*An Asymptotic Expansion of the Whittaker Function $W_{k,m}(z)$ **

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In solving a transonic flow problem by the hodograph method, a family of particular solutions of Chaplygin's second equation for a hypothetical gas were found [1]. The general form of these solutions is $W_{k,m}(z) \sin n\theta$, where θ is the angle of the velocity vector and the parameters of the Whittaker function can be written $k = -k_0 n + k_1$, $m = im_0 n + m_1 + O\left(\frac{1}{n}\right)$, $z = nz_0$ for all values of $n \geq 2$, (k_0, m_0, z_0 are all real, positive quantities). We propose to determine the asymptotic expansion of $W_{k,m}(z)$, which shall now be designated $W_{k,m}^{(n)}(z)$ to indicate that all parameters are functions of n for large values of n .† The asymptotic expansion turns out to be basically different in the cases‡ when

$$\Delta_0^2 = 1 + 4k_0/z_0 - 4m_0^2/z_0^2$$

is positive, zero or negative, corresponding respectively to subsonic, sonic and supersonic flow. Actually $\Delta_0^2 = 0$ corresponds to the transition point of the Whittaker equation at large n , as shown later.

For low subsonic flow ($\Delta_0^2 \rightarrow 1$), high supersonic flow ($\Delta_0^2 \rightarrow -\infty$) and sonic flow ($\Delta_0^2 = 0$) the corresponding asymptotic expansions are obtained by Debye's method of steepest descent [3]. For transonic cases (Δ_0 is nearly 0) this method is impractical, and then Langer's method is used [4].

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† See Appendix.

‡ Similar to those for Bessel functions, see for example [2].

The integral representation for the Whittaker function is [5]

$$W_{k,m}^{(n)}(z) = -\frac{1}{2\pi i} \Gamma(k + \frac{1}{2} - m) e^{-(z/2)} z^k \cdot \int_{\infty}^{(0+)} (-t')^{-k-\frac{1}{2}+m} \left(1 + \frac{t'}{z}\right)^{k-\frac{1}{2}+m} e^{-t'} dt'. \quad (1)$$

The contour of integration must start at $+\infty$, proceed in a counterclockwise direction around the origin, and return to $+\infty$ in such a way that the branch point $t' = -z$ is excluded. The integrand is rendered one-valued by taking $-\pi \leq \arg(-t') < \pi$ and that value of $\arg\left(1 + \frac{t'}{z}\right)$ which tends to zero as $t' \rightarrow 0$ on a path lying inside the contour.

It is convenient to perform the transformation $t' = zt$. Equation (1) yields

$$W_{k,m}^{(n)}(z) = -\frac{1}{2\pi i} \Gamma(k + \frac{1}{2} - m) e^{-(z/2)} z^{\frac{1}{2}+m} \cdot \int_{\infty}^{(0+)} (-t)^{-k-\frac{1}{2}+m} (1+t)^{k-\frac{1}{2}+m} e^{-zt} dt. \quad (2)$$

The branch point is now $t = -1$. Let the integral part of the expression be I :

$$I = \int_{\infty}^{(0+)} \exp[f(t)] dt,$$

where

$$f(t) = (-k - \frac{1}{2} + m) \log(-t) + (k - \frac{1}{2} + m) \log(1+t) - zt. \quad (3)$$

If the asymptotic expansion of I can be found, then that of $W_{k,m}^{(n)}(z)$ can be written immediately because the asymptotic expansion of the gamma function is well known.

The saddle points, t_1, t_2 , are given by the roots of the equation

$$f'(t) = z(t - t_1)(t - t_2)t^{-1}(t + 1)^{-1} = 0, \quad (4)$$

whence

$$t_1, t_2 = \left(\frac{m}{z} - \frac{1}{2}\right) - \frac{1}{2z} \pm \frac{1}{2}\Delta, \quad (5)$$

where

$$\begin{aligned} \Delta^2 &= \Delta_0^2 + [4im_0(2m_1 - 1) - 4k_1 z_0]/nz_0^2 + O(n^{-2}), \\ \Delta_0^2 &= 1 + 4k_0/z_0 - 4m_0^2/z_0^2. \end{aligned} \quad (6)$$

For $\Delta_0 \neq 0$ the saddle points can be written

$$t_1, t_2 = \lambda_{1,2}^{(0)} + \lambda_{1,2}^{(1)} n^{-1} + O(n^{-2}),$$

where

$$\begin{aligned} \lambda_{1,2}^{(0)} &= im_0/z_0 - \frac{1}{2} \pm \Delta_0/2, \\ \lambda_{1,2}^{(1)} &= [(m_1 - \frac{1}{2})\Delta_0 z_0 \pm \{im_0(2m_1 - 1) - k_1 z_0\}]/\Delta_0 z_0^2. \end{aligned} \quad (7)$$

As $n \rightarrow \infty$, t_1, t_2 become $\lambda_{1,2}^{(0)}$. In an ϵ neighborhood of the saddle points, $f(t)$ can be expanded in Taylor's series:

$$\begin{aligned} f(t) &= f(t_1) - \frac{z}{2} \frac{t_1 - t_2}{t_1(t_1 + 1)} (t - t_1)^2 + \dots, \\ f(t) &= f(t_2) - \frac{z}{2} \frac{t_2 - t_1}{t_2(t_2 + 1)} (t - t_2)^2 + \dots. \end{aligned} \quad (8)$$

It is assumed that the value $|\Delta_0|$ is nearly one or greater and that ϵ is small enough that the third and higher order terms are negligible. The contributions to the value of the integral I given near the saddle points, where γ_1, γ_2 (restricted to be $< \pi$) are chosen as directions of steepest descent according to Debye, are

$$\begin{aligned} I_1 &= \exp f(t_1) e^{i\gamma_1} \int_{-\epsilon}^{\epsilon} \exp \left\{ -\frac{z}{2} \frac{t_1 - t_2}{t_1(t_1 + 1)} s^2 e^{i\gamma_1} \right\} ds \\ &= \exp f(t_1) \sqrt{\frac{2\pi}{z} \frac{t_1(t_1 + 1)}{t_1 - t_2}}, \\ I_2 &= \exp f(t_2) e^{i\gamma_2} \int_{-\epsilon}^{\epsilon} \exp \left\{ -\frac{z}{2} \frac{t_2 - t_1}{t_2(t_2 + 1)} s^2 e^{i\gamma_2} \right\} ds \\ &= \exp f(t_2) \sqrt{\frac{2\pi}{z} \frac{t_2(t_2 + 1)}{t_2 - t_1}}, \end{aligned} \quad (9)$$

where the variable $t - t_j$ has been replaced by $se^{i\gamma_j}$ ($j = 1, 2$).

Now consider $\Delta_0^2 > 0$, remembering the integration contour. The best path of integration is through t_1 and in the valleys (Fig. 1-a), since along this path almost the whole contribution to I is given by the part near the saddle point (I_1). The sense of integration $\int_{\infty}^{(0+)}$ is opposite to that of I_1 so $I = -I_1$. Including the asymptotic expansion of the gamma function by Stirling's formula we obtain

$$W_{k,m}^{(n)}(z) \sim -i[\exp f(t_1)]z^m(k - \frac{1}{2} - m)^{k-m}e^{-(k+\frac{1}{2}(s-1)-m)}\sqrt{\frac{t_1(t_1+1)}{t_1-t_2}}. \quad (10)$$

FIG. 1. Contour and path of steepest descent in t -plane when $\Delta_0^2 \gtrless 0$.

In the case where $m_1 = 0$ (m is pure imaginary), this expression should reduce to a real quantity, which it does.

$$\begin{aligned}
 W_{k,m}^{(n)}(z) &\sim -\frac{n^k z_0^k}{\sqrt{\Delta_0}} \left\{ \frac{\frac{1}{2} + \frac{k_0}{z_0} + \frac{\Delta_0}{z}}{\frac{1}{2} + \frac{k_0}{z_0} - \frac{\Delta_0}{z}} \left(\frac{k_0^2}{z_0^2} + \frac{m_0^2}{z_0^2} \right) \right\}^{\frac{1}{2}k} \\
 &\exp \left\{ nk_0 - nz_0 \frac{\Delta_0}{z} - nm_0 \tan^{-1} \left[\frac{\frac{m_0}{z_0} \frac{k_0}{z_0} (\Delta_0 - 1) + \frac{2m_0^3}{z_0^3}}{\frac{k_0^2}{z_0^2} \frac{m_0^2}{z_0^2} \Delta_0 - 2 \frac{k_0}{z_0} \frac{m_0^2}{z_0^2}} \right] \right\} \\
 &\cdot \left[1 + O\left(\frac{\log n}{n}\right) \right].
 \end{aligned} \tag{11}$$

Consider then $\Delta_0^2 < 0$. Here the integration path $I = -I_1 + I_2$ (Fig. 1-c). It follows that

$$W_{k, m}^{(n)}(z) \sim iz^m (k - \frac{1}{2} - m)^{k-m} \exp \left\{ - \left(k + \frac{z-1}{2} - m \right) \right\} \cdot \left\{ -\exp f(t_1) \sqrt{\frac{t_1(t_1+1)}{t_1-t_2}} + \exp f(t_2) \sqrt{\frac{t_2(t_2+1)}{t_2-t_1}} \right\}. \quad (12)$$

When m is purely imaginary, this expression reduces to the real form ($\Delta_0 = i\Delta_1$)

$$W_{k, m}^{(n)}(z) \sim -2 \frac{n^k z_0^k}{\Delta_1} \left\{ \frac{k_0^2}{z_0^2} + \frac{m_0^2}{z_0^2} \right\}^{k/2} \exp \left\{ nk_0 - nm_0 \tan^{-1} \left(\frac{m_0}{-k_0} \right) \right\} \cdot \cos \left[nm_0 \log \left(\frac{-k_0}{z_0} + \frac{2m_0}{z_0^2} + \frac{m_0 \Delta_1}{z_0} \right) \right. \\ \left. - \frac{nz_0 \Delta_1}{2} - (k + \frac{1}{4})\pi + 2k \tan^{-1} 2 \left(\frac{m_0}{z_0} + \frac{\Delta_1}{2} \right) + k \tan^{-1} \frac{m_0}{z_0} \right] \cdot \left[1 + O \left(\frac{\log n}{n} \right) \right]. \quad (13)$$

Now consider the case where $\Delta_0 = 0$:

$$t_1, t_2 = \mu^{(0)} + \mu_{1,2}^{(1)} n^{-\frac{1}{2}} + \mu^{(1)}(n^{-1}) + \mu_{1,2}^{(1)}(n^{-\frac{1}{2}}) + O(n^{-2}),$$

where

$$\mu^{(0)} = -\frac{1}{2} + i \frac{m_0}{z_0}, \quad \mu_{1,2}^{(1)} = \pm \{ im_0(2m_1 - 1)/z_0^2 - k_1/z_0 \}^{\frac{1}{2}}, \quad (14) \\ \mu^{(1)} = \frac{m_1}{z_0} - \frac{1}{2z_0}, \quad \mu_{1,2}^{(1)} = \pm \frac{1}{z_0} \frac{(2m_1 - 1)^2 + 1}{[im_0(2m_1 - 1) - k_1 z_0]^{\frac{1}{2}}}.$$

As $n \rightarrow \infty$, $t_1 = t_2 = \mu^{(0)}$. In this case, since the saddle points coincide, the second order term of the Taylor expansion vanishes and the third order term is necessary (higher order terms are assumed negligible).

$$f(t) = f(t_1) - \frac{z}{3} \frac{(t - t_1)^3}{t_1(t_1 + 1)} + \dots \quad (15)$$

A consideration of the complex t -plane (Fig. 1-b) where the paths of integration are lines of steepest descent gives $I = -I_A + I_B$ where

$$\begin{aligned}
 I_A &= \exp f(t_1) e^{i(\pi/3)} \int_0^\epsilon \exp \left\{ -\frac{z}{3} \frac{1}{t_1(t_1 + 1)} s^3 e^{i\pi} \right\} ds \\
 &= \exp f(t_1) \Gamma(\tfrac{1}{3}) e^{i(\pi/3)} \sqrt[3]{\frac{|t_1(t_1 + 1)|}{9z}}, \\
 I_B &= \exp f(t_1) e^{-i(\pi/3)} \int \exp \left\{ -\frac{z}{3} \frac{1}{t_1(t_1 + 1)} s^3 e^{-i\pi} \right\} ds \\
 &= \exp f(t_1) \Gamma(\tfrac{1}{3}) e^{-i(\pi/3)} \sqrt[3]{\frac{|t_1(t_1 + 1)|}{9z}}.
 \end{aligned} \tag{16}$$

Hence

$$\begin{aligned}
 W_{k,m}^{(n)}(z) &\sim \frac{1}{\sqrt{2\pi}} \Gamma(\tfrac{1}{3}) (k - \tfrac{1}{2} - m)^{k-m} \\
 &\cdot \left(\frac{z}{3}\right)^{1/6} e^{-(k+(z-1)/2-m)z^m} \exp f(t_1) \sqrt[3]{|t_1(t_1 + 1)|}.
 \end{aligned} \tag{17}$$

For $m_1 = 0$, the real form is

$$\begin{aligned}
 W_{k,m}^{(n)}(z) &\sim \frac{n^{k+1/6}}{\sqrt{2\pi}} \Gamma(\tfrac{1}{3}) \left(\frac{z_0}{3}\right)^{1/6} z_0^k e^{\{nk_0 - 2nm_0 \tan^{-1}(2m_0/z_0)\}} \\
 &\cdot \left(\frac{m_0^2}{z_0^2} + \tfrac{1}{4}\right)^{k-1/6} \left[1 + O\left(\frac{1}{n}\right)\right].
 \end{aligned} \tag{18}$$

The equation satisfied by the Whittaker function $W_{k,m}^{(n)}(z)$ is

$$\frac{d^2 F}{d\sigma^2} - n^2 K(\sigma) F = 0 \tag{19}$$

where $\sigma = \int_a^* \rho_0 dq/(\rho q)$, a monotonically decreasing function of velocity q (ρ_0/ρ is a known function q) (σ greater than zero, equal to zero, less than zero correspond respectively to subsonic, sonic and supersonic flow) and

$$K(\sigma) = a\sigma(1 + b\sigma)(1 + c\sigma)^{-2},$$

where a, b, c are real positive constants determined by the hypothetical gas chosen, k, m, z being related to a, b, c by

$$\begin{aligned}
 k &= -n(c - 2b)2^{-1}c^{-2}a^{\frac{1}{2}}b^{-\frac{1}{2}}, \\
 m &= [2^{-2} - n^2 a(c - b)c^{-4}]^{\frac{1}{2}}, \\
 z &= 2n(ab)^{\frac{1}{2}}c^{-2}(1 + c\sigma).
 \end{aligned} \tag{20}$$

According to Langer's theory [4], the two asymptotic solutions of the differential equation are

$$\begin{aligned}\mu_1(\sigma) &= |s|^{1/6} |K(\sigma)|^{-1/2} (ns)^{1/2} J_{-1/2}(ns)(2\pi)^{1/2}, \\ \mu_2(\sigma) &= |s|^{5/6} |K(\sigma)|^{-1/2} \operatorname{sgn} \sigma (ns)^{-1/2} J_{1/2}(ns)(2\pi)^{1/2},\end{aligned}\quad (21)$$

where $s(\sigma) = \int_0^\sigma \sqrt{-K(\sigma)} d\sigma$. The asymptotic expansion of any solution $F(n, \sigma)$ of the differential equation can be written as some linear combination of $\mu_1(\sigma)$ and $\mu_2(\sigma)$. When s is not small and $n \rightarrow \infty$ so that $|ns|$ also $\rightarrow \infty$, the universal asymptotic expansion of the Bessel function may be substituted into (21). It follows then that the asymptotic expansion of any solution of the differential equation can be written as some linear combinations of the functions

$$\begin{aligned}v_1(\sigma) &= |K(\sigma)|^{-1/2} \exp\left(in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right), \\ v_2(\sigma) &= |K(\sigma)|^{-1/2} \exp\left(-in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right),\end{aligned}\quad (22)$$

provided that s is not too small. In other words

$$\begin{aligned}F(n, \sigma) \sim |K(\sigma)|^{-1/2} \Big\{ f_1(n) \exp\left(+in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right) \\ + f_2(n) \exp\left(-in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right) \Big\}.\end{aligned}\quad (23)$$

The Whittaker function $W_{k,m}^{(n)}(z)$ is defined as that solution of the differential equation which remains finite as $\sigma \rightarrow \infty$. Hence, if we have the understanding that $F(n, \sigma)$ henceforth designate $W_{k,m}^{(n)}(z)$, then $f_2(n) = 0$. Thus

$$F(n, \sigma) = W_{k,m}^{(n)}(z) \sim |K(\sigma)|^{-1/2} f_1(n) \exp\left(+in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right). \quad (24)$$

To find $f_1(n)$, let $\sigma \rightarrow \infty$. Equating the right hand side to the equivalent asymptotic expression for the Whittaker function [5],

$$W_{k,m}^{(n)}(z) \sim e^{-z/2} z^k = (2n\sigma)^{-[(c-2b)/2c^2] \sqrt{(a/b)n}} e^{-n/c} e^{-n\sigma}, \quad (25)$$

we find that

$$\begin{aligned}f_1(n) &= \left(\frac{2b}{n}\right)^{n[(c-2b)/2c^2] \sqrt{(a/b)n}} \exp\left\{n \left(\frac{c-2b}{2bc}\right) \right. \\ &\quad \left. - \frac{n}{c} \sqrt{\frac{c-b}{b}} \left(\sin^{-1} \frac{c-2b}{c} + \frac{\pi}{2}\right)\right\}.\end{aligned}\quad (26)$$

To find the continuation of the asymptotic formula to other ranges of velocities (that is, other ranges of σ) consider the original $\mu_1(\sigma)$, $\mu_2(\sigma)$, where the general

expression is

$$F(n, \sigma) = W_{k, m}^{(n)}(z) = g_1(n)\mu_1(\sigma) + g_2(n)\mu_2(\sigma). \quad (27)$$

For $\sigma > 0$, s is imaginary ($s = i |s|$), since $K(\sigma) < 0$. Moreover, when σ is not small $|ns|$ will be very large as $n \rightarrow \infty$, so

$$\begin{aligned} \mu_1(\sigma) &\sim |K(\sigma)|^{-\frac{1}{2}} \{e^{n|s|} n^{-1/6} + e^{-n|s|} (-n)^{-1/6}\}, \\ \mu_2(\sigma) &\sim |K(\sigma)|^{-\frac{1}{2}} \{e^{n|s|} n^{-5/6} + e^{-n|s|} (-n)^{-5/6}\}, \end{aligned} \quad (28)$$

or

$$F(n, \sigma) = W_{k, m}^{(n)}(z) \sim |K(\sigma)|^{-\frac{1}{2}}$$

$$\begin{aligned} &\exp\left(n \int_0^\sigma |\sqrt{K(\sigma)}| d\sigma\right) \left\{g_1(n)n^{-1/6} + g_2(n)n^{-5/6}\right\} \\ &+ |K(\sigma)|^{-\frac{1}{2}} \exp\left(-n \int_0^\sigma |\sqrt{K(\sigma)}| d\sigma\right) \left\{g_1(n)(-n)^{-1/6} + g_2(n)(-n)^{-5/6}\right\}, \end{aligned} \quad (29)$$

so that

$$g_1(n)n^{-1/6} + g_2(n)n^{-5/6} = 0; \quad g_1(n)(-n)^{-1/6} + g_2(n)(-n)^{-5/6} = f_1(n),$$

or

$$g_1(n) = n^{1/6} 3^{-\frac{1}{2}} f_1(n); \quad g_2(n) = -(n)^{5/6} 3^{-\frac{1}{2}} f_1(n). \quad (30)$$

A general asymptotic form for the Whittaker function, valid for *all values of* σ , is

$$\begin{aligned} W_{k, m}^{(n)}(z) &\sim \left(\frac{2\pi}{3}\right)^{\frac{1}{2}} f_1(n) |K(\sigma)|^{-\frac{1}{2}} \\ &\{ |ns|^{1/6} (ns)^{\frac{1}{2}} J_{-1/3}(ns) - \operatorname{sgn} \sigma |ns|^{5/6} (ns)^{-\frac{1}{2}} J_{1/3}(ns) \}. \end{aligned} \quad (31)$$

Now we simplify this result for various values of σ .

1) *Low subsonic flow* ($\sigma \rightarrow \infty$).

$$W_{k, m}^{(n)}(z) \sim f_1(n) \exp\left(in \int_0^\sigma \sqrt{-K(\sigma)} d\sigma\right) |K(\sigma)|^{-\frac{1}{2}}. \quad (32)$$

2) *High supersonic flow* ($\sigma \rightarrow -1/c$). Here s will not be small so $|ns|$ will be large when $n \rightarrow \infty$. Remembering that

$$z^{-\nu} J_\nu(z) (2\pi)^{\frac{1}{2}} \sim e^{iz} (iz)^{-\nu-\frac{1}{2}} + e^{-iz} (-iz)^{-\nu-\frac{1}{2}} \quad \text{as } z \rightarrow \infty \quad (33)$$

and that s is real and positive for $\sigma < 0$, we reduce equation (31) to

$$W_{k, m}^{(n)}(z) \sim 2f_1(n) |K(\sigma)|^{-\frac{1}{2}} \cos\left[n \int_0^\sigma \sqrt{-K(\sigma)} d\sigma + \frac{\pi}{4}\right]. \quad (34)$$

3) *Sonic flow* ($\sigma = 0$). When $\sigma \rightarrow 0$, $s \rightarrow 0$; thus

$$W_{k,m}^{(n)}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{3}\right) \left| \frac{n}{3a} \right|^{1/6} f_1(n). \quad (35)$$

For all three of these cases if k_0, m_0, z_0 are expressed in terms of a, b, c and $k_1 = m_1 = 0$, the asymptotic expansions found previously reduce to these forms.

4) *High subsonic flow* (σ positive, $\neq 0$). Here s is small and $s = i |s|$, and since

$$I_{-}(z) - I_{+}(z) = \frac{2}{\pi} \sin \nu \pi K_{+}(z),$$

equation (31) can be simplified to

$$W_{k,m}^{(n)}(z) \sim \sqrt{\frac{2}{\pi}} f_1(n) |K(\sigma)|^{-1/2} K_{1/2}(n |s|). \quad (36)$$

5) *Low supersonic flow* (σ negative, $\neq 0$). Remembering that s is now a real negative number and that

$$Y_{+}(z) = [J_{+}(z) \cos \nu \pi - J_{-}(z)] / \sin \nu \pi, \quad H_{+}(z) = J_{+}(z) + iY_{+}(z),$$

we reduce equation (31) to the simpler form

$$W_{k,m}^{(n)}(z) \sim \sqrt{\frac{\pi}{2}} f_1(n) |ns|^{\frac{1}{2}} |K(\sigma)|^{-1/2} e^{i(\pi/6)} H_{1/2}(ns). \quad (37)$$

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Appendix

There are a number of asymptotic expansions of the Whittaker function for various conditions on the parameters k, m, z . Some of these are tabulated below.

Author	Year	k	m	z	Restriction	Method
Whittaker & Watson		complex bounded	complex bounded	complex $\rightarrow \infty$		
Perron	1921	complex $\rightarrow \infty$	complex bounded	complex bounded		Saddlepoint
Frenkel	1935	complex $\rightarrow \infty$	$\pm \frac{1}{2}$	complex bounded		Saddlepoint
Schwid	1935	complex $\rightarrow \infty$	$\pm \frac{1}{2}$	any value		Langer's
Erdélyi	1937	real, positive $\rightarrow \infty$	complex bounded	complex bounded		Saddlepoint
Erdélyi	1937	real, positive $\rightarrow \infty$	complex constant	real, positive $\rightarrow \infty$	$z/4k = \text{const.} < 1$	Saddlepoint
		imaginary $\rightarrow -i\infty$	complex constant	imaginary $\rightarrow -i\infty$	$z/4k = \text{const.} < 0$	Saddlepoint
Erdélyi	1938	real, positive $\rightarrow \infty$	real, positive $\rightarrow \infty$	real, positive $\rightarrow \infty$	$m > k$ $z/k, m/z$ constant	Saddlepoint
		real $ k \rightarrow \infty$	real $ m \rightarrow \infty$	real, positive $\rightarrow \infty$	$m > k$ $z/k, m/z$ constant	Saddlepoint
Taylor	1939	complex $\rightarrow \infty$	real, positive constant	any value		Langer's
Badarau	1940	real constant	half odd integer	real, positive $\rightarrow \infty$		Saddlepoint
Lauwerier	1950	real, positive $\rightarrow \infty$	zero	real, positive $\rightarrow \infty$		Saddlepoint

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