Two Measures of Vorticity

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In Memoriam P. F. NEMÉNYI

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1. Origin of the problem. In the course of his brilliant researches on Hugoniot's theory of wave motions Hadamard (1903) discovered that oblique passage through a curved shock front destroys the irrotational character of a uniform gas flow. Many methods of approximate solution of the hydrodynamical equations are based upon the existence of a velocity-potential, so that unless it can be shown that Hadamard's vorticity is actually negligible, these methods therefore become inapplicable to the region behind a curved shock. Although the more accurate treatments do indeed take account of the vorticity, many calculations of supersonic flows neglect their rotational character. Now the vorticity itself, not being dimensionless, cannot be taken as a meaningful measure of the amount of rotation present. We may therefore set the following problem: *to find a quantitative measure of the amount of rotation in a motion.*

We might of course select some particular motion as a standard and rate all others in terms of it. This procedure is used for the choice of physical units: when we say a time is "very short" we mean usually that it is much smaller than the 51,400th part of the period of the earth's rotation about its axis. When we say a motion is "slightly rotational," however, we do not really mean merely that it is much less rotational than some other perfectly arbitrarily selected motion. We desire rather some absolute criterion of rotationality. A particular class of motions will then indeed appear as a unit or standard, but the precision of this class should be an object of research rather than a matter of mere definition.

In fact, the need for such a measure is older than theoretical gas dynamics. The first exact solution of the hydrodynamical equations to represent surface waves was obtained by Gerstner (1802), but it is not a potential flow and thus cannot be generated through the action of gravity or of impulsive pressures. Some have tried to overcome this objection by suggesting that possibly the rotation is "small" enough to be negligible. In taking up this question Levi-Civita (1940) in one of his last papers proposed the measure

$$\Omega(t) \equiv \frac{1}{2} \int_0^t \mathbf{w} \, dt, \quad (1.1)$$

where $\mathbf{w}$ is the vorticity:

$$\mathbf{w} \equiv \text{curl } \mathbf{v}, \quad (1.2)$$

$\mathbf{v}$ being the velocity and t the time, and the space co-ordinates are held constant during the integration. After some difficult approximations Levi-Civita concluded that in terms of this measure Gerstner's waves are strongly rotational.

The measure (1.1) is itself not satisfactory, however. For a rigid rotation at constant angular velocity $\frac{1}{2}\mathbf{w}$ we obtain $2\Omega = \mathbf{w}t$, and thus the *measure* of the rotation increases with the time without limit, while the *character* of the motion is absolutely unchanged: surely it is not more rotational at $t = 10$ than

at $t = 0$. If Ω be calculated for two different rigid rotations, its value for that at greater angular speed exceeds its value for that at lesser angular speed, but certainly any two rigid rotations are equally rotational in character: the measure of rotation should indicate not the relative angular speeds but the *rotational quality* or *degree*, and thus for steady rigid motions should be constant in time and independent of the angular speed. Ω is a vector, but if possible the rotational measure should be a simple number: in particular, the degree of rotation of a rigid motion should be independent of the direction of the axis of rotation. Finally, according to Levi-Civita's criterion every motion has the measure $\Omega = 0$, the same as that for an irrotational motion, at some perfectly arbitrary initial instant $t = 0$.

We shall therefore seek some other dimensionless measure of vorticity.

2. The kinematical vorticity number $\mathcal{W}_K$. A fundamental theorem of Cauchy (1841) and Stokes (1845) states that any continuous motion of a medium at a given instant is composed of a translation, a rigid rotation, and a pure deformation. A vector representing the rotation is the vorticity $\mathbf{w}$, and a scalar indication of the amount of rotation is accordingly its magnitude w . Euler (1770) showed that the numerical magnitude of the rate of deformation of any line segment can be calculated from the tensor

$$d_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}), \quad (2.1)$$

where v_i is the velocity vector. Since $d^{ij}d_{ij} > 0$ if any one component d_{ij} is non-vanishing, a scalar indication of the amount of deformation is accordingly the invariant

$$(kd^{ij}d_{ij})^{\frac{1}{2}}, \quad (2.2)$$

where k is a dimensionless constant. Both w and $(kd^{ij}d_{ij})^{\frac{1}{2}}$ are of the dimension $\mathbf{T}^{-1}$, where $\mathbf{T}$ is a unit of time. Consequently

$$\frac{w}{(kd^{ij}d_{ij})^{\frac{1}{2}}} \quad (2.3)$$

is a dimensionless measure of the ratio of the magnitude of the rotation to the magnitude of the deformation. In order to find a suitable value for k , let us consider the special case when the deformation reduces to a pure shearing of the $y = \text{const.}$ planes. Then the mean χ of the instantaneous rates of rotation of the $x = \text{const.}$ and $y = \text{const.}$ planes is given by $\chi = d_{xy}$, while the local angular velocity magnitude as always is given by $\omega = \frac{1}{2}w$, so that (2.3) becomes

$$\frac{w}{(2kd_{xy}^2)^{\frac{1}{2}}} = \frac{2\omega}{\sqrt{2k}\chi}. \quad (2.4)$$

Now it would seem most natural if our measure of the proportion of rotation in a motion should in this special case reduce to the ratio ω/χ of the one angular speed

to the other. From (2.4) it follows that (2.3) will indeed reduce to ω/χ if and only if $k = 2$. Accordingly we define the **kinematical vorticity number** $\mathcal{W}_K$ as the dimensionless scalar

$$\mathcal{W}_K \equiv \frac{w}{(2d^{ij}d_{ij})^{\frac{1}{2}}}. \quad (2.5)$$

This measure may be calculated from the instantaneous velocity field at a point. It is generally a function both of place and of time.

Since a rigid rotation is characterized by $d_{ij} = 0$, $w \neq 0$, while an irrotational non-rigid motion is characterized by $w = 0$, $d^{ij}d_{ij} \neq 0$, from (2.5) we have

Theorem 1. *At a given point, a rotational motion is instantaneously rigid if and only if*

$$\mathcal{W}_K = \infty,$$

while a non-rigid motion is instantaneously irrotational if and only if

$$\mathcal{W}_K = 0.$$

Only when the velocity gradient vanishes does the measure (2.5) fail to exist. Thus all possible motions with the sole exception of rigid translations are assigned a numerical degree of rotationality on a scale from 0 to ∞ , a rigid motion being the most rotational type of motion possible. A *nearly irrotational motion* is one for which

$$\mathcal{W}_K \ll 1. \quad (2.6)$$

3. Analogues. Evidently in the case of deformation in which the displacement gradients are infinitesimal it is possible to measure the relative amount of local rotation by forming a number analogous to $\mathcal{W}_K$ but with w replaced by the magnitude of the infinitesimal rotation and d_{ij} replaced by the infinitesimal strain tensor. We do not develop the properties of this local rotation number for two reasons. First, a necessity analogous to that impelling us to introduce $\mathcal{W}_K$ is wanting: the rotation vector is itself dimensionless and a *very small rotation* is a perfectly well-defined concept. Second, there appear to be no applications of interest in the theory of elasticity: the ordinary theory requires that the rotation be infinitesimal, and then neglects it altogether, but if the rotation be large the whole theory must be modified: the displacement components do not constitute a vector field and their commutators fail to furnish a measure of the rotation.

Professor S. Corrsin pointed out to me in 1950 that if a *mean vorticity number* be defined by (2.5) with all quantities replaced by averages, then in isotropic turbulence this number has always the value 1.

4. Alternative expressions for $\mathcal{W}_K$. We return now to the formula (2.5). First, the rotation tensor ω_{ij} is defined by

$$\omega_{ij} \equiv \frac{1}{2}(v_{i,j} - v_{j,i}). \quad (4.1)$$

Then $v_{i,j} = d_{ij} + \omega_{ij}$ and $-2\omega^{ij}\omega_{ji} = w^2$. We have

$$\begin{aligned} 2d^{ij}d_{ij} &= 2d^{ij}v_{j,i}, \\ &= 2(v^{i,j} - \omega^{ij})v_{j,i}, \\ &= 2v^{i,j}v_{j,i} - 2\omega^{ij}\omega_{ji}, \\ &= 2v^{i,j}v_{j,i} + w^2. \end{aligned} \quad (4.2)$$

Hence when $w \neq 0$ (2.5) may be put into the form

$$\mathcal{W}_K = \left(1 + \frac{2v^{i,j}v_{j,i}}{w^2}\right)^{-\frac{1}{2}}. \quad (4.3)$$

Second, in terms of the principal rates of extension d_1, d_2, d_3 we may write the squared intensity of deformation as

$$d^{ij}d_{ij} = (d_1)^2 + (d_2)^2 + (d_3)^2. \quad (4.4)$$

The first and second principal invariants I and II of d_{ij} are

$$\begin{aligned} I &\equiv d^k_k = v^k_{,k} = d_1 + d_2 + d_3 = \operatorname{div} \mathbf{v}, \\ II &\equiv \frac{1}{2}\delta^{jk}_{lm}d^l_jd^m_k = d_2d_3 + d_3d_1 + d_1d_2. \end{aligned} \quad (4.5)$$

From (4.4) and (4.5) we have

$$d^{ij}d_{ij} = -2II + I^2, \quad (4.6)$$

and (2.5) becomes

$$\mathcal{W}_K = \frac{w}{(2I^2 - 4II)^{\frac{1}{2}}}. \quad (4.7)$$

Third, it is easy to show that

$$K \equiv \frac{1}{2}\delta^{ij}_k d^k_{,i}v^i_{,j} = II + \frac{1}{4}w^2. \quad (4.8)$$

Hence (4.7) becomes

$$\mathcal{W}_K = \left(1 + \frac{2I^2 - 4K}{w^2}\right)^{-\frac{1}{2}}. \quad (4.9)$$

Fourth, let $\mathbf{a}$ be the acceleration vector. Then by taking the divergence of Euler's acceleration formula,

$$a^i = \frac{\partial v^i}{\partial t} + v^i_{,j}v^j, \quad (4.10)$$

we obtain

$$a^{i,i} = \frac{\partial v^{i,i}}{\partial t} + v^{i,ji} v^j + v^{i,j} v_{j,i}, \quad (4.11)$$

and hence

$$v^{i,j} v_{j,i} = \operatorname{div} \mathbf{a} - \dot{I}, \quad (4.12)$$

where the dot denotes the material derivative. From this result and (4.3) follows

$$\mathcal{W}_K = \left(1 + 2 \frac{\operatorname{div} \mathbf{a} - \dot{I}}{w^2} \right)^{-\frac{1}{2}}, \quad (4.13)$$

a form of particular interest since it involves vectors and scalars only.

Fifth, let the acceleration be expressed in terms of its scalar and vector potentials:

$$\mathbf{a} = -\operatorname{grad} \Phi + \operatorname{curl} \mathbf{\Pi}. \quad (4.14)$$

Then since $\operatorname{div} \mathbf{a} = -\nabla^2 \Phi$, (4.13) becomes

$$\mathcal{W}_K = \left(1 - 2 \frac{\nabla^2 \Phi + \dot{I}}{w^2} \right)^{\frac{1}{2}}, \quad (4.15)$$

a form noteworthy since it shows that *the functional form of $\mathcal{W}_K$ is independent of the functional form of the vector potential $\mathbf{\Pi}$ of the acceleration.*

Sixth, by taking the divergence of Lagrange's acceleration formula,

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \operatorname{grad} \frac{1}{2} v^2, \quad (4.16)$$

we obtain

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{\partial \operatorname{div} \mathbf{v}}{\partial t} + \mathbf{v} \cdot \operatorname{curl} \mathbf{w} - w^2 + \nabla^2 \frac{1}{2} v^2, \\ &= \frac{\partial \operatorname{div} \mathbf{v}}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \operatorname{div} \mathbf{v} - \mathbf{v} \cdot \nabla^2 \mathbf{v} + \nabla^2 (\frac{1}{2} v^2) - w^2, \\ &= \dot{I} - \mathbf{v} \cdot \nabla^2 \mathbf{v} + \nabla^2 (\frac{1}{2} v^2) - w^2. \end{aligned} \quad (4.17)$$

From (4.13) we have then

$$\mathcal{W}_K = \left(\frac{\nabla^2 v^2 - 2\mathbf{v} \cdot \nabla^2 \mathbf{v}}{w^2} - 1 \right)^{-\frac{1}{2}}. \quad (4.18)$$

5. Motions with vorticity number unity. Our next problem is to determine the unit of rotationality, that is, to characterize the class of all motions in which $\mathcal{W}_K = 1$. From the reasons motivating our choice $k = 2$ in the fundamental definition (2.5) it follows that a simple shearing motion is a member of

this class, and indeed any generalized Poiseuille flow $v_x = f(y, z)$, $v_y = 0$, $v_z = 0$ may easily be seen to have $\mathcal{W}_K = 1$ everywhere. More generally, however, from the formulae of the preceding section we can solve the general problem at a glance:

Theorem 2. *In a rotational motion, necessary and sufficient that*

$$\mathcal{W}_K = 1$$

is any one of the six equivalent conditions

$$v^{i,j} v_{j,i} = 0, \quad (5.1)$$

$$4II + w^2 = 2I^2, \quad (5.2)$$

$$2K = I^2, \quad (5.3)$$

$$\operatorname{div} \mathbf{a} = \dot{I}, \quad (5.4)$$

$$\nabla^2 \Phi = -\dot{I}, \quad (5.5)$$

$$\nabla^2 v^2 - 2\nabla \cdot \nabla^2 \mathbf{v} = 2w^2. \quad (5.6)$$

The generalized Poiseuille flows appear as the simplest special cases satisfying these conditions.

A motion in which material volumes are conserved, *i.e.* in which $I = 0$, is called *isochoric*. From (5.4) we may conclude a result of intermediate generality:

Corollary 1. *In any rotational isochoric motion without acceleration¹*

$$\mathcal{W}_K = 1.$$

The generalized Poiseuille flows are again the simplest special case.

The case of isochoric motions in general presents especial interest. From (5.4) follows

Theorem 3. *In a rotational isochoric motion*

$$\mathcal{W}_K = 1$$

if and only if the acceleration field be solenoidal.

From (5.2) and (5.3) follow the equivalent condition

$$K = II + \frac{1}{4}w^2 = 0. \quad (5.7)$$

Now the class of isochoric motions satisfying (5.7) has been characterized in an elegant way by Bilimovitch (1948)². Expressed in the terminology of the present paper, the result of Bilimovitch is equivalent to

¹ An interesting analysis of motions without acceleration has been given by CALDONAZZO (1947), but his results are not employed here.

² In my review [Math. Rev. **10**, 517 (1949)] of this paper the factor $\frac{1}{4}$ is erroneously omitted from the condition. Motions in which $K = 0$ had been studied earlier, first kinematically and then for the case of a viscous liquid, by HAMEL (1936).

Theorem 4. *In an isochoric motion, let $\mathcal{V}_S$ be the volume inclosed by an arbitrary closed surface $\mathcal{S}$, let $\mathcal{V}_V$ be the volume inclosed by the surface swept out by the termini of the velocity vectors at all points on $\mathcal{S}$, and let $\mathcal{V}_H$ be the volume inclosed by the surface swept out by these same vectors when their origins are all put at the origin of co-ordinates. Then a necessary and sufficient condition that*

$$\mathcal{W}_K = 1$$

at all points is

$$\mathcal{V}_V = \mathcal{V}_S + \mathcal{V}_H. \quad (5.8)$$

For proof the reader is referred to the paper of Bilimovitch, or to a paper of mine (1950) where the result of Bilimovitch is generalized.

For isochoric motions in general (4.18) is equivalent to

$$\mathcal{W}_K = \left(\frac{\nabla^2 v^2 + 2\mathbf{v} \cdot \text{curl } \mathbf{w}}{w^2} - 1 \right)^{-\frac{1}{2}}. \quad (5.9)$$

6. First example: flow in a rotating cylinder. If $\text{curl } \mathbf{w} = 0$ the formula (5.9) reduces to

$$\mathcal{W}_K = \left(\frac{\nabla^2 v^2}{w^2} - 1 \right)^{-\frac{1}{2}}. \quad (6.1)$$

We may apply this result to isochoric flows in a rotating rigid cylindrical vessel, for we have $\mathbf{v} = \mathbf{k} \times \text{grad } \psi$, where $\psi = \psi(x, y)$ and $\nabla^2 \psi = 2\omega$, ω being the angular speed of the rotating vessel. An easy calculation yields $w = 2\omega$,

$$\begin{aligned} \nabla^2 v^2 &= \nabla^2 \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right], \\ &= 2 \left[\left(\frac{\partial^2 \psi}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 \right], \end{aligned} \quad (6.2)$$

so that (6.1) becomes

$$\mathcal{W}_K = \left[1 + \frac{\left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 - \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2}}{\omega^2} \right]^{-\frac{1}{2}}. \quad (6.3)$$

Any of the cylinders $\psi = \text{const.}$ may be taken as the boundary. For a vessel whose cross-section is a conic, $\psi = \frac{1}{2}(\omega + A)x^2 + \frac{1}{2}(\omega - A)y^2 + Kx + Ly + M$, and (6.3) yields

$$\mathcal{W}_K = \left| \frac{\omega}{A} \right|. \quad (6.4)$$

Thus $\mathcal{W}_K$ is constant throughout the flow, and for different types of vessels assumes all possible values from 0 to ∞ . The case $A = 0$ yields $\mathcal{W}_K = \infty$, since it represents a rigid rotation of a circular cylinder; for an elliptic vessel, $\mathcal{W}_K > 1$; for a parabolic vessel, $\mathcal{W}_K = 1$; for a hyperbolic vessel, $\mathcal{W}_K < 1$. More specifically, for the elliptical vessel $x^2/a^2 + y^2/b^2 = 1$ we obtain

$$\mathcal{W}_K = \left| \frac{a^2 + b^2}{a^2 - b^2} \right|, \quad (6.5)$$

while for the hyperbolic vessel $x^2/a^2 - y^2/b^2 = 1$ we obtain

$$\mathcal{W}_K = \left| \frac{a^2 - b^2}{a^2 + b^2} \right|. \quad (6.6)$$

In summary of these results we have

Theorem 5. *In an isochoric flow of uniform vorticity 2ω within a rigid cylindrical vessel rotating at angular speed ω , if the cross-section of the vessel be a conic the vorticity number is uniform and independent of ω , being equal to 1, inferior to 1, or superior to 1 according as the vessel is parabolic, hyperbolic, or elliptic.*

It would be interesting to characterize the class of flows whose vorticity number is uniform.

7. Second example: Beltrami motion. A simple expression for $\mathcal{W}_K$ results when

$$\mathbf{w} \times \mathbf{v} = 0, \quad w \neq 0. \quad (7.1)$$

Motions satisfying these conditions were first studied in detail by Gromeka and Beltrami and are now generally called *Beltrami motions*. For Beltrami motion, (4.16) put into (4.13) yields

$$\nabla^2 v^2 = w^2 \left(\frac{1}{\mathcal{W}_K^2} - 1 \right) + 2\mathbf{v} \cdot \text{grad } I. \quad (7.2)$$

In this memoir we shall have occasion to use the simplest properties of subharmonic and superharmonic functions. We recall that a twice continuously differentiable function f is superharmonic (subharmonic) in a region if and only if $\nabla^2 f \leq 0$ ($\nabla^2 f \geq 0$) at all interior points, and that a function which is superharmonic (subharmonic) in a region cannot experience a minimum (maximum) value at an interior point.

From these facts and from (7.2) follows

Theorem 6. *In a Beltrami motion the squared speed is*

$$\left. \begin{array}{l} \text{subharmonic} \\ \text{harmonic} \\ \text{superharmonic} \end{array} \right\} \quad \text{if and only if} \quad \left\{ \begin{array}{l} \mathcal{W}_K \leq \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \\ \mathcal{W}_K = \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \\ \mathcal{W}_K \geq \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \end{array} \right.$$

Hence at an interior point of a region where

$$\left. \begin{array}{l} \mathcal{W}_K \leq \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \\ \mathcal{W}_K = \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \\ \mathcal{W}_K \geq \left(1 - \frac{2\mathbf{v} \cdot \text{grad } I}{w^2} \right)^{-1} \end{array} \right\} \text{the speed cannot experience a} \left\{ \begin{array}{l} \text{maximum} \\ \text{maximum or minimum.} \\ \text{minimum} \end{array} \right.$$

Sufficient for the first alternative in the condition for the theorem is the pair of weaker statements $\mathcal{W}_K \leq 1$, $\mathbf{v} \cdot \text{grad } I \geq 0$. The latter states that the expansion does not decrease in the direction of motion along a stream-line. By formulating similar sufficient conditions in the other two cases also, we obtain the following

Corollary. *At an interior point of a Beltrami motion where $\mathcal{W}_K \leq 1$ ($\mathcal{W}_K = 1$) ($\mathcal{W}_K \geq 1$) and moreover the expansion does not decrease (change) (increase) in the direction of motion along a stream-line, the speed cannot experience a maximum (maximum or minimum) (minimum).*

For an isochoric Beltrami motion, of course, the condition of the Corollary is expressed in terms of $\mathcal{W}_K$ only.

An example illustrating the harmonic case is included in one of the families of flows given by Beltrami (1889), viz. $v_r = 0$, $v_\theta = K$, $v_z = K(1 - 2 \log r)^\frac{1}{2}$, where r , θ , z are cylindrical coordinates and $K = \text{const}$. The stream- (vortex-) lines are ordinary circular helices of pitch $\text{Arctan } (1 - 2 \log r)^\frac{1}{2}$.

The case $\mathcal{W}_K = 1$ may be obtained directly from (5.4) if we prefer, since for an isochoric Beltrami motion $\text{div } \mathbf{a} = 0$ if and only if $\nabla^2 v^2 = 0$. From (7.2) we observe in passing that in any isochoric Beltrami motion the relation

$$\nabla^2 v^2 \geq -w^2 \quad (7.3)$$

is satisfied, the equality holding only when the motion is rigid and consequently being impossible throughout a region. From this inequality we might have started; for in an isochoric irrotational motion, as is well known, the speed cannot ex-

perience a maximum at an interior point, and we might have asked in how far this property carries over into Beltrami motions.³

The result for the irrotational case will be generalized in Corollary 2 to Theorem 8, §10.

8. Motions with steady vorticity number. In general the vorticity number $\mathcal{W}_K$ is a function of both time and locality. In a steady motion of course the vorticity number is also steady, but the converse does not hold. In fact for motions of the type

$$\mathbf{v}(\mathbf{r}, t) = T(t)\mathbf{v}(\mathbf{r}) \quad (8.1)$$

it follows at once from (2.5) that the vorticity number is steady. Special cases of this interesting class of flows, introduced by d'Alembert (1752) and analysed by Lagrange (1762) and Cauchy (1823), have been shown by Trkal (1919) and Taylor (1923) (*cf.* also Kampé de Fériet (1923), Berker (1936)) to be dynamically possible for viscous liquids, and have been used to illustrate the manner in which viscosity causes a motion to die out. That the stream-line pattern does not change with time may seem somewhat strange and different from the effect of other types of friction, but may perhaps be easier understood in view of the fact that the balance between deformation and vorticity at any one point of the flow is also constant in time. Thus these flows are of a relatively special type, and can hardly be expected to furnish typical illustrations of the process of decay.

9. The range of $\mathcal{W}_K$ in a motion in a finite domain.⁴ In an earlier paper (1948) I employed the identity

$$v^{i,j}v_{j,i} = (v^{i,j}v_j)_{,i} - (Iv^i)_{,i} + I^2. \quad (9.1)$$

Comparison of this result with (4.2) yields

$$2d^{i,j}d_{ij} - w^2 = 2I^2 + 2(v^{i,j}v_j - Iv^i)_{,i}, \quad (9.2)$$

whence by (2.5) we obtain

$$w^2 \left(\frac{1}{\mathcal{W}_K^2} - 1 \right) = 2I^2 + 2(v^{i,j}v_j - Iv^i)_{,i}, \quad (9.3)$$

provided $w \neq 0$. Integration over a volume $\mathfrak{B}$ yields

$$\int_{\mathfrak{B}} w^2 \left(\frac{1}{\mathcal{W}_K^2} - 1 \right) dv = 2 \int_{\mathfrak{B}} I^2 dv + 2 \oint_{\mathfrak{S}} (v^{i,j}v_j - Iv^i) dS_i. \quad (9.4)$$

³ The material of this section is thus approached in my note (1952b).

⁴ The results of this section constitute a reformulation of some given in my paper (1950).

For a motion within finite boundaries to which the material adheres without slipping ($v^i = 0$ on $\mathfrak{S}$) the surface integral vanishes and we then obtain the not inelegant identity

$$\int_{\mathfrak{S}} w^2 \left(\frac{1}{\mathfrak{W}_K^2} - 1 \right) dv = 2 \int_{\mathfrak{S}} I^2 dv. \quad (9.5)$$

Since the right hand side is essentially positive, we may conclude

Theorem 7. *In a rotational⁵ motion confined by stationary walls to which the material adheres without slipping, it is necessary that there be at each instant some volume where*

$$\mathfrak{W}_K \leq 1; \quad (9.6)$$

if at a given instant there be some volume where the density of the substance is changing ($I \neq 0$), then there must also be a volume where

$$\mathfrak{W}_K < 1. \quad (9.7)$$

Corollary (Truesdell (1950)). *In an isochoric motion confined by stationary walls to which the material adheres without slipping the following formula, expressing an average balance between vorticity and deformation, holds:*

$$\int_{\mathfrak{S}} w^2 \left(\frac{1}{\mathfrak{W}_K^2} - 1 \right) dv = 0. \quad (9.8)$$

Hence in such a motion either we have

- a. $\mathfrak{W}_K = 1$ throughout; or
- b. There exists at each instant a volume where $\mathfrak{W}_K > 1$ and another where $\mathfrak{W}_K < 1$.

10. A general extremal theorem. Let us write (4.15) in the form

$$\nabla^2 \Phi = \frac{1}{2} w^2 \left(1 - \frac{1}{\mathfrak{W}_K^2} \right) - \dot{I}. \quad (10.1)$$

An equivalent result in the special case $I = 0$ was discovered by Rowland (1880). Hence follows the *general extremal theorem*:

⁵ An *isochoric* motion in a finite stationary domain to whose walls the material adheres is necessarily rotational. This and stronger results are proved by SUPINO (1949) and in a forthcoming work (1953). DUHEM (1903) made a similar statement without the restriction to isochoric motion, but his proof is not convincing and indeed it can be seen from a counter-example that the result cannot be carried over to motions in which there are volume changes, at least not without some rather strong restrictions.

Theorem 8. *Necessary and sufficient that the scalar potential Φ of the acceleration be*

$$\left. \begin{array}{l} \text{subharmonic} \\ \text{harmonic} \\ \text{superharmonic} \end{array} \right\} \text{ is } \left\{ \begin{array}{l} \frac{1}{2}w^2 \left(1 - \frac{1}{\mathcal{W}_K^2} \right) - \dot{I} \geq 0 \\ \frac{1}{2}w^2 \left(1 - \frac{1}{\mathcal{W}_K^2} \right) - \dot{I} = 0. \\ \frac{1}{2}w^2 \left(1 - \frac{1}{\mathcal{W}_K^2} \right) - \dot{I} \leq 0 \end{array} \right.$$

In the first case, then, the function Φ cannot suffer a maximum in the region; in the second, neither a maximum nor a minimum; in the third, a minimum.

Corollary 1. *In a region where*

$$\left. \begin{array}{l} \mathcal{W}_K \geq 1, \dot{I} \leq 0 \\ \mathcal{W}_K = 1, \dot{I} = 0 \\ \mathcal{W}_K \leq 1, \dot{I} \geq 0 \end{array} \right\} \text{the function } \Phi \text{ cannot experience a } \left\{ \begin{array}{l} \text{maximum} \\ \text{maximum or minimum.} \\ \text{minimum} \end{array} \right.$$

To prove this corollary we need only observe that the conditions stated are sufficient according to the Theorem. The kinematical interpretation of the condition $\dot{I} \leq 0$ ($\dot{I} \geq 0$) is that the expansion for a given particle shall not increase (decrease). In the special case when $\dot{I} = 0$, isochoric motion being included, the criterion indicated by the Corollary is expressed wholly in terms of the vorticity number.

The case $\mathcal{W}_K = 0$ merits special attention. In an irrotational motion we have the formula of Euler (1757a) and Vessiot (1911):

$$\Phi = \frac{\partial \phi}{\partial t} - \frac{1}{2}v^2 + f(t), \quad (10.2)$$

where ϕ is the velocity-potential. But

$$\nabla^2 \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t} \nabla^2 \phi = -\frac{\partial I}{\partial t}. \quad (10.3)$$

Combining this identity with (10.1) and (10.2) yields

$$\nabla^2(\tfrac{1}{2}v^2) = \mathbf{v} \cdot \text{grad } I + d^{ij} d_{ij}, \quad (10.4)$$

and hence if $\mathbf{v} \cdot \text{grad } I \geq 0$ it follows that $\nabla^2 v^2 \geq 0$. Thus we have

Corollary 2. *In a region of irrotational motion where the expansion does not decrease in the direction of motion along the streamlines, the greatest speed cannot occur at an interior point.*

The general extremal theorem has important application to fluid dynamics. First, an ideal fluid obeys Euler's equation,

$$\rho \mathbf{a} = -\text{grad } p + \rho \mathbf{f}; \quad (10.5)$$

when the extraneous force $\mathbf{f}$ is derivable from a potential v and when a barotropic relation $f(p, \rho) = 0$ holds, (10.3) yields a resolution of the type (4.14) with

$$\Phi = \int \frac{dp}{\rho} + v. \quad (10.6)$$

When the force potential v is harmonic, the function Φ may be replaced by $\int dp/\rho$ in the statements of Theorem 8. Since $\rho > 0$ it follows that $\int dp/\rho$ and p can experience a maximum or a minimum only simultaneously. Noting Euler's continuity equation

$$I = -\overline{\log \rho}, \quad (10.7)$$

we thus obtain

Corollary 3. *Given a flow of an ideal fluid in which either $\rho = \text{const.}$ or $p = p(\rho)$ and further the extraneous force is derivable from a harmonic potential, then in the interior of a region where*

$$\left. \begin{array}{ll} \mathcal{W}_K \geq 1, & \overline{\log \rho} \geq 0 \\ \mathcal{W}_K = 1, & \overline{\log \rho} = 0 \\ \mathcal{W}_K \leq 1, & \overline{\log \rho} \leq 0 \end{array} \right\} \text{the pressure cannot experience a } \left\{ \begin{array}{l} \text{maximum} \\ \text{maximum or minimum.} \\ \text{minimum} \end{array} \right.$$

In so far as it applies to ideal fluids, with the aid of Bernoulli's equation Corollary 2 may be derived from the third case in Corollary 3; in this formulation it was given by Bouligand (1927).

Second, a homogeneous viscous incompressible fluid obeys Navier's equation

$$\rho \mathbf{a} = -\text{grad } p + \rho \mathbf{f} - \mu \text{curl } \mathbf{w}, \quad (10.8)$$

$\rho = \text{const.}$, $\mu = \text{const.}$ When the extraneous force is derivable from a potential, (10.8) yields a relation of the type (4.14) with

$$\Phi = \frac{p}{\rho} + v, \quad (10.9)$$

irrespective of the value of μ . The resulting special case of Corollary 1 was discovered by Hamel (1936) and expressed by him in a rather different way, but we shall formulate it as

Corollary 4. (Theorem of Hamel). *Let a homogeneous incompressible fluid be*

subject to extraneous force derivable from a harmonic potential; then, whatever the value of the viscosity μ , in the interior of a region where

$$\left. \begin{array}{l} \mathfrak{W}_K \geq 1 \\ \mathfrak{W}_K = 1 \\ \mathfrak{W}_K \leq 1 \end{array} \right\} \text{the pressure cannot experience a} \left\{ \begin{array}{l} \text{maximum} \\ \text{maximum or minimum.} \\ \text{minimum} \end{array} \right.$$

The case when v is the potential of Newtonian gravitation is of interest in the theory of figures of equilibrium. Then the vorticity is uniform and

$$\nabla^2 v = 4\pi\rho G, \quad (10.10)$$

where G is the gravitational constant. Since (10.9) is a special case of (10.6), we may consider both cases simultaneously and conclude from (10.1) that

$$\frac{1}{2\omega^2} \nabla^2 \int \frac{dp}{\rho} = 1 - \frac{2\pi\rho G}{\omega^2} - \left(\frac{1}{\mathfrak{W}_K^2} + \frac{\dot{I}}{2\omega^2} \right), \quad (10.11)$$

where ω is the local angular velocity. For a rigid rotation $\omega = \text{const.}$, $\mathfrak{W}_K = \infty$, $I = 0$, and from (10.11) follows a celebrated theorem of Poincaré (*e.g.* Lamb (1932, §372)), which may be generalized as follows:

Theorem 9. *Given a motion of a fluid subject to its own Newtonian attraction, any further extraneous force being supposed derivable from a harmonic potential; assume that either or both of the following conditions hold:*

a. *The fluid is ideal and the flow is barotropic; or*

b. *The fluid is homogeneous, incompressible, and endowed with viscosity μ .*

Then necessary and sufficient conditions that $\int dp/\rho$ be

$$\left. \begin{array}{l} \text{subharmonic} \\ \text{harmonic} \\ \text{superharmonic} \end{array} \right\} \text{ are } \left\{ \begin{array}{l} \frac{2\pi\rho G}{\omega^2} \leq 1 - \frac{\dot{I}}{2\omega^2} - \frac{1}{\mathfrak{W}_K^2} \\ \frac{2\pi\rho G}{\omega^2} = 1 - \frac{\dot{I}}{2\omega^2} - \frac{1}{\mathfrak{W}_K^2} \\ \frac{2\pi\rho G}{\omega^2} \geq 1 - \frac{\dot{I}}{2\omega^2} - \frac{1}{\mathfrak{W}_K^2} \end{array} \right. \quad (10.12)$$

The reader will readily formulate more special sufficient conditions. For example, if $\omega^2 < 2\pi\rho G$ and $\dot{I} \geq 0$ the third case holds. In the case of rigid rotation the first condition becomes $\omega^2 \geq 2\pi\rho G$, whence follows Poincaré's corollary that a fluid of uniform density which cannot sustain tension cannot rotate rigidly in equilibrium unless $\omega^2 < 2\pi\rho G$. The classical argument, of course, is that since p cannot experience a maximum in the interior of a region where $\omega^2 \geq 2\pi\rho G$, and since also $p = 0$ on the complete boundary, it must follow that $p < 0$ at some point in the interior. The present result shows that deformation of the fluid mass alters this conclusion, as expressed in the following:

Corollary 1. *Under the conditions of Theorem 9, a mass of fluid which cannot sustain tension cannot persist in a flow in vacuo such that the two conditions*

$$\frac{\dot{I}}{2\omega^2} + \frac{1}{\mathfrak{A}_K^2} < 1, \quad (10.13)$$

$$\omega^2 \geq \frac{2\pi\rho G}{1 - \frac{\dot{I}}{2\omega^2} - \frac{1}{\mathfrak{A}_K^2}}, \quad (10.14)$$

are satisfied at all points.

Thus when $\dot{I} \geq 0$ the effect of deformation is to *increase* the possible range of (local) angular speeds; but in a highly rotational flow of a fluid whose expansion is sharply decreasing ($\frac{1}{2}\dot{I}\omega^{-2} \ll -1$) the range is *decreased*. When $\mathfrak{A}_K \leq 1$ and $\dot{I} \geq 0$ the first condition is never satisfied and the Corollary never applies. We may state its counterpart as

Corollary 2. *Under the conditions of Theorem 9, if*

$$\frac{\dot{I}}{2\omega^2} + \frac{1}{\mathfrak{A}_K^2} \geq 1 \quad (10.15)$$

at all points in a mass of fluid moving in vacuo, then the pressure is non-negative at all interior points.

For under this condition the third possibility in (10.12) holds for all values of ω ; hence $\oint dp/\rho$ is superharmonic; and hence the minimum pressure occurs only upon the boundary, where $p = 0$. Corollary 2 specifies an extensive class of flows in which cavitation arising from negative pressures will not occur. Sufficient that (10.13) hold are the weaker conditions

$$\dot{I} \geq 0, \quad \mathfrak{A}_K \leq 1. \quad (10.16)$$

The special case $I = 0$ both in Corollary 1 and in Corollary 2 is sufficiently interesting to be stated as:

Corollary 3. *In a motion in vacuo of a homogeneous incompressible fluid of viscosity μ , subject to its own Newtonian attraction and to any solenoidal and lamellar extraneous force field:*

(a). *If $\mathfrak{A}_K > 1$ at all points, then if the fluid cannot sustain tension the local angular velocity must satisfy*

$$\omega^2 < \frac{2\pi\rho G}{1 - \frac{1}{\mathfrak{A}_K^2}}. \quad (10.17)$$

(b). *If $\mathfrak{A}_K \leq 1$ at all points, then $p \geq 0$ at all points.*

Poincaré's theorem and Bouligand's theorem now appear as the limiting cases $\mathfrak{A}_K = \infty$ and $\mathfrak{A}_K = 0$ of cases (a) and (b), respectively, of the last Corollary. Thus both these results—the one stating that cavitation must occur in rigid

rotations at too great an angular speed, the other that cavitation never occurs in irrotational flows—are fitted into a general scheme of kinematical extremal theorems.

As is well known, Poincaré's upper bound $\omega^2 = 2\pi\rho G$ has been halved by Crudeli and Nikliborc (1929), and at the same time the restriction to motions *in vacuo* has been lifted. It would be interesting to extend these sharper results to the case when deformation is present. Possibly also further properties of sub- and superharmonic functions might yield other interesting corollaries of Theorems 8 and 9.

11. The number $\mathcal{W}_K$ in a fluid motion. For any particular type of substance (4.13) may be converted at once into dynamical form. We give here the explicit formulae for certain types of fluids, generalizing some of the expressions of §10. For example, in ordinary hydrodynamics governed by Euler's equation (10.3) we obtain

$$\mathcal{W}_K = \left[1 + \frac{2}{w^2} \left(\ddot{\log \rho} - \text{grad } \frac{1}{\rho} \cdot \text{grad } p - \frac{1}{\rho} \nabla^2 p + \text{div } \mathbf{f} \right) \right]^{-1}, \quad (11.1)$$

while for homogeneous viscous incompressible fluids obeying Navier's equation (10.8) we have

$$\mathcal{W}_K = \left[1 + \frac{2}{w^2} \left(\text{div } \mathbf{f} - \frac{1}{\rho} \nabla^2 p \right) \right]^{-1}. \quad (11.2)$$

It is interesting that $\mathcal{W}_K$ is not explicitly dependent upon the viscosity μ , a result foreshadowed by (4.15).

From (11.2) follows

Theorem 10. *Let a homogeneous incompressible fluid of viscosity μ be in rotational flow subject to a solenoidal field of force. Then*

$$\mathcal{W}_K = 1$$

at all points if and only if the pressure be harmonic:

$$\nabla^2 p = 0. \quad (11.3)$$

Since (11.3) is satisfied by all "slow" motions (*e.g.* Lamb (1932, §§336, 342)), both in the original theory of Stokes and in the modified theory of Oseen, it follows that $\mathcal{W}_K = 1$ in all these flows, an indication of their very restricted and special nature.

The result of Theorem 10 may be generalized in a purely kinematical way. Since in a "slow" motion the convective acceleration is neglected:

$$\mathbf{a} \approx \frac{\partial \mathbf{v}}{\partial t}, \quad (11.4)$$

by inspection of (5.4) follows

Theorem 11. *In any rotational isochoric "slow" motion*

$$\mathfrak{U}_K = 1$$

at all points.

Corollary 1. *In a "slow" motion in stationary domain to whose walls the material adheres, (9.8) is satisfied by the identical vanishing of the integrand.*

Corollary 2. *In a motion consisting of infinitesimal shear waves, $\mathfrak{U}_K = 1$ at all points.*

12. Third example: Gerstner's waves. We now revert to one of the two problems, mentioned in §1, which gave rise to this whole investigation. The Gerstner waves are given by the material description (*e.g.* Appell (1921)).

$$x = \alpha + \frac{1}{k} e^{-k\beta} \sin k(\alpha + ct), \quad y = \beta + \frac{1}{k} e^{-k\beta} \cos k(\alpha + ct), \quad (12.1)$$

where the arbitrary wave-length is $\lambda = 2\pi/k$, the speed of propagation is $c = (g/k)^{\frac{1}{2}}$, and α and β are functions of the material co-ordinates X, Y defined by the implicit relations

$$X = \alpha + \frac{1}{k} e^{-k\beta} \sin k\alpha, \quad Y = \beta + \frac{1}{k} e^{-k\beta} \cos k\alpha. \quad (12.2)$$

The Bernoulli equation assumes the form

$$\frac{p}{\rho} = g\beta + \frac{1}{2}c^2 e^{-2k\beta} + C; \quad (12.3)$$

thus the surfaces $\beta = \text{const.}$ are surfaces of constant pressure, the free surface may be taken as $\beta = \beta_0$, where β_0 is any constant, and by choice of the constant C the pressure may be given any desired value $p = p_0$ when $\beta = \beta_0$. The condition that the free surface be without loops is $\beta_0 \geq 0$, where the extreme case $\beta_0 = 0$ yields a cusped wave form. The flow is confined to the region $\beta \geq \beta_0$, and (12.3) indicates that if h be the depth then for large values of h we have $\beta = O(h)$. If by the amplitude a of the wave at the surface $\beta = \text{const.}$ we agree to mean one half the height of the crests above the troughs at that surface we have

$$a = \frac{1}{k} e^{-k\beta}, \quad (12.4)$$

and hence

$$\frac{2\pi a}{\lambda} = e^{-k\beta} \quad (12.5)$$

In particular, the amplitude A at the free surface is given by

$$\frac{2\pi A}{\lambda} = e^{-k\beta_0}, \quad (12.6)$$

and the requirement $\beta_0 \geq 0$ then assumes the form

$$A \leq \frac{\lambda}{2\pi} : \quad (12.7)$$

that is, Gerstner waves of any amplitude and wave length exist, providing the former do not exceed about 16% of the latter.

Now (12.1) is an exact solution of Euler's dynamical equation. If it is to serve as a suitable representation for the surface waves which occur in actual fluids, however, it must be only very slightly rotational. In order to determine the physical utility of the solution, then, we need to calculate $\mathcal{M}_K$.

Since

$$v_x = ce^{-k\beta} \cos k(\alpha + ct), \quad v_y = -ce^{-k\beta} \sin k(\alpha + ct), \quad (12.8)$$

and since in general

$$\begin{aligned} J \frac{\partial v_x}{\partial x} &= \frac{\partial(v_x, y)}{\partial(\alpha, \beta)}, & J \frac{\partial v_x}{\partial y} &= \frac{\partial(x, v_x)}{\partial(\alpha, \beta)}, \\ J \frac{\partial v_y}{\partial x} &= \frac{\partial(v_y, y)}{\partial(\alpha, \beta)}, & J \frac{\partial v_y}{\partial y} &= \frac{\partial(x, v_y)}{\partial(\alpha, \beta)}, \end{aligned} \quad (12.9)$$

where $J \equiv \partial(x, y)/\partial(\alpha, \beta)$, a ready calculation yields

$$\begin{aligned} e^{k\beta} \frac{J}{kc} \frac{\partial v_x}{\partial x} &= -\sin k(\alpha + ct), & e^{k\beta} \frac{J}{kc} \frac{\partial v_x}{\partial y} &= -\cos k(\alpha + ct) - e^{-k\beta}, \\ e^{k\beta} \frac{J}{kc} \frac{\partial v_y}{\partial x} &= -\cos k(\alpha + ct) + e^{-k\beta}, & e^{k\beta} \frac{J}{kc} \frac{\partial v_y}{\partial y} &= \sin k(\alpha + ct). \end{aligned} \quad (12.10)$$

By substitution into (2.5) and use of (12.5) we then obtain the remarkably simple expression

$$\mathcal{M}_K = e^{-k\beta} = \frac{2\pi a}{\lambda}, \quad (12.11)$$

which at a glance lays bare the rotational character of the Gerstner waves. Our results are summarized in

Theorem 12.1. *Since*

$$\mathcal{W}_K \leq \frac{2\pi A}{\lambda} \leq 1, \quad (12.12)$$

Gerstner's waves are nowhere more rotational than a Poiseuille flow.

2. *The vorticity number $\mathcal{W}_K$ is constant upon each surface of constant pressure.*3. *The vorticity number $\mathcal{W}_K$ is greatest at the free surface, is a monotone decreasing function of depth, and for fairly large depths decreases exponentially with the depth.*4. *The vorticity number $\mathcal{W}_K$ assumes the value 1 at the free surface in the extreme case when $\lambda = 2\pi A$ and the wave form is cusped.*5. *For a given wave length, the Gerstner waves become more significantly rotational with increasing amplitude. For example, in order for the vorticity number $\mathcal{W}_K$ to be less than 10% it is necessary that $A < 1.5\% \lambda$. That is, the Gerstner waves are strongly rotational except for very small amplitudes, amplitudes indeed small enough that the linearized theory of surface waves may be employed.*

This examination confirms the now generally accepted belief that the theory of Stokes and Levi-Civita gives the only acceptable representation of surface waves of finite amplitude.

13. The dynamical vorticity number $\mathcal{W}_D$. The magnitude of $\mathcal{W}_K$ indicates the amount of vorticity relative to deformation. When the aerodynamicist says that the rotation in a particular flow is negligible, however, he is not directly concerned with deformation. What he really means is that the vorticity term in the equation of motion may be neglected. Now this equation is

$$\rho \mathbf{a} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \text{grad } \frac{1}{2}v^2 \right] = \dots, \quad (13.1)$$

where the dots indicate an expression which varies according to whether or not gravity, viscosity, or other moving forces are taken into account. The equation resulting when the effect of vorticity on the inertia of the material is neglected is

$$\rho \mathbf{a} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \text{grad } \frac{1}{2}v^2 \right] = \dots. \quad (13.2)$$

Thus the number

$$\mathcal{W}_D = \frac{|\mathbf{w} \times \mathbf{v}|}{\left| \frac{\partial \mathbf{v}}{\partial t} + \text{grad } \frac{1}{2}v^2 \right|} \quad (13.3)$$

is a proper measure of the dynamical importance of vorticity, and we shall call it the **dynamical vorticity number**, although of course it is determined by purely

kinematical quantities. Unlike the kinematical vorticity number $\mathcal{W}_K$, it depends not only upon the spatial configuration of the velocity field but also upon its local time rate of change. The criterion for *dynamically negligible rotation* is

$$\mathcal{W}_D \ll 1. \quad (13.4)$$

We must take care to note, however, that while $\mathcal{W}_K$ is a Galilean invariant, $\mathcal{W}_D$ is not. Thus $\mathcal{W}_K$ can have physical significance, while $\mathcal{W}_D$ cannot. The magnitude of $\mathcal{W}_D$ indicates the relative importance of the vorticity in the acceleration as observed in a particular frame; while the acceleration itself has the same value in all inertial frames, the relative contributions made by the three portions on the right hand side of (4.16) will generally vary from one inertial frame to another. The value of $\mathcal{W}_D$ gives an indication of the importance of the rotation term in the dynamical equations as written in *one particular frame*.

It is hardly necessary to remark that in neglecting the rotational term in an approximate solution one is expressing his confidence that its *effect* on the motion, not merely its *magnitude*, is small. While in the absence of a proper approximation theorem for partial differential equations we cannot positively assert, yet, following the universal custom in works on physics, we may with some confidence expect, that when a term is relatively small it is permissible to drop it from the equation. With this qualification we may then say that the condition $\mathcal{W}_D \ll 1$ in a certain frame justifies the mathematical neglect of the rotation term when solving the dynamical equations written in that particular frame. In some other inertial frame such neglect may not be justified.

It is interesting to note that there is a class of motions without acceleration in which $\mathcal{W}_D$ is not defined, namely those irrotational or Beltrami motions for which $\partial \mathbf{v} / \partial t + \text{grad } \frac{1}{2} v^2 = 0$. Such motions have been studied by Neményi & Prim (1950). See also Prim (1952). If this class be excepted, however, the condition $\mathcal{W}_D = 0$ characterizes the case $\mathbf{w} \times \mathbf{v} = 0$, whence follows

Theorem 13. *In motions for which $\partial \mathbf{v} / \partial t + \text{grad } \frac{1}{2} v^2 \neq 0$, we have*

$$\mathcal{W}_D = 0$$

for and only for an irrotational or a Beltrami motion.

In a steady rigid motion we have $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is a constant. Then $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, $\mathbf{w} \times \mathbf{v} = 2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ and hence from (13.3) follows

Theorem 14. *In a steady rigid rotation*

$$\mathcal{W}_D = 2.$$

The condition $\mathcal{W}_D = 2$ is not sufficient, however, that the motion be rigid. For example, in the motion $v_x = a(x + ky)$, $v_y = a$, $a > 0$, $k > 0$, which is nowhere irrotational and nowhere rigid, we have $\mathcal{W}_D = k$ at the point $x = k^{-1}$, $y = 1$, and in particular we may take $k = 2$.

Equally evident is

Theorem 15. *In a steady motion which is neither an irrotational nor a Beltrami motion, if the speed be uniform then*

$$\mathcal{W}_D = \infty.$$

In these motions, of which an example will be given in §15, the rotation term $\mathbf{w} \times \mathbf{v}$ is the entire acceleration, and to neglect it would be to neglect the inertia of the material altogether.

Directly from (13.3) follows

Theorem 16. *In order that*

$$\mathcal{W}_D = 1$$

it is necessary and sufficient that the vectors $\mathbf{a}$ and $\mathbf{w} \times \mathbf{v}$ form the base and one leg, respectively, of an isosceles triangle.

Corollary. *In any motion without acceleration $\mathcal{W}_D = 1$.*

The simplest case included by this Corollary is that of a Poiseuille motion. That this type of motion again appears as strongly rotational is to be expected: if one attempted to neglect its rotation in solving the appropriate dynamical equations one would write $\mathbf{a} \approx \text{grad } \frac{1}{2}v^2$ instead of the correct result $\mathbf{a} = 0$, and the resulting solution would be completely wrong.

Again a dynamical expression for $\mathcal{W}_D$ is easy to obtain when some particular medium is considered. For example, in gas dynamics we shall have

$$\mathcal{W}_D = \frac{|\mathbf{w} \times \mathbf{v}|}{\left| \mathbf{w} \times \mathbf{v} + \frac{1}{\rho} \text{grad } p \right|} \quad (13.5)$$

Inspection of this formula shows that if $\mathbf{w} \times \mathbf{v} \neq 0$ then $\mathcal{W}_D = 1$ if and only if

$$\text{grad } p \cdot \left[\mathbf{w} \times \mathbf{v} + \frac{1}{\rho} \text{grad } p \right] = 0. \quad (13.6)$$

Interpretation of this condition yields

Theorem 17. *For an inviscid fluid subject to no extraneous force, in a steady flow which is neither an irrotational flow nor a Beltrami flow in order that $\mathcal{W}_D = 1$ at a point it is necessary and sufficient that either*

- a. *The pressure gradient vanishes; or*
- b. *The length of the projection of $\mathbf{w} \times \mathbf{v}$ onto the direction of $\text{grad } p$ has the magnitude $|\text{grad } p|/\rho$.*

We note also the evident bounds

$$\frac{1}{1 + \frac{|\text{grad } p|}{\rho |\mathbf{w} \times \mathbf{v}|}} \leq \mathcal{W}_D \leq \frac{1}{\left| 1 - \frac{|\text{grad } p|}{\rho |\mathbf{w} \times \mathbf{v}|} \right|}, \quad (13.7)$$

$$\left| 1 - \frac{|\text{grad } p|}{\rho |\text{grad } \frac{1}{2}v^2|} \right| \leq \mathcal{W}_D \leq 1 + \frac{|\text{grad } p|}{\rho |\text{grad } \frac{1}{2}v^2|}. \quad (13.8)$$

14. Comparison of the significances of the two vorticity numbers. From the example of a Beltrami flow we see by (7.2) that it is possible in a motion which is strongly rotational in the kinematical sense ($\mathcal{W}_K \geq 1$) that the rotation be dynamically negligible ($\mathcal{W}_D \approx 0$). Conversely, in §15 we shall find it to be equally possible that in a motion only slightly rotational ($\mathcal{W}_K \ll 1$), what rotation there is may be of considerable dynamic significance ($\mathcal{W}_D \approx 1$). Both dynamically and kinematically *the rotational isochoric motion without acceleration is of unit degree of rotation*:

$$\mathcal{W}_K = \mathcal{W}_D = 1$$

at all points, all generalized Poiseuille motions and all steady "slow" isochoric motions being included as special cases. A rigid rotation has the highest kinematical rating ($\mathcal{W}_K = \infty$), and on the dynamical scale it again appears as strongly rotational ($\mathcal{W}_D = 2$). In the following section these results will be illustrated by a specific class of flows in which both $\mathcal{W}_K$ and $\mathcal{W}_D$ vary over the full possible range, 0 to ∞ .

15. Fourth example: simple vortex. Consider the class of motions in which the stream-lines are concentric circles rotating at angular speed ω :

$$v_x = -y\omega(r), \quad v_y = x\omega(r), \quad v_z = 0, \quad (15.1)$$

where $r^2 \equiv x^2 + y^2$. Since Euler's first great memoirs on analytical fluid dynamics, repeatedly these simple flows have furnished concrete illustrations of the general theorems of the subject. For the vorticity we have

$$w = 2\omega + r\omega', \quad (15.2)$$

and hence the two vorticity numbers $\mathcal{W}_K$ and $\mathcal{W}_D$ are given by

$$\mathcal{W}_K = \left| 1 + \frac{2\omega}{r\omega'} \right|, \quad \mathcal{W}_D = \left| 1 + \frac{1}{1 + \frac{r\omega'}{\omega}} \right|. \quad (15.3)$$

Set

$$n \equiv 2 + \frac{r\omega'}{\omega}. \quad (15.4)$$

In general, n will be a function of r , and indeed corresponding to any prescribed n we have the two values

$$\omega = \pm \exp \int \frac{n-2}{r} dr \quad (15.5)$$

for ω , so that to consider the whole class of flows (15.1) it is sufficient to let n

run from $-\infty$ to $+\infty$. For n to be constant it is necessary and sufficient that the flow be an n^{th} power vortex:

$$\omega = \Omega r^{-n-2}, \quad (15.6)$$

where Ω is a constant.

In terms of n the expressions (15.3) become

$$\mathcal{W}_K = \left| \frac{n}{n-2} \right|, \quad \mathcal{W}_D = \left| \frac{n}{n-1} \right|. \quad (15.7)$$

These two relations are presented graphically in Fig. 1. For large $|n|$ both $\mathcal{W}_K$ and $\mathcal{W}_D$ approach 1. The vertical asymptote for $\mathcal{W}_K$ and the value 2 for

FIG. 1. Kinematical and dynamical vorticity numbers for different types of simple vortex

$\mathcal{W}_D$ when $n = 2$ reflect the fact that the motion is then rigid. The vertical asymptote for $\mathcal{W}_D$ at $n = 1$ indicates that for this value of n the speed is uniform and hence the vorticity term $\mathbf{w} \times \mathbf{v}$ is the entire acceleration; that this motion is merely of unit degree of rotation on the kinematical scale follows from the solenoidal character of the velocity and the acceleration. If $n = \frac{3}{2}$ the two measures are equal, each having the value 3. Of some interest is the region $n \approx 0$, since when $n = 0$ the motion is irrotational. The graph indicates that there is a range of values of n yielding nearly irrotational flows which are *not* flows of dynamically negligible rotation. For example, if $n = \frac{1}{3}$ we obtain $\mathcal{W}_K = \frac{3}{2}$, $\mathcal{W}_D = \frac{3}{4}$, so that the flow is only about 10% rotational on the kinematical scale, but the dynamical rating 25% indicates that what rotation there is is too important to be neglected.

These results indicate that our definitions of $\mathcal{W}_K$ and $\mathcal{W}_D$ and our theorems concerning flows in which these numbers assume various values are not vacuous for fluid dynamics, since, as was originally shown by Euler (1757b) and is now obvious, any simple vortex flow, being circulation-preserving, is a dynamically possible flow for a large class of fluids. We may put this observation as follows:

Theorem 18. *Given any non-negative number k , there exists a dynamically possible flow of an inviscid fluid, subject to conservative extraneous force, such that $\mathcal{W}_D = k$ or $\mathcal{W}_K = k$, as we please, at an arbitrarily assignable point. Nay more, even for a homogeneous incompressible inviscid fluid, subject to no extraneous force, there exists a flow which fills all space but for a single line, and in which $\mathcal{W}_D = k$ or $\mathcal{W}_K = k$, as we please, at all places and times.*

The simultaneous values of $\mathcal{W}_D$ and $\mathcal{W}_K$ for a single flow is also a matter of interest. From the general theorems of §§5 and 13 it is plain that a flow in which $\mathcal{W}_D = k_1$, $\mathcal{W}_K = k_2$, where k_1 and k_2 are arbitrary, is not possible. For the special class of flows considered in this section there is a specific relation between $\mathcal{W}_K$ and $\mathcal{W}_D$. From (15.6) it is not difficult to show that

$$\mathcal{W}_D = \frac{2\mathcal{W}_K}{\mathcal{W}_K + 1} \quad \text{when } n > 2 \quad \text{or } n < 0, \quad (15.8)$$

with $\mathcal{W}_K > 1$, $2 > \mathcal{W}_D > 1$ when $n > 2$, while $\mathcal{W}_K < 1$, $\mathcal{W}_D < 1$ when $n < 0$; that

$$\mathcal{W}_D = \frac{2\mathcal{W}_K}{\mathcal{W}_K - 1} \quad \text{when } 1 < n < 2, \quad (15.9)$$

with $\mathcal{W}_K > 1$, $\mathcal{W}_D > 2$; and that

$$\mathcal{W}_D = \frac{2\mathcal{W}_K}{1 - \mathcal{W}_K} \quad \text{when } 0 < n < 1, \quad (15.10)$$

with $\mathcal{W}_K < 1$. The three curves (15.8)–(15.10) are portions of three hyperbolae, each pair of which has either a common tangent or a common asymptote. They are shown in Fig. 2. For ease of statement restricting ourselves to the case $n = \text{const.}$, we have

Theorem 19. *For an arbitrary positive number $k \neq 1$, there exist exactly two n^{th} power vortices such that $\mathcal{W}_K = k$. If $k > 1$ then $n > 1$, and for one of the two flows $\mathcal{W}_D > 2$ and $n < 2$, while for the other $n > 2$ and $1 < \mathcal{W}_D < 2$. If $k < 1$, then $n > 0$ for one, $n < 0$ for the other; for the former $\mathcal{W}_D > \mathcal{W}_K$, while for the latter, $\mathcal{W}_K < \mathcal{W}_D < 1$. $\mathcal{W}_K = 1$ if and only if $n = 1$; $\mathcal{W}_K = 0$ if and only if $n = 0$.*

Theorem 20. *For an arbitrary positive number $k \neq 1$, there exist exactly two n^{th} power vortices such that $\mathcal{W}_D = k$. If $k > 1$ then for one of these $\frac{1}{2} < n < 1$, for the other $n > 1$; for the former $\frac{1}{3} < \mathcal{W}_K < 1$, while for the latter $\mathcal{W}_K > 1$. If $k < 1$ then $\frac{1}{2} > n > 0$ for one, $n < 0$ for the other; for the former $\mathcal{W}_K < \frac{1}{3}$,*

and for both $\mathcal{W}_K < \mathcal{W}_D$. $\mathcal{W}_D = 1$ if and only if $n = \frac{1}{2}$; $\mathcal{W}_D = 0$ if and only if $n = 0$.

FIG. 2. Correlation between $\mathcal{W}_D$ and $\mathcal{W}_K$ for a simple vortex

16. Fifth example: Couette flow. A Couette flow is obtained by superposition of the irrotational vortex upon a rigid rotation about the axis of the vortex:

$$\omega = \Omega \left(1 - \frac{a^2}{r^2} \right). \quad (16.1)$$

This motion is dynamically possible for viscous liquids which adhere to solid boundaries. We may visualize it as generated by revolving the two cylinders $r = R_1 \neq 0$ and $r = R_2 > R_1$ at suitable angular speeds. If $R_2 = a$ the outer cylinder is at rest; if $R_1 = a$ the inner cylinder is at rest; if $R_2 < a$ both cylinders rotate in the same sense and the angular speed of the inner exceeds that of the outer; if $R_1 > a$ both cylinders rotate in the same sense and the angular speed of the outer exceeds that of the inner; if $R_1 < a < R_2$ the senses of rotation are opposite. The arbitrary constant a determines the strength of the vortex relative to the angular speed of the rigid rotation. In all cases the vorticity is constant throughout the flow: $w = 2\Omega$. It is quite otherwise with the vorticity numbers, however, for from (14.3) we obtain

$$\mathcal{W}_K = \frac{r^2}{a^2}, \quad \mathcal{W}_D = \frac{2r^2}{r^2 + a^2} = \frac{2\mathcal{W}_K}{\mathcal{W}_K + 1}. \quad (16.2)$$

Fig. 3 presents these results in graphical form.

FIG. 3. Kinematical and dynamical vorticity numbers for a Couette flow

The values of W_K and W_D are plotted against those of $\beta \equiv r^2/a^2$. At the stationary cylinder $\beta = 1$, in the region interior to it $\beta < 1$, while in that exterior $\beta > 1$.

The various types of Couette flows are now distinguished kinematically one from another in a striking way; as expressed in

Theorem 21. *In a Couette flow,*

1. *If both cylinders rotate in the same sense and the angular speed of the inner exceed that of the outer, then*

$$W_K < 1, \quad W_D < 1$$

throughout the flow.

2. *If the outer cylinder be at rest then*

$$W_K \leq 1, \quad W_D \leq 1,$$

the values 1 being attained at the outer cylinder.

3. *If the cylinders rotate in opposite senses then in the inner part of the flow we have*

$$W_K < 1, \quad W_D < 1,$$

on an intermediate cylinder we have

$$W_K = W_D = 1,$$

and in the outer part of the flow we have

$$W_K > 1, \quad W_D > 1.$$

4. *If the inner cylinder be at rest then*

$$W_K \geq 1, \quad W_D \geq 1,$$

the values 1 being attained at the inner cylinder.

5. *If both cylinders rotate in the same sense and the angular speed of the outer exceed that of the inner, then*

$$\mathcal{W}_K > 1, \quad \mathcal{W}_D > 1.$$

throughout the flow.

The five possible types of Couette flow are thus graded in order of increasing rotationality. One characteristic all have in common: $\mathcal{W}_K$ and $\mathcal{W}_D$ always attain their minimum values at the inner cylinder and increase steadily as the outer is approached.

The stability of Couette flows was analysed by Rayleigh (1916). His result, based solely upon consideration of the inertia of the fluid, is as follows: Let Ω_1, Ω_2 be the angular speeds and R_1, R_2 the radii of the inner and outer cylinders, and let R be defined by

$$R \equiv \frac{\Omega_2 R_2^2}{\Omega_1 R_1^2}; \quad (16.3)$$

then a Couette flow is stable if and only if

$$R > 1. \quad (16.4)$$

We shall now express Rayleigh's result in terms of vorticity numbers, putting $\mathcal{W}_1, \mathcal{W}_2$ for the values of $\mathcal{W}_K$ at the inner and outer cylinders. From (16.1) we have

$$\Omega_i = \Omega \left(1 - \frac{a^2}{R_i^2} \right), \quad i = 1, 2, \quad (16.5)$$

and hence by (16.2)₁

$$\Omega_i = \Omega \left(1 - \frac{1}{\mathcal{W}_i} \right). \quad (16.6)$$

Thus

$$\frac{\Omega_2}{\Omega_1} = \frac{1 - \frac{1}{\mathcal{W}_2}}{1 - \frac{1}{\mathcal{W}_1}}, \quad (16.7)$$

so that Rayleigh's ratio (16.3) becomes

$$R = \frac{\Omega_2 \frac{R_2^2}{a^2}}{\Omega_1 \frac{R_1^2}{a^2}} = \frac{\Omega_2 \mathcal{W}_2}{\Omega_1 \mathcal{W}_1} = \frac{\mathcal{W}_2 - 1}{\mathcal{W}_1 - 1} = 1 + \frac{\mathcal{W}_2 - \mathcal{W}_1}{\mathcal{W}_1 - 1}. \quad (16.8)$$

We have seen that $\mathcal{W}_2 > \mathcal{W}_1 > 0$. Hence in order that $R > 1$ it is necessary and sufficient that $\mathcal{W}_1 > 1$. Since $\mathcal{W}_1$ is the minimum value of $\mathcal{W}_K$, it follows that

Rayleigh's stability criterion may be put thus: *a Couette flow is stable if and only if*

$$\mathcal{M}_K > 1 \quad (16.9)$$

at all points. Equivalently,

$$\mathcal{M}_D > 1 \quad (16.10)$$

at all points.

I have endeavored in vain to show that this result is more than a mere curiosity. It is not impossible that the condition $\mathcal{M}_K > 1$ is a sufficient general condition for stability of fluid flows at infinite Reynolds number.

17. Sixth example: the rotation generated by a curved shock. We now revert to the problem mentioned in the opening phrases of this memoir.

Generalizing a result of T. Y. Thomas (1947), I have shown elsewhere (1952a) that the velocity gradient $v_{\alpha,\beta}$ behind a curved shock in steady plane flow of an inviscid fluid, devoid of heat flux and subject to no extraneous force, can be expressed in terms of the shock curvature κ by the formula

$$v_{\alpha,\beta} = \kappa v_n B_{ij} D_{i\alpha} D_{j\beta}, \quad (17.1)$$

where v_n is the component of velocity in the direction normal to the shock, evaluated behind the shock, and the dimensionless quantities B_{ij} and $D_{i\alpha}$ are expressible in terms of the shock inclination and the discontinuities in v_α , p , and ρ . Throughout this section we take the coordinates x_α as rectangular Cartesian, and we employ the summation convention of Cartesian tensors. The hypotheses just stated regarding the fluid and the flow hold throughout the section and will not be repeated in the enunciations of the succeeding theorems. Quantities with subscript 1 shall denote evaluations at the front of the shock; quantities without subscripts, evaluations behind it. Denoting by v_i the component of velocity in the direction tangent to the shock, let

$$\delta \equiv \frac{[\rho]}{\rho_1}, \quad \tau \equiv \frac{v_t}{v_{1n}}, \quad \sigma \equiv \text{Arctan } \tau \quad (17.2)$$

be the shock strength, obliquity, and angle, respectively. Then

$$\chi \equiv \frac{v_t}{v_n} = D\tau, \quad \text{where} \quad D \equiv 1 + \delta. \quad (17.3)$$

Of the quantities $D_{i\alpha}$ we shall here require to know only the facts that $\det D_{i\alpha} = 1$, that

$$\mathbf{X} \equiv \parallel D_{i\alpha} D_{j\alpha} \parallel = \begin{vmatrix} 1 + \chi^2 & -\chi \\ -\chi & 1 \end{vmatrix}, \quad (17.4)$$

and that $v_\alpha = v_n C_{\alpha 2}$, where $\|C_{\alpha i}\|$ is the inverse of $\|D_{i\alpha}\|$. General expressions for the B_{ij} are elaborate. For the case when the flow ahead of the shock is uniform we have the universal formula

$$w = \kappa v_n \tau \delta^2 \quad (17.5)$$

for the vorticity behind the shock. If, moreover, the fluid be a perfect gas with constant ratio γ of specific heats, which shall be assumed to exceed 1, then we have

$$\|B_{ij}\| = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = \begin{vmatrix} \delta & R\tau \\ A\tau & -B + C\tau^2 \end{vmatrix}, \quad (17.6)$$

where

$$\begin{aligned} A &= R + \delta^2, & R &= \frac{4(1 + \delta)}{\gamma + 1}, & B &= \delta + \frac{2}{\gamma + 1}, \\ C &= MBD, & M &= \frac{1}{\gamma + 1} \left(\frac{8}{\delta} + 7 - \gamma \right). \end{aligned} \quad (17.7)$$

The range of possible shock strengths δ is $0 \leq \delta \leq 2/(\gamma - 1)$.

We shall now evaluate $\mathcal{W}_D$ and $\mathcal{W}_K$ behind the shock, first in full generality in terms of the B_{ij} , and second in the special case given by (17.6) and (17.7).

From (13.3) and (17.1) we have

$$\begin{aligned} \mathcal{W}_D &= \frac{vw}{(v_{\alpha,\beta} v_\alpha v_{\gamma,\beta} v_\gamma)^{\frac{1}{2}}}, \\ &= \frac{2vB_{ij} D_{i[2} D_{j]1}}{(B_{ij} B_{kl} D_{i\alpha} D_{j\beta} D_{k\gamma} D_{l\beta} v_\alpha v_\gamma)^{\frac{1}{2}}}, \\ &= \frac{2vB_{[21]} \det D_{i\alpha}}{v_n (B_{ij} B_{kl} D_{i\alpha} D_{j\beta} D_{k\gamma} D_{l\beta} C_{\alpha 2} C_{\gamma 2})^{\frac{1}{2}}}, \\ &= \frac{2vB_{[21]}}{v_n (B_{2j} B_{2l} D_{j\beta} D_{l\beta})^{\frac{1}{2}}}. \end{aligned} \quad (17.8)$$

By using (17.4) we then obtain

Theorem 22. *If $\kappa = 0$, $\mathcal{W}_D$ is zero or indeterminate. If $\kappa \neq 0$, however, we have*

$$\mathcal{W}_D = \frac{2B_{[21]}(1 + \chi^2)^{\frac{1}{2}}}{(B_{21}^2 + \{B_{22} - \chi B_{21}\}^2)^{\frac{1}{2}}}. \quad (17.9)$$

Thus for curved shocks $\mathcal{W}_D$ is a function of χ and of the B_{ij} only. For a perfect gas in uniform flow ahead of the shock, consequently, $\mathcal{W}_D$ is a function of δ , τ , and γ only.

We shall now study the expression (17.9) for the special case (17.6), stating

here once and for all that in so doing we assume the gas to be perfect and the flow ahead of the shock to be uniform. Further, we assume always that $\kappa \neq 0$. It occasions no loss in generality to suppose $\sigma \geq 0$. The range of physical interest for the variables becomes then $0 \leq \sigma \leq \frac{1}{2}\pi$, $0 \leq \delta \leq 2/(\gamma - 1)$, and our problem is to study the nature of the surface $\mathcal{W}_D = f(\sigma, \delta)$ above this finite rectangle in the σ - δ plane. We shall consider also the change in form of this surface incident upon change of the parameter γ in the range $1 < \gamma < \infty$, with special attention to the sub-range $1 < \gamma \leq \frac{5}{3}$.

If $\sigma < \frac{1}{2}\pi$ we have $0 \leq \chi < \infty$; hence by (17.9) $\mathcal{W}_D$ is finite unless $B_{21} = B_{22} = 0$, an eventuality which (17.6) makes plain to be impossible. For normal shocks, since $\tau = 0$, we have $B_{21} = B_{[21]} = 0$, but since $B > 0$, we have also $B_{22} < 0$, whence it follows that $\mathcal{W}_D = 0$; for oblique shocks of zero strength we have $B_{21} > 0$ but $B_{[21]} = 0$, and hence $\mathcal{W}_D = 0$. From (17.9) it is plain also that if $\mathcal{W}_D = 0$ we must have $B_{[21]} = 0$, and hence $\tau\delta = 0$. Thus we have established

Theorem 23. $\mathcal{W}_D$ is a finite continuous function of both δ and σ when $0 \leq \delta \leq 2/(\gamma - 1)$, $0 \leq \sigma < \frac{1}{2}\pi$. $\mathcal{W}_D = 0$ if and only if $\sigma = 0$ or $\delta = 0$.

For glancing shocks we have $\sigma = \frac{1}{2}\pi$, $\tau = \infty$. To study these it is easiest to put $x \equiv \tau^{-2}$ and thus from (17.9) to obtain in general

$$\frac{\mathcal{W}_D^2}{\delta^4} = \frac{D^2 + x}{F + Ex + B^2 x^2}, \quad (17.10)$$

where

$$E \equiv A^2 + 2B(DA - C), \quad F \equiv (AD - C)^2. \quad (17.11)$$

It follows that

$$\begin{aligned} \mathcal{W}_D|_{\sigma=\frac{1}{2}\pi} &= \frac{D\delta^2}{\sqrt{F}} = \frac{\delta^2}{\left| \begin{array}{c} C \\ D - A \end{array} \right|}, \\ &= \frac{\delta^2}{|MB - A|}. \end{aligned} \quad (17.12)$$

Now

$$(\gamma + 1)^2 \delta(MB - A) = 16 + 2(\gamma + 9)\delta + (3 - \gamma)(\gamma + 1)\delta^2 - (\gamma + 1)^2 \delta^3. \quad (17.13)$$

The cubic polynomial on the right has a single positive root, whatever the value of γ . To see whether this root occurs in the range $0 \leq \delta \leq 2/(\gamma - 1)$, put $\delta = 2/(\gamma - 1)$, thus obtaining

$$(MB - A)|_{\delta=2/(\gamma-1)} = \frac{8\gamma(\gamma - 2)}{(\gamma + 1)(\gamma - 1)^2}. \quad (17.14)$$

Consequently when $\gamma < 2$, $MB - A$ passes from a positive value when $\delta = 0$ to a negative value when $\delta = 2/(\gamma - 1)$, but when $\gamma > 2$, $MB - A$ is positive

for all possible δ . In the case when there is a root δ^* , it is given by annulling the right hand side of (17.13):

$$\gamma + 1 = \frac{1 + 2\delta^*}{\delta^*(1 + \delta^*)} \left[1 + \sqrt{1 + \frac{16(1 + \delta^*)^2}{(1 + 2\delta^*)^2}} \right]. \quad (17.15)$$

This relation is shown graphically in Fig. 4. Some of the pairs of values (γ, δ^*)

FIG. 4. Strength δ^* at which $\mathcal{W}_D = \infty$ for glancing shocks ($\sigma = \frac{1}{2}\pi$)

are: (1, 3.05), (1.40, 2.53), (1.67, 2.27), (2, 2), and in fact δ^* is a monotone decreasing function of γ in the range $1 \leq \gamma \leq 2$. Furthermore, from (17.13) it is easy to show that $(MB - A)\delta^{-2}$ is a monotone decreasing function of δ . Taking into account the absolute value sign in (17.12), we conclude

Theorem 24. *For glancing shocks ($\sigma = \frac{1}{2}\pi$), two cases must be distinguished.*

(a) *If $1 \leq \gamma < 2$, $\mathcal{W}_D$ increases with δ until it becomes infinite when δ attains the value δ^* given by (17.15); thereafter it decreases steadily.*

(b) *If $\gamma \geq 2$, $\mathcal{W}_D$ increases steadily with δ ; if $\gamma = 2$, $\mathcal{W}_D$ becomes infinite for shocks of maximum strength, while if $\gamma > 2$, $\mathcal{W}_D$ remains bounded.*

In both cases

$$\mathcal{W}_D \big|_{\sigma=\frac{1}{2}\pi, \delta=2/(\gamma-1)} = \frac{\gamma + 1}{2\gamma |2 - \gamma|}. \quad (17.16)$$

To gain some insight into the behavior of $\mathcal{W}_D$ as a function of γ , let us hold δ at any fixed value and again set $\sigma = \frac{1}{2}\pi$. By differentiating (17.13) it is easy to verify the following

Lemma. For any fixed $\delta \geq 0$, $(MB - A)\delta^{-2}$ is a monotone decreasing function of γ .

To follow the succeeding argument it is essential to recall that $\delta \leq 2/(\gamma - 1)$. For fixed δ , it is equivalent to say that γ varies in the range $1 \leq \gamma \leq 1 + 2/\delta$.

Suppose first $\delta \geq \delta^*$, where $\delta^* = 3.05 \dots$ is the positive root δ^* of (17.15) when $\gamma = 1$. Then $\delta \geq \delta^*$ for all possible values of γ , so $MB - A \leq 0$ for all such γ . By the lemma and (17.12), then, $\mathcal{U}_D|_{\sigma=\frac{1}{2}\pi}$ decreases steadily to the limit value (17.16), where γ is to be put equal to $1 + 2/\delta$.

Next, suppose $\delta^* > \delta > 2$. Then for $\gamma = 1$ we have $\delta < \delta^*$. As γ increases from 1 to $1 + 2/\delta$, the point δ^* moves leftward and crosses through the value δ . Hence $\mathcal{U}_D$ increases to ∞ , then falls off again to the value (17.16).

Finally, suppose $\delta \leq 2$. Then $\delta \leq \delta^*$ for all γ . As γ increases from 1 to $1 + 2/\delta$, the point δ^* again moves leftward, but now it fails to reach δ in the permitted range of variation for γ . Hence $\mathcal{U}_D$ increases steadily to the limit value (17.16).

In summary of this analysis we have

Theorem 25. Assign δ a fixed positive value, and put $\sigma = \frac{1}{2}\pi$. Then let γ increase from 1 to $1 + 2/\delta$.

- (a) If $\delta \geq \delta^*$, $\mathcal{U}_D$ decreases.
- (b) If $2 < \delta < \delta^*$, $\mathcal{U}_D$ increases until at the value of γ given by (17.15) with $\delta^* = \delta$, it becomes infinite; then it decreases.
- (c) If $0 < \delta \leq 2$, $\mathcal{U}_D$ increases.

In all cases

$$\mathcal{U}_D|_{\sigma=\frac{1}{2}\pi, \gamma=1+(2/\delta)} = \frac{\delta + 1}{(\delta + 2)|\delta - 2|}. \quad (17.17)$$

This theorem suggests the following conjecture. In the σ - δ plane there exist curves C_1 and C_2 joining the points $(\frac{1}{2}\pi, \delta^*)$ and $(\frac{1}{2}\pi, 2)$ to a certain point $(0, \delta')$, such that in the region interior to C_1 , C_2 and the segment $\delta^* \geq \delta \geq 2$, $\sigma = \frac{1}{2}\pi$, the number $\mathcal{U}_D$ for fixed σ and δ experiences a single maximum as a function of γ , while to the right of C_1 $\mathcal{U}_D$ is a decreasing function, to the left of C_2 an increasing function.

We now consider $\mathcal{U}_D$ as a function of σ when δ is held fixed. From (17.10) we may calculate and annul $d\mathcal{U}_D/dx$, thus obtaining a quadratic for x , whose roots are

$$x = D^2 \left[\pm \sqrt{1 + \frac{F - ED^2}{B^2 D^4}} - 1 \right]. \quad (17.18)$$

The minus sign cannot yield a real value for τ . To see whether the plus sign can, note that

$$\begin{aligned} (\gamma + 1)^3 \delta^2 \frac{F - ED^2}{BD^2} &= 128 + 256\delta + 2(-\gamma^2 + 14\gamma + 79)\delta^2 \\ &+ (\gamma + 1)(-\gamma^2 - 14\gamma + 19)\delta^3 - 2(\gamma + 1)^2(\gamma + 5)\delta^4 - 2(\gamma + 1)^3\delta^5. \end{aligned} \quad (17.19)$$

The coefficient of δ^2 is non-negative when $\gamma \leq 7 + 8\sqrt{2}$. Hence, for such values of γ , there is one change of sign in the coefficients, so that by Descartes' rule it follows that the left hand side may vanish for at most one positive value of δ . Since when $\delta = 0$ the right hand side is positive, but when $\delta = \infty$ it is negative, there must be exactly one positive root δ_0 .

Does this root fall in the range $0 \leq \delta \leq 2/(\gamma - 1)$? We have in fact

$$(\gamma - 1)^5 (\gamma + 1)^3 \delta^2 \frac{F - ED^2}{BD^2} \Big|_{\delta=2/(\gamma-1)} = 16\gamma(\gamma + 1)^2 (7\gamma^2 - 22\gamma + 7). \quad (17.20)$$

The right hand side is non-positive when $7\gamma \leq 11 + 6\sqrt{2}$, becoming positive for greater values of γ . Hence $F - ED^2$ experiences a single change of sign in the former case, no such change in the latter case. For the case when such a change occurs, the corresponding shock strength δ_0 is shown graphically in Fig. 5. Some pairs of values (γ, δ_0) are (1, 2.08), (1.40, 1.75), (1.67, 1.57), (2, 1.41).

FIG. 5. Upper curve: strength δ_0 below which $\mathcal{W}_D$ as a function of σ suffers a single maximum, above which it is a monotone increasing function. Lower curve: corresponding value $\mathcal{W}_{Dmm}$ of $\mathcal{W}_D$ when $\sigma = \frac{1}{2}\pi$.

Choosing then the plus sign in (17.18) and putting the resulting expression for x into (17.10), after some reductions we obtain for the corresponding value $\mathcal{W}_{Dm}$ of $\mathcal{W}_D$

$$\mathcal{W}_{Dm} = \frac{\delta^2}{\sqrt{E + 2B^2 x}}. \quad (17.21)$$

When $\delta = \delta_0$, the angle σ_m where the maximum occurs becomes $\sigma_m = \frac{1}{2}\pi$ and

of course as $\delta \rightarrow \delta_0 - 0$ also $\sigma_m \rightarrow \frac{1}{2}\pi - 0$. For $\delta = \delta_0$ we have both $x = 0$ and $E = F/D^2$, so (17.21) yields for the corresponding value $\mathcal{W}_{Dmm}$ of $\mathcal{W}_D$

$$\begin{aligned}\mathcal{W}_{Dmm} &= \frac{\delta_0^2 D}{\sqrt{F}} = \frac{\delta_0^2 D}{|AD - C|}, \\ &= \frac{\delta_0^2}{|A - MB|}, \\ &= \frac{1}{\left| 1 - \frac{16 + 2(\gamma + 9)\delta_0 + (3 - \gamma)(\gamma + 1)\delta_0^2}{\delta_0^3(\gamma + 1)^2} \right|}.\end{aligned}\tag{17.22}$$

This quantity also is shown graphically in Fig. 5. Its value is never so great as 1, and it is a nearly constant monotone decreasing function of γ .

In summary of the foregoing analysis we have

Theorem 26. *In considering $\mathcal{W}_D$ as a function of σ for fixed strength δ , two cases must be distinguished.*

a. *If $1 < \gamma \leq (11 + 6\sqrt{2})/7$, then there exists a critical value δ_0 of the shock strength δ , given as the single positive root of the polynomial on the right hand side of (17.19), such that when $\delta < \delta_0$ $\mathcal{W}_D$ experiences a single maximum, but when $\delta \geq \delta_0$ $\mathcal{W}_D$ is monotone increasing. For such δ , the maximum occurs when*

$$\tau = \tau_m = \frac{1}{\sqrt{x}}, \quad x = D^2 \left[\sqrt{1 + \frac{F - ED^2}{B^2 D^4}} - 1 \right], \tag{17.23}$$

the value of the maximum being given by (17.21). Further, as $\delta \rightarrow \delta_0 - 0$, $\sigma_m \rightarrow \frac{1}{2}\pi - 0$. The value of $\mathcal{W}_D$ when $\delta = \delta_0$, $\sigma = \frac{1}{2}\pi$ is given by (17.22); this value is inferior to 1. If $\delta = \delta_0$, the curve of $\mathcal{W}_D$ as a function of σ is flat at $\sigma = \frac{1}{2}\pi$; if $\delta > \delta_0$, the curve of $\mathcal{W}_D$ as a function of δ has positive slope at $\sigma = \frac{1}{2}\pi$.

b. *If, however, $(11 + 6\sqrt{2})/7 < \gamma \leq 7 + 8\sqrt{2}$, then $\mathcal{W}_D$ is monotone increasing in σ for any fixed value of δ .*

Values of γ in excess of $7 + 8\sqrt{2}$, being devoid of physical interest, are not considered.

I conjecture further that in case (a) both τ_m and $\mathcal{W}_{Dm}$ are monotone increasing functions of δ and monotone decreasing functions of γ . I conjecture also that when $\gamma < 2$ there exists a critical shock angle σ_0 such that when $\sigma < \sigma_0$ $\mathcal{W}_D$ is a monotone increasing function of δ , but when $\sigma \geq \sigma_0$ $\mathcal{W}_D$ experiences a single maximum as a function of δ ; that there exists a $\gamma_0 > 2$ such that when $2 \leq \gamma < \gamma_0$, there is a second critical angle σ_1 , having the property that when $\sigma_0 \leq \sigma \leq \sigma_1$ $\mathcal{W}_D$ experiences a maximum, but when $\sigma < \sigma_0$ or $\sigma > \sigma_1$ $\mathcal{W}_D$ is monotone; that finally, when $\gamma = \gamma_0$ $\mathcal{W}_D$ is a monotone increasing function of δ for all shock angles.

For the case $\gamma = 1.40$, the surface $\mathcal{W}_D = f(\sigma, \delta)$ is shown in isometric projection in Fig. 6.

We note also the limit formulae

$$\mathcal{W}_D = \frac{\delta^3(1 + \tau^2)^{\frac{1}{2}}(\gamma + 1)^2}{16\tau} + O(\delta^5), \quad \tau = \text{const.} \neq 0, \quad \delta \rightarrow 0. \quad (17.24)$$

$$\mathcal{W}_D = \frac{\tau\delta^2}{\delta + \frac{\gamma + 1}{2}} \left(1 + \frac{DB^2 - E}{2B^2} \tau^2 \right) + O(\tau^5), \quad \delta = \text{const.}, \quad \tau \rightarrow 0. \quad (17.25)$$

$$\mathcal{W}_D = \left[\frac{1}{\frac{16 + 2(\gamma + 9)\delta + (3 - \gamma)(\gamma + 1)\delta^2}{\delta^3(\gamma + 1)^2} - 1} \right] \left(1 + \frac{F - ED^2}{2FD^2\tau^2} \right) + O\left(\frac{1}{\tau^4}\right), \quad \delta = \text{const.}, \quad \tau \rightarrow \infty. \quad (17.26)$$

To obtain bounds for $\mathcal{W}_D$, we begin by noting that from the steps leading to (17.8) and from a formula for $|\text{grad } p|$ which I have given elsewhere (1952a) we obtain from (13.8)

$$\left| 1 - \sqrt{\frac{B_{12}^2 + (\chi B_{12} - B_{22})^2}{B_{21}^2 + (\chi B_{21} - B_{22})^2}} \right| \leq \mathcal{W}_D \leq 1 + \sqrt{\frac{B_{12}^2 + (\chi B_{12} - B_{22})^2}{B_{21}^2 + (\chi B_{21} - B_{22})^2}}. \quad (17.27)$$

We could take up special cases in which these bounds can be simplified, but in these cases somewhat sharper estimates can be obtained directly.

When $C\tau^2 < B$, or, equivalently,

$$\tau^2 < \frac{1}{MD} = \frac{\gamma + 1}{(1 + \delta) \left(\frac{8}{\delta} + 7 - \gamma \right)}, \quad (17.28)$$

we have $B_{22} < 0$, and hence, since $B_{21} > 0$,

$$B_{21}^2 + (\chi B_{21} - B_{22})^2 > (1 + \chi^2)B_{21}^2. \quad (17.29)$$

By (17.9), then,

$$\mathcal{W}_D < \frac{\tau\delta^2}{B_{21}} = \frac{1}{1 + \frac{4(1 + \delta)}{(\gamma + 1)\delta^2}}. \quad (17.30)$$

The bound on the right is monotone increasing with δ from 0 to γ^{-1} , and hence *a fortiori* (17.28) insures that $\mathcal{W}_D < 1$. One notes that for fixed δ , as γ increases so does the right hand side of (17.28); that is, increase in γ increases the range of shock angles for which $\mathcal{W}_D < 1$. This fact supports the conjecture made above regarding the behavior of $\mathcal{W}_D$ as a function of γ when δ and σ are held fixed.

It is permissible to replace the signs " $<$ " or " $>$ " by " $=$ " in (17.28), (17.29)

and (17.30). (17.28) then yields a curve on which the value of $\mathfrak{W}_D$ is the especially simple expression on the right hand side of (17.30), and this curve is the boundary of the region in which the bound (17.30) is valid.

To obtain a lower bound, first try to find a constant $\alpha > 0$ such that there exist a range of τ and δ satisfying

$$-\alpha B_{21} \leq B_{22} - \chi B_{21} \leq \alpha B_{21}. \quad (17.31)$$

Equivalently,

$$(D\tau - \alpha)A\tau \leq B(-1 + MD\tau^2) \leq (D\tau + \alpha)A\tau. \quad (17.32)$$

Analysis of this condition, using the results obtained already concerning the sign of $MB - A$, shows that when $\delta \leq \delta^*$ it is equivalent to

$$\begin{aligned} \frac{\alpha A}{2B} \left[-1 + \sqrt{1 + \frac{4BD(MB - A)}{A^2 \alpha^2}} \right] &\leq \frac{1}{\tau} \\ &\leq \frac{\alpha A}{2B} \left[1 + \sqrt{1 + \frac{4BD(MB - A)}{A^2 \alpha^2}} \right], \end{aligned} \quad (17.33)$$

for arbitrary α , but that when $\delta \geq \delta^*$ we must choose α large enough that

$$4BD(A - MB) \leq \alpha^2 A^2 \quad (17.34)$$

for the value of δ in question, while τ must satisfy

$$\begin{aligned} \frac{\alpha A}{2B} \left[1 - \sqrt{1 + \frac{4BD(MB - A)}{A^2 \alpha^2}} \right] &\leq \frac{1}{\tau} \\ &\leq \frac{\alpha A}{2B} \left[1 + \sqrt{1 + \frac{4BD(MB - A)}{A^2 \alpha^2}} \right]. \end{aligned} \quad (17.35)$$

The region thus obtained is a disc in the σ - τ plane, bounded away from the lines $\sigma = 0$ and $\delta = 0$, while touching $\sigma = \frac{1}{2}\pi$ only when $\delta = \delta^*$. A part of the disc lies on either side of the line $\delta = \delta^*$. On the left side, α remains arbitrary, while on the right side α must be chosen large enough that (17.33) is satisfied. Under these conditions, then, we have (17.30).

From (17.30) follows

$$B_{21}^2 + (B_{22} - \chi B_{21})^2 \leq (1 + \alpha^2) B_{21}^2, \quad (17.36)$$

and hence by (17.9)

$$\begin{aligned} \mathfrak{W}_D &\geq \frac{2B_{[21]}(1 + \chi^2)^{\frac{1}{2}}}{B_{21}(1 + \alpha^2)^{\frac{1}{2}}}, \\ &= \frac{1}{1 + \frac{4(1 + \delta)}{\delta^2(\gamma + 1)}} \sqrt{\frac{1 + \tau^2(1 + \delta)^2}{1 + \alpha^2}}, \end{aligned} \quad (17.37)$$

which is the required lower bound. One notes that for the region $\delta \leq \delta^*$ by choice of α one can find a curve (17.33) which with the line $\delta = \delta^*$ bounds a region in which the right side of (17.30) furnishes a *lower* bound for $\mathcal{W}_D$, just as it furnishes an upper bound in the region (17.28).

Two special cases may be noted. First, choose $\alpha = 0$, $\delta \leq \delta^*$. Then the region (17.33) reduces to the curve

$$\frac{1}{\tau} = \sqrt{D \left(M - \frac{A}{B} \right)}, \quad (17.38)$$

a curve stretching across the σ - δ plane from the origin $(0, 0)$ to the point $(\frac{1}{2}\pi, \delta^*)$. On this curve (17.37) yields

$$\mathcal{W}_D = \frac{\sqrt{1 + \frac{1 + \delta}{M - \frac{A}{B}}}}{1 + \frac{4(1 + \delta)}{\delta^2(1 + \gamma)}}, \quad (17.39)$$

a function of δ which for $\delta > 0$ always exceeds (17.31) and which increases steadily from the value 0 at $(0, 0)$ to the value ∞ at $(\frac{1}{2}\pi, \delta^*)$.

For the second special case, suppose $\delta \geq \delta^*$ and choose α to satisfy the equality in (17.35). Then the region (17.36) reduces to the curve

$$\frac{1}{\tau} = \sqrt{D \left(\frac{A}{B} - M \right)}, \quad (17.40)$$

a curve which, when $\gamma \leq 2$, stretches across the σ - δ plane from the point $(\frac{1}{2}\pi, \delta^*)$ to the point $(\text{Arctan} [(\gamma - 1)\{2(2 - \gamma)(\gamma + 1)\}^{-\frac{1}{2}}], 2/[\gamma - 1])$. On this curve $B_{22} - \chi B_{21} = -\alpha B_{21}$, so that

$$\mathcal{W}_D = \frac{1}{1 + \frac{4(1 + \delta)}{\delta^2(\gamma + 1)}} \sqrt{\frac{1 + \epsilon}{1 + \frac{4B^2}{A^2} \epsilon}}, \quad \epsilon \equiv \frac{D}{\frac{A}{B} - M}. \quad (17.41)$$

Finally, notice that on the curve $B_{22} - \chi B_{21} = +\chi B_{21}$ the value of $\mathcal{W}_D$ is the quantity on the right hand side of (17.31). The equation of this curve is

$$\tau^2 = \frac{B}{D(MB - 2A)}; \quad (17.42)$$

hence it intersects the line $\sigma = \frac{1}{2}\pi$ at a value of δ which is less than δ^* . This curve and the one obtained by replacing “<” by “=” in (17.28), in the part of the σ - δ plane where both exist, lie before and behind, respectively, the curve of the maxima of $\mathcal{W}_D$ as a function of σ . At the points of intersection of lines $\delta = \text{const.}$ with these two curves, $\mathcal{W}_D$ has the same value.

We now turn to $\mathcal{W}_K$. By (17.1) we have

$$\begin{aligned} d_{\alpha\beta} d_{\alpha\beta} &= v_{(\alpha,\beta)} v_{\alpha,\beta}, \\ &= \kappa^2 v_n^4 B_{ij} B_{kl} D_{i(\alpha} D_{j\beta)} D_{k\alpha} D_{l\beta}, \\ &= \kappa^2 v_n^4 B_{(ij)} B_{(kl)} D_{i\alpha} D_{j\beta} D_{k\alpha} D_{l\beta}. \end{aligned} \quad (17.43)$$

Writing $\mathbf{B} \equiv \| B_{(ij)} \|$, we have by (17.4)

$$d_{\alpha\beta} d_{\alpha\beta} = \kappa^2 v_n^4 \text{trace } (\mathbf{B}\mathbf{X})^2. \quad (17.44)$$

Substituting in (2.5) yields

$$\mathcal{W}_K = \frac{\sqrt{2} B_{[21]}}{\{\text{trace } (\mathbf{B}\mathbf{X})^2\}^{\frac{1}{2}}}. \quad (17.45)$$

By using (17.4) to put this result into explicit form, we obtain

Theorem 26. *If $\kappa = 0$, $\mathcal{W}_K$ is zero or indeterminate. If $\kappa \neq 0$, however, we have*

$$\mathcal{W}_K = \frac{\sqrt{2} B_{[21]}}{\{B_{11}^2 + 2(\chi B_{11} - B_{(12)})^2 + (B_{22} - 2\chi B_{(12)} + \chi^2 B_{11})^2\}^{\frac{1}{2}}}. \quad (17.46)$$

Thus for curved shocks $\mathcal{W}_K$ is a function of χ and of the B_{ij} only. For a perfect gas in uniform flow ahead of the shock, consequently, $\mathcal{W}_K$ is a function of δ , τ , and γ only.

From (17.46) it follows that for $\mathcal{W}_K = \infty$ it is necessary that $B_{11} = B_{(12)} = B_{22} = 0$. When (17.6) and (17.7) hold, these values of the B_{ij} are impossible, so that $\mathcal{W}_K$ must remain finite.

For the special case of a perfect gas flow which is uniform ahead of the shock, an explicit expression for $\mathcal{W}_K$ as a function of τ , δ , and γ is obtainable from

$$2\mathcal{W}_K^2 = \frac{\tau^2 \delta^4}{U + V\tau^2 + W\tau^4}, \quad (17.47)$$

where

$$\begin{aligned} U &= \delta^2 + B^2, & S &= \tfrac{1}{2}(A + R), \\ V &= 2S^2 - 2MD \left(B^2 - \frac{2\delta}{\gamma + 1} \right), \\ W &= \left(\frac{2MD}{\gamma + 1} \right)^2. \end{aligned} \quad (17.48)$$

The following approximate formulae are easily obtained:

$$\mathcal{W}_K = \frac{(\gamma + 1)^2 \delta^3}{16 \sqrt{2} \tau} + O(\delta^4), \quad \tau = \text{const.} \neq 0, \quad \delta \rightarrow 0. \quad (17.49)$$

$$\mathcal{W}_K = \frac{\tau \delta^2}{2 \sqrt{\delta^2 + \frac{2\delta}{\gamma + 1} + \frac{2}{(\gamma + 1)^2}}} + O(\tau^3), \quad \delta = \text{const.}, \quad \tau \rightarrow 0. \quad (17.50)$$

$$\mathcal{W}_K = \frac{(\gamma + 1)^2 \delta^3}{2\sqrt{2} \tau(1 + \delta)(8 + [7 - \gamma]\delta)} + O\left(\frac{1}{\tau^3}\right), \quad \delta = \text{const.}, \quad \tau \rightarrow \infty. \quad (17.51)$$

From the second and third of these expressions it follows that $\mathcal{W}_K$, considered as a function of τ when δ is held constant, vanishes both when $\tau = 0$ and when $\tau = \infty$, and hence that it must experience at least one maximum. In fact by calculus one easily concludes that but a single maximum exists, the corresponding obliquity $\tau = \tau_m$ being given by

$$\tau_m^4 = \frac{U}{W} = \frac{(\gamma + 1)^4 \left[\left(1 - \frac{1}{1 + \delta}\right)^2 + \left(1 - \frac{\gamma - 1}{(\gamma + 1)(1 + \delta)}\right)^2 \right]}{4 \left(\frac{8}{\delta} + 7 - \gamma\right)^2}, \quad (17.52)$$

with the value

$$2\mathcal{W}_{K_m}^2 = \frac{\delta^4}{V + 2\sqrt{UW}}. \quad (17.53)$$

The formula (17.52) shows that τ_m is a monotone increasing function of δ , and that when $\delta = 2/(\gamma - 1)$ we have $\tau = \tau_{mm}$, where

$$9\tau_{mm}^4 = 1 + \left(\frac{2\gamma}{\gamma + 1}\right)^2. \quad (17.54)$$

More specifically, for any δ we have the bounds

$$\frac{1}{M^2 D^2} < \tau_m^4 < \frac{1 + \left(\frac{2\gamma}{\gamma + 1}\right)^2}{M^2}. \quad (17.55)$$

For fixed δ , τ_m is a monotone increasing function of γ , provided $\gamma < 7$. Some pairs of values (γ, τ_{mm}) are (1, .714), (1.40, .715), (1.67, .730), (2, .745). Thus for all gases σ_{mm} is close to 36° . The value $\mathcal{W}_{K_{mm}}$ of $\mathcal{W}_K$ corresponding to (17.54) is given by

$$\mathcal{W}_{K_{mm}}^2 = \frac{\gamma + 1}{-5\gamma^3 + 9\gamma^2 - 6\gamma + 4 + 3(\gamma - 1)^2 \sqrt{5\gamma^2 + 2\gamma + 1}}. \quad (17.56)$$

$\mathcal{W}_{Kmm}$ as a function of γ is plotted in Fig. 7. The single sharp maximum occurs when $\gamma = 1.49$. Some pairs of values $(\gamma, \mathcal{W}_{Kmm})$ are (1, 1), (1.40, 1.35) (1.67, 1.29) (2, 1).

FIG. 7. The height of the peak in $\mathcal{W}_K$ as a function of γ for shocks of maximum strength

In summary of these facts, we have

Theorem 27. *The kinematical vorticity number $\mathcal{W}_K$ is a finite, continuous function of δ , σ , and γ , vanishing if and only if $\sigma = 0$, $\sigma = \frac{1}{2}\pi$, or $\delta = 0$. For any fixed δ and γ , it experiences a single maximum at the angle σ_m given by (17.52), where its value is given by (17.53). The angle σ_m increases steadily from 0 for $\delta = 0+$ to the value given by (17.54) (about 36° for all perfect gases) for shocks of maximum strength. Moreover, for fixed δ the angle σ_m is a monotone increasing function of γ , provided $\gamma < 7$. The value $\mathcal{W}_{Kmm}$ of the maximum corresponding to a shock of maximum strength is given by (17.56); for $1 \leq \gamma \leq 2$ we have $1 \leq \mathcal{W}_{Kmm} \leq 1.92$, the value 1.92 being achieved when $\gamma = 1.49$.*

Numerical exploration suggests also the following conjectures:

- a) $\mathcal{W}_K$ is a monotone increasing function of δ for any fixed σ and γ .
- b) For fixed δ and σ , $\mathcal{W}_K$ is a monotone decreasing function of γ .
- c) For fixed γ , $\mathcal{W}_K$ as a function of δ and σ experiences its greatest value $\mathcal{W}_D = \mathcal{W}_{Kmm}$ at $\delta = 2/(\gamma - 1)$, $\sigma = \sigma_m$, i.e., for shocks of maximum strength, and of inclination about 36° for all gases. Hence $\mathcal{W}_K \leq 1.92$ for any shock in any gas ($1 < \gamma \leq \frac{5}{3}$).

For the case $\gamma = 1.40$, the surface $\mathcal{W}_K = f(\sigma, \delta)$ is shown in isometric projection in Fig. 8.

Comparison of Fig. 6 and Fig. 8 reveals a striking difference in the behaviors of $\mathcal{W}_K$ and $\mathcal{W}_D$. In §15 we noted examples when $\mathcal{W}_K$ was small but $\mathcal{W}_D$ moderately large. Here, however, we see that at the point $(\frac{1}{2}\pi, \delta_0)$ we have $\mathcal{W}_K = 0$, but $\mathcal{W}_D = \infty$! For all but the weakest of nearly glancing shocks, while the actual rotation in the flow is very slight, what there is furnishes the major part of the acceleration.

Fig. 8. σ_k as a function of σ and δ , when $\gamma = 1.40$

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