range, but is double valued in the mixed subsonic-supersonic range. The stream function satisfies the quasi-linear partial differential equation,

$$(4) \quad \left(1 - \frac{\psi_y^2}{(\rho c)^2} \right) \psi_{xx} + \frac{2\psi_x \psi_y}{(\rho c)^2} \psi_{xy} + \left(1 - \frac{\psi_x^2}{(\rho c)^2} \right) \psi_{yy} = 0,$$

in which ρ and c are considered functions of ψ_x and ψ_y defined implicitly by relations (1) and (2). This non-dimensional form of the equation for ψ contains no parameters of the flow (such as free stream values of the flow quantities) and therefore may be used to describe all flows of the fluid without reference to constants of the flow.

We now make several observations essential to our later proofs. If ψ and $\bar{\psi}$ are stream functions for two purely subsonic flows defined in the same region of the plane, then in this region $\omega \equiv \bar{\psi} - \psi$ is a solution of a linear elliptic partial differential equation of the form

$$(5) \quad a\omega_{xx} + 2b\omega_{xy} + c\omega_{yy} + d\omega_x + e\omega_y = 0.$$

Although the formal device by which this result is achieved is well known we provide the details here for sake of later reference. Consider any equation of the form

$$(6) \quad au_{xx} + 2bu_{xy} + cu_{yy} + d = 0$$

where $a = a[u] \equiv a(x, y, u_x, u_y)$ and also $b[u]$, $c[u]$, $d[u]$, are functions possibly of the coordinates and of the first derivatives u_x , u_y but not of u (*e.g.* equation (4) is of this form.) Let u and v be two solutions and set $u - v \equiv \omega$. Then

$$\begin{aligned} 0 &= a[v + \omega](v_{xx} + \omega_{xx}) + \cdots - (a[v]v_{xx} + \cdots) \\ &= a[v]\omega_{xx} + 2b[v]\omega_{xy} + c[v]\omega_{yy} + \{a[v + \omega] - a[v]\}u_{xx} + \cdots \\ &\quad + \{d[v + \omega] - d[v]\}. \end{aligned}$$

If the mean value theorem is now applied to the expressions in braces we obtain an equation of the form (5), in which the coefficients a , b , c , considered as function of x and y are dependent for their values only on the solution v (or on the solution u if we had so chosen) while d and e are given by mean value expressions,

$$(7) \quad \begin{aligned} d(x, y) &= \frac{\partial a}{\partial p} u_{xx} + 2\frac{\partial b}{\partial p} u_{xy} + \frac{\partial c}{\partial p} u_{yy} + \frac{\partial d}{\partial p}, \\ e(x, y) &= \frac{\partial a}{\partial q} u_{xx} + 2\frac{\partial b}{\partial q} u_{xy} + \frac{\partial c}{\partial q} u_{yy} + \frac{\partial d}{\partial q}; \end{aligned}$$

the derivatives in the right members are evaluated at some intermediate value of the first derivatives p , q ,

$$p = u_x + t(u_x - v_x), \quad q = u_y + t(u_y - v_y), \quad 0 < t(x, y) < 1.$$

Since any elliptic differential equation of the form (5) obeys the maximum principle, that is, its non-constant solutions cannot have an interior maximum or minimum, this simple formal device establishes the maximum principle for

the difference ω of any two subsonic stream functions. This fact plays a central role in our later comparisons of subsonic flows.

From the fact that only one of the solutions, say ψ , determines the value of the coefficients a, b, c in (5), it might appear at first glance that $\omega \equiv \bar{\psi} - \psi$ is the solution of an elliptic differential equation whether or not $\bar{\psi}$ makes (4) elliptic everywhere. However, if the latter equation were anywhere of hyperbolic type for $\bar{\psi}$ the above analysis would break down at application of the mean value theorem since the coefficients of (4) are double valued in the mixed elliptic-hyperbolic range. This comment would not apply to solutions of the equation for the velocity potential since its coefficients are single valued.

In the following we limit discussion entirely to subsonic flows, and furthermore we require that *the maximum Mach number¹ in each flow be less than unity.*

By a *regular boundary point* of a region we shall mean a boundary point lying on the circumference of some circle whose interior falls entirely within the region. For a boundary point to be regular it obviously suffices that the boundary have bounded curvature near the point.

We shall need the following lemma frequently.

Boundary point lemma: *Let the equation*

$$L(u) \equiv au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0$$

have continuous bounded coefficients in the neighborhood of a regular point P on the boundary of its region of definition and, for some δ , let $ac - b^2 \geq \delta > 0$ in the neighborhood of P . If $u(x, y)$ is a solution which is everywhere positive in the neighborhood while $u(P) = 0$, then the inward drawn normal derivative obeys the strict inequality

$$\frac{\partial u}{\partial n}(P) > 0.^2$$

For a simple proof of this result we refer to E. Hopf [5].

The boundary point lemma will be applied later to equation (5) satisfied by the difference of two subsonic stream functions. To insure boundedness of coefficients d and e we must require, as (7) shows, that the second derivatives of the stream function, or equivalently, that the acceleration components be bounded in the neighborhood of the subject boundary point. This can be guaranteed by sufficient smoothness of the boundary. At all events, we make the assumption henceforth in application of the lemma that the second derivatives of the stream function remain bounded in the neighborhood of the boundary point.

Monotonic dependence on free stream velocity. In this section we prove several results concerning the monotonic dependence of the flow speed on its free stream value. First we prove a comparison theorem.

¹ I.e., the least upper bound of the local Mach numbers.

² The normal derivative is assumed to exist; otherwise it can be taken to mean the lower normal derivate at the point P .

Theorem 1. Let two subsonic flows be defined in regions D and $\bar{D}$, each being bounded by a streamline extending to $x = \pm\infty$, and such that $D \subset \bar{D}$ (Fig. 1). Let the two flows be uniform at infinity with respective free stream velocities $q_0, \bar{q}_0$ satisfying the inequality $q_0 < \bar{q}_0$. Then, if the two bounding streamlines have a regular point P in common, the respective flow speeds $q, \bar{q}$ satisfy at P the inequality,

$$q(P) < \bar{q}(P).$$

Proof. Let us designate by ψ and $\bar{\psi}$ the stream functions of the two flows normalized so that $\psi = 0$ on the boundary γ of D and $\bar{\psi} = 0$ on the boundary $\bar{\gamma}$ of $\bar{D}$ (Fig. 1). We have by hypothesis that $\psi_x \rightarrow 0, \bar{\psi}_x \rightarrow 0$ at infinity while

FIG. 1

$\psi_y \rightarrow Q_0, \bar{\psi}_y \rightarrow \bar{Q}_0$, where Q_0 and $\bar{Q}_0$ are the free stream mass flows, and $Q_0 < \bar{Q}_0$. In polar coordinates we obtain for the difference $\omega \equiv \bar{\psi} - \psi$,

$$\frac{1}{r} \frac{\partial \omega}{\partial \theta} = (\bar{Q}_0 - Q_0) \cos \theta + \epsilon_1(r, \theta), \quad \epsilon_1 \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Let $(x, y) = (r, \theta)$ be any point in D . We integrate the above equation along the arc $r = \text{const.}$, starting from (r, θ) and following the arc in the direction of decreasing θ if $\theta \leq \pi/2$ and in the direction of increasing θ if $\theta > \pi/2$. If r is sufficiently large then the arc of integration will meet γ at a first point $\gamma_r = (r, \theta_r) = (x_r, y_r)$ such that $y > y_r$. We have,

$$\omega(r, \theta) - \omega(\gamma_r) \geq r[(\bar{Q}_0 - Q_0)(\sin \theta - \sin \theta_r) - \epsilon(r) |\theta - \theta_r|]$$

where

$$\epsilon(r) = \max_{\theta} |\epsilon_1(r, \theta)| \rightarrow 0 \text{ as } r \rightarrow \infty;$$

from this we obtain the inequality,

$$\omega(r, \theta) - \omega(\gamma_r) \geq [(\bar{Q}_0 - Q_0) - \alpha \epsilon(r)](y - y_r),$$

in which α is a constant depending only on D such that

$$0 < \frac{|\theta - \theta_r|}{\sin \theta - \sin \theta_r} \leq \alpha$$

for all $(r, \theta) \in D$ outside of some circle. Since $\bar{\psi} \geq 0$ on γ and $\psi = 0$ on γ , we get

$$(8) \quad \omega(r, \theta) \geq [(\bar{Q}_0 - Q_0) - \alpha\epsilon(r)](y - y_r), \quad \epsilon(r) \rightarrow 0 \text{ as } r \rightarrow \infty,$$

so that for sufficiently large r , $\omega(r, \theta) \geq 0$ in D . It follows from the maximum principle for ω that

$$\omega \geq 0 \text{ throughout } D.$$

By hypothesis $\psi \equiv \bar{\psi}$, so that the maximum principle enforces strict inequality in D , $\omega > 0$. The boundary point lemma applied at the point P now gives the result,

$$0 < \frac{\partial \omega}{\partial n}(P) = \frac{\partial \bar{\psi}}{\partial n}(P) - \frac{\partial \psi}{\partial n}(P);$$

from this and (3) follows

$$q(P) < \bar{q}(P). \quad \text{Q.E.D.}$$

For the case that $D = \bar{D}$ we have as an immediate corollary the theorem that the flow speed on the boundary is a monotonically increasing function of the flow speed at infinity. Let us designate as a *flow past a wall* the type of flow considered in the preceding theorem, namely, a flow uniform at infinity having as boundary a single streamline extending to $x = \pm \infty$. Because of its intrinsic interest we state the corollary as a theorem.

Theorem 2. *In the subsonic flow past a wall the flow speed at any regular boundary point is a strictly increasing function of the flow speed at infinity.*

This theorem holds, in particular, for the regular points on a body in a symmetric flow,³ the wall in this case consisting of the upper half boundary of the body and of the portion of the axis of symmetry external to the body.

The analogue of the preceding theorem for incompressible fluids is of course a triviality. The result itself is not surprising or unexpected, but it seems to have gone without mention in the mathematical literature on compressible fluids.

The velocity components of a flow satisfy an equation of the form (5), as is readily seen by differentiation of the equation for the velocity potential. Consequently the maximum speed in a subsonic flow is attained on the boundary, or possibly at infinity. Combining this observation with Theorem 2, we obtain,

Theorem 3. *In the subsonic flow past a wall the maximum speed of the flow is achieved on the boundary or at infinity, and is a strictly increasing function of the free stream speed.*

To assert the strict inequality this result presupposes that the boundary point of maximum speed is regular; this can be assured by imposing obvious smoothness conditions on the boundary.

³ Throughout this paper symmetry will be understood with respect to the x -axis, which will always be taken as the direction of the free stream velocity.

Uniqueness of flows past a wall. In this section we prove several uniqueness theorems, in particular the uniqueness of symmetric flows past finite obstacles. Our method of proof rests on the following general theorem of Phragmén-Lindelöf type for elliptic differential equations.

Theorem 4. *Let the equation*

$$(9) \quad L(u) \equiv au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

have continuous coefficients and be of elliptic type in an infinite region D lying in a half-plane.⁴ Let the coefficients a, b, c approach limits a_0, b_0, c_0 , respectively, at infinity, with $a_0 c_0 - b_0^2 > 0$, and let $u(x, y)$ be a bounded solution in D which is continuous in the closure of D . Then if u has the limit u_0 when infinity is approached along the boundary of D , it has this same limit when infinity is approached from the interior of D ; that is,

$$\lim_{(x, y) \rightarrow \infty} u(x, y) = u_0, \quad (x, y) \in D.$$

This theorem generalizes to elliptic partial differential equations a known result for harmonic functions. The result can be extended to include equations with terms in the first derivative, but we shall not be concerned here with this detail.

Proof. By means of an appropriate linear coordinate transformation we may replace (9) by an equation of the same form in which the transformed coefficients a, b, c are such that

$$a(x, y) \rightarrow 1, \quad b(x, y) \rightarrow 0, \quad c(x, y) \rightarrow 1 \quad \text{as } (x, y) \rightarrow \infty.$$

We may take the transformed domain D as lying in the half plane $x \geq x_0 > 0$. For convenience we assume $u_0 = 0$ and take M to be an upper bound on the magnitude of u in D . Let $\epsilon > 0$ be fixed during the following discussion. Now choose $r_0(\epsilon)$ so large that $u(x, y) \geq -\epsilon$ on the portion $\bar{D}$ of the boundary lying outside the circle $r = r_0(\epsilon)$, and also so large that the quantities $a - 1, b, c - 1$ differ from zero by a sufficiently small quantity (to be determined later) outside this circle. We set

$$v_\lambda(z) \equiv \frac{\lambda}{z},$$

$$\phi_\lambda(x, y) \equiv \operatorname{Re} v_\lambda(z) = \frac{\lambda x}{r^2}$$

$$> 0 \quad \text{in } D,$$

⁴ By a slight modification of the proof the theorem can be shown to hold for any region contained in a sector of angle less than 2π .

where $\lambda = \lambda(\epsilon)$ is real and positive, and will later be chosen suitably large. We now seek to determine a real function

$$F(x, y) \equiv f(\phi_\lambda(x, y))$$

with the following properties:

- (i) $L[F(x, y) + u(x, y)] \leq 0$ for $r > r_0(\epsilon)$;
- (ii) $F(x, y) + u(x, y) \geq -\epsilon$ on $r = r_0(\epsilon)$ and on $\dot{D}$;
- (iii) $F(x, y) + u(x, y)$ is bounded for $r \geq r_0(\epsilon)$;
- (iv) $F(x, y) \rightarrow 0$ as $(x, y) \rightarrow \infty$.

From these several properties of F it will follow that

$$\liminf_{(x, y) \rightarrow \infty} u(x, y) \geq -\epsilon$$

and from this the theorem.

In order to satisfy condition (i), we seek to choose $f(\phi)$ so that

$$(10) \quad L[f(\phi)] = f''(a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2) + f'[L(\phi)] \leq 0;$$

this will be satisfied if both $f' > 0$ and

$$\frac{f''}{f'} \leq -\frac{|L(\phi)|}{a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2}.$$

(The subscript λ is dropped temporarily without ambiguity.) For $r > r_0(\epsilon)$,

$$\begin{aligned} |L(\phi)| &= |(a-1)\phi_{xx} + 2b\phi_{xy} + (c-1)\phi_{yy}| \\ &\leq [(a-1)^2 + 2b^2 + (c-1)^2]^{\frac{1}{2}}(\phi_{xx}^2 + 2\phi_{xy}^2 + \phi_{yy}^2)^{\frac{1}{2}} \leq \delta_1 v''(z) \end{aligned}$$

and

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 \geq (1 + \delta_2)(\phi_x^2 + \phi_y^2) = (1 + \delta_2) |v'(z)|^2;$$

here δ_1 and $|\delta_2|$ are positive constants which can be arbitrarily small if $r_0(\epsilon)$ is large enough; in particular, $r_0(\epsilon)$ can be chosen so that

$$0 < \sigma = \frac{2\delta_1}{1 + \delta_2} < 1.$$

From the definition of $v(z)$

$$\left| \frac{v''}{v'^2} \right| = 2 \left| \frac{z}{\lambda} \right| = \frac{2}{|v|} \leq \frac{2}{\phi}.$$

Thus,

$$-\frac{\sigma}{\phi} \leq -\frac{\delta_1}{1+\delta_2} \left| \frac{v''}{v'^2} \right| \leq -\frac{|L(\phi)|}{a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2}.$$

Now choose $f(\phi)$ so that

$$\frac{f''}{f'} = -\frac{\sigma}{\phi}, \quad \text{and} \quad f' > 0;$$

for any such $f(\phi)$ the inequality (10) is automatically satisfied. In particular, take

$$(11) \quad F(x, y) \equiv f(\phi_\lambda) = \phi_\lambda^{1-\sigma} = \left(\frac{\lambda x}{r^2} \right)^{1-\sigma}.$$

Finally, let us choose $\lambda = \lambda(\epsilon)$ so large that the inequality $F(x, y) \geq M$ is satisfied at all points of the arc $r = r_0(\epsilon)$, $x \geq x_0$. With this selection of $F(x, y)$ we observe that conditions (i)–(iv) are fulfilled: (i) follows from the construction of F ; and (ii) from the choice of $r_0(\epsilon)$ and from the fact that $F \geq M$ on $r = r_0(\epsilon)$ while $|u| \leq M$ in D ; (iii) is a consequence of the boundedness of both u and F ; and (iv) is satisfied because

$$F(x, y) = \left(\frac{\lambda x}{r^2} \right)^{1-\sigma} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty.$$

Conditions (i)–(iii) allow us to infer from the Phragmén-Lindelöf theorem for elliptic differential equations [4] that

$$F(x, y) + u(x, y) \geq -\epsilon$$

in D for all $r > r_0(\epsilon)$. Since $F \rightarrow 0$ as $(x, y) \rightarrow \infty$ we have finally that

$$\liminf_{(x, y) \rightarrow \infty} u(x, y) \geq -\epsilon.$$

But ϵ was arbitrary to begin with, so we conclude

$$\liminf_{(x, y) \rightarrow \infty} u(x, y) \geq 0.$$

Applying the same argument to $-u(x, y)$, we obtain also

$$\limsup_{(x, y) \rightarrow \infty} u(x, y) \leq 0$$

and hence in conclusion,

$$\lim u(x, y) = 0 \quad \text{as} \quad (x, y) \rightarrow \infty.$$

This proves the theorem. With it we are able to obtain the following result.

Theorem 5. *Let $\psi(x, y)$ be the stream function of a subsonic flow past a wall with equal finite ordinates at $x = \pm \infty$, and let $\psi_y \rightarrow Q_0$ at infinity. Then the function $\psi(x, y) - Q_0 y$ has a limit as (x, y) approaches infinity from within the flow region.⁵*

Proof. For convenience let us take $\psi = 0$ on the wall and the ordinate at infinity to be zero, and let y_M and y_m be respectively the maximum and minimum ordinates on the body; (it is possible that one of these values is not achieved at a finite point). Consider the linear elliptic equation

$$(12) \quad au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

whose coefficients are the functions of x, y obtained by inserting ψ in the coefficients of the equation for the stream function. Of course ψ is a solution of this equation in the flow region D and so is $\alpha Q_0(y - y_M)$ for any constant α . The latter can be considered the stream function of the uniform flow in the half plane $y > y_M$ with free stream mass flow αQ_0 . The same considerations as in the proof of Theorem 1 give us the result that if $0 < \alpha < 1$ then

$$\psi - \alpha Q_0(y - y_M) > 0 \text{ for } y > y_M;$$

this inequality holds *a fortiori* in the whole of the flow region D . Letting $\alpha \rightarrow 1$ we obtain in the limit

$$(13) \quad \psi(x, y) - Q_0(y - y_M) \geq 0, \quad (x, y) \in D.$$

Similarly, by considering the solution $\psi - \alpha Q_0(y - y_m)$ for $\alpha > 1$, we obtain

$$(14) \quad \psi(x, y) - Q_0(y - y_m) \leq 0, \quad (x, y) \in D.$$

Thus,

$$-Q_0 y_M \leq \psi(x, y) - Q_0 y \leq -Q_0 y_m, \quad (x, y) \in D,$$

and we see that the function $\psi(x, y) - Q_0 y$ is bounded in D . We have also that

$$\psi(x, y) - Q_0 y \rightarrow 0$$

as (x, y) approaches infinity along the boundary. The conditions of Theorem 4 are therefore satisfied in D by the solution $\psi(x, y) - Q_0 y$ of equation (12), and from this follows the theorem.

We remark that the proof of (13) and (14) does not require that the wall have equal ordinates at $x = \pm \infty$, but only that the maximum (minimum) ordinate be finite. As incidental consequence of (13) and (14) we have that *the flow speed at a finite point of maximum (minimum) ordinate on a wall is greater (less) than the free stream value unless the flow is constant*. Therefore, if the wall has a finite point of maximum (minimum) ordinate the *maximum (minimum) speed*

⁵ This result is proved by SHIFFMAN ([6], p. 632) for flow past an arbitrary finite obstacle provided the Dirichlet integral of the flow is finite.

of the flow is achieved on the wall and not at infinity. In particular, in the symmetric flow past a finite body the maximum speed of the flow is attained on the body.

Now let ψ and $\bar{\psi}$ be two stream functions satisfying the conditions of the preceding theorem, and let both ψ and $\bar{\psi}$ be normalized to zero on the wall. Then by Theorem 5

$$\psi - Q_0 y \rightarrow 0, \quad \bar{\psi} - Q_0 y \rightarrow 0 \quad \text{as } (x, y) \rightarrow \infty,$$

and hence $\bar{\psi} - \psi \rightarrow 0$ at infinity. But since $\bar{\psi} - \psi = 0$ on the wall the maximum principle gives the result,

$$\bar{\psi} \equiv \psi,$$

and therefore the flows are identical. Thus we have the uniqueness theorem,

Theorem 6. *Let C be a wall with equal ordinates at $x = \pm \infty$. Then there can be at most one subsonic flow past C with prescribed free stream Mach number. In particular, a symmetric subsonic flow past a finite body is uniquely determined by its free stream Mach number.*

We note that the proof of this theorem requires no explicit assumptions as to the smoothness of the boundary or concerning the behavior of the flow near the boundary.

This result answers a question raised by Shiffman ([6] p. 608) in his work on the existence of subsonic flows. The flows he constructs are uniquely determined by the free stream Mach number, provided the admissible flows have finite Dirichlet integral over the flow region. Whether every flow has this character is left an open question. Theorem 6 provides an affirmative answer in the case of symmetric flows past a finite body, since Shiffman's solutions must be the only solutions.

In subsonic flow problems it is not always reasonable to assign the free stream Mach number, since the corresponding flow cannot in general be achieved. From this point of view, a better formulation of the subsonic flow problem is one fixing the maximum local Mach number of the flow. With the problem posed in this way we again have uniqueness for the class of flows considered in Theorem 6, and more generally, for all domains in which the flows are uniquely determined by the free stream Mach number.

Theorem 7. *The flows considered in Theorem 6 are uniquely determined by the maximum value of the local Mach number, and the symmetric flow past a finite body is uniquely determined by the maximum Mach number on the body. More generally, if the subsonic flow past any wall is uniquely determined by its free stream Mach number then it is also uniquely determined by the maximum value of the local Mach number.*

Proof. This theorem is proved under assumptions which allow application of the boundary point lemma and is therefore slightly more restricted than Theorem 6. It suffices to prove that two flows with the same maximum flow speeds have equal free stream velocities. If, on the contrary, the two free stream values are

unequal, then Theorem 3 on the monotonic dependence of the maximum speed on the free stream speed states that the two maximum speeds must be unequal, which contradicts the hypothesis.

Comparison theorem. Let us consider the limiting case of Theorem 1 when the two flows have the same free stream velocity, the other hypotheses remaining unchanged. This section will be devoted to proof of the comparison theorem that in this limiting case, as in Theorem 1, the inequality $q(P) < \bar{q}(P)$ prevails, with the possible exception that now the equality $q(P) = \bar{q}(P)$ may hold if and only if the two flows are identical. This theorem is the exact analogue for subsonic flows of the Lavrentieff comparison theorem⁶ for incompressible fluids, and while both these results are probably true in equal generality we are able to establish the theorem here for subsonic flows only under some restriction of the class of admissible flows.

The theorem is easily proved in two special situations. In the first we suppose it known that the second derivatives of the stream function for either of the two comparison flows are $O(1/r^{1+\epsilon})$ as $r \rightarrow \infty$. Then the coefficients d and e in (5) also vanish to this order. Since we have $Q_0 = \bar{Q}_0$ in the present theorem, inequality (8) for the difference ω of the stream functions becomes

$$\begin{aligned}\omega &\geq -\epsilon(r)(y - y_r), & \epsilon(r) &\rightarrow 0 \text{ as } r \rightarrow \infty, \\ &= o(r) & &\text{as } r \rightarrow \infty.\end{aligned}$$

The hypotheses of the Phragmén-Lindelöf theorem [4] for equation (5) are therefore fulfilled. This allows us to conclude from the inequality $\omega \geq 0$ on the boundary of D that the same inequality holds throughout D . The comparison theorem follows by application of the boundary point lemma. Although the stated limitation on the second derivatives is restrictive, it is undoubtedly fulfilled in most flows of interest.

The second case we treat is that for which both flow regions D and $\bar{D}$ are of the type considered in Theorems 5 and 6. If ψ and $\bar{\psi}$ are the respective stream functions, each being zero on its boundary, then by Theorem 5,

$$\psi - Q_0 y \rightarrow -Q_0 y_0, \quad \bar{\psi} - Q_0 y \rightarrow -Q_0 \bar{y}_0 \quad \text{as } (x, y) \rightarrow \infty;$$

here $y_0, \bar{y}_0$ are the ordinates of the two boundaries at infinity, and obey the inequality $y_0 \geq \bar{y}_0$. Subtracting, we get

$$\bar{\psi} - \psi \rightarrow Q_0(y_0 - \bar{y}_0) \geq 0,$$

which leads to the conclusion, $\bar{\psi} - \psi \geq 0$ throughout D . The comparison theorem now follows in the usual way from the boundary point lemma. This proves the comparison theorem in particular for the important case of symmetric flows past each of two finite bodies, one being contained within the other. At a common

⁶ See references [1], [2], [3].

regular point of contact of both boundaries, the flow speed is larger for the inner body.

Our main effort will be directed to proving the comparison theorem under the condition that at least one of the two flow regions D and $\bar{D}$, say D , allows a *finite Dirichlet integral*. By this we shall mean that there exists a continuous piecewise smooth function $f(x, y)$ defined in D and vanishing on its boundary, such that the integral

$$\iint_D [f_x^2 + (f_y - Q_0)^2] dx dy$$

is finite.⁷ In his proof of the existence of subsonic flows past finite bodies Shiffman [6] bases his argument on the fact that his flow regions allow a finite Dirichlet integral. However, this argument carries over without modification to any flow region allowing finite Dirichlet integral and whose boundary satisfies his smoothness condition of bounded curvature. Under these assumptions Shiffman's variational theory provides a family of uniquely determined solutions $\Psi(x, y; Q)$ in D depending continuously on the free stream mass flow Q . While the solutions of the variational problem for arbitrary Q do not necessarily define *flows*, that is, solutions of (4), throughout D , they nevertheless satisfy an elliptic differential equation

$$(15) \quad A(\Psi_x, \Psi_y)\Psi_{xx} + 2B(\Psi_x, \Psi_y)\Psi_{xy} + C(\Psi_x, \Psi_y)\Psi_{yy} = 0$$

which coincides with (4) over the range of values of Ψ_x, Ψ_y in

$$0 \leq \sqrt{\Psi_x^2 + \Psi_y^2} \leq b_1 < b_0,$$

where b_0 is the value of the mass flow at Mach number 1, and b_1 may be arbitrary provided $b_1 < b_0$. In particular, b_1 can be chosen so large that any given subsonic solution of the flow equation (4) also satisfies (15). Let $\psi(x, y)$ be such a stream function defining a flow in D , and let its free stream mass flow be Q_0 . Since the functions $\Psi(x, y; Q)$ have the limit behavior,

$$\Psi_x \rightarrow 0, \quad \Psi_y \rightarrow Q \quad \text{as } (x, y) \rightarrow \infty \text{ in } D,$$

while the difference $\Psi(x, y; Q) - \psi(x, y)$ enjoys the maximum principle in D , we see as in the proof of Theorem 1 that the inequality $Q_1 > Q_0 > Q_2$ implies

$$\Psi(x, y; Q_1) > \psi(x, y) > \Psi(x, y; Q_2), \quad (x, y) \in D.$$

Now letting $Q_1, Q_2 \rightarrow Q_0$ and recalling the continuous dependence of the functions Ψ on Q , we have in the limit

$$(16) \quad \Psi(x, y; Q_0) \equiv \psi(x, y).$$

⁷ A sufficient condition for the domain to allow a finite Dirichlet integral is that under inversion the boundary curve have curvature at the origin satisfying a Hölder condition. While this includes a broad class of domains, it excludes, for example, regions with parabolic boundary behavior at infinity.

In other words, the solution $\Psi(x, y; Q_0)$ of the variational problem is actually a flow and must coincide with the given solution in D . In addition, the functions $\Psi(x, y; Q)$ define flows in D if $|Q - Q_0|$ is sufficiently small. We note that (16) extends Theorem 6 by enlarging the class of flows past a wall for which uniqueness is proved.

We turn again to the proof of the comparison theorem for flows $\psi, \bar{\psi}$ in $D, \bar{D}$ having the same free stream values $Q_0 = \bar{Q}_0$. Let $\Psi(x, y; Q_1)$ be one of the continuous family of flows in D discussed above, and let $Q_1 < Q_0$. As before we have

$$\bar{\psi}(x, y) - \Psi(x, y; Q_1) \geq 0 \quad \text{in } D;$$

from this and (16) we obtain in the limit as $Q_1 \rightarrow Q_0$,

$$\bar{\psi}(x, y) - \psi(x, y) \equiv \bar{\psi}(x, y) - \Psi(x, y; Q_0) \geq 0, \quad (x, y) \in D.$$

This conclusion is obviously unchanged if $\bar{D}$ instead of D allows a finite Dirichlet integral. Applying the boundary point lemma at the common boundary point P of the two flows, we then have the comparison theorem.

We observe that the essence of the proof is to exploit an appropriate continuous family of functions, which in the preceding argument consists of the functions $\Psi(x, y; Q)$. Thus, dropping the condition on the Dirichlet integral, we can state more generally that if the flow in D or $\bar{D}$ is imbedded in a family of flows depending continuously on the free stream velocity, the theorem follows in the same way. Indeed, in the incompressible case it suffices to take simply the constant multiples $\alpha\psi(x, y)$. Since the assumption of a continuous family is one that we can expect to be fulfilled under any reasonable circumstances, we embody this in our statement of the comparison theorem.

Theorem 8. *Let two subsonic flows having the same free stream velocity be defined in regions D and $\bar{D}$, each being bounded by a streamline extending to $x = \pm\infty$. We suppose that the flow in D or $\bar{D}$ is one of a family of such flows depending continuously on the free stream velocity; this condition is satisfied, in particular, if the region allows a finite Dirichlet integral. Then if $D \subset \bar{D}$, and if the two bounding streamlines have a regular point P in common, the respective flow speeds q and $\bar{q}$ satisfy at P the inequality*

$$q(P) \leq \bar{q}(P);$$

the equality holds if and only if the two flows are identical and $D = \bar{D}$.

It would be desirable to have a proof of this theorem which is valid without express limitation on the domain or on the derivatives at infinity, and which preferably does not rest on a deep existence theory. As indicated above, such a proof, entirely of elementary character, is available for incompressible fluids.

Flows in a channel. We now extend the preceding results to flows in a channel. As we shall see, the proofs are now both simpler and more general in certain respects. To be specific, by a flow in a channel we shall mean a flow in a region

bounded by two streamlines, each having its ends at infinity, and such that the flow is uniform at both extremities of the region; the two velocities at the ends of the channel are not required to have the same direction. We prove the following theorems simultaneously.

Theorem 9. Let D and $\bar{D}$ be two channels bounded by the pairs of curves γ_1, γ_2 and $\bar{\gamma}_1, \bar{\gamma}_2$ respectively. Suppose $\bar{\gamma}_1$ and γ_2 together bound a region containing γ_1 and $\bar{\gamma}_2$ in its closure (Fig. 2); (γ_1 and $\bar{\gamma}_1, \gamma_2$ and $\bar{\gamma}_2$ may, but need not, coincide in whole or in part). Consider subsonic flows through D and $\bar{D}$ for which the total flux Q through D and $\bar{Q}$ through $\bar{D}$ satisfy the inequality $Q \leq \bar{Q}$. If P is a regular bound-

FIG. 2

ary point common to γ_1 and $\bar{\gamma}_1$ then the respective flow speeds q and $\bar{q}$ obey at P the inequality,

$$q(P) \leq \bar{q}(P);$$

furthermore, the equality holds if and only if the two flows are identical and $D = \bar{D}$.

Theorem 10. The flow through a channel is uniquely determined by the total flux or equivalently, by the flow speed at either end. The flow speed at any regular boundary point is a strictly increasing function of the flux through the channel.

Corollary. In the symmetric flow past a body immersed in a channel, the flow is uniquely determined by the flux through the channel and also by the maximum flow speed on the body.

Proof. Let ψ and $\bar{\psi}$ be the respective stream functions in D and $\bar{D}$, with

$$\psi = 0 \text{ on } \gamma_1, \quad \psi = Q \text{ on } \gamma_2, \quad 0 < \psi < Q \text{ in } D,$$

$$\bar{\psi} = 0 \text{ on } \bar{\gamma}_1, \quad \bar{\psi} = \bar{Q} \text{ on } \bar{\gamma}_2, \quad 0 < \bar{\psi} < \bar{Q} \text{ in } \bar{D}.$$

We observe that $\bar{\psi} - \psi \geq 0$ on γ_1 and on $\bar{\gamma}_2$. Consider the behavior of the streamlines γ_1 and $\bar{\gamma}_2$. If $\bar{\gamma}_2$ intersects γ_1 on both sides of P then there is a subregion of D bounded by subarcs of $\bar{\gamma}_2$ and γ_1 , the latter containing P in its interior. On the boundary of this subregion, $\bar{\psi} - \psi \geq 0$, and therefore the strict inequality holds in the interior. Theorem 9 follows after application of the boundary point lemma.

We may therefore suppose that γ_1 extends to infinity in at least one direction without intersecting $\bar{\gamma}_2$. Let us fix our attention on this end of the channel D and for convenience take the limit direction, which is the same for both flows, to be parallel to the x -axis. Set the limit width of D and $\bar{D}$ to be d and $\bar{d}$, respectively, and Δ to be that of the union of D and $\bar{D}$. For large x , let $y_0 = y_0(x)$ and $\bar{y}_0 = \bar{y}_0(x)$ be the ordinates on γ_1 and $\bar{\gamma}_1$. At any point (x, y) in the intersection of D and $\bar{D}$, we have

$$\psi(x, y) = \frac{Q}{d} (y - y_0) + \epsilon_1(x, y), \quad \bar{\psi}(x, y) = \frac{\bar{Q}}{\bar{d}} (y - \bar{y}_0) + \bar{\epsilon}_1(x, y),$$

where $\epsilon_1, \bar{\epsilon}_1 \rightarrow 0$ as $x \rightarrow \infty$. Now,

$$\frac{\bar{Q}}{\bar{d}} (y - \bar{y}_0) \geq \frac{\bar{Q}}{\Delta} (y - \bar{y}_0) \geq \frac{Q}{\Delta} (y - \bar{y}_0) = \frac{Q(y - y_0 + y_0 - \bar{y}_0)}{d + y_0 - \bar{y}_0} \geq \frac{Q}{d} (y - y_0).$$

Therefore

$$\begin{aligned} \bar{\psi}(x, y) - \psi(x, y) &\geq \left(\frac{\bar{Q} - Q}{\Delta} \right) (y - y_0) + \bar{\epsilon}_1 - \epsilon_1 \\ &\geq -\epsilon(x) \end{aligned}$$

where

$$\epsilon(x) = \max_y |\bar{\epsilon}_1(x, y) - \epsilon_1(x, y)| \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

A similar conclusion holds at the other end of the channel if γ_1 and $\bar{\gamma}_2$ do not intersect. The maximum principle now tells us that $\bar{\psi} - \psi \geq 0$ in the intersection of D and $\bar{D}$. The theorems and the corollary follow in the usual way.

We observe that the problem of channel flows is essentially simpler and the results more general than for flows past a wall. Our proof does not presuppose an existence theory (as in Theorem 8), nor does it require results on differential equations any deeper than the elementary maximum principle (such as Theorem 4).

Axially symmetric flows. The fact that comparison methods deal directly with the physical plane suggests the means of generalization of our results to the case of axial symmetry. In this respect at least, comparison methods have an advantage over those resting on the hodograph transformation. While at present our results for axially symmetric flows are not quite so complete as for plane

ones, we shall see that substantially the same theorems hold for both classes of flows.

The equation for the stream function in axial symmetry is

$$(17) \quad \left(1 - \frac{\psi_y^2}{(\rho cy)^2}\right) \psi_{xx} + \frac{2\psi_x \psi_y}{(\rho cy)^2} \psi_{xy} + \left(1 - \frac{\psi_x^2}{(\rho cy)^2}\right) \psi_{yy} - \frac{\psi_y}{y} = 0,$$

where x is the axial and y the radial variable, $y > 0$. As in the case of plane flows, the difference of two subsonic stream functions will again satisfy the maximum principle. The boundary point lemma holds provided the boundary point P does not lie on the axis of symmetry, $y = 0$.⁸ It is therefore convenient to modify the definition of regular boundary point so as now to exclude points on the axis of symmetry, leaving the definition otherwise the same. Theorems 1 and 2 on the monotonic dependence of the boundary speed on the free stream speed then remain true for axially symmetric flows as well. This we show as follows.

As in the proof of Theorem 1, let $\psi, \bar{\psi}$ be the stream functions in the flow region $D, \bar{D}$ for which the respective free stream mass flows are $Q_0, \bar{Q}_0$. In polar coordinates, we obtain for the difference $\omega \equiv \bar{\psi} - \psi$,

$$\frac{1}{r^2 \sin \theta} \frac{\partial \omega}{\partial \theta} = (\bar{Q}_0 - Q_0) \cos \theta + \epsilon_1(r, \theta), \quad \epsilon_1(r, \theta) \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

By integration along the arc $r = \text{const.}$ from the point (r, θ) to the boundary of D , on which $\omega \geq 0$, we obtain the analogue of (8):

$$(18) \quad \omega(r, \theta) \geq [(\bar{Q}_0 - Q_0) - \beta \epsilon(r)] \left(\frac{y^2 - y_r^2}{2} \right), \quad \epsilon(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

where β is a constant depending only on the domain D , and y_r has the same meaning as before. Hence, for sufficiently large r we see that $\omega > 0$ in D . The consequence is that the inequality holds everywhere in D . Application of the boundary point lemma at the common regular boundary point P , whose ordinate is say $y_P > 0$, gives the inequality,

$$0 < \frac{1}{y_P} \frac{\partial \omega}{\partial n}(P) = \frac{1}{y_P} \frac{\partial \bar{\psi}}{\partial n}(P) - \frac{1}{y_P} \frac{\partial \psi}{\partial n}(P);$$

this implies $\bar{q}(P) > q(P)$, and the theorem is proved.

The theorem on monotonic dependence of the boundary flow speed on the free stream speed follows as a corollary.

That the maximum flow speed is achieved on the boundary of the flow region or infinity is true in axial symmetry as well as in the plane. In the case of flow past a finite body or past a wall with an interior point of greatest ordinate the maximum speed is furthermore achieved on the body. This we establish as in

⁸ For points on the axis of symmetry the appropriate lemma, not yet established, would state $\partial^2 \omega(P) / \partial y^2 > 0$.

the plane case by observing that at the point of maximum ordinate the flow must have a larger speed than at infinity unless the flow is constant. For, let ψ be the stream function of the flow, normalized to zero on the boundary, and let Q_0 be its free stream mass flow. If y_M is the maximum ordinate on the body or wall, let $\frac{1}{2}\alpha Q_0(y^2 - y_M^2)$ be the stream function of the constant axially symmetric flow of mass flow αQ_0 . Then, as above, we have in the half plane $y > y_M$ for any $\alpha < 1$,

$$\psi(x, y) - \frac{1}{2}\alpha Q_0(y^2 - y_M^2) > 0.$$

Letting $\alpha \rightarrow 1$ we obtain in the limit,

$$\psi(x, y) \geq \frac{1}{2}Q(y^2 - y_M^2),$$

from which the equality is excluded if ψ is not the uniform flow. Now applying the boundary point lemma at the point P of maximum ordinate $y_P = y_M$ we see that

$$\frac{1}{y_P} \frac{\partial \psi}{\partial y}(P) > Q_0$$

and therefore that the flow speed at P is greater than the free stream speed.

The same argument yields the result that the flow speed at a point of minimum ordinate is less than the free stream value provided the minimum ordinate is not zero. If, however, the minimum ordinate is zero, we cannot assert a strict inequality for lack of a boundary point lemma on the axis of symmetry. In this case then, we may state with certainty only that the flow speed on the axis of symmetry is *not greater* than the free stream value. Since the maximum speed of the flow occurs either on the boundary at a point of non-zero ordinate or at infinity, we may assert that the maximum speed of the flow is a strictly increasing function of the free stream speed.

At present the principal gaps in our treatment of the axially symmetric flow problem are the counterparts of Theorems 5 and 8. If we could show, in analogy to Theorem 5, that the function

$$\psi(x, y) - \frac{Q_0 y^2}{2}$$

has a definite limit at infinity, then the results corresponding to Theorems 6 and 7 would follow immediately.

The Lavrentieff comparison theorem very likely can be extended in equal generality to plane and axially symmetric flows. Under the assumption of a continuous family of flows, the result follows as in Theorem 8; however, knowledge of the existence of such a continuous family for a domain allowing finite Dirichlet integral is at present lacking in the case of axial symmetry. It is not difficult to show—although we omit the details here—that the theorem is true also under

the assumption that one of the two comparison flows has a stream function whose second derivatives approach zero as $1/r^{1+\epsilon}$ as $r \rightarrow \infty$.

Extension of the results on plane channel flows to the case of axial symmetry introduces no new difficulties. Indeed, Theorems 9 and 10 and the corollary can be taken over word for word while the proof requires only inessential changes appropriate to axial symmetry. We may conclude, in summary, that Theorems 1, 2, 3, 9, 10 hold unchanged for flows with axial symmetry, while Theorems 7 and 8 remain true at least in part.

Remark on flows with free boundaries. In the work of Lavrentieff [1], Serrin [2], and the author [3] on incompressible fluids, comparison methods have been successfully applied to a variety of uniqueness problems for flows with free boundaries. It requires but little variation from the procedure in these proofs to obtain the same results for compressible fluids, and it seems unnecessary therefore to enter into the details here. We remark that the uniqueness theorems for finite cavities and jet flows from a channel carry over without essential change. However, the proof of uniqueness for the Helmholtz infinite cavity remains incomplete in both the plane and the axially symmetric case for lack of a sufficiently general proof of Lavrentieff's comparison theorem and also because the formula for asymptotic shape of the free boundary has not been proved for compressible fluids.

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